Proc. Camb. Phil. Soc. (1971), **70**, 31
PCPS 70–8
Printed in Great Britain

A diophantine problem on groups. III

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(*Received* 16 *April* 1970)

Abstract. The following generalization of a theorem of Weyl appeared in part I of this series of papers. Let G be a locally compact Abelian group with dual group $\hat{G}$. Let $\Lambda = (\lambda_n)_{n=1}^{\infty}$ be a sequence in $\hat{G}$, not too slowly growing in a certain precise sense. Then, provided $\hat{G}$ has 'not too many' elements of finite order, the sequences

$$\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x) \ldots$$

are uniformly distributed on the circle, for almost all x in G.

In this paper we find the exact structural conditons on G needed for this result to be true. We also show that under these conditions it can never be a 'best possible' statement about the exceptional set E of x. A more refined estimate of the size of E is given for compactly generated G, and the ideas are carried further in the case when G is the real line.

1. *Introduction.* Let G be a locally compact Abelian group, and $\Lambda = (\lambda_n)_{n=1}^{\infty}$ a sequence in the dual group $\hat{G}$. We write $\Lambda(x)$ for the sequence $(\lambda_n(x))_{n=1}^{\infty}$, and we say that x is a *normal element* of G (for Λ) if $\Lambda(x)$ is uniformly distributed on the circle group T. We write $E(\Lambda)$ for the set of non-normal elements.

As we shall see in section 2, locally compact Abelian groups G divide into two classes:

1. Those for which, under very mild growth restrictions on Λ, the set $E(\Lambda)$ is locally null. We call these *Weyl groups*.

2. Those for which no growth condition on Λ ensures that $E(\Lambda)$ is locally null.

(A locally null set A is a measurable set in G for which $m(A \cap K) = 0$ for each compact set K; m denotes Haar measure on G.)

Clearly, the first class forms the most suitable domain for a metric theory of uniform distribution in groups, because we can go on to consider finer properties of $E(\Lambda)$. One result of this sort was given in Baker (2).

In (1) we showed that, for example, connected groups belong in the first class; and that if $\hat{G}$ has an infinite discrete subgroup of bounded order, G belongs in the second class. In this paper we give an exact structural description of the two classes of groups (section 2, Theorems 1 and 2). Some information about the structure of locally compact Abelian groups is obtained by the way.

In section 3 we show that '$E(\Lambda)$ is locally null' is never the best possible assertion when G is a Weyl group. We give more precise results, in terms of Fourier–Stieltjes

† This research was supported by the Science Research Council.

https://doi.org/10.1017/S0305004100049641 Published online by Cambridge University Press

transforms, for compact and compactly generated Weyl groups. These results are, we believe, new even in the classical cases $G = T$ and $G = R$, the real line. In section 4 we are able to go considerably further in these two cases, by adapting a method of Erdös and Koksma (3). In particular, we show that if $G = T$, $E(\Lambda)$ supports no measure $\mu \geqslant 0$ such that

$$\hat{\mu}(n) = 0\left(\frac{1}{(\log|n|)^{5+\delta}}\right) \quad \text{as} \quad |n| \to \infty, \tag{1·1}$$

where $\delta > 0$. This goes some way in the direction of Kahane and Salem's conjecture ((8) or (7)) that $E(\Lambda)$ supports no measure $\mu \geqslant 0$ such that

$$\hat{\mu}(n) = o(n) \quad \text{as} \quad |n| \to \infty.$$

2. *Characterization of Weyl groups.* We begin with some definitions, some of which occur in (1). G will always denote a locally compact Abelian group with dual $\hat{G}$.

2·1. DEFINITION. Let $\Lambda = (\lambda_n)_{n=1}^{\infty}$ be a sequence in G and let $1 \leqslant \phi(1) \leqslant \phi(2) \leqslant \ldots$ be a nondecreasing sequence of real numbers. We say that Λ *satisfies the growth condition* $\phi(n)$ if there is a neighbourhood V of 0 in G such that whenever

(i) $1 \leqslant n_1 < n_2 < \ldots < n_k \leqslant m$ are integers,

(ii) $\lambda_{n_i} - \lambda_{n_j} \in V$ for $i, j = 1, 2, \ldots, k$,

then

(iii) $k \leqslant \phi(m)$.

We point out that, with slightly different terms, the same definition could be made if G were any topical space with topology given by a uniformity. So growth conditions are entirely non-algebraic.

If Λ satisfies some condition

$$\phi(n) = \frac{An}{(\log n)^{1+\epsilon}} \quad \text{with} \quad A > 0$$

and $\epsilon > 0$, Λ is called an *N-sequence* (see (1)). Plainly, the slower the growth of ϕ, the more restrictive the growth condition.

2·2. DEFINITION. A locally compact Abelian group G is said to be a *Weyl group* if $E(\Lambda)$ is locally null for each N-sequence Λ in $\hat{G}$.

This is slightly different from the definition in (1) where the word 'locally' is omitted.

2·3. DEFINITION. If for each $m = 1, 2, \ldots, \hat{G}$ has finitely many elements of order m, then $\hat{G}$ is said to be *almost torsion free*.

2·4. DEFINITION. Let G_1 be a compact subgroup of G. We say that G_1 is a *compact structure subgroup* of G if there is an integer $a \geqslant 0$ such that $G_1 \times R^a$ is an open subgroup of G (up to topological isomorphism).

Here R^a is Euclidean a-space. It is well known (see e.g. (9), section 2·4) that G always possesses a compact structure subgroup, and that there may well be a number of essentially different compact structure subgroups of a single group G.

We now come to our first main results.

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THEOREM 1. *If G has a compact structure subgroup G_1 with almost torsion free dual $\hat{G}_1$, then G is a Weyl group.*

THEOREM 2. *If G has a compact structure group G_1 whose dual $\hat{G}_1$ is not almost torsion free, then G is not a Weyl group. Further, if $\phi(n)$ is any growth condition, there is a sequence Λ in $\hat{G}$ satisfying $\phi(n)$ and an open subgroup G_0 of G such that $\{\Lambda(x): x \in G_0\}$ is contained in a finite subgroup of T.*

We have the following corollary of Theorems 1 and 2.

THEOREM 3. *If one compact structure subgroup of G has almost torsion free dual, then so does every compact structure subgroup.*

In the light of this result, we have characterized Weyl groups as *those locally compact Abelian groups for which the compact structure subgroup has almost torsion free dual.*

We require a number of lemmas.

