THE CALCULATION OF TRANSIENTS IN DYNAMICAL SYSTEMS

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Communicated by R. A. Rankin

Received 15 April 1952; *and in revised form* 19 March 1953

ABSTRACT. The paper shows that the calculation of transients by Tricomi's method using Laguerre functions is a practical alternative to the use of partial fractions. By taking numerical examples of Laplace transforms ranging from quadratic to sixth-power polynomials it shows that the approximation offered by this method would be acceptable for many practical uses. It analyses the composition of the coefficients of the Laguerre series, thus disclosing the conditions for quick convergence; for some transforms these conditions are not attainable.

1. *Introduction.* The problem of predicting the motion of some element of a dynamical system during the time interval following a sudden disturbance is one which arises in many branches of applied science and engineering. It is commonly solved by building up a Laplace transform of the motion and by inverting this in partial fractions. Thus most of the problems which arise in practice lead to subsidiary problems in finding the poles of a rational function in which the denominator, in current technique, may be of the sixth to the twelfth power.

The present paper deals with the application of Tricomi's method by which a Laplace transform may be inverted without finding its poles. The method will be described with particular reference to the motions produced by a step-function disturbance, and it will be assumed that single or multiple poles at the origin, representing constant values of displacement, velocity, etc., have been removed before the inversion of the transform is attempted.

It will appear below that there are advantages in inverting such a transform in orthogonal functions. If the functions $\phi_1, \phi_2, \ldots$ form a closed, normalized, orthogonal set over the interval 0 to $+\infty$, and if for some function $F(t)$ the coefficients a_n $(n = 0, 1, 2, \ldots)$ are given by the integral

$$a_n = \int_0^\infty \phi_n F(t)\, dt,$$

supposed convergent, then under suitable conditions the sum

$$a_0\phi_0 + a_1\phi_1 + \ldots + a_n\phi_n + \ldots \tag{1}$$

converges to the sum $F(t)$. In practice, such a method shows no saving of labour over the use of exponential functions if the series (1) is slow to converge; its interest is, therefore, that it offers an approximate solution which is based on the first few terms of an infinite series. It may be shown ((2), p. 44) that where inversion is in a series of orthogonal functions and a limited number of terms only are taken, the mean square error of the approximation is at a minimum when such coefficients as are used are given the values of those in the infinite series.

https://doi.org/10.1017/S0305004100029078 Published online by Cambridge University Press

2. *Method due to F. Tricomi.* Of the several orthogonal functions which are available the Laguerre polynomial ((2), pp. 79, 284, 324) is the most attractive because its Laplace transform is simple. Its use for the inversion of transforms in general was first described by F. Tricomi in the year 1935 (7). He used the definition

$$L_n(t) = \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{(-t)^\nu}{\nu!},$$

so that the polynomials of the lowest order are

$$L_0(t) = 1,$$
$$L_1(t) = (1-t)/1!,$$
$$L_2(t) = (t^2-4t+2)/2!,$$
$$L_3(t) = (-t^3+9t^2-18t+6)/3!,$$
$$L_4(t) = (t^4-16t^3+72t^2-96t+24)/4!.$$

The Laplace transform (3) of this function is

$$\mathscr{L} L_n(t) = \frac{1}{s}\left(\frac{s-1}{s}\right)^n,$$

and therefore

$$\mathscr{L} e^{-ht} L_n(t) = \frac{1}{s+h}\left(\frac{s+h-1}{s+h}\right)^n,$$

where h is a real constant. If then the transform which is to be inverted be $F_1(s)$ and be written in the form

$$F_1(s) = \frac{1}{s+h} \sum_{n=0}^{\infty} a_n \left(\frac{s+h-1}{s+h}\right)^n,$$

the appropriate inversion is

$$F(t) = e^{-ht} \sum_{n=0}^{\infty} a_n L_n(t).$$

The coefficients a_n are therefore given by the series

$$(s+h)\, F_1(s) = \sum_{n=0}^{\infty} a_n \sigma^n = F_2(\sigma),$$

where

$$\sigma = \frac{s+h-1}{s+h}. \tag{2}$$

This equation transforms any circle $|\sigma| = r$ in the σ plane into a circle in the s plane which may be made to enclose all the singularities of $(s+h)\, F_1(s)$ by taking suitable values of r and h; for stable dynamical systems this gives $r > 1$. The function $(s+h)\, F_1(s)$ may thus be expanded in powers of σ with a radius of convergence $r > 1$.

