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Relativization of sensitivity in minimal systems

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Abstract. Let $\pi : (X, T) \to (Y, S)$ be an extension between minimal systems; we consider its relative sensitivity. We obtain that π is relatively n-sensitive if and only if the relative n-regionally proximal relation contains a point whose coordinates are distinct; and the structure of π which is relatively n-sensitive but not relatively $(n+1)$-sensitive is determined. Let $\mathcal{F}_t$ be the families consisting of thick sets. We introduce notions of relative block $\mathcal{F}_t$-sensitivity and relatively strong $\mathcal{F}_t$-sensitivity. Let $\pi : (X, T) \to (Y, S)$ be an extension between minimal systems. Then the following Auslander–Yorke type dichotomy theorems are obtained: (1) π is either relatively block $\mathcal{F}_t$-sensitive or $\phi : (X, T) \to (X^{\pi}_{\mathrm{eq}}, T_{\mathrm{eq}})$ is a proximal extension where $(X^{\pi}_{\mathrm{eq}}, T_{\mathrm{eq}}) \to (Y, S)$ is the maximal equicontinuous factor of π. (2) π is either relatively strongly $\mathcal{F}_t$-sensitive or $\phi : (X, T) \to (X^{\pi}_{d}, T_{d})$ is a proximal extension where $(X^{\pi}_{d}, T_{d}) \to (Y, S)$ is the maximal distal factor of π.

Key words: relative sensitivity, minimality, equicontinuous factor, distal factor
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1. *Introduction*

Throughout this paper (X, T) denotes a *topological dynamical system* (t.d.s.), where X is a compact metric space and $T : X \to X$ is a homeomorphism. In this section, we first discuss the motivations of our research and then state the main results of the article.

The notion of sensitivity (sensitive dependence on initial conditions) was first used by Ruelle [**17**]. It is in the kernel of the definition of Devaney's chaos. According to Auslander and Yorke [**3**] a t.d.s. (X, T) is called *sensitive* if there exists $\delta > 0$ such that for every opene (open and non-empty) subset U, there exist $x_1, x_2 \in U$ and $m \in \mathbb{N}$ with $d(T^m x_1, T^m x_2) > \delta$. Auslander and Yorke [**3**] proved the following dichotomy theorem:

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a minimal system is either equicontinuous or sensitive. A similar result obtained by Glasner and Weiss [**7**] states that a transitive system is either almost equicontinuous or sensitive.

Xiong [**20**] introduced a new notion called n-sensitivity, Shao, Ye and Zhang [**18**, **22**] studied the properties of n-sensitivity, particularly for minimal systems. Zhang [**24**] introduced the notion of relative sensitivity and proved that $\pi : (X, T) \to (Y, S)$ an extension between minimal systems is either equicontinuous or relatively sensitive. In this paper, we will first introduce the notion of relative n-sensitivity for an extension $\pi : (X, T) \to (Y, S)$ and study the properties of relative n-sensitivity. It is proved that a minimal system is relatively n-sensitive if and only if the relative n-regionally proximal relation contains a point whose coordinates are distinct. Moreover, the structure of π which is relatively n-sensitive but not relatively $(n+1)$-sensitive is determined.

The notion of sensitivity was generalized by measuring the set of integers for which the sensitivity occurs [**6**, **8**, **9**, **11**–**13**, **16**, **21**, **23**, **25**]. For a subset A of $\mathbb{Z}$, we say that A is *thick* if for every $d \in \mathbb{N}$ there is $n_d \in \mathbb{Z}$ such that $\{n_d, n_d + 1, \ldots, n_d + d\} \subset A$, and denote the family of thick sets by $\mathcal{F}_t$. Ye and Yu [**21**] introduced notions called block $\mathcal{F}_t$-sensitivity and strong $\mathcal{F}_t$-sensitivity and proved that: (1) a minimal system is either block $\mathcal{F}_t$-sensitive or a proximal extension of its maximal equicontinuous factor; (2) a minimal system is either strongly $\mathcal{F}_t$-sensitive or a proximal extension of its maximal distal factor.

Inspired by previous work, particularly by Ye and Yu [**21**], we define and study some notions of sensitivity related to an extension $\pi : (X, T) \to (Y, S)$. We say that π is *relatively block $\mathcal{F}_t$-sensitive* if there is $\delta > 0$ such that for any opene set $U \subset X$ and $l \in \mathbb{N}$ there are $u_l, v_l \in U$ with $\pi(u_l) = \pi(v_l)$ and $\{n \in \mathbb{Z} : d(T^n u_l, T^n v_l) > \delta\}$ containing $\{m+1, \ldots, m+l\}$ for some $m \in \mathbb{Z}$. We say that π is *relatively strongly $\mathcal{F}_t$-sensitive* if there is $\delta > 0$ such that for any opene set $U \subset X$ there are $u, v \in U$ with $\pi(u) = \pi(v)$ and $\{n \in \mathbb{Z} : d(T^n u, T^n v) > \delta\}$ a thick set.

We have the following main results.

THEOREM A. *Let $\pi : (X, T) \to (Y, S)$ be an extension between minimal systems with $R_\pi^n = \{(x_1, \ldots, x_n) \in X^n : \pi(x_i) = \pi(x_j),\ \forall\ 1 \le i, j \le n\}$ having dense minimal points for any $n \in \mathbb{N}$. Let $\psi : (X_\pi^{\mathrm{eq}}, T_{\mathrm{eq}}) \to (Y, S)$ be the maximal equicontinuous factor of π and $\phi : (X, T) \to (X_\pi^{\mathrm{eq}}, T_{\mathrm{eq}})$ with $\psi \circ \phi = \pi$. If π is relatively n-sensitive but not relatively $(n+1)$-sensitive for some $n \ge 2$, then ϕ is a finite-to-one extension with $n \le \max_{z \in X_\pi^{\mathrm{eq}}} \mathrm{Card}(\phi^{-1}(z)) \le n^2$.*

THEOREM B. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between minimal systems. Let $\psi : (X_\pi^{\mathrm{eq}}, T_{\mathrm{eq}}) \to (Y, S)$ be the maximal equicontinuous factor of π and $\phi : (X, T) \to (X_\pi^{\mathrm{eq}}, T_{\mathrm{eq}})$ with $\psi \circ \phi = \pi$. Then the following statements are equivalent.*

(1) *π is relatively block $\mathcal{F}_t$-sensitive.*
(2) *$\phi : (X, T) \to (X_\pi^{\mathrm{eq}}, T_{\mathrm{eq}})$ is not proximal.*

THEOREM C. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between minimal systems. Let $\psi : (X_\pi^d, T_d) \to (Y, S)$ be the maximal distal factor of π and $\phi : (X, T) \to (X_\pi^d, T_d)$ with $\psi \circ \phi = \pi$. Then the following statements are equivalent.*

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(1) *π is relatively strongly $\mathcal{F}_t$-sensitive.*
(2) *$\phi : (X, T) \to (X_\pi^d, T_d)$ is not proximal.*

This paper is organized as follows. In §2 we recall some definitions and some related theorems. In §3 we discuss relative n-sensitivity and prove Theorem A. In §4 we study relative block $\mathcal{F}_t$-sensitivity and prove Theorem B. In §5 we investigate relative strong $\mathcal{F}_t$-sensitivity and prove Theorem C.

2. *Preliminaries*

2.1. *Topological dynamical systems.* In this article, the sets of integers, non-negative integers and natural numbers are denoted by $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$, respectively. By a topological dynamical system we mean a pair (X, T) where X is a compact metric space with a metric d and $T : X \to X$ is a homeomorphism. A non-vacuous closed invariant subset $Y \subseteq X$ defines naturally a subsystem (Y, T) of (X, T). A system (X, T) is called *minimal* if it contains no proper subsystem. Each point belonging to some minimal subsystem of (X, T) is called a *minimal point*. The *orbit* of a point $x \in X$ is the set $\mathrm{Orb}(x, T) = \{T^n x : n \in \mathbb{Z}\}$.

For $x \in X$ and subsets $U, V \subset X$, put

$$N(x, U) = \{n \in \mathbb{Z} : T^n x \in U\} \quad \text{and} \quad N(U, V) = \{n \in \mathbb{Z} : U \cap T^{-n} V \neq \emptyset\}.$$

Recall that a dynamical system (X, T) is called *topologically transitive* (or just *transitive*) if for every two opene subsets U, V of X the set $N(U, V)$ is infinite. Any point with dense orbit is called a *transitive point*. Denote the set of all transitive points by $\mathrm{Trans}(X, T)$. It is well known that for a transitive system, $\mathrm{Trans}(X, T)$ is a dense G_δ subset of X.

Given a t.d.s. (X, T) and an integer $n \geq 2$, the nth product system is the t.d.s. $(X^n, T^{(n)})$, where X^n is the cartesian product of X with itself n times and $T^{(n)}$ represents the simultaneous action of T in each coordinate of X^n. The diagonal of X^n is denoted by $\Delta_n(X) = \{(x, \ldots, x) \in X^n : x \in X\}$. We also denote $\Delta(X) = \Delta_2(X)$ for $n = 2$.