2·5 LEMMA. *Let $G = G_1 \times R^a$ where G_1 is compact, $a \geqslant 0$ and $\hat{G}_1$ is almost torsion free. Let $\Lambda = (\lambda_n)_{n=1}^{\infty}$ be any N-sequence in $\hat{G}$ and $(a_n)_{n=1}^{\infty}$ any sequence of complex numbers of modulus one. Then the sequence*

$$a_1\lambda_1(x), a_2\lambda_2(x), \ldots, a_n\lambda_n(x), \ldots \tag{2·1}$$

is uniformly distributed on the unit circle except for a null set of x in G.

Proof. Throughout section 2 and section 3 we write all group operations as addition. Following Rudin (9) we write (x, λ) for $\lambda(x)$ whenever $x \in G$ and $\lambda \in \hat{G}$. Thus

$$(x+y, \lambda) = (x, \lambda).(y, \lambda) \quad \text{and} \quad (x, \lambda+\mu) = (x, \lambda).(x, \mu) \quad (x, y \in G; \lambda, \mu \in \hat{G}).$$

The proof breaks into two stages. We merely outline these, as a slightly less general result is proved in (1).

(i) We first prove the result when G is compact ($a = 0$). By Weyl's criterion a sequence $(z_n)_{n=1}^{\infty}$ is uniformly distributed on the unit circle if and only if

$$\frac{1}{n}\sum_{k=1}^{n}(z_k)^m = o(1) \quad (n \to \infty) \quad (m = 1, 2, \ldots).$$

It is clearly enough to show, then, that for any given m,

$$f_n(x) = \frac{1}{n}\sum_{k=1}^{n} a_k^m (x, m\lambda_k) = o(1) \quad (n \to \infty)$$

for almost all x. We have, where dx denotes normalized Haar measure on G,

$$\int_G |f_n(x)|^2 dx = \frac{1}{n^2}\sum_{i,j=1}^{n} a_i^m \overline{a}_j^m \int_G (x, m\lambda_i - m\lambda_j) dx$$

$$= \frac{1}{n^2}\sum_{(i,j)} a_i^m \overline{a}_j^m$$

by orthogonality, the second sum being taken over all pairs of integers (i, j) for which $m\lambda_i = m\lambda_j$. By the hypothesis that $\hat{G}$ is almost torsion free and that $(\lambda_n)_{n=1}^{\infty}$ is an

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N-sequence, the number of such pairs is $O(n^2/(\log n)^{1+\epsilon})$. It follows that for the subsequence $(n_v)_{v=1}^{\infty}$ of the integers defined by

$$n_v = [\{\exp v^{2/(2+\epsilon)}\}]$$

we have

$$\sum_{v=1} \int_G |f_{n_v}(x)|^2 dx < \infty$$

and consequently $f_{n_v}(x) = o(1)$ as $v \to \infty$ for almost all x. We then use $n_{v+1}/n_v \to 1$ as $v \to \infty$ to extend this to the full sequence $(f_n(x))_{n=1}^{\infty}$.

(ii) If G is non-compact, let $\epsilon > 0$ and define the neighbourhood V of 0 in $\hat{G}$ by

$$V = \left\{\lambda \in \hat{G};\ |\lambda(x) - 1| < \frac{\epsilon}{m} \quad \text{for all} \quad x \in K_0\right\},$$

where K_0 is any compact subset of G. There is (cf. (1)) a discrete subgroup K of G such that

$$K + V = \{\lambda + \mu: \lambda \in K, \mu \in V\} = G.$$

Choose $\mu_1, \mu_2, \ldots$ in K such that

$$\mu_n - \lambda_n \in V \quad (n = 1, 2, \ldots). \tag{†}$$

We can regard μ_i as characters of the compact group G/H where

$$H = \{x \in G: (x, \lambda) = 1 \quad \text{for all} \quad \lambda \in K\}.$$

(see (9), section 2·1). Moreover K is almost torsion free and it follows from (†) that $(\lambda_n)_{n=1}^{\infty}$ is an N-sequence in K. By (i),

$$g_n(x + H) = \frac{1}{n} \sum_{k=1}^{n} a_k^m(x, m\mu_k) = o(1) \quad (n \to \infty)$$

for almost all $x + H \in G/H$. By a general lemma, given in (1),

$$g_n'(x) = \frac{1}{n} \sum_{k=1}^{n} a_k^m(x, m\mu_k) = o(1) \quad (n \to \infty)$$

for almost all $x \in G$. For such an x, chosen from K_0, and defining $f_n(x)$ as in (i),

$$|f_n(x)| \leqslant \left|\frac{1}{n} \sum_{k=1}^{n} a_k^m(x, m\mu_k)\right| + \left|\frac{1}{n} \sum_{k=1}^{n} a_k^m\{(x, m\mu_k) - (x, m\lambda_k)\}\right|$$

and clearly

$$\overline{\lim_{n \to \infty}} |f_n(x)| \leqslant \sup_{x \in K,\ \lambda \in V} |(x, m\lambda) - 1| < \epsilon.$$

Since G is σ-compact, an easy argument now shows that $f_n(x) = o(1)$ for almost all x in G. As in (i), this proves the result and the lemma is completely proved.

Similar arguments are given in more detail in section 3.

2·6. LEMMA. *Let H be a compact subgroup of the locally compact Abelian dual group $\hat{G}$. Let $\Lambda = (\lambda_n)_{n=1}^{\infty}$ be an N-sequence in $\hat{G}$. Let $\phi: \hat{G} \to \hat{G}/H$ be the natural homomorphism. Then $(\phi(\lambda_n))_{n=1}^{\infty}$ is an N-sequence in $\hat{G}/H$.*