3. *Application of the method.* The application is governed by the properties of the transformation (2). Three cases have received attention:

(*a*) $h = 0$; $\sigma = (s-1)/s$. This transforms the half-plane $\Re(s) < 0$ into the half-plane $\Re(\sigma) > 1$, and $F_2(\sigma)$ converges in the remainder of the complex plane. The inversion in Laguerre functions was given by M. Picone (6) in a comment on Tricomi's paper; the inversion is

$$F(t) = \Sigma a_n L_n(t).$$

(*b*) $h = \frac{1}{2}$; $\sigma = (s-\frac{1}{2})/(s+\frac{1}{2})$. This transforms the half-plane $\Re(s) < 0$ into the area outside the unit circle $|\sigma| = 1$ and the other half-plane $\Re(s) > 0$ into the area inside

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the same circle; it is the symmetrical case. $F_2(\sigma)$ converges within the unit circle. The Laplace transform was listed in 1940 by K. W. Wagner ((8), transform 63), but its application in a series is due to Aigrain and Williams (1). The inversion is

$$F(t) = e^{-\frac{1}{2}t}\Sigma a_n L_n(t).$$

(*c*) $h = 1$; $\sigma = s/(s+1)$. This transforms the left half-plane of s into a circle in the right half-plane of σ. The Laplace transform was listed by Wagner ((8), transform 61), but it cannot be used for a series expansion about $\sigma = 0$, since the circle of convergence has its centre at $\sigma = +\frac{1}{2}$ and its radius is $\frac{1}{2}$.

The above survey endorses the judgement of Aigrain and Williams who chose the value $h = \frac{1}{2}$; the product $e^{-\frac{1}{2}t}L_n(t)$ is a Laguerre function and these authors used the definition

$$B_n(t) = e^{-t}L_n(2t) = \mathscr{L}^{-1}\frac{1}{s+1}\left(\frac{s-1}{s+1}\right)^n. \tag{3}$$

It will appear below that these functions are convenient for numerical calculation; they have forms similar to the motions of stable dynamical systems in that they resemble damped sine waves. A short table of these functions is given by Aigrain and Williams, computed from the equation

$$B_n(t) = e^{-t}\sum_{\nu=0}^{n}\binom{n}{\nu}\frac{(-2t)^\nu}{\nu!}, \tag{4}$$

and a more extensive table is given by Jahnke and Emde (4).

However, the definition (3) used by these authors is not as flexible as Wagner's form

$$\mathscr{L}B_n(bt) = \frac{1}{s+b}\left(\frac{s-b}{s+b}\right)^n, \tag{5}$$

in which the real constant b will be shown to have a powerful effect on the convergence of the series. In this form

$$\sigma = \frac{s-b}{s+b},$$

therefore

$$s = b\left(\frac{1+\sigma}{1-\sigma}\right) \tag{6}$$

and

$$\frac{1}{s+b} = \frac{1-\sigma}{2b}. \tag{7}$$

The procedure for inverting a transform is then as follows:

(*a*) Given the Laplace transform $F_1(s)$ of the transient, substitute $s = b\left(\frac{1+\sigma}{1-\sigma}\right)$.

(*b*) Manipulate the resulting function of σ to give the form

$$F_2(\sigma) = \left(\frac{1-\sigma}{2b}\right)(a_0 + a_1\sigma + a_2\sigma^2 + \ldots).$$

(*c*) Then the time function of the transient is

$$F(t) = a_0B_0(bt) + a_1B_1(bt) + a_2B_2(bt) + \ldots.$$

4. *Numerical examples.* The following simple examples show how the method may be used; the constant b will be given the value unity in the first two cases:

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(*a*) Let the transient have the Laplace transform

$$F_1(s) = \frac{1}{(s+1)(s+2)}. \tag{8}$$

Writing $s = (1+\sigma)/(1-\sigma)$,

$$F_2(\sigma) = \frac{1}{\left(\frac{1+\sigma}{1-\sigma}\right)^2 + 3\left(\frac{1+\sigma}{1-\sigma}\right) + 2}$$
$$= \frac{1}{3}\left(\frac{1-\sigma}{2}\right)\left(1 - \frac{2\sigma}{3} - \frac{2\sigma^2}{9} - \ldots\right).$$

Hence $F(t) = \frac{1}{3}B_0(t) - \frac{2}{3}(\frac{1}{3}B_1(t) + \frac{1}{9}B_2(t) + \frac{1}{27}B_3(t) + \ldots)$.

By the usual method of partial fractions

$$F(t) = e^{-t} - e^{-2t}.$$

Values obtained from the two functions are given together in Table 1.

Table 1. *Transient represented by* $F_1(s) = \dfrac{1}{(s+1)(s+2)}$

t	True values	Laguerre series (seven terms)	Difference $10^{-5}\times$
0·0	+0·00000	+0·00112	+112
0·05	+0·04639	+0·04655	− 16
0·1	+0·08611	+0·08608	+ 3
0·2	+0·14841	+0·14824	+ 17
0·3	+0·19201	+0·19185	+ 16
0·5	+0·23865	+0·23866	− 1
0·6	+0·24762	+0·24770	− 8
0·75	+0·24924	+0·24936	− 12
0·85	+0·24473	+0·24486	− 13
1·0	+0·23254	+0·23263	− 9
1·5	+0·17334	+0·17326	+ 8
2·0	+0·11702	+0·11695	+ 7
2·5	+0·07535	+0·07538	− 3
3·0	+0·04731	+0·04739	− 8
4·0	+0·01798	+0·01797	+ 1

(*b*) Let the transform of the transient be

$$F_1(s) = \frac{s+0{\cdot}8}{s^2+0{\cdot}8s+1}, \tag{9}$$

which might represent the motion of a simple system having inertia, compliance and damping:

$$F_2(\sigma) = 2\left(\frac{1-\sigma}{2}\right)\left[\frac{1{\cdot}8+0{\cdot}2\sigma}{2{\cdot}8+1{\cdot}2\sigma^2}\right]$$
$$= \left(\frac{1-\sigma}{2}\right)(1{\cdot}285714 + 0{\cdot}142857\sigma - 0{\cdot}551020\sigma^2 - 0{\cdot}061224\sigma^3$$
$$+ 0{\cdot}236152\sigma^4 + 0{\cdot}026239\sigma^5 - 0{\cdot}101208\sigma^6 \ldots), \tag{10}$$

whence $$F(t) = 1{\cdot}285714B_0(t) + 0{\cdot}142857B_1(t) - \ldots. \tag{11}$$

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Table 2 shows some values of this function for comparison with the solution by exponential functions.

Table 2. *Transient represented by* $F_1(s) = \dfrac{s+0{\cdot}8}{s^2+0{\cdot}8s+1}$

t	True values	Laguerre series (seven terms)	Difference $10^{-4}\times$
0·0	+1·0000	+0·9775	+225
0·5	+0·8923	+0·8905	+ 18
1·0	+0·6401	+0·6361	+ 40
1·5	+0·3418	+0·3487	− 69
2·0	+0·0729	+0·0762	− 33
2·5	−0·1221	−0·1276	+ 55
2·75	−0·1861	−0·1937	+ 76
3·0	−0·2281	−0·2354	+ 73
3·5	−0·2532	−0·2544	+ 12
4·0	−0·2189	−0·2128	− 61
4·5	−0·1517	−0·1428	− 89
5·0	−0·0761	−0·0703	− 58

(*c*) Let the Laplace transform be

$$F_1(s) = \frac{s+8}{s^2+8s+100}.$$

This transform, when treated in the same way, gives

$$F_2(\sigma) = \frac{18}{109}\left(\frac{1-\sigma}{2}\right)(1{\cdot}00+1{\cdot}04\sigma+1{\cdot}03\sigma^2+0{\cdot}99\sigma^3+0{\cdot}92\sigma^4+0{\cdot}82\sigma^5\ldots).$$