2.2. *Proximal, distal and equicontinuous extensions.* Let (X, T) and (Y, S) be two dynamical systems. If there is a continuous surjection $\pi : X \to Y$ with $\pi \circ T = S \circ \pi$, then we say that π is a *factor map*, the system (Y, S) is a *factor* of (X, T) or (X, T) is an *extension* of (Y, S). Denote $R_\pi^n = \{(x_1, \ldots, x_n) \in X^n : \pi(x_i) = \pi(x_j), \forall\, 1 \leq i, j \leq n\}$ for $n \geq 2$. We also denote $R_\pi = R_\pi^2$ for $n = 2$.

Let $\pi : (X, T) \to (Y, S)$ be an extension. We say that $\psi : (Z, W) \to (Y, S)$ is a *factor of an extension* π if there exists a extension $\phi : (X, T) \to (Z, W)$ with $\psi \circ \phi = \pi$.

Let (X, T) be a dynamical system. A pair $(x_1, x_2) \in X \times X$ is said to be *proximal* if for any $\varepsilon > 0$, there exists $n \in \mathbb{Z}$ such that $d(T^n x_1, T^n x_2) < \varepsilon$. Let $P(X, T)$ denote the collection of all proximal pairs. Then $P(X, T)$ is a reflexive symmetric T-invariant relation, but it is in general not transitive or closed. If (x_1, x_2) is not proximal, it is said to be a *distal pair*. A system (X, T) is called *distal* if any pair of distinct points in (X, T) is a distal pair.

An extension $\pi : (X, T) \to (Y, S)$ is *proximal* if $R_\pi \subset P(X, T)$ and is *distal* if $R_\pi \cap P(X, T) = \Delta(X)$. An extension $\pi : (X, T) \to (Y, S)$ is *equicontinuous* if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any $(x_1, x_2) \in R_\pi$ with $d(x_1, x_2) < \delta$, we have $d(T^n x_1, T^n x_2) < \varepsilon$ for any $n \in \mathbb{Z}$.

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2.3. *The maximal distal (equicontinuous) factor of an extension.* Let $\pi : (X, T) \to (Y, S)$ be an extension between systems. A pair $(x_1, x_2) \in X \times X$ is said to be *relative proximal* of π if $(x_1, x_2) \in R_\pi$ and for any $\varepsilon > 0$, there exists $n \in \mathbb{Z}$ such that $d(T^n x_1, T^n x_2) < \varepsilon$. Let $P_\pi(X, T)$ denote the collection of all relative proximal pairs of π. Then $P_\pi(X, T)$ is a reflexive symmetric T-invariant relation, but is in general not transitive or closed. If $P_\pi(X, T)$ is closed, then $P_\pi(X, T)$ is already an equivalence relation.

A pair $(x_1, x_2) \in X \times X$ is said to be *relative regionally proximal of* π if $(x_1, x_2) \in R_\pi$ and for each $\varepsilon > 0$ and each open neighborhood U_i of x_i, $i = 1, 2$, there are $x_i' \in U_i$, $i = 1, 2$, with $(x_1', x_2') \in R_\pi$ and $n \in \mathbb{Z}$ with $d(T^n x_1', T^n x_2') < \varepsilon$. Let $Q_\pi(X, T)$ denote the collection of all relative regionally proximal pairs of π. Then $Q_\pi(X, T)$ is a reflexive symmetric T-invariant closed relation, but is in general not transitive.

Let $\pi : (X, T) \to (Y, S)$ be an extension between systems. Then there is a maximal distal factor $\psi : (X_\pi^d, T_d) \to (Y, S)$ of π and a maximal equicontinuous factor $\phi : (X_\pi^{\mathrm{eq}}, T_{\mathrm{eq}}) \to (Y, S)$ of π. That is, $\psi : (X_\pi^d, T_d) \to (Y, S)$ is distal extension and a factor of π; and every distal extension $\psi' : (Z, W) \to (Y, S)$ of π is a factor of $\psi : (X_\pi^d, T_d) \to (Y, S)$. $(X_\pi^{\mathrm{eq}}, T_{\mathrm{eq}})$ has the corresponding property for equicontinuous factors of π. Thus there are closed T-invariant equivalence S_π^d and S_π^{eq} such that $X/S_\pi^d = X_\pi^d$ and $X/S_\pi^{\mathrm{eq}} = X_\pi^{\mathrm{eq}}$. S_π^d is the smallest closed T-invariant equivalence relation containing $P_\pi(X, T)$, and S_π^{eq} is the smallest closed T-invariant equivalence relation containing $Q_\pi(X, T)$.

2.4. *Families.* Let $\mathcal{P} = \mathcal{P}(\mathbb{Z})$ be the collection of all subsets of $\mathbb{Z}$. A subset $\mathcal{F}$ of $\mathcal{P}$ is a *family* if it is hereditary upwards, that is, $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. A family $\mathcal{F}$ is *proper* if it is a proper subset of $\mathcal{P}$, that is, neither empty nor all of $\mathcal{P}$. It is easy to see that $\mathcal{F}$ is proper if and only if $\mathbb{Z} \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$. A family $\mathcal{F}$ has the *Ramsey property* if $F \in \mathcal{F}$ and $F = F_1 \cup F_2$ imply that $F_i \in \mathcal{F}$ for some $i \in \{1, 2\}$. Any subset $\mathcal{A}$ of $\mathcal{P}$ generates a family

$$[\mathcal{A}] = \{F \in \mathcal{P} : F \supset A \text{ for some } A \in \mathcal{A}\}.$$

If a proper family $\mathcal{F}$ is closed under intersection, then $\mathcal{F}$ is called a *filter*. For a family $\mathcal{F}$, the *dual family* is

$$\mathcal{F}^* = \{F \in \mathcal{P} : \mathbb{Z} \setminus F \notin \mathcal{F}\} = \{F \in \mathcal{P} : F \cap F' \neq \emptyset \text{ for all } F' \in \mathcal{F}\}.$$

$\mathcal{F}^*$ is a family, proper if $\mathcal{F}$ is. It is well known that a proper family has the Ramsey property if and only if its dual $\mathcal{F}^*$ is a filter [**5**]. Clearly,

$$(\mathcal{F}^*)^* = \mathcal{F} \quad \text{and} \quad \mathcal{F}_1 \subset \mathcal{F}_2 \Rightarrow \mathcal{F}_2^* \subset \mathcal{F}_1^*.$$

We say that a subset F of $\mathbb{Z}$ is:

(1) *thick* if it contains arbitrarily long blocks of consecutive integers, that is, for every $d \geq 1$ there is $n_d \in \mathbb{Z}$ such that $\{n_d, n_d + 1, \ldots, n_d + d\} \subset F$;

(2) *syndetic* if it has bounded gaps, that is, there is $N \in \mathbb{N}$ such that for every $k \in \mathbb{Z}$ we have $\{k, k + 1, \ldots, k + N\} \cap A \neq \emptyset$.

The collection of all syndetic (respectively, thick) subsets is denoted by $\mathcal{F}_s$ (respectively, $\mathcal{F}_t$). Note that $\mathcal{F}_s^* = \mathcal{F}_t$ and $\mathcal{F}_t^* = \mathcal{F}_s$.

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3. *Relative n-sensitivity*

In this section, we will study relative n-sensitivity and prove Theorem A.

Definition 3.1. Let $\pi : (X, T) \to (Y, S)$ be a factor map between systems. We say that:

(1) π is *relatively n-sensitive* if there exists $\delta > 0$ such that for any opene set U there exists $(x_1, \ldots, x_n) \in R_\pi^n \cap \prod_{l=1}^n U$ and $m \in \mathbb{Z}$ such that $d(T^m x_i, T^m x_j) > \delta$ for $1 \le i \ne j \le n$;

(2) $(x_1, \ldots, x_n) \in X^n$ is a *relative sensitive n-tuple* if $(x_1, \ldots, x_n) \in R_\pi^n \setminus \Delta_n(X)$, and for any open neighborhood U_i of x_i and any opene subset U of X there exist $k \in \mathbb{Z}$ and $(x_1', \ldots, x_n') \in R_\pi^n \cap \prod_{l=1}^n U$ with $T^k x_i' \in U_i$ for $i = 1, 2, \ldots, n$.

Denote by $S_\pi^n(X, T)$ the set of all relatively sensitive n-tuples. It is not hard to see that $S_\pi^n(X, T) \cup \Delta_n(X)$ is a closed $T^{(n)}$-invariant subset of R_π^n. We say that $(x_1, \ldots, x_n) \in S_\pi^n(X, T)$ is *essential* (denoted by $(x_1, \ldots, x_n) \in S_\pi^{n,e}(X, T)$) if $x_i \ne x_j$ for each $1 \le i < j \le n$.