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Proof. $\hat{G}/H$ is the dual group of G_0, the annihilator of H in G. Since (λ_n) is an N-sequence in $\hat{G}$ there is a compact neighbourhood U of 0 in $\hat{G}$ and numbers $C > 0$, $\epsilon > 0$ such that if $1 \leqslant n_1 < n_2 < \ldots < n_k \leqslant m$ and if

$$\lambda_{n_i} - \lambda_{n_j} \in U \quad \text{for all} \quad i, j = 1, 2, \ldots, k$$

then

$$k \leqslant \frac{Cm}{(\log m)^{1+\epsilon}}.$$

Since ϕ is an open mapping, $\phi(U)$ is a compact neighbourhood of 0 in $\hat{G}/H$. Suppose that $1 \leqslant n_1 < n_2 < \ldots < n_k \leqslant m$ and that

$$\phi(\lambda_{n_i}) - \phi(\lambda_{n_j}) \in \phi(U)$$

for all $i, j = 1, 2, \ldots, m$. Then $\quad \lambda_{n_i} - \lambda_{n_j} \in U + H$

since H is the kernel of ϕ. Now $U + H$ is compact. Let V be a symmetric neighbourhood of 0 in $\hat{G}$ such that $V - V \subset U$. Let $\{V + \gamma_1, \ldots, V + \gamma_p\}$ be a finite covering of $U + H$ by translates of V. Then at least k/p of $\lambda_{n_1} - \lambda_{n_1}, \lambda_{n_1} - \lambda_{n_2}, \ldots, \lambda_{n_1} - \lambda_{n_k}$ are in some fixed $V + \gamma_i$, say $V + \gamma_1$. Renumbering these as

$$\lambda_{n_1} - \lambda_{n_{i_1}}, \lambda_{n_1} - \lambda_{n_{i_2}}, \ldots, \lambda_{n_1} - \lambda_{n_{i_q}} \quad (q \geqslant k/p),$$

with $1 \leqslant n_{i_1} < n_{i_2} < \ldots < n_{i_q} \leqslant m$, we have

$$\lambda_{n_{i_r}} - \lambda_{n_{i_s}} \in V - V \subset U \quad \text{for} \quad r, s = 1, 2, \ldots, q.$$

This proves, by hypothesis, that $q \leqslant \frac{Cm}{(\log m)^{1+\epsilon}}$. Hence

$$k \leqslant \frac{Cpm}{(\log m)^{1+\epsilon}}$$

and the lemma is proved. Note that $\phi(\lambda_n) = \lambda_n$ as functions on G_0.

2·7. Lemma. *Let G be a locally compact Abelian group and let G_0 be a σ-compact open subgroup of G. Let m and m_0 denote the Haar measure of G and G_0 respectively. If E is a Borel set in G_0 and $m_0(E) = 0$ then $m(E) = 0$.*

We have no reference for this result, so we offer a sketch of a proof. If f is a continuous function on G with compact support K let $G_0 + a_1, \ldots, G_0 + a_k$ be the distinct cosets intersecting K. Define

$$T(f) = \sum_{i=1}^{k} \int_G f(x)\, dm_{0,a_i}(x)$$

where $m_{0,a}$ denotes the translated measure (regarding m_0 as a measure on G)

$$m_{0,a}(B) = m_0(B - a) \quad (B \text{ a Borel set}, a \in G).$$

It is not hard to prove that T defines a non-zero, positive, translation invariant linear functional on $K(G)$, the space of continuous functions on G with compact support. By the uniqueness of Haar measure,

$$T(f) = \lambda \int_G f(x)\, dm(x)$$

where $\lambda > 0$ is a constant.

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Since m is regular it may be assumed that E is compact. Using regularity of m_0 find an open set $U \supset E$, $m_0(U) < \epsilon$ where ϵ is any positive number. We can find a continuous function f with compact support with

$$f(x) = 1 \quad \text{on} \quad E,$$

$$f(x) = 0 \quad \text{outside} \quad U.$$

But then $$\int_{G_0} f(x)\, dm_0(x) < \epsilon.$$

Thus $$mE < \int_G f(x)\, dm(x) = \lambda^{-1} T(f) = \lambda^{-1} \int_{G_0} f(x)\, dm_0(x) < \lambda^{-1}\epsilon.$$

Since $\epsilon > 0$ is arbitrary, $mE = 0$.

2·8. *Proof of Theorem* 1. Let $G_0 = G_1 \times R^a$ be an open subgroup of G where G_1 is as in the hypothesis of the theorem. Note that G/G_0 is discrete and that G_0 is also a closed subgroup. If H is the annihilator of G_0, then H may be identified with the dual group of G/G_0, so H is compact.

Let $\Lambda = (\lambda_n)_{n=1}^{\infty}$ be any N-sequence in $\hat{G}$. Let ϕ be as in Lemma 2·6; then $(\phi(\lambda_n))_{n=1}^{\infty}$ is an N-sequence in $\hat{G}/H$, the dual group of G_0. With m_0 as in Lemma 2·7, we deduce that for any sequence $(a_n)_{n=1}^{\infty}$ on the circle, $(a_n \lambda_n(x))$ is uniformly distributed except for a set $E \subset G_0$, for which $m_0(E) = 0$. (Lemma 2·5.) E is plainly a Borel set.

By Lemma 2·7, which applies since G_0 is σ-compact, $m(E) = 0$. We have established in particular that $m(E(\Lambda) \cap G_0) = 0$.

Since G_0 is open, to prove that $m(E(\Lambda) \cap K) = 0$ for any compact subset K of G it is enough to prove that

$$m(E(\Lambda) \cap (G_0 + x_0)) = 0 \tag{2·2}$$

for any fixed x_0 in G. Put $a_n = \lambda_n(x_0)$, $n = 1, 2, \ldots$; a_n lies on T. Let

$$E = \{x \in G_0 \,|\, (a_n \lambda_n(x))_{n=1}^{\infty} \text{ is not uniformly distributed}\}.$$

We observe that $$E(\Lambda) \cap (G_0 + x_0) \subset E + x_0$$

as one may easily verify. By what we established above,

$$m(E + x_0) = m(E) = 0$$

and (2·2) follows. This completes the proof of Theorem 1.