It will be seen that the convergence is so slow as to increase greatly the labour of calculation. The convergence may be improved by giving the constant b a value other than unity. Substituting $s = 10(1+\sigma)/(1-\sigma)$, then

$$F_2(\sigma) = \left(\frac{1-\sigma}{2{\cdot}10}\right)(1{\cdot}285714+0{\cdot}142857\sigma-0{\cdot}551020\sigma^2-0{\cdot}061224\sigma^3+\ldots),$$

which is the function of equation (10) to a modified scale. Its inversion is

$$F(t) = 1{\cdot}285714B_0(10t)+0{\cdot}142857B_1(10t)-0{\cdot}551020B_2(10t)-\ldots,$$

which correctly depicts the same function as equation (11) on a scale ten times enlarged.

This example makes it clear that rapid convergence of the coefficients a_n depends upon the choice of the scale constant b. It is necessary therefore to examine this matter.

5. *Composition of the coefficients.* If we write

$$F_1(s) = \sum_{m=1}^{p}\left(\frac{k_m}{s+c_m}\right), \tag{12}$$

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and for the mth pole we make the substitution (6), then

$$\frac{k_m}{s+c_m} = \left[\frac{k_m}{b\left(\frac{1+\sigma}{1-\sigma}\right)+c_m}\right]$$

$$= \frac{2bk_m}{b+c_m}\left(\frac{1-\sigma}{2b}\right)\left[\frac{1}{1+\sigma\left(\frac{b-c_m}{b+c_m}\right)}\right]$$

$$= \frac{2bk_m}{c_m+b}\left(\frac{1-\sigma}{2b}\right)\left[1+\sum_{n=0}^{\infty}\left(\frac{c_m-b}{c_m+b}\right)^n\sigma^n\right].$$

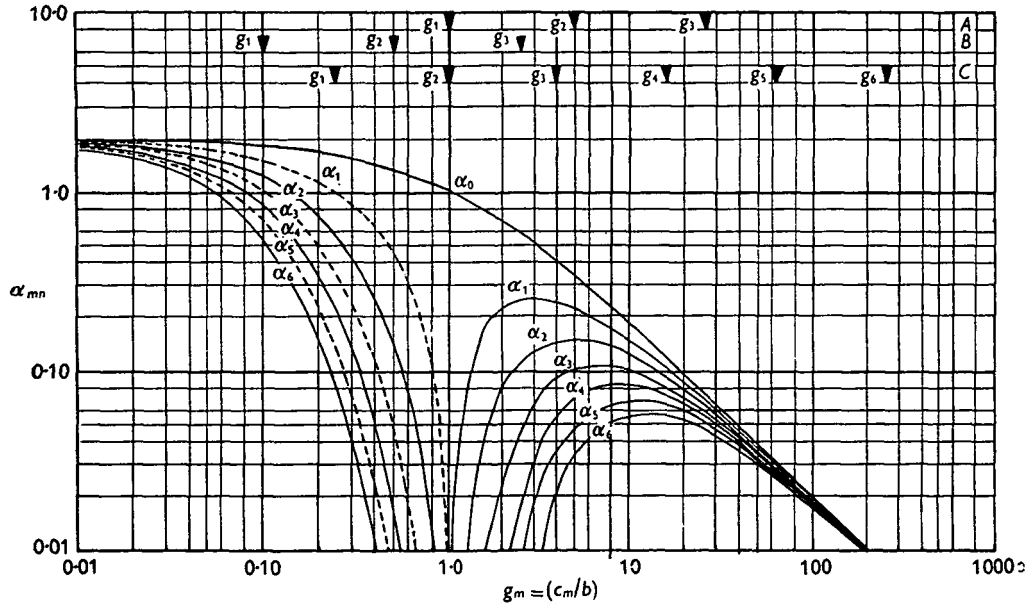


Fig. 1. Component α_{mn} (broken lines represent negative values).