Inspired by [**10**, Proposition 2.7] and [**22**, Theorem 3.7], we get the relation between relative n-sensitivity and relatively sensitive n-tuples.

PROPOSITION 3.2. *Let $\pi : (X, T) \to (Y, S)$ be a factor map, $n \ge 2$.*

(1) *If $S_\pi^{n,e}(X, T) \ne \emptyset$, then π is relatively n-sensitive.*

(2) *If (X, T) is transitive and π is relatively n-sensitive, then $S_\pi^{n,e}(X, T) \ne \emptyset$.*

Proof. (1). This follows directly from the definition.

(2) Since π is relatively n-sensitive, there is $\varepsilon > 0$ such that for any opene subset U of X we can find $(z_1, z_2, \ldots, z_n) \in R_\pi^n \cap \prod_{l=1}^n U$ and $m \in \mathbb{Z}$ with $d(T^m z_i, T^m z_j) > \varepsilon$ for each $1 \le i < j \le n$.

Assume to the contrary that $S_\pi^{n,e}(X, T) = \emptyset$. Let

$$R_\pi^{n,\varepsilon} = \Big\{(x_1, x_2, \ldots, x_n) \in R_\pi^n : \min_{1 \le i < j \le n} d(x_i, x_j) \ge \varepsilon\Big\}.$$

Clearly, $R_\pi^{n,\varepsilon}$ is closed in R_π^n. For any $\overline{x} = (x_1, x_2, \ldots, x_n) \in R_\pi^{n,\varepsilon}$, since $S_\pi^{n,e}(X, T) = \emptyset$, there exist an open neighborhood $U_i(\overline{x})$ of x_i and an opene subset $V(\overline{x})$ of X with

$$(T^{(n)})^m \Big(\prod_{i=1}^n V(\overline{x})\Big) \cap \Big(\prod_{i=1}^n U_i(\overline{x})\Big) \cap R_\pi^n(X, T) = \emptyset$$

for each $m \in \mathbb{Z}$.

Since $\{U_1(\overline{x}) \times U_2(\overline{x}) \times \cdots \times U_n(\overline{x}) : \overline{x} \in R_\pi^{n,\varepsilon}\}$ is an open cover of $R_\pi^{n,\varepsilon}$, there exist $\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_k \in R_\pi^{n,\varepsilon}$ such that

$$\bigcup_{i=1}^k U_1(\overline{y}_i) \times U_2(\overline{y}_i) \times \cdots \times U_n(\overline{y}_i) \supseteq R_\pi^{n,\varepsilon}.$$

Since (X, T) is transitive, there exist $m_1 < m_2 < \cdots < m_k$ such that $V := \bigcap_{i=1}^k T^{m_i} V(\overline{y}_i)$ is an opene subset of X. Then clearly for $m \in \mathbb{Z}$ and $1 \le i \le k$ one has

$$(T^{(n)})^m \Big(\prod_{j=1}^n V\Big) \cap \Big(\prod_{j=1}^n U_j(\overline{y}_i)\Big) \cap R_\pi^n \subseteq (T^{(n)})^{m+m_i} \Big(\prod_{j=1}^n V(\overline{y}_i)\Big) \cap \Big(\prod_{j=1}^n U_j(\overline{y}_i)\Big) \cap R_\pi^n = \emptyset.$$

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This implies that for each $m \in \mathbb{Z}$ one has

$$(T^{(n)})^m(V \times V \times \cdots \times V) \cap R_\pi^{n,\varepsilon} = \emptyset.$$

Hence we cannot find $(x_1, x_2, \ldots, x_n) \in R_\pi^n \cap \prod_{l=1}^n V$ and $m \in \mathbb{Z}$ such that $d(T^m x_i, T^m x_j) > \varepsilon$ for each $1 \le i < j \le n$, a contradiction. □

Definition 3.3. Let $\pi\colon (X, T) \to (Y, S)$ be a factor map between systems. We say that $(x_1, \ldots, x_n) \in R_\pi^n$ is a:

(1) *relative n-regionally proximal tuple* (denoted by $(x_1, \ldots, x_n) \in Q_\pi^n(X, T)$) if for any $\varepsilon > 0$ there are $(x_1', \ldots, x_n') \in R_\pi^n$ and $m \in \mathbb{Z}$ such that $d(x_i, x_i') < \varepsilon$ for $1 \le i \le n$ and $d(T^m x_i', T^m x_j') < \varepsilon$ for $1 \le i \ne j \le n$;

(2) *relative n-proximal tuple* (denoted by $(x_1, \ldots, x_n) \in P_\pi^n(X, T)$) if for any $\varepsilon > 0$ there is $m \in \mathbb{Z}$ such that $d(T^m x_i, T^m x_j) < \varepsilon$ for $1 \le i \ne j \le n$.

THEOREM 3.4. *Let $\pi : (X, T) \to (Y, S)$ be a factor map. Then $S_\pi^n(X, T) \subset Q_\pi^n(X, T) \setminus \Delta_n(X)$, and the converse is true when (X, T) is minimal.*

Proof. Suppose that $(x_1, \ldots, x_n) \in S_\pi^n(X, T)$, Assume that $x \in X$ and, for each $k \ge 1$, U_k is a neighborhood of x with $\operatorname{diam}(U_k) < 1/k$. Let V_i^k is a neighborhood of x_i with $\operatorname{diam}(V_i^k) < 1/k$. Then for each $k \ge 1$ there are $l_k \in \mathbb{Z}$ and $y_i^k \in U_k$ with $(y_1^k, \ldots, y_n^k) \in R_\pi^n$ and $T^{l_k} y_i^k \in V_i$ for $1 \le i \le n$. Thus $(x_1, \ldots, x_n) \in Q_\pi^n(X, T)$.

Now assume (X, T) is minimal. Suppose $(x_1, \ldots, x_n) \in Q_\pi^n(X, T) \setminus \Delta_n(X)$. For each $k \ge 1$ and each neighborhood U_i of x_i there are $y_i^k \in U_i$ with $(y_1^k, \ldots, y_n^k) \in R_\pi^n$ and $l_k \in \mathbb{Z}$ with

$$\max\{d(T^{l_k} y_i^k, T^{l_k} y_j^k) : 1 \le i < j \le n\} \le \frac{1}{k}.$$

Without loss of generality we assume that $\lim_{k\to\infty} T^{l_k}(y_i^k) = x$ for $i = 1, \ldots, n$. For each opene subset U of X, there is $m \in \mathbb{Z}$ such that $T^m x \in U$. Then for k large enough, we have $T^{l_k+m} y_i^k \in U$. Since $y_i^k \in U_i$ with $(T^{l_k+m} y_1^k, \ldots, T^{l_k+m} y_n^k) \in R_\pi^n$, we have $(x_1, \ldots, x_n) \in S_\pi^n(X, T)$. □

To get the structure of relative n-sensitivity, we will prove the relative case of [**14**, Theorem 3.9].

THEOREM 3.5. *Let $\pi : (X, T) \to (Y, S)$ be an extension between minimal systems with R_π^n having dense minimal points for any $n \in \mathbb{N}$. Then for any $x_1, \ldots x_n \in X$, if $(x_1, \ldots x_n)$ is minimal and $(x_i, x_{i+1}) \in Q_\pi^2(X, T)$ for $1 \le i \le n-1$, then $(x_1, \ldots, x_n) \in Q_\pi^n(X, T)$.*

Proof. If $n = 2$ the result is obvious. We assume that the result holds for n and we show that it holds for $n + 1$.

Let $(x_1, \ldots x_n, x_{n+1})$ be a minimal point in $R_\pi^{n+1}(X, T)$ with $(x_i, x_{i+1}) \in Q_\pi^2(X, T)$ for $1 \le i \le n$. Since the result holds for n, we know $(x_1, \ldots x_n) \in Q_\pi^n(X, T)$. Since $(x_1, \ldots x_n, x_{n+1})$ is minimal, there is an idempotent $u \in M$ such that $u(x_1, \ldots x_n, x_{n+1}) = (x_1, \ldots, x_n, x_{n+1})$ where M is the universal minimal system. We show that for any $\varepsilon > 0$ and open neighborhoods U_i of x_i for $1 \le i \le n+1$, there are $m \in \mathbb{Z}$ and $(\widetilde{x}_1, \ldots, \widetilde{x}_n, \widetilde{x}_{n+1}) \in \prod_{i=1}^{n+1} U_i \cap R_\pi^{n+1}(X, T)$ such that $\operatorname{diam}\{T^m \widetilde{x}_1, \ldots, T^m \widetilde{x}_{n+1}\} < 2\varepsilon$.

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Let $W = \{\xi \in M : \xi(x_1, \ldots, x_{n+1}) \in \prod_{i=1}^{n+1} U_i \cap R_\pi^{n+1}\}$, where M is the universal minimal system. Since $u \in W$, one obtains that W is a non-empty open set of M. Since the map $M \to X$, $p \mapsto px_n$ is semi-open, $V = \mathrm{int}(Wx_n) \neq \emptyset$.