2·9. LEMMA. *Let G_0 be an open subgroup of the locally compact Abelian group G. Let $(\lambda_n)_{n=1}^{\infty}$ be an N-sequence in $\hat{G}_0$, and let $(\lambda_n')_{n=1}^{\infty}$ be any extensions whatever of the respective λ_n to continuous characters of G. Then $(\lambda_n')_{n=1}^{\infty}$ is an N-sequence in $\hat{G}$.*

Proof. There is a neighbourhood V of 0 in $\hat{G}_0$ and constants $A > 0$, $\epsilon > 0$ such that if $1 \leqslant n_1 < n_2 < \ldots < n_k \leqslant m$ and

$$\lambda_{n_i} - \lambda_{n_j} \in V \quad (1 \leqslant i, j \leqslant k)$$

then $$k \leqslant \frac{Am}{(\log m)^{1+\epsilon}}.$$

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V can be assumed to be of the form

$$V = \{\gamma \in \hat{G}_0 : |1-(x,\gamma)| < \delta \quad \text{for all} \quad x \in K\}$$

where K is a compact subset of G_0, hence of G, and $\delta > 0$. Thus

$$V' = \{\gamma' \in \hat{G} : |1-(x,\gamma')| < \delta \quad \text{for all} \quad x \in K\}$$

is a neighbourhood of 0 in $\hat{G}$. Now, if $1 \leqslant n_1 < n_2 < \ldots < n_k \leqslant m$ and

$$\lambda'_{n_i} - \lambda'_{n_j} \in V' \quad \text{for all} \quad i,j, \ldots, 1, 2, \ldots, k$$

then $|(x, \lambda_{n_i} - \lambda_n) - 1| < \epsilon$ for all $x \in K$ and $i,j, = 1, 2, \ldots, k$ and so for these i, j,

$$\lambda_{n_i} - \lambda_{n_j} \in V.$$

Consequently $k \leqslant Am/(\log m)^{1+\epsilon}$. So (λ'_n) is indeed an N-sequence.

The same proof applies if $Am/\{(\log m)^{1+\epsilon}\}$ is replaced by any $\phi(m)$ as in (2·1).

2·10. *Proof of Theorem* 2. Let $G_0 = G_1 \times R^a$ be an open subgroup of G. Since $\hat{G}_1$ is not almost torsion free, there is an integer $m \geqslant 1$ such that $\hat{G}_1$ contains distinct elements $\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots$ whose order divides m. That is

$$(x, \gamma_k)^m = 1 \quad \text{for all} \quad x \in G_1 \quad \text{and} \quad k = 1, 2, \ldots.$$

Let $\lambda_k(x,y) = \gamma_k(x)$ for $(x,y) \in G_1 \times R^a$. Plainly $(\lambda_k)_{k=1}^{\infty}$ is a sequence in $\hat{G}_0$ satisfying any growth condition $\phi(n)$. By Theorem 2.1.4 of Rudin (9), each λ_k has an extension to a continuous character λ'_k of G. By Lemma 2·9, $\Lambda = (\lambda'_k)_{k=1}^{\infty}$ satisfies $\phi(n)$ as a sequence in $\hat{G}$. But each λ'_k maps G_0 into a finite subgroup F of T. Since G_0 is open, G_0 has positive Haar measure, and Theorem 2 is completely proved.

2·11. *Remarks on structure theory.* (*a*) Consider groups G which are topologically isomorphic to a product

$$G = G_1 \times R^a \times D, \tag{2·3}$$

where G_1 is compact and D is discrete. This is quite a wide class, although it excludes some locally compact Abelian groups (see, e.g. (9), 2·4·7). For groups of the form (2·3) it is clear that the Weyl groups are precisely those for which $\hat{G}_1$ *is almost torsion free.* By Theorem 3, *this property of* $\hat{G}_1$ *is independent of the choice of* G_1 *in a factorization of* G *of form* (2·1).

This special case shows, roughly, that it is not so much the number of elements of $\hat{G}$ with finite order, as how well spaced out they are, that determines whether G is a Weyl group. In (2·3) the number of elements of the compact subgroup $\hat{D}$ of $\hat{G}$ with finite order is irrelevant to the question of whether G is a Weyl group.

(*b*) Let G be a locally compact Weyl group and let F_m be the subgroup:

$$F_m = \{\lambda \in G | m\lambda = 0\}.$$

Obviously F_m is closed. The hypothesized property of G shows that F_m is compact; if not, we could pick out 'well-spaced' characters mapping G onto a finite group, as in (1). We deduce:

THEOREM 4. *If the compact structure subgroup of* G *has almost torsion free dual, each group* F_m *is compact.*

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A simple direct proof of this result is not apparent, because of the difficulty of saying whether elements of F_m are in distinct cosets of a discrete subgroup of $\hat{G}$. It seems probable that the converse of Theorem 4 is true (it is, for example, in the case of groups (2·3)) but we have not been able to establish this.

(*c*) Finally, in all the preceding work we could confine ourselves to one integer $m \geqslant 1$ instead of considering all of $m = 1, 2, \ldots$ together. Our results would be somewhat different. We content ourselves with writing down one such theorem; m is any positive integer.

THEOREM 5. *If, for one compact structure subgroup G_1 of the locally compact Abelian group G, the dual $\hat{G}_1$ has finitely many elements of order m, then the same is true of the dual of* every *compact structure subgroup of G.*

3. *On the set of non-normal elements in a Weyl group.* We begin with a 'tighter' version of Theorem 2·4 of (1). We have been obliged to impose more restrictive growth conditions in this theorem and in the others in this section.

If G is a locally compact Abelian group, $M(G)$ denotes the space of complex regular Borel measures on G; $M^+(G)$ the positive members of $M(G)$; and let $L^p(G)$ is the usual space $L^p(G, m)$ where m is Haar measure ($1 \leqslant p < \infty$).

3·1. THEOREM. *Let G be a compact Abelian Weyl group. Let μ be in $M^+(G)$, and suppose $\hat{\mu} \in L^p(\hat{G})$ for some p, $2 < p < \infty$. Let $\Lambda = (\lambda_n)_{n=1}^{\infty}$ be a sequence in $\hat{G}$ satisfying some growth condition $\phi(n) = Cn^{\delta}$, where $C > 0$ and $0 < \delta < 1$. Then*

$$\mu(E(\Lambda)) = 0. \tag{3·1}$$

Proof. Here $\hat{\mu}$ is the Fourier–Stieltjes transform

$$\hat{\mu}(\lambda) = \int_G \lambda(x)\, d\mu(x) \quad (\lambda \in \hat{G}). \tag{3·2}$$

It is enough to show for a fixed integer $m \geqslant 1$ that

$$\mu(E_m) = 0, \tag{3·3}$$

where $E_m = \left\{x \in G: \sum_{k=1}^{n} (x, m\lambda_k) \neq o(n)\right\}$ since by Weyl's famous criterion

$$E \subset \bigcup_{m=1}^{\infty} E_m.$$

We set
$$f_n(x) = \frac{1}{n} \sum_{k=1}^{n} (x, m\lambda_k) \tag{3·4}$$

and estimate the integral
$$I(n) = \int_G |f_n(x)|^2 d\mu(x). \tag{3·5}$$

Now
$$|f_n(x)|^2 = \frac{1}{n^2} \sum_{i,j=1}^{n} (x, m\lambda_i - m\lambda_j).$$