This relation shows that each coefficient a_n in the power series in σ contains a contribution $k_m\alpha_{mn}$ from each of the partial fractions $\left(\frac{k_m}{s+c_m}\right)$. Thus

$$a_n = \sum_{m=1}^{p} k_m\alpha_{mn}$$

and if $g_m = c_m/b$, then

$$\alpha_{mn} = \frac{2b}{c_m+b}\left[\frac{c_m-b}{c_m+b}\right]^n$$

$$= \frac{2}{g_m+1}\left[\frac{g_m-1}{g_m+1}\right]^n. \qquad (13)$$

The transform $F_1(s)$ may contain both real and complex poles.

5·1. *Poles of $F_1(s)$ wholly real.* In the transient function $F(t) = \Sigma a_n B_n(bt)$ the coefficients are

$$a_n = \sum_{m=1}^{p} k_m\alpha_{mn} = \sum_{m=1}^{p}\frac{2k_m}{g_m+1}\left[\frac{g_m-1}{g_m+1}\right]^n.$$

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The component α_{mn} is shown in Fig. 1 plotted in double logarithmic coordinates as a function of g_m.

Good agreement has been obtained between the values a_n given by the long division of two polynomials in σ and those calculated from equation (13) or read off from a graph.

The double-logarithmic coordinates have two practical advantages; one is that the components $k_m\alpha_{mn}$ contributed by any one pole may be read off directly from a logarithmic scale laid along the g_m ordinate; their other advantage is that the effect of the scale factor upon the convergence may be explored by using a grid of parallel lines spaced apart by distances $\log(c_{m+1}/c_m)$ which may be moved across the chart parallel to the g_m axis.

This shows at once the mechanism which is at work as b is varied. The neighbourhood of $g_m = 1$ affords the quickest convergence, and the aim in choosing b is to place the pattern of roots in such a way that those having the largest values of k_m are balanced astride the line $g_m = 1$. It is clear that the roots remote from $g_m = 1$ will contribute components α_{mn} which will converge slowly. Two practical examples, representing transients in servo-control systems, have been explored by this method.

(*d*) Let the transform of the transient be

$$F_1(s) = \frac{s^2}{s^3+31s^2+155s+125}$$

$$= \frac{0{\cdot}010417}{s+1} - \frac{0{\cdot}312500}{s+5} + \frac{1{\cdot}302083}{s+25}. \tag{14}$$

As given by the usual method the transient is

$$F(t) = +0{\cdot}010417e^{-t} - 0{\cdot}312500e^{-5t} + 1{\cdot}302083e^{-25t}.$$

Using Laguerre functions and taking $b = 1$ the values of g_m will be $g_1 = 1$, $g_2 = 5$, $g_3 = 25$; they are shown at A in Fig. 1. It is clear that the powerful root at $g_3 = 25$ will prevent rapid convergence. The coefficients are:

$$a_0 = 0{\cdot}00641, \quad a_1 = 0{\cdot}02301, \quad a_2 = 0{\cdot}03905, \quad a_3 = 0{\cdot}04791, \quad \text{etc.}$$

The value $b = 10$ promises better; the positions of the roots are shown at B in Fig. 1 and the resulting coefficients are:

$$a_0 = 0{\cdot}346320, \quad a_1 = 0{\cdot}442271, \quad a_2 = 0{\cdot}103044, \quad a_3 = 0{\cdot}063628,$$

$$a_4 = 0{\cdot}028444, \quad a_5 = 0{\cdot}005528, \quad a_6 = 0{\cdot}009720.$$

Table 3 shows the exponential solution together with points calculated from the first seven terms of the Laguerre series.

(*e*) Let the transform of the transient be

$$F_1(s) = \frac{s^5+42{\cdot}65620ds^4+363{\cdot}64450d^2s^3+734{\cdot}947265d^3s^2}{(s+1)(s+4)(s+16)(s+64)(s+256)(s+1024)}, \tag{15}$$

where $d = 32$. Then

$$F_1(s) = \left(\frac{0{\cdot}032062}{s+1}\right) - \left(\frac{0{\cdot}651723}{s+4}\right) + \left(\frac{2{\cdot}262167}{s+16}\right) - \left(\frac{0{\cdot}683401}{s+64}\right) + \left(\frac{0{\cdot}041350}{s+256}\right) - \left(\frac{0{\cdot}000455}{s+1024}\right).$$