Choose $x_n' \in V$. From the definition of V, there is $\xi_1 \in W$ such that $x_n' = \xi_1 x_n$. Put $x_i' = \xi_1 x_i$ for $1 \leq i \leq n-1$; then by induction $(x_1', \ldots, x_n') \in \xi_1 Q_\pi^n \subset Q_\pi^n$. Since the minimal points of R_π^n are dense, there exists a minimal point $(x_1'', \ldots, x_n'') \in \prod_{i=1}^{n-1} U_i \times V \cap R_\pi^n$ such that

$$N((x_1'', \ldots, x_n''), \Delta_\varepsilon^n(\pi)) \neq \emptyset,$$

where $\Delta_\varepsilon^n(\pi) = \{(z_1, \ldots, z_n) \in R_\pi^n : \mathrm{diam}\{z_1, \ldots, z_n\} < \varepsilon\}$ is an opene subset of R_π^n. Moreover, the minimality of $(x_1'', \ldots, x_n'')$ implies $N((x_1'', \ldots, x_n''), \Delta_\varepsilon^n(\pi))$ is syndetic. Let L be a gap of $N((x_1'', \ldots, x_n''), \Delta_\varepsilon^n(\pi))$. Choose $\delta > 0$ such that for any $x, y \in X$ with $d(x, y) < \delta$ one has $d(T^m x, T^m y) < \varepsilon$ for $0 \leq m \leq L$.

Since $(x_1'', \ldots, x_n'')$ is minimal, there is an idempotent $v \in M$ such that $v(x_1'', \ldots, x_n'') = (x_1'', \ldots, x_n'')$. Let $\widetilde{W} = \{\xi \in M : \xi(x_1'', \ldots, x_n'') \in \prod_{i=1}^{n-1} U_i \times V \cap R_\pi^n\}$. Since $v \in \widetilde{W}$ and $M \to X$, $p \mapsto px_n''$ is semi-open, then $\widetilde{W}$ is an opene set and $\widetilde{V} = \mathrm{int}(\widetilde{W}x_n'') \neq \emptyset$.

Choose $x_n''' \in \widetilde{V}$. Observe that $\widetilde{V} \subset \widetilde{W}x_n'' \subset V \subset Wx_n$, and there is $\xi_2 \in W$ such that $x_n''' = \xi_2 x_n$. Let $x_{n+1}''' = \xi_2 x_{n+1} \in U_{n+1}$. One deduces $(x_n''', x_{n+1}''') = \xi_2(x_n, x_{n+1}) \in \xi_2 Q_\pi (X, T) \subset Q_\pi(X, T)$. Hence there is a minimal point $(\widetilde{x}_n, \widetilde{x}_{n+1}) \in \widetilde{V} \times U_{n+1} \cap R_\pi^2$ such that

$$N((\widetilde{x}_n, \widetilde{x}_{n+1}), \Delta_\delta^2(\pi)) \neq \emptyset.$$

Since $(\widetilde{x}_n, \widetilde{x}_{n+1})$ is minimal, $N((\widetilde{x}_n, \widetilde{x}_{n+1}), \Delta_\delta^2(\pi))$ is syndetic and by the definition of δ, the set

$$\{m \in \mathbb{Z} : \{m, \ldots, m + L - 1\} \subset N((\widetilde{x}_n, \widetilde{x}_{n+1}), \Delta_\varepsilon^2(\pi))\}$$

is syndetic.

Since $\widetilde{x}_n \in \widetilde{V} \subset \widetilde{W}x_n''$, there is $\xi_3 \in \widetilde{W}$ such that $\widetilde{x}_n = \xi_3 x_n''$. Let $\widetilde{x}_i = \xi_3 x_i'' \in U_i$ for $1 \leq i \leq n-1$. Since $(x_1'', \ldots, x_n'')$ is a minimal point in R_π^n, it follows that $(\widetilde{x}_1, \ldots, \widetilde{x}_n)$ is minimal too. Since $N((x_1'', \ldots, x_n''), \Delta_\varepsilon^n(\pi)) \neq \emptyset$, we have

$$N((\widetilde{x}_1, \ldots, \widetilde{x}_n), \Delta_\varepsilon^n(\pi)) \neq \emptyset,$$

and it is syndetic with gap L. Since $\{m \in \mathbb{Z} : \{m, \ldots, m + L - 1\} \subset N((\widetilde{x}_n, \widetilde{x}_{n+1}), \Delta_\varepsilon^2(\pi))\}$ is syndetic, we know that

$$N((\widetilde{x}_1, \ldots, \widetilde{x}_n), \Delta_\varepsilon^n(\pi)) \cap N((\widetilde{x}_n, \widetilde{x}_{n+1}), \Delta_\varepsilon^2(\pi)) \neq \emptyset.$$

Thus there is $m \in \mathbb{Z}$ such that $\mathrm{diam}\{T^m \widetilde{x}_1, \ldots, T^m \widetilde{x}_{n+1}\} < 2\varepsilon$. Since $(\widetilde{x}_n, \widetilde{x}_{n+1}) \in R_\pi^2$ and $(\widetilde{x}_1, \ldots, \widetilde{x}_n) \in R_\pi^n$, we know that $(\widetilde{x}_1, \ldots, \widetilde{x}_n, \widetilde{x}_{n+1}) \in R_\pi^{n+1}$. Since $(\widetilde{x}_1, \ldots, \widetilde{x}_n, \widetilde{x}_{n+1}) \in \prod_{i=1}^{n+1} U_i$, we have $(x_1, \ldots, x_{n+1}) \in Q_\pi^{n+1}$. □

Proof of Theorem A. Since R_π has dense minimal points and (X, T) is minimal, by [**15**] or [**19**, p. 549], $Q_\pi(X, T)$ is an equivalence relation. So $(X_\pi^{\mathrm{eq}}, T_{\mathrm{eq}}) = (X/Q_\pi(X, T), T_{\mathrm{eq}})$.

Since π is relatively n-sensitive, by Theorem 3.4 and Proposition 3.2, there is a tuple $(z_1, \ldots, z_n) \in Q_\pi^n(X, T)$ with $z_i \neq z_j$ for $1 \leq i \neq j \leq n$, so we have $n \leq \max_{z \in X_\pi^{\mathrm{eq}}} \mathrm{Card}(\phi^{-1}(z))$. We next prove $\max_{z \in X_\pi^{\mathrm{eq}}} \mathrm{Card}(\phi^{-1}(z)) \leq n^2$.

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Let u be a minimal idempotent and $z \in X_\pi^{\text{eq}}$. If $\text{Card}(u\phi^{-1}(z)) > n$, then there are distinct $x_1, \ldots, x_{n+1} \in u\phi^{-1}(z)$. Since $(X_\pi^{\text{eq}}, T_{\text{eq}}) = (X/Q_\pi(X,T), T_{\text{eq}})$, we have $(x_i, x_{i+1}) \in Q_\pi(X,T)$ for $1 \le i \le n$. Since $(x_1, \ldots, x_n, x_{n+1})$ is a minimal point, by Theorem 3.5 $(x_1, \ldots, x_n, x_{n+1}) \in Q_\pi^{n+1}(X,T)$. By Theorem 3.4 and Proposition 3.2, π is relatively $(n+1)$-sensitive, a contradiction. Thus one has $\text{Card}(u\pi^{-1}(z)) \le n$.

Now we show that $\text{Card}(\phi^{-1}(z)) \le n^2$. If this is not the case, then there is $X_0 \subset \phi^{-1}(z)$ such that $\text{Card}(X_0) = n+1$ and $u(X_0)$ is a singleton. Let $X_0 = \{x_1, \ldots, x_{n+1}\}$, then it is clear that $(x_1, \ldots, x_n, x_{n+1}) \in P_\pi^{n+1}(X,T) \setminus \Delta_{n+1}(X) \subset Q_\pi^{n+1}(X,T) \setminus \Delta_{n+1}(X)$. Again by Theorem 3.4 and Proposition 3.2, (X,T) is $(n+1)$-sensitive, a contradiction. So ϕ is a finite-to-one extension with $\max_{z \in X_\pi^{\text{eq}}} \text{Card}(\phi^{-1}(z)) \le n^2$. □

COROLLARY 3.6. *Let $\pi : (X,T) \to (Y,S)$ be an extension between minimal systems with R_π^n having dense minimal points for any positive integer n. Let $\psi : (X_\pi^{\text{eq}}, T_{\text{eq}}) \to (Y,S)$ be the maximal equicontinuous factor of π and $\phi : (X,T) \to (X_\pi^{\text{eq}}, T_{\text{eq}})$ with $\psi \circ \phi = \pi$. If $\max_{z \in X_\pi^{\text{eq}}} \text{Card}(\phi^{-1}(z)) < \infty$, then π is either equicontinuous or there is $n \ge 2$ such that π is relatively n-sensitive but not relatively $(n+1)$-sensitive. If $\max_{z \in X_\pi^{\text{eq}}} \text{Card}(\phi^{-1}(z)) = \infty$, then π is relatively n-sensitive for any $n \ge 2$*

4. *Relative block $\mathcal{F}_t$-sensitivity*

In this section, we will study relative block $\mathcal{F}_t$-sensitivity and prove Theorem B.