Let $\gamma_1, \gamma_2, \ldots$ be an enumeration of all the distinct elements of $\hat{G}$ of the form $m\lambda_i - m\lambda_j$ $(i, j = 1, 2, \ldots)$. Then

$$|f_n(x)|^2 = \frac{1}{n^2} \sum_{\nu=1}^{n} \alpha_\nu \gamma_\nu(x), \tag{3·6}$$

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where N depends on n and α_ν is a non-negative integer, the number of solutions (i,j) with $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$ of

$$m\lambda_i - m\lambda_j = \gamma_\nu.$$

Since G is a Weyl group, $\hat{G}$ has M elements whose order divides m, where $M < \infty$. We have

$$\sum_{\nu=1}^{N} \alpha_\nu = n^2; \quad \alpha_\nu \leqslant CMn^{\delta+1}. \tag{3·7}$$

The first assertion of (3·7) is obvious. For the second, if $m\lambda_i - m\lambda_j = \gamma_\nu$ then $\lambda_i - \lambda_j$ is one of at most M elements of $\hat{G}$, as one may easily verify. Hence if i is given, λ_j is one of at most M elements of $\hat{G}$. By our growth hypothesis this restricts j to be one of at most MCn^δ indices. Since i can take n values, the second assertion in (3·7) follows.

Putting (3·6) into (3·5),

$$I(n) = \frac{1}{n^2}\int_G \sum_{\nu=1}^{N} \alpha_\nu \gamma_\nu(x)\, d\mu(x) = \frac{1}{n^2}\sum_{\nu=1}^{N} \alpha_\nu \hat{\mu}(\gamma_\nu).$$

By Hölder's inequality, if $p' = p/(p-1)$,

$$\begin{aligned} I(n) &\leqslant \frac{1}{n^2}\left(\sum_{\nu=1}^{N} \alpha_\nu^{p'}\right)^{1/p'} \left(\sum_{\nu=1}^{N} |\hat{\mu}(\gamma_\nu)|^p\right)^{1/p} \\ &\leqslant \frac{\|\hat{\mu}\|_p}{n^2}\left(\max_{\nu \leqslant N} \alpha_\nu\right)^{(p'-1)/p'} \left(\sum_{\nu=1}^{N} \alpha_\nu\right)^{1/p'} \\ &\leqslant \frac{\|\hat{\mu}\|_p}{n^2}\,(CM)^{1/p}\, n^{(\delta+1)/p}\, n^{2/p'} \end{aligned} \tag{3·8}$$

using (3·7). Now the index of n on the right-hand side of (3·8) is

$$-2 + \frac{\delta+1}{p} + \frac{2}{p'} = \frac{\delta}{p} + \frac{1}{p'} - 1 = -\alpha < 0.$$

So there is a constant A independent of Λ such that

$$I(n) \leqslant An^{-\alpha}.$$

Let s be a fixed integer larger than $2/\alpha$, say. Then

$$\sum_{n=1}^{\infty} I(n^s) \leqslant A \sum_{n=1}^{\infty} \frac{1}{n^{s\alpha}} < \infty.$$

Thus

$$\int_G \sum_{n=1}^{\infty} |f_{n^s}(x)|^2\, d\mu(x) = \sum_{n=1}^{\infty} I(n^s) < \infty.$$

It follows that $\sum_{n=1}^{\infty} |f_{n^s}(x)|^2\, dx < \infty$ almost everywhere with respect to μ, and in particular

$$\lim_{n\to\infty} f_{n^s}(x) = 0 \quad \text{almost everywhere with respect to } \mu. \tag{3·9}$$

Because

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^s = 1,$$

it is not hard to deduce from (3·9) that

$$\lim_{n\to\infty} f_n(x) = 0 \quad \text{almost everywhere with respect to } \mu. \tag{3·10}$$

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(Compare, for example, the end of Theorem 2·4 in (1).) Going back to the definition of E_m, we have proved that (3·3) holds, and theorem 3·1 is proved.

Remarks. (*a*) Note that in the discrete group $\hat{G}$ any sequence of *distinct* elements satisfies any growth condition $\phi(n)$.

(*b*) Theorem 3·1 is inspired by a theorem of Salem (10). He proved that, if $(\lambda_n)_{n=1}^{\infty}$ is a sequence of distinct integers, i.e. characters of the group T, and if

$$\lambda_n = O(n^q) \quad \text{for some} \quad q \geqslant 1 \tag{3·11}$$

then $\mu(E(\Lambda)) = 0$ for every $\mu \in M^+(T)$ such that

$$\sum_{\substack{\nu=-\infty \\ \nu \neq 0}} \frac{|\hat{\mu}(\nu)|^2}{|\nu|^{1-\alpha}} < \infty \quad (\alpha > 0,\ 1-(1/q) < \alpha \leqslant 1). \tag{3·12}$$

Or, in other words, the Hausdorff dimension of $E(\Lambda)$ is at most $1-(1/q)$, under the hypothesis (3·11). It seems difficult to generalize this result to other groups. The simple fact that $\hat{\mu} \in L^p(Z)$ implies (3·12) for $\alpha < 2/p$ means that Theorem 3·1 is already known for $G = T$, $\lambda_1, \lambda_2, \ldots$ all distinct and $\lambda_n = O(n^q)$ with $q < p/(p-2)$. In section 4, however, we shall find results for $G = T$ which are considerably better than Theorem 3·1, and cannot be partially subsumed under Salem's theorem in the above sense.

We now extend Theorem 3·1 to non-compact groups, analogously to the way in which Theorem 3·2 of (1) extends Theorem 2·4 of that paper.

3·2. DEFINITION. Let Γ be a locally compact Abelian group. A family $\mathscr{A}$ of subgroups Γ is called a *dense class of subgroups* if for each neighbourhood V of 0 in Γ there is an H in $\mathscr{A}$ such that $V + H = \Gamma$.

We observed in (1) that the dual of G of a compactly generated group G has a dense class of *discrete* subgroups, and we shall apply this below. There are groups Γ, however, which have no dense class of discrete subgroups, for example $\Gamma = \hat{Q}$, where Q denotes the discrete rationals.