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In exponential functions the transient is given by

$$F(t) = 0{\cdot}032062e^{-t} - 0{\cdot}651723e^{-4t} + 2{\cdot}262167e^{-16t} - 0{\cdot}683401e^{-64t} + 0{\cdot}041350e^{-256t} - 0{\cdot}000455e^{-1024t}.$$

By Laguerre functions, choosing by inspection the value $b = 4{\cdot}0$ which places the roots as shown at C in Fig. 1, the transient is given by the series

$$F(t) = 0{\cdot}225312B_0 + 0{\cdot}442429B_1 + 0{\cdot}282816B_2 + 0{\cdot}130294B_3 + 0{\cdot}076305B_4 + 0{\cdot}024458B_5 + 0{\cdot}007721B_6 - \ldots.$$

Values calculated from these terms are given in Table 4 together with the exponential solution.

Table 3. *Transient represented by* $F_1(s) = \dfrac{s^2}{(s+1)(s+5)(s+25)}$

t	True values	Laguerre series (seven terms)	Difference $10^{-5}\times$
0·0	+1·00000	+0·99896	+104
0·01	+0·72712	+0·72741	− 29
0·02	+0·51720	+0·51766	− 46
0·03	+0·35620	+0·35638	− 18
0·04	+0·23316	+0·23301	+ 15
0·05	+0·13959	+0·13921	+ 38
0·075	−0·00543	−0·00575	+ 32
0·1	−0·07323	−0·07309	− 14
0·125	−0·10087	−0·10043	− 44
0·15	−0·10803	−0·10764	− 39
0·2	−0·09766	−0·09788	+ 22
0·25	−0·07891	−0·07938	+ 47
0·3	−0·06129	−0·06142	+ 13

Table 4. *Transient represented by equation* (15)

t	True values	Laguerre series (seven terms)	Difference $10^{-3}\times$
0·0	+1·000	+1·189	−189
0·025	+0·820	+0·763	+ 57
0·05	+0·486	+0·451	+ 35
0·075	+0·223	+0·227	− 4
0·1	+0·048	+0·069	− 21
0·15	−0·125	−0·109	− 16
0·2	−0·174	−0·174	0
0·25	−0·173	−0·182	+ 9
0·3	−0·154	−0·163	+ 9
0·4	−0·106	−0·108	+ 2
0·5	−0·068	−0·064	− 4
0·625	−0·036	−0·033	− 3
0·75	−0·017	−0·017	0

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5·2. *Complex poles of* $F_1(s)$. If we let c_m and $\bar{c}_m$ be a pair of conjugate poles, making g_m and $\bar{g}_m$ also conjugate, their contributions to the coefficient a_n will have a sum which will be wholly real and equal to

$$\alpha_{mn} = 2 \times \Re\left\{\frac{2}{g_m+1}\left(\frac{g_m-1}{g_m+1}\right)^n\right\}.$$

We write $$\left(\frac{2}{g_m+1}\right) = K_1 e^{j\psi_1}$$

and $$\left(\frac{g_m-1}{g_m+1}\right) = K_2 e^{j\psi_2}.$$

Then $$\alpha_{mn} = 2\,|\,K_1\,|\,|\,K_2\,|^n \cos(\psi_1 + n\psi_2).$$

This component of a_n converges at a rate largely controlled by $|\,K_2\,|$. If $|\,K_2\,|$ be taken as a measure of the rate of convergence, then b should be chosen to make $|\,g_m\,|$ approach unity. If the Laplace transform of two conjugate roots be written as

$$F_1(s) = \frac{1}{s^2 + 2xs + \omega_1^2},$$

then b should approximate to ω_1.

6. *Choice of the value of* b. The advantage of Tricomi's method is that it can be used without finding the poles of the transform; the coefficients of the denominator of the transform are therefore the only information available to guide the choice of b; in view of the mechanism shown in Fig. 1 it is not easy to formulate any relation between these coefficients and the best value of b. The arithmetic mean of the roots, $\frac{1}{p}\Sigma c_m$, is readily found, but it can differ greatly from the desired value, as Fig. 1 shows. A considerable body of work has been done on the question of the bounds of $|\,c_m\,|$ as defined by the coefficients (5), but this information also is not directly relevant.