Definition 4.1. $\pi : (X,T) \to (Y,S)$ is called *relatively block $\mathcal{F}_t$-sensitive* if there is $\delta > 0$ such that for any opene set $U \subset X$ and $l \in \mathbb{N}$ there are $u_l, v_l \in U$ with $(u_l, v_l) \in R_\pi$ and $\{n \in \mathbb{Z} : d(T^n u_l, T^n v_l) > \delta\}$ containing $\{m+1, \ldots, m+l\}$ for some $m \in \mathbb{Z}$.

PROPOSITION 4.2. *Let $\pi : (X,T) \to (Y,S)$ be a factor map between minimal systems. Let $\psi : (X_\pi^{\text{eq}}, T_{\text{eq}}) \to (Y,S)$ be the maximal equicontinuous factor of π and $\phi : (X,T) \to (X_\pi^{\text{eq}}, T_{\text{eq}})$ with $\psi \circ \phi = \pi$. If π is relatively block $\mathcal{F}_t$-sensitive, then $\phi : (X,T) \to (X_\pi^{\text{eq}}, T_{\text{eq}})$ is not proximal.*

Proof. Let the metric of (X,T) (respectively, $(X_\pi^{\text{eq}}, T_{\text{eq}})$) be d (respectively, d_1). Since $\psi : (X_\pi^{\text{eq}}, T_{\text{eq}}) \to (Y,S)$ is a equicontinuous extension between minimal systems, we can assume that for any $(u,v) \in R_\psi$, $d_1(T_{\text{eq}}u, T_{\text{eq}}v) = d_1(u,v)$.

Let $\varepsilon_k > 0$ with $\varepsilon_k \to 0$. Then for each $k \in \mathbb{N}$, there is $0 < \tau_k < \varepsilon_k$ such that if $w_1, w_2 \in X$ with $d(w_1, w_2) < 2\tau_k$ then $d_1(\phi(w_1), \phi(w_2)) < \varepsilon_k$.

Fix $x \in X$ and let $U_k = B(x, \tau_k)$. By the assumption π is relatively block $\mathcal{F}_t$-sensitive with sensitivity constant $\delta > 0$, thus for each $l \in \mathbb{N}$, there are $u_{k,l}, v_{k,l} \in U_k$ with $(u_{k,l}, v_{k,l}) \in R_\pi$ and $a_{k,l} \in \mathbb{Z}$ such that $F = \{n \in \mathbb{Z} : d(T^n u_{k,l}, T^n v_{k,l}) \ge \delta\}$ containing $\{a_{k,l} - l, a_{k,l} - l + 1, \ldots, a_{k,l}, a_{k,l} + 1, \ldots, a_{k,l} + l\}$.

Without loss of generality we assume that $T^{a_{k,l}} u_{k,l} \to u_k$ and $T^{a_{k,l}} v_{k,l} \to v_k$ when $l \to \infty$. It is clear that $d(T^n u_k, T^n v_k) \ge \delta$ for any $n \in \mathbb{Z}$ and $(u_k, v_k) \in R_\pi$. Now let $u = \lim_{k\to\infty} u_k$ and $v = \lim_{k\to\infty} v_k$. We have $(u,v) \in R_\pi$ and $d(T^n u, T^n v) \ge \delta$ for any $n \in \mathbb{Z}$. So (u,v) is a distal pair.

Now we show that $(u,v) \in R_\phi$. Since $u_{k,l}, v_{k,l} \in U_k$, we have $d_1(\phi(u_{k,l}), \phi(v_{k,l})) < \varepsilon_k$. Since $(\phi(u_{k,l}), \phi(v_{k,l})) \in R_\psi$, we thus have $d_1(T_{\text{eq}}^n \phi(u_{k,l}), T_{\text{eq}}^n \phi(v_{k,l})) < \varepsilon_k$ for each

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$n \in \mathbb{Z}$. In particular,

$$d_1(T_{\text{eq}}^{a_{k,l}}\phi(u_{k,l}), T_{\text{eq}}^{a_{k,l}}\phi(v_{k,l})) < \varepsilon_k$$

for each $l \in \mathbb{N}$. This implies that $d_1(\phi(u_k), \phi(v_k)) \le \varepsilon_k$, and hence $d_1(\phi(u), \phi(v)) = 0$. So we have proved $\phi(u) = \phi(v)$. Thus $\phi : (X, T) \to (X_\pi^{\text{eq}}, T_{\text{eq}})$ is not proximal. □

PROPOSITION 4.3. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between minimal systems. If $P_\pi(X, T) \neq Q_\pi(X, T)$, then $\pi : (X, T) \to (Y, S)$ is relatively block $\mathcal{F}_t$-sensitive.*

Proof. Since $P_\pi(X, T) \neq Q_\pi(X, T)$, there are $x_1, x_2 \in X$ such that $(x_1, x_2) \in Q_\pi(X, T)$ with (x_1, x_2) is a distal pair. Let $l = \inf_{n\in\mathbb{Z}} d(T^n x_1, T^n x_2) > 0$. By Theorem 3.4, $(x_1, x_2) \in Q_\pi(X, T) \setminus \Delta_2(X) = S_\pi(X, T)$. For any $m \in \mathbb{N}$, let U_1^m, U_2^m be neighborhoods of x_1, x_2 respectively such that $d(T^k U_1^m, T^k U_2^m) \ge (l/2)$ for $k = 0, 1, \ldots, m$. For any opene set U, there is $s \in \mathbb{Z}$ and $(v_1, v_2) \in U \times U \cap R_\pi$ such that $T^s v_i \in U_i^m$. So $T^s T^k v_i \in T^k U_i^m$ and $d(T^{s+k} v_1, T^{s+k} v_2) \ge l/2$ for $k = 0, 1, \ldots, m$. Thus $\pi : (X, T) \to (Y, S)$ is relatively block $\mathcal{F}_t$-sensitive. □

We will now prove the following theorem which covers Theorem B.

THEOREM 4.4. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between minimal systems. Let $\psi : (X_\pi^{\text{eq}}, T_{\text{eq}}) \to (Y, S)$ be the maximal equicontinuous factor of π. Let $\phi : (X, T) \to (X_\pi^{\text{eq}}, T_{\text{eq}})$ with $\psi \circ \phi = \pi$. Then the following statements are equivalent.*

(1) *π is relatively block $\mathcal{F}_t$-sensitive.*
(2) *$\phi : (X, T) \to (X_\pi^{\text{eq}}, T_{\text{eq}})$ is not proximal.*
(3) *$P_\pi(X, T) \neq Q_\pi(X, T)$.*

Proof. (1) ⇒ (2) is Proposition 4.2.

(2) ⇒ (3) If $P_\pi(X, T) = Q_\pi(X, T)$, then Q_π is an equivalence relation [**19**, p. 423]. So $\phi : (X, T) \to (X_\pi^{\text{eq}}, T_{\text{eq}})$ is proximal, a contradiction.

(3) ⇒ (1) is Proposition 4.3. □

5. *Relative strong $\mathcal{F}_t$-sensitivity*

In this section, we will study relatively strong $\mathcal{F}_t$-sensitivity and prove Theorem C.

Definition 5.1. $\pi : (X, T) \to (Y, S)$ is called *relatively strongly $\mathcal{F}_t$-sensitive* if there is $\delta > 0$ such that for any opene set $U \subset X$ there are $x_1, x_2 \in U$ with $(x_1, x_2) \in R_\pi^2$ and $\{n \in \mathbb{Z} : d(T^n x_1, T^n x_2) > \delta\} \in \mathcal{F}_t$.

PROPOSITION 5.2. *Let $\pi : (X, T) \to (Y, S)$ be a proximal extension. If $\pi_2 : (Y, S) \longrightarrow (Z, R)$ is not relatively strongly $\mathcal{F}_t$-sensitive, then neither is $\pi_2\pi : (X, T) \longrightarrow (Z, R)$.*

Proof. Let d, d_1 be the compatible metric of X, Y, respectively. Since $\pi_2 : (Y, S) \to (Z, R)$ is not relatively strongly $\mathcal{F}_t$-sensitive, there are $\delta_k \to 0$ and opene subsets U_k of Y such that if $(y_k, y_k') \in U_k \times U_k \cap R_{\pi_2}$ then there is a syndetic subset F (depending on y_k, y_k') with $d_1(S^n y_k, S^n y_k') < \delta_k$ for every $n \in F$.