3·3. THEOREM. *Let G be a Weyl group and suppose $\hat{G}$ has a dense class $\mathscr{A}$ of discrete subgroups. Let $\mu \in M^+(G)$ and suppose that for some p, $2 < p < \infty$,*

$$\sum_{\gamma \in D} |\hat{\mu}(\gamma)|^p < \infty \tag{3·13}$$

for each D in $\mathscr{A}$. Let $\Lambda = (\lambda_n)_{n=1}^{\infty}$ be a sequence in $\hat{G}$ satisfying some growth condition Cn^{δ}, where $C > 0$ and $0 < \delta < 1$. Then

$$\mu(E(\Lambda)) = 0. \tag{3·14}$$

Proof. Let $D \in \mathscr{A}$ and suppose first that $\lambda_1, \lambda_2, \ldots$, belong to D. It is clear by (3·13) that we can repeat the entire proof of Theorem 3·1. Let m be a fixed integer $\geqslant 1$, let

$$f_n(x) = \frac{1}{n} \sum_{k=1}^{n} (x, m\lambda_k), \tag{3·15}$$

then $\lim_{n \to \infty} f_n(x) = 0$ almost everywhere with respect to μ. Theorem 3·3 is true, then, for such sequences.

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Now suppose $\Lambda = (\lambda_n)_{n=1}^{\infty}$ is any sequence in $\hat{G}$ satisfying a growth condition $\phi(n) = Cn^{\delta}$. Since μ is regular and finite, (3·14) will follow if we fix a compact set K in G and prove

$$\mu(E(\Lambda) \cap K) = 0.$$

Defining E_m as in Theorem 3·1, it is thus enough to prove for fixed m that

$$\mu(E_m \cap K) = 0.$$

Let
$$K_{m,p} = \left\{x \in E_m : \overline{\lim_{n\to\infty}} |f_n(x)| \geqslant \frac{1}{p}\right\},$$

where $f_n(x)$ is defined by (3·15). Since

$$E_m = \bigcup_{p=1}^{\infty} K_{m,p}$$

we confine ourselves to proving for a single p that

$$\mu(K_{m,p} \cap K) = 0. \tag{3·16}$$

Let W_1 be a neighbourhood of 0 in $\hat{G}$ such that

(i) $1 \leqslant n_1 < n_2 < \ldots < n_k \leqslant m$ and

(ii) $\lambda_{n_i} - \lambda_{n_j} \in W_1$ for $1 \leqslant i, j \leqslant k$ implies

(iii) $k \leqslant Cn^{\delta}$.

Let $W_2 = \{\gamma \in \hat{G} : |1 - (x, \gamma)| < 1/2mp \quad \text{for all} \quad x \in K\}$.

Then W_2 is a neighbourhood of 0 in G. Let U be a symmetric neighbourhood of 0 in $\hat{G}$ such that $U + U \subset W_1 \cap W_2$. Let D in $\mathscr{A}$ be such that $D + U = \hat{G}$.

Now choose $\gamma_1, \gamma_2, \ldots$ in D such that $\lambda_n - \gamma_n \in U$, $n = 1, 2, \ldots$. Then $(\gamma_n)_{n=1}^{\infty}$ satisfies the growth condition Cn^{δ} in D. For we can take V to be $\{0\}$ in Definition (2·1) so that $\gamma_{n_i} - \gamma_{n_j} \in V$ means $\gamma_{n_i} = \gamma_{n_j}$. If we have (i) and

$$\gamma_{n_1} = \gamma_{n_2} = \ldots = \gamma_{n_k},$$

then for $1 \leqslant i, j \leqslant k$,

$$\lambda_{n_i} - \lambda_{n_j} = \lambda_{n_i} - \gamma_{n_i} + \gamma_{n_j} - \lambda_{n_j} \in U + U \subset W_1$$

and so $k \leqslant Cn^{\delta}$ by the hypothesis on $(\lambda_n)_{n=1}^{\infty}$. This proves our assertion. Let

$$E = \{x \in K : \lim_{n\to\infty} |g_n(x)| > 0\}$$

where
$$g_n(x) = \frac{1}{n} \sum_{k=1}^{n} (x, m\gamma_k).$$

By the case discussed at the start of the proof, $\mu(E) = 0$. To prove (3·16) it is enough to prove

$$K_{m,p} \cap K \subset E. \tag{3·17}$$

Let $x \in K$, $x \notin E$. Then

$$\begin{aligned} |f_n(x)| &\leqslant |f_n(x) - g_n(x)| + |g_n(x)| \\ &\leqslant \left| \frac{1}{n} \sum_{k=1}^{n} (x, m\lambda_k) - \frac{1}{n} \sum_{k=1}^{n} (x, m\gamma_k) \right| + \epsilon_n \end{aligned}$$

where $\epsilon_n \to 0$ as $n \to \infty$. Clearly

$$|f_n(x)| \leqslant \epsilon_n + m \max_{k \leqslant n} |\lambda_k(x) - \gamma_k(x)|.$$

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Since $x \in K$ and $\lambda_k - \gamma_k \in U$ $(k = 1, \ldots, n)$ and $U \subset W_2$,

$$|f_n(x)| \leqslant \epsilon_n + \frac{m}{2mp}$$

and

$$\overline{\lim_{n\to\infty}} |f_n(x)| \leqslant \frac{1}{2p}. \tag{3·18}$$

Therefore $x \notin K_{m,n}$, so (3·17) holds and the proof of the theorem is complete.

Remarks. (*a*) This proof resembles Theorem 3·2 of (1) but it seems worth giving full details for the sake of clarity.

(*b*) Note that in the proof p could depend on D. We know of no example of such a measure μ.

3·4. LEMMA. *Let G be a non-discrete compactly generated, locally compact Abelian group. Let $p > 0$, $4 < p < \infty$. Then there are continuous singular measures μ in $M^+(T)$ which satisfy* (3·13) *for a dense class $\mathscr{A}$ of discrete subgroup D of G.*

Proof. Here 'singular' means with respect to Haar measure m on G. Now by a theorem of van Kampen, G is topologically isomorphic with $R^a \times Z^b \times F$ for some non-negative integers a and b and some compact Abelian group F. We shall identify G with this group and $\hat{G}$ with $R^a \times T^b \times \hat{F}$. (See e.g. (6), Theorem 9·8.) As in (1), we take the class $\mathscr{A}$ to consist of all subgroups.

$$D = \left(\frac{1}{m} Z^a\right) \times (Z(m))^b \times \hat{F} \quad (m = 1, 2, \ldots), \tag{3·19}$$

where $(1/m)\, Z^a$ is self-explanatory and $Z(m)$ is the subgroup of T with m elements.