The best choice of b may be regarded as the centroid in logarithmic units of the positions of the roots c_m, each being weighted according to its associated numerator k_m. That is

$$\log b = \frac{\sum_{m=1}^{p} |\,k_m\,| \log c_m}{\sum_{m=1}^{p} |\,k_m\,|}, \tag{16}$$

but the coefficients of the denominator throw little light on these quantities.

It is clear from equation (13) that no positive real value of b can result in coefficients which are not valid. The best course in practice seems to be therefore to use the geometric mean of the roots and to improve this estimate if possible; this mean is available as the pth root of the constant term of the denominator. It differs from the definition of equation (16) only in the fact that the roots are not weighted according to their associated values of k_m. Should it result in slowly converging terms of like sign it may be deduced that a powerful root has fallen in the region $g_m > 5$ (see Fig. 1) and b should be made larger; conversely, slow convergence and alternating sign are caused by roots in the region $g_m < 0{\cdot}3$ and b should be made smaller.

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Fig. 1 also shows that if a transform has two roots spaced widely apart (e.g. in the ratio 200 : 1 or greater) and both have large numerators k_m, then no value of b can be found which will give rapid convergence.

7. *Number of terms needed.* The error given by taking a finite number of terms of the Laguerre series has been investigated for the basic case of a single real pole for which the sum to infinity is the exponential e^{-ct}. Taking any partial fraction

$$F_1(s) = (s+c)^{-1}$$

and writing

$$\alpha_n = \frac{2}{g+1}\left(\frac{g-1}{g+1}\right)^n = \frac{K_1}{\rho^n},$$

then the associated time function computed to the qth term is

$$F(t) = K_1 \sum_{n=0}^{q} \frac{B_n}{\rho^n}.$$

Values of this function and of the sum to infinity are given in Table 5 to four terms for $g = 1{\cdot}5$ and in Table 6 to eleven terms for $g = 4$. It is clear that to keep the error within a given limit the number of terms needed rises sharply as the ratio g departs from unity.

Table 5. *Inversion of* $F_1(s) = (s+c)^{-1}$

ct	True values	Laguerre series, $g = 1{\cdot}5$ (four terms)	Difference $10^{-5} \times$
0·0	1·00000	0·99840	+ 160
0·75	0·47237	0·47293	− 56
1·5	0·22313	0·22289	+ 24
2·25	0·10540	0·10496	+ 44
3·0	0·04979	0·04966	+ 13
3·75	0·02352	0·02373	− 21
4·5	0·01111	0·01147	− 36
5·25	0·00525	0·00557	− 32
6·0	0·00248	0·00266	− 18
6·75	0·00117	0·00117	0
7·5	0·00055	0·00040	+ 15

Table 6. *Inversion of* $F_1(s) = (s+c)^{-1}$

ct	True values	Laguerre series, $g = 4$ (eleven terms)	Difference $10^{-5} \times$
0·0	1·00000	0·99637	+ 363
1·0	0·36788	0·36842	− 54
2·0	0·13534	0·13437	+ 97
3·0	0·04979	0·04982	− 3
4·0	0·01832	0·01904	− 72
5·0	0·00674	0·00715	− 41
6·0	0·00248	0·00225	+ 23
7·0	0·00091	+ 0·00036	+ 55
8·0	+ 0·00034	− 0·00012	+ 46

https://doi.org/10.1017/S0305004100029078 Published online by Cambridge University Press

The error in the final transient is a function of the number of poles, their respective ratios g_m and their associated numerators k_m. The data of these tables suggest that, in a technical application where the applied disturbance is a unit function and the transform has from six to twelve poles, the error might in a typical case be of the order of 0·05 if ten terms of the series were used.

The author is anxious to record his gratitude for a number of discussions on the subject with Prof. G. J. Kynch and with Prof. R. A. Rankin, who drew attention to the contributions of the partial fractions to the coefficients, and also for the kind help of Dr D. C. Martin, of the Royal Society, who furnished tables of the Laguerre functions.

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https://doi.org/10.1017/S0305004100029078 Published online by Cambridge University Press