Assume to the contrary that $(X, T) \to (Z, R)$ is relatively strongly $\mathcal{F}_t$-sensitive with a sensitivity constant $\delta > 0$. Then for each opene subset U of X there are $(x, x') \in U \times U \cap R_{\pi_2\pi}$ such that $\{n \in \mathbb{Z} : d(T^n x, T^n x') > \delta\} \in \mathcal{F}_t$. Thus there are $(u_k, v_k) \in \pi^{-1}(U_k) \times$

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$\pi^{-1}(U_k) \cap R_{\pi_2\pi}$ such that $F_k := \{n \in \mathbb{Z} : d(T^n u_k, T^n v_k) > \delta\} \in \mathcal{F}_t$. Note that $E_k := \{n \in \mathbb{Z} : d_1(S^n\pi(u_k), S^n\pi(v_k)) < \delta_k\}$ is a syndetic set. This implies that there exists $b_k \in \mathbb{Z}$ such that

$$d_1(S^{b_k}\pi(u_k), S^{b_k}\pi(v_k)) < \delta_k \quad \text{and} \quad d(T^j(u_k), T^j(v_k)) > \delta$$

for $j \in [b_k - k, b_k + k]$. Without loss of generality, we assume that $T^{b_k}u_k \to u$, $T^{b_k}v_k \to v$. Then we have $d(T^n u, T^n v) \geq \delta$ for all $n \in \mathbb{Z}$. Since $\pi(T^{b_k}u_k) \to \pi(u)$, $\pi(T^{b_k}v_k) \to \pi(v)$ and $d_1(S^{b_k}\pi(u_k), S^{b_k}\pi(v_k)) < \delta_k$, we conclude that $\pi(u) = \pi(v)$, a contradiction. This indicates that $\pi_2\pi : (X, T) \longrightarrow (Z, R)$ is not relatively strongly $\mathcal{F}_t$-sensitive. □

PROPOSITION 5.3. *Let $\pi : (X, T) \to (Y, S)$ be a distal extension between minimal systems. Then $\pi : (X, T) \to (Y, S)$ is not relatively strongly $\mathcal{F}_t$-sensitive.*

Proof. Since $\pi : (X, T) \to (Y, S)$ is a distal extension of minimal systems, $\pi^{-1}(y)$ is an almost periodic set for every $y \in Y$ [**1**, p. 141]. So for any δ, we can choose an opene subset U of X with $\operatorname{diam}(U) < \delta$; then for any $(x, x') \in U \times U \cap R_\pi$, $N((x, x'), U \times U)$ is a syndetic set, thus $\pi : (X, T) \to (Y, S)$ is not relatively strongly $\mathcal{F}_t$-sensitive. □

PROPOSITION 5.4. *Let $\pi : (X, T) \to (Y, S)$ be an extension between minimal systems. If $\pi_1 : (X, T) \to (X_\pi^d, T_d)$ is a proximal extension, then $\pi : (X, T) \to (Y, S)$ is not relatively strongly $\mathcal{F}_t$-sensitive.*

Proof. Since $\pi_2 : (X_\pi^d, T_d) \to (Y, S)$ is a distal extension, $\pi_2 : (X_\pi^d, T_d) \to (Y, S)$ is not relatively strongly $\mathcal{F}_t$-sensitive by Proposition 5.3.

Since $\pi_1 : (X, T) \to (X_\pi^d, T_d)$ be a proximal extension of minimal systems, by Proposition 5.2, $\pi : (X, T) \to (Y, S)$ is not relatively strongly $\mathcal{F}_t$-sensitive. □

So we have proved one direction of Theorem C. We will prove the other direction using a structure theorem for minimal systems. First we need some preparations.

Recall that an extension $\pi : X \to Y$ of minimal systems is a *relatively incontractible* (RIC) extension if it is open and for every $n \geq 1$ the minimal points are dense in the relation R_π^n. The extension $\pi : (X, T) \to (Y, S)$ is *weakly mixing* when R_π is topologically transitive, and is *weakly mixing of all orders* when R_π^n is topologically transitive for any $n \in \mathbb{N}$.

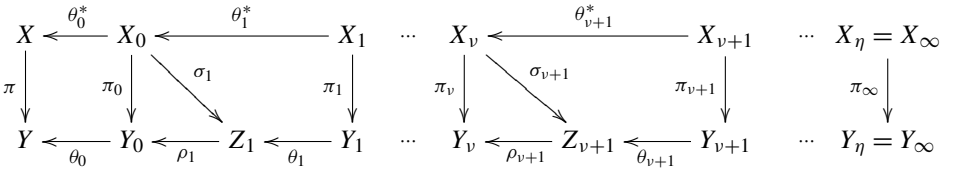
We say that an extension $\pi : (X, T) \to (Y, S)$ between minimal systems is a *strictly Proximal-Isometric (PI) extension* if there are an ordinal η (which is countable when X is metrizable) and a family of systems $\{(W_\iota, w_\iota)\}_{\iota \leq \eta}$ such that: (i) $W_0 = Y$; (ii) for every $\iota < \eta$ there exists a homomorphism $\phi_\iota : W_{\iota+1} \to W_\iota$ which is either proximal or equicontinuous; (iii) for a limit ordinal $\nu \leq \eta$ the system W_ν is the inverse limit of the systems $\{W_\iota\}_{\iota<\nu}$; and (iv) $W_\eta = X$. We say that an extension $\pi : (X, T) \to (Y, S)$ between minimal systems is a *PI extension* if there exist a strictly PI extension $\phi : \tilde{X} \to Y$ and a proximal extension $\theta : \tilde{X} \to X$.

We have the following structure theorem for minimal systems.

LEMMA 5.5. (Structure theorem for minimal systems, [**4**]) *Given an extension $\pi : X \to Y$ of minimal dynamical system, there exist an ordinal η (countable when X is metrizable)*

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and a canonically defined commutative diagram (the canonical PI tower)



where for each $\nu \le \eta$, π_ν is RIC, ρ_ν is equicontinuous, θ_ν, θ_ν^* are proximal and π_∞ is RIC and weakly mixing of all orders. For a limit ordinal ν, X_ν, Y_ν, π_ν etc. are the inverse limits (or joins) of X_ι, Y_ι, π_ι etc. for $\iota < \nu$.

We know that X_∞ is a proximal extension of X and an RIC weakly mixing extension of Y_∞, and $Y_\infty \to Y$ is a strictly PI extension. So the extension π_∞ is an isomorphism (so that $X_\infty = Y_\infty$) if and only if $\pi : X \to Y$ is a PI extension.

PROPOSITION 5.6. *Let $\pi : (X, T) \to (Y, S)$ be an extension of minimal systems. If $\pi : (X, T) \to (Y, S)$ is not a PI extension, then $\pi : (X, T) \to (Y, S)$ is relatively strongly $\mathcal{F}_t$-sensitive.*

Proof. Since $\pi : (X, T) \to (Y, S)$ is not a PI extension, $\theta^* : X_\infty \to X$ is proximal, and $\pi_\infty : X_\infty \to Y_\infty$ is weakly mixing, RIC and not an isomorphism. Since $\pi_\infty : X_\infty \to Y_\infty$ is RIC, the minimal points are dense in R_{π_∞}. Thus there exist $x_1, x_2 \in X_\infty$ such that (x_1, x_2) is a distal pair with $\pi_\infty(x_1) = \pi_\infty(x_2)$. Let $\inf_{n\in\mathbb{Z}} d(T^n x_1, T^n x_2) = 2\delta$.

For any $m \in \mathbb{N}$, let U_m^1, U_m^2 be closed neighborhoods of x_1, x_2 respectively such that $d(T^l U_m^1, T^l U_m^2) > \delta$ for $l = 0, 1, \ldots, m$.

Let $V_m = U_m^1 \times U_m^2 \cap R_{\pi_\infty}$. For any opene set $U \subset X_\infty$, let $W_1 = U \times U \cap R_{\pi_\infty}$. Since R_{π_∞} is transitive, $N(W_1, V_1) \neq \emptyset$. Choose $k_1 \in N(W_1, V_1)$, and let $W_2 = W_1 \cap T^{-k_1} V_1$.

Since R_{π_∞} is transitive, $N(W_2, V_2) \neq \emptyset$. Choose $k_2 \in N(W_2, V_2)$, and let $W_3 = W_2 \cap T^{-k_2} V_2$.

Now assume that by induction we defined W_m. Since R_{π_∞} is transitive, $N(W_m, V_m) \neq \emptyset$. Choose $k_m \in N(W_m, V_m)$, and let $W_{m+1} = W_m \cap T^{-k_m} V_m$.

Let $(x_1, x_2) \in \cap_{m=1}^\infty W_m$. From the above construction, it is easy to obtain

$$\{n \in \mathbb{Z} : d(T^n x_1, T^n x_2) > \delta\} \supset \bigcup_{m=0}^{\infty} \{k_m, k_m + 1, \ldots, k_m + m\}.$$

Thus $\pi_\infty : X_\infty \to Y_\infty$ is relatively strongly $\mathcal{F}_t$-sensitive. Let $\rho : Y_\infty \to Y$; then $\rho\pi_\infty : X_\infty \to Y$ is relatively strongly $\mathcal{F}_t$-sensitive. Since $\theta^* : X_\infty \to X$ is proximal, $\pi : X \to Y$ is relatively strongly $\mathcal{F}_t$-sensitive. □

Let $E(X, T)$ be the enveloping semigroup of (X, T).