By a theorem of Hewitt and Zuckerman (5), if F is infinite there is a measure $\lambda \in M^+(F)$ such that λ is continuous and singular and $\lambda^* \lambda$ is absolutely continuous with Radon–Nikodym derivative in $L^r(F)$ for all r, $1 < r < 2$. It follows by the Hausdorff–Young theorem (which applies equally well to locally compact Abelian groups; see the proof in (12), Vol. II, Ch. XII, (2·3), where the modifications needed are obvious) that $(\lambda * \lambda)^\wedge = (\hat{\lambda})^2$ belongs to $L^{r'}(G)$ for $2 < r' < \infty$ and so $\hat{\lambda}$ belongs to $L^p(\hat{G})$ for $4 < p < \infty$.

By a theorem of Wiener and Wintner (11), there is a continuous singular measure $\nu \in M^+(R)$ whose Fourier–Stieltjes transform satisfies

$$\hat{\nu}(u) = O(|u|^{-2/p}) \quad \text{as} \quad |u| \to \infty. \tag{3·20}$$

We construct μ as follows. Let ϵ_0 be the unit mass at 0 in Z^b. If $a > 0$ and F is infinite, let μ be the product measure

$$\mu = \nu \times \nu \times \ldots \times \nu \times \epsilon_0 \times \lambda \quad (a \text{ factors } \nu). \tag{3·21}$$

If $a = 0$ and F is infinite identify G with $Z^b \times F$ and let

$$\mu = \epsilon_0 \times \lambda. \tag{3·22}$$

If $a > 0$ and F is finite let

$$\mu = \nu \times \nu \times \ldots \times \nu \times \epsilon_0 \times \delta_0 \quad (a \text{ factors } \nu) \tag{3·23}$$

where δ_0 is the unit mass at 0 in F.

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In case (3·21), elementary calculations show that $\mu \in M^+(G)$, $\mu \neq 0$, that μ is continuous and singular, and that

$$\hat{\mu}(x_1, \ldots, x_a, t_1, \ldots, t_b, d) = \hat{\nu}(x_1) \ldots \hat{\nu}(x_a)\, \hat{\lambda}(d) \tag{3·24}$$

where $(x_1, x_a) \in R^a$, $(t_1, t_b) \in T^b$ and $d \in \hat{F}$.

Let D be as in (3·19). Then as one may easily verify

$$\sum_{\gamma \in D} |\hat{\mu}(\gamma)|^p = m^b \|\hat{\lambda}\|_p \left[\sum_{k=-\infty}^{\infty} \left| \hat{\nu}\left(\frac{k}{m}\right) \right|^p \right]^a < \infty \tag{3·25}$$

since $|\hat{\nu}(k/m)|^p = O(1/k^2)$ as $|k| \to \infty$ by (3·20). This completes the proof in the case (3·21). The other two cases are similar but easier.

Remarks. (*a*) A study of (9) shows that there is considerable freedom of choice of λ to satisfy the requirements of the proof. As is well-known, there is also considerable freedom of choice of ν.

(*b*) The above proof actually works when $G = R^a \times D \times F$, where D is any discrete group such that D has a dense class of finite subgroups. An open question is whether the lemma holds for any locally compact Abelian group that admits a dense class of discrete subgroups.

(*c*) We shall use a special case of Lemma 3·4 in Theorem 3·6 below, but its primary purpose is to show to what extent Theorem 3·3 is a valid refinement of the basic assertion that $E(\Lambda)$ has measure zero.

It seems worth singling out the consequences of Theorem 3·3 for Euclidean space.

3·5. THEOREM. *Let $\mu \in M^+(R^a)$ and suppose $\hat{\mu}$ admits a majorization*

$$|\hat{\mu}(u)| \leqslant w(\|u\|) \quad (\|u\| = \textit{Euclidean norm})$$

where w is a monotonically decreasing function on the positive real line such that for some p, $2 < p < \infty$,

$$\int_0^\infty w(t)^p \, dt < \infty.$$

Let $\Lambda = (\lambda_n)_{n=1}^\infty$ be any sequence in R^a such that

$$\|\lambda_1\| \leqslant \|\lambda_2\| \leqslant \ldots$$

and

$$\|\lambda_{n+k}\| \geqslant \|\lambda_n\| + c \quad \textit{for} \quad k \geqslant Cn^\delta$$

where $c > 0$, $C > 0$ and δ, $0 < \delta < 1$ are constants. Then

$$\mu(E(\Lambda)) = 0$$

regarding the λ_n as characters of the group R^a.

Proof. It is easy to verify that $(\lambda_n)_{n=1}^\infty$ satisfies the growth condition Cn^δ and that $\hat{\mu}$ satisfies (3·13). The result then follows from Theorem 3·3. We omit the details.

3·6. THEOREM. *Let G be any Weyl group. Then there are sets E in G, of Haar measure zero, such that*

$$E \not\subset E(\Lambda)$$

for every sequence Λ in $\hat{G}$ satisfying a growth condition Cn^δ, where $C > 0$ and $0 < \delta < 1$.

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Proof. Let $G_0 = G_1 \times R^a$ be an open subgroup of G, where G_1 is compact. Let μ be a continuous singular measure in $M^+(G_0)$, $\mu(G_0) = 1$, which satisfies (3·13) for a dense class of discrete subgroups of $\hat{G}_0$. We know by Lemma 3·4 that μ exists. Let E be a Borel set of Haar measure zero in G_0 on which μ is concentrated. By Lemma 2·7, E has Haar measure zero in G.

Let $\Lambda = (\lambda_n)_{n=1}^{\infty}$ be any sequence in $\hat{G}$ satisfying a growth condition Cn^{δ}. As in the proof of Lemma 2·6, the restrictions of λ_n, regarded as a sequence in $\hat{G}_0$, satisfy the growth condition Cn^{δ}. By Theorem 3·3,

$$\mu(E'(\Lambda)) = 0,$$

where $E'(\Lambda)$ is the set of x in G_0 for which $(\lambda_n(x))_{n=1}^{\infty}$ is not uniformly distributed. Since μ is concentrated on G_0,

$$\mu(E(\Lambda)) = 0.$$

Since $\mu(E) = 1$, we obviously cannot have $E \subset E(\Lambda)$ and indeed $E \backslash E(\Lambda)$ is non-denumerable. This completes the proof.