LEMMA 5.7. *Let $\pi : X \longrightarrow Y$ be a distal factor map between two minimal systems. Then π is open and $\pi^{-1}(py) = p\pi^{-1}(y)$ for any $y \in Y$ and any $p \in E(X)$.*

THEOREM 5.8. *Let (X_3, T) be minimal and $Y \overset{\pi_0}{\leftarrow} X_1 \overset{\pi_1}{\leftarrow} X_2 \overset{\pi_2}{\leftarrow} X_3$, where π_0 is a distal extension, π_1 is a non-trivial proximal extension, and π_2 is a non-trivial distal extension. If $P_\pi = P(X_3) \cap R_\pi$ is not closed, where $\pi = \pi_0\pi_1\pi_2$, then $X_3 \to Y$ is relatively strongly $\mathcal{F}_t$-sensitive.*

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Proof. Since $P(X_3) \cap R_\pi$ is not closed, there are a distal pair $(x^1, x^2) \in X_3 \times X_3 \cap R_\pi$ and proximal pairs $(x_i^1, x_i^2) \in X_3 \times X_3 \cap R_\pi$ such that $(x_i^1, x_i^2) \to (x^1, x^2)$ for $i \to \infty$. Let $\pi' = \pi_1\pi_2$. Since π_0 is distal extension and $\pi(x_i^1) = \pi(x_i^2)$, we have $\pi'(x_i^1) = \pi'(x_i^2)$. This implies that $\pi'(x^1) = \pi'(x^2)$. Moreover, we may assume that (x^1, x^2) is a minimal point. As $(\pi_2(x^1), \pi_2(x^2))$ is proximal and $(\pi_2(x^1), \pi_2(x^2))$ is a minimal point we know that $\pi_2(x^1) = \pi_2(x^2)$. Choose $p_i \in E(X_3)$ satisfying $p_i x^1 = x_i^1$. Since $E(X_3)$ is a compact Hausdorff space, there exists a subnet $p_{h(s)}$ converging to a point $p \in E(X_3)$ where h is a function from a directed set $(S, \geq)$ to $\mathbb{N}$. So $px^1 = \lim p_{h(s)}x^1 = \lim x_{h(s)}^1 = x^1$. By Lemma 5.7, $p\pi_2^{-1}(\pi_2(x^1)) = \pi_2^{-1}(p\pi_2(x^1)) = \pi_2^{-1}(\pi_2(x^1))$. So there exists $z \in \pi_2^{-1}(\pi_2(x^1))$ such that $pz = x^2$. We know that $(p_{h(s)}z, p_{h(s)}x^1)$ are distal pairs and $\lim p_{h(s)}z = pz = x^2$, $\lim p_{h(s)}x^1 = \lim x_{h(s)}^1 = x^1$. We also have that $(p_{h(s)}x^1, x_{h(s)}^2) = (x_{h(s)}^1, x_{h(s)}^2)$ are proximal pairs. Let $3\delta = \inf_{n\in\mathbb{Z}} d(T^n x^1, T^n z)$; then $3\delta \leq \inf_{n\in\mathbb{Z}} d(T^n p_{h(s)}x^1, T^n p_{h(s)}z)$.

Since $(p_{h(s)}x^1, x_{h(s)}^2) = (x_{h(s)}^1, x_{h(s)}^2)$ are proximal pairs, $\{n \in \mathbb{Z} : d(T^n p_{h(s)}x^1, T^n x_{h(s)}^2) < \delta\} \in \mathcal{F}_t$. Then we have $\{n \in \mathbb{Z} : d(T^n p_{h(s)}z, T^n x_{h(s)}^2) > \delta\} \in \mathcal{F}_t$. Since $(p_{h(s)}z, x_{h(s)}^2) \to (x^2, x^2)$ and $\pi(p_{h(s)}z) = \pi(p_{h(s)}x^1) = \pi(x_{h(s)}^2)$, for any open neighborhood $U \subset X_3$ of x_2, we can choose $s \in S$ such that $(p_{h(s)}z, x_{h(s)}^2) \in U \times U$, $\{n \in \mathbb{Z} : d(T^n p_{h(s)}z, T^n x_{h(s)}^2) > \delta\} \in \mathcal{F}_t$ and $\pi(p_{h(s)}z) = \pi(x_{h(s)}^2)$. Since X_3 is minimal, we know that $X_3 \to Y$ is relatively strongly $\mathcal{F}_t$-sensitive. □

LEMMA 5.9. *Let (Z_{n+1}, T) be minimal and consider the strictly PI tower $Y \overset{\theta_0}{\leftarrow} Y_0 \overset{\rho_1}{\leftarrow} Z_1 \overset{\theta_1}{\leftarrow} Y_1 \overset{\rho_2}{\leftarrow} Z_2 \overset{\theta_2}{\leftarrow} Y_2 \overset{\rho_3}{\leftarrow} Z_3 \overset{\theta_3}{\leftarrow} \cdots \overset{\rho_n}{\leftarrow} Z_n \overset{\theta_n}{\leftarrow} Y_n \overset{\rho_{n+1}}{\leftarrow} Z_{n+1}$, where θ_i is a proximal extension (non-trivial for $i \geq 1$) and ρ_i is a distal extension (non-trivial for $i \geq 2$). If $(Z_{n+1}, T) \to (Y, S)$ is not strongly $\mathcal{F}_t$-sensitive, then $P(Z_{n+1}) \cap R_\pi$ is closed where $\pi = \theta_0\rho_1\theta_1 \cdots \theta_n\rho_{n+1}$.*

Proof. There are three cases: (1) θ_0 is trivial; (2) θ_0 is non-trivial, ρ_1 is non-trivial; (3) θ_0 is non-trivial, ρ_1 is trivial.

Case 1. In this case $Y = Y_0$, and we prove this lemma by induction on n. By Theorem 5.8, the lemma holds for $n = 1$. Now we assume that the lemma holds for $n \leq k-1$, and we prove it still holds for $n = k$. Let $Z_\pi^d \to Y_0$ be the maximal distal factor of $Z_2 \to Y_0$ and let $\pi_1 : Z_2 \to Z_\pi^d$. Since $P(Z_2) \cap R_{\rho_1\theta_1\rho_2}$ is closed, π_1 is proximal. So $\pi_2 = \pi_1\theta_2 : Y_2 \to Z_\pi^d$ is proximal. Let $\pi_0 : Z_\pi^d \to Y_0$ and consider the new PI tower

$$Y_0 \overset{\pi_0}{\leftarrow} Z_\pi^d \overset{\pi_2}{\leftarrow} Y_2 \overset{\rho_3}{\leftarrow} Z_3 \overset{\theta_3}{\leftarrow} \cdots \overset{\rho_k}{\leftarrow} Z_n \overset{\theta_k}{\leftarrow} Y_k \overset{\rho_{k+1}}{\leftarrow} Z_{k+1}$$

and that the lemma holds for $n \leq k-1$; we know that $P(Z_{k+1}) \cap R_\pi$ is closed, that is, the lemma holds for $n = k$. So the lemma holds for case 1.

Case 2. Let $\rho_0 : (Y, S) \to (Y, S)$ be the trivial distal extension. Consider the new PI tower

$$Y \overset{\rho_0}{\leftarrow} Y \overset{\theta_0}{\leftarrow} Y_0 \overset{\rho_1}{\leftarrow} Z_1 \overset{\theta_1}{\leftarrow} Y_1 \overset{\rho_2}{\leftarrow} Z_2 \overset{\theta_2}{\leftarrow} Y_2 \overset{\rho_3}{\leftarrow} Z_3 \overset{\theta_3}{\leftarrow} \cdots \overset{\rho_n}{\leftarrow} Z_n \overset{\theta_n}{\leftarrow} Y_n \overset{\rho_{n+1}}{\leftarrow} Z_{n+1}.$$

Then case 2 has the same form as case 1. Since the lemma holds for case 1, it also holds for case 2.

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Case 3. Since ρ_1 is trivial, $\theta_0\rho_1\theta_1 : Y_1 \to Y$ is non-trivial proximal extension, and $\rho_2 : Z_2 \to Y_1$ is non-trivial distal extension, consider the new PI tower

$$Y \overset{\theta_0\rho_1\theta_1}{\leftarrow} Y_1 \overset{\rho_2}{\leftarrow} Z_2 \overset{\theta_2}{\leftarrow} Y_2 \overset{\rho_3}{\leftarrow} Z_3 \overset{\theta_3}{\leftarrow} \cdots \overset{\rho_n}{\leftarrow} Z_n \overset{\theta_n}{\leftarrow} Y_n \overset{\rho_{n+1}}{\leftarrow} Z_{n+1}.$$

Then case 3 has the same form as case 2. Since the lemma holds for case 2, it also holds for case 3. □

LEMMA 5.10. *Let $\pi_1 : X \to Y$ and $\pi_2 : Y \to Z$ be factor maps between minimal systems, and denote $\pi = \pi_2\pi_1$.*

(1) *If $P_\pi(X)$ is closed, then $P_{\pi_2}(Y)$ is closed.*

(2) *If π_1 is proximal and $P_{\pi_2}(Y)$ is closed, then $P_\pi(X)$ is closed.*

Proof. Statement (1) follows from [**2**, Lemma 2].

(2) Let $(x_i, x_i') \in R_\pi \cap P(X) = P_\pi(X)$ such that $(x_i, x_i') \to (x, x') \in R_\pi$ for $i \to \infty$. Then $(\pi_1(x_i), \pi_1(x_i')) \in R_{\pi_2} \cap P(Y) = P_{\pi_2}(Y)$ such that $(\pi_1(x_i), \pi_1(x_i')) \to (\pi_1(x), \pi_1(x'))$. Since $P_{\pi_2}(Y)$ is closed, $(\pi_1(x), \pi_1(x')) \in P_{\pi_2}(Y)$. So there exists $p \in E(X)$ (where $E(X)$ is the Ellis semigroup of X) such that $p\pi_1(x) = p\pi_1(x')$, that is, $\pi_1(px) = \pi_1(px')$. Since π_1 is proximal, there exists $q \in E(X)$ such that $qpx = qpx'$, that is, $(x, x') \in P(X)$. So $P_\pi(X)$ is closed. □

LEMMA 5.11. *Let X be an inverse limit of minimal systems $\{(X_i, T_i)\}_{i=1}^\infty$ and let $\pi_i : X_i \to Y$ and $\pi : X \to Y$. If $P_{\pi_i}(X_i)$ is closed for each $i \in \mathbb{N}$, then $P_\pi(X)$ is closed.*

Proof. By the definition of inverse limit, $P_\pi(X)$ is closed. □

THEOREM 5.12. *Let $\pi : (X, T) \to (Y, S)$ be an extension of minimal systems. If $\pi : (X, T) \to (Y, S)$ is not relatively strongly $\mathcal{F}_t$-sensitive, then $\pi : (X, T) \to (X_\pi^d, T_d)$ is proximal.*

Proof. By Proposition 5.6, $\pi : (X, T) \to (Y, S)$ is a PI extension. Consider the strictly PI tower in the structure theorem,

$$Y \overset{\theta_0}{\leftarrow} Y_0 \overset{\rho_1}{\leftarrow} Z_1 \overset{\theta_1}{\leftarrow} Y_1 \overset{\rho_2}{\leftarrow} Z_2 \overset{\theta_2}{\leftarrow} Y_2 \overset{\rho_3}{\leftarrow} Z_3 \overset{\theta_3}{\leftarrow} \cdots Y_\infty = X_\infty$$

where θ_i is a proximal extension (non-trivial for $i \geq 1$) and ρ_i is a distal extension (non-trivial for $i \geq 2$).

By Proposition 5.2 the factor map $\tau : X_\infty \to Y$ is not relatively strongly $\mathcal{F}_t$-sensitive. So each finite tower

$$Y \overset{\theta_0}{\leftarrow} Y_0 \overset{\rho_1}{\leftarrow} Z_1 \overset{\theta_1}{\leftarrow} Y_1 \overset{\rho_2}{\leftarrow} Z_2 \overset{\theta_2}{\leftarrow} Y_2 \overset{\rho_3}{\leftarrow} Z_3 \overset{\theta_3}{\leftarrow} \cdots \overset{\rho_n}{\leftarrow} Z_n \overset{\theta_n}{\leftarrow} Y_n \overset{\rho_{n+1}}{\leftarrow} Z_{n+1}$$

is not relatively strongly $\mathcal{F}_t$-sensitive.

Let $\tau_{n+1} : Z_{n+1} \to Y$, $\tau_n' : Y_n \to Y$. Then by Lemma 5.9, $P_{\tau_{n+1}}(Z_{n+1})$ is closed. So $P_{\tau_n'}(Y_n)$ is closed by Lemma 5.10. By Lemma 5.11, $P_\tau(X_\infty)$ is closed. By Lemma 5.10, $P_\pi(X)$ is closed. So $P_\pi(X)$ is an equivalence relation [**19**, p. 478]. □

Proof of Theorem C. (1) $\Rightarrow$ (2) is Proposition 5.4.

(2) $\Rightarrow$ (1) is Theorem 5.12. □

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REFERENCES

[**1**] J. Auslander. *Minimal Flows and Their Extensions (North-Holland Mathematics Studies 153)*. Elsevier, Amsterdam, 1988.
[**2**] J. Auslander and B. Horelick. Regular minimal sets. II. The proximally equicontinuous case. *Compos. Math.* **22** (1970), 203–214.
[**3**] J. Auslander and J. A. Yorke. Interval maps, factors of maps, and chaos. *Tôhoku Math. J. (2)* **32**(2) (1980), 177–188.
[**4**] R. Ellis, S. Glasner and L. Shapiro. Proximal-isometric flows. *Adv. Math.* **17** (1975), 213–260.
[**5**] H. Furstenberg. *Recurrence in Ergodic Theory and Combinatorial Number Theory (M. B. Porter Lectures)*. Princeton University Press, Princeton, NJ, 1981.
[**6**] F. García-Ramos, J. Li and R. Zhang. When is a dynamical system mean sensitive. *Preprint*, 2017, arXiv:1708.01987. *Ergod. Th. & Dynam. Sys.* to appear.
[**7**] E. Glasner and B. Weiss. Sensitive dependence on initial conditions. *Nonlinearity* **6**(6) (1993), 1067–1075.
[**8**] W. Huang, D. Khilko, S. Kolyada and G. Zhang. Dynamical compactness and sensitivity. *J. Differential Equation* **260**(9) (2016), 6800–6827.
[**9**] W. Huang, S. Kolyada and G. Zhang. Analogues of Auslander-Yorke theorems for multi-sensitivity. *Ergod. Th. & Dynam. Sys.* **38**(2) (2018), 651–665.
[**10**] W. Huang, P. Lu and X. Ye. Measure-theoretical sensitivity and equicontinuity. *Israel J. Math.* **183** (2011), 233–283.
[**11**] J. Li. Measure-theoretic sensitivity via finite partitions. *Nonlinearity* **29**(7) (2016), 2133–2144.
[**12**] J. Li, S. Tu and X. Ye. Mean equicontinuity and mean sensitivity. *Ergod. Th. & Dynam. Sys.* **35**(8) (2015), 2587–2612.
[**13**] K. Liu and X. Zhou. Auslander-Yorkes type dichotomy theorems for stronger version r-sensitivity, preprint.
[**14**] A. Maass and S. Shao. Structure of bounded topological-sequence-entropy minimal systems. *J. Lond. Math. Soc. (2)* **76**(3) (2007), 702–718.
[**15**] D. C. McMahon. Relativized weak disjointness and relatively invariant measures. *Trans. Amer. Math. Soc.* **236** (1978), 225–237.
[**16**] T. K. Subrahmonian Moothathu. Stronger forms of sensitivity for dynamical systems. *Nonlinearity* **20**(9) (2007), 2115–2126.
[**17**] D. Ruelle. *Dynamical Systems with Turbulent Behavior, Mathematical Problems in Theoretical Physics (Proc. Internat. Conf., Univ. Rome, Rome, 1977) (Lecture Notes in Physics, 80)*. Springer, Berlin, 1978, pp. 341–360.
[**18**] S. Shao, X. Ye and R. Zhang. Sensitivity and regionally proximal relation in minimal systems. *Sci. China* **51**(6) (2008), 987–994.
[**19**] J. de Vries. *Elements of Topological Dynamics*. Kluwer Academic Publishers, Dordrecht, 1993.
[**20**] J. Xiong. Chaos in topological transitive systems. *Sci. China* **48**(7) (2005), 929–939.
[**21**] X. Ye and T. Yu. Sensitivity, proximal extension and high order almost automorphy. *Trans. Amer. Math. Soc.* **370**(5) (2018), 3639–3662.
[**22**] X. Ye and R. Zhang. On sensitive sets in topological dynamics. *Nonlinearity* **21**(7) (2008), 1601–1620.
[**23**] T. Yu. Measurable sensitivity via Furstenberg families. *Discrete Contin. Dyn. Syst.* **37**(8) (2017), 4543–4563.
[**24**] G. Zhang. Relativization of complexity and sensitivity. *Ergod. Th. & Dynam. Sys.* **27**(4) (2007), 1349–1371.
[**25**] Y. Zou. Stronger version sensitivity, almost finite to one extension and maximal pattern entropy. *Commun. Math. Stat.* **5**(2) (2017), 123–139.

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