Remarks. (*a*) Ignoring the rather more restrictive growth condition, Theorem 3·6 in fact says that the generalized Weyl theorem is never 'best possible'. As we have already seen, and as we shall find further in section 4, a good deal more can be said in the classical cases, $G = R$ or T.

4. *On the set of non-normal numbers in the classical groups.* In this section we prove the following result.

THEOREM 4·1. *Let $(\lambda_n)_{n=1}$ be an increasing sequence of real numbers and suppose*

$$\lambda_{n+1} - \lambda_n \geqslant c > 0 \quad (n = 1, 2, \ldots). \tag{1}$$

Let μ be a positive finite regular Borel measure on R. If

$$\hat{\mu}(y) = 0\left(\frac{1}{(\log|y|)^{5+\delta}}\right) \quad (|y| \to \infty) \tag{2}$$

for some $\delta > 0$, then

$$\lambda_1 x, \lambda_2 x, \ldots \tag{3}$$

is uniformly distributed (mod 1) *for μ-almost all x.*

To put this in the language of section 3, we have only to map R onto r via the 1–1 correspondence

$$\lambda \to \lambda' \quad \text{where} \quad \lambda(x) = e^{2\pi i \lambda' x} \quad (\lambda \in \hat{R},\ \lambda' \in R).$$

The conclusion of the theorem is then just

$$\mu(E(\Lambda)) = 0. \tag{4}$$

Theorem 4·1 is essentially stronger than Theorem 3·5, and we shall say a few words about this at the end of section 4. Our method is a fairly straightforward modification of that of Erdös and Koksma (3); but our result is quite different, and it seems worthwhile to write out the proof in detail.

We use two quite deep theorems as lemmas.

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LEMMA 1. (Erdös–Turán.) Let $u_1, u_2, \ldots$ be a real sequence and let $D(N)$ denote its discrepancy. Then for each integer $m \geqslant 1$, we have

$$N.D(N) \leqslant K\left(\frac{N}{m+1} + \sum_{h=1}^{m} \frac{1}{h}\left|\sum_{n=1}^{N} e^{2\pi i u_n}\right|\right)$$

where K is a numerical constant.

A reference for this result may be found in (3).

LEMMA 2. *Let $F(M,N) = F(M,N;\theta)$ denote a bounded measurable function of θ on R for each pair of integers $M \geqslant 0$, $N \geqslant 0$, such that*

$$|F(M,N)| \leqslant |F(M,N_1)| + |F(M+N_1, N-N_1)| \tag{5}$$

whenever $M \geqslant 0$, $N \geqslant 0$, and $N_1 \leqslant N$. Suppose further that

$$\int_R |F(M,N;\theta)|^2 \, d\mu(\theta) \leqslant K_0 . N^2 . (\log N)^{-(3+\delta)} \quad (N \geqslant 2) \tag{6}$$

where μ is a probability measure on R, K_0 and $\delta > 0$ being constants. Then

$$F(0,N;\theta) = o(N) \tag{7}$$

for μ-almost all θ.

This lemma is a special case of a theorem of Gál and Koksma (4). (In the notation of that paper, the special case is $\Phi(M,N) = K_0 N^2 (\log N)^{-(3+\delta)}$, $N \geqslant 2$, $\phi(N) = (\log N)^{3+\delta/2}$ $(N \geqslant 2)$, $p = 2$, $\psi(k) = 2^k$ and $\Omega_k(h) = 2^h$.)

We now imitate the analysis of Erdös and Koksma.

LEMMA 3. *Let $(\lambda_n)_{n=1}$ be a sequence of real numbers satisfying* (1) *and μ a regular positive Borel measure on R satisfying* (2). *Then*

$$\int_R N^2 . D^2(N,\theta) \, d\mu(\theta) \leqslant K_0 . N^2 . (\log N)^{-(3+\delta)} \quad (N \geqslant 2), \tag{8}$$

where K_0 depends only on μ and c.

Proof. Put $m = [\sqrt{N}]$. Then, by Lemma 1,

$$N^2 . D^2(N,\theta) \leqslant K^2\left(N + 2\sqrt{N}\sum_{h=1}^{[\sqrt{N}]} \frac{1}{h}\left|\sum_{n=1}^{N} e^{2\pi i h \lambda_n \theta}\right|\right)$$
$$+ K^2\left(\sum_{h=1}^{[\sqrt{N}]}\sum_{k=1}^{[\sqrt{N}]} \frac{1}{hk}\left|\sum_{n=1}^{N} e^{2\pi i h \lambda_n \theta}\right|\left|\sum_{n=1}^{N} e^{2\pi i k \lambda_n \theta}\right|\right). \tag{9}$$

Integrating with respect to $d\mu$,

$$\int_R N^2 D^2(N,\theta) \, d\mu(\theta) \leqslant K^2\left(N\mu(R) + 2\sqrt{N}\sum_{h=1}^{[\sqrt{N}]} \frac{1}{h}\int_R \left|\sum_{n=1}^{N} e^{2\pi i h \lambda_n \theta}\right| d\mu(\theta)\right)$$
$$+ K^2\left(\sum_{h=1}^{[\sqrt{N}]}\sum_{k=1}^{[\sqrt{N}]} \frac{1}{hk}\int_R \left|\sum_{n=1}^{N} e^{2\pi i h \lambda_n \theta}\right|\left|\sum_{n=1}^{N} e^{2\pi i k \lambda_n \theta}\right| d\mu(\theta)\right) \tag{10}$$

and by the Cauchy–Schwartz inequality for integrals

$$\leqslant K^2\left(N\mu(R) + 2\sqrt{N}\sum_{h=1}^{[\sqrt{N}]} \frac{1}{h}\left\{\int_R l^2 \, d\mu(\theta) . \int_R \left|\sum_{n=1}^{N} e^{2\pi i h \lambda_n \theta}\right|^2 d\mu(\theta)\right\}^{\frac{1}{2}}\right)$$
$$+ K^2\left(\sum_{h=1}^{[\sqrt{N}]}\sum_{k=1}^{[\sqrt{N}]} \frac{1}{hk}\left\{\int_R \left|\sum_{n=1}^{N} e^{2\pi i h \lambda_n \theta}\right|^2 d\mu(\theta) . \int_R \left|\sum_{n=1}^{N} e^{2\pi i k \lambda_n \theta}\right|^2 d\mu(\theta)\right\}^{\frac{1}{2}}\right)$$

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Now
$$\int_R \left| \sum_{n=1}^{N} e^{2\pi i k \lambda_n \theta} \right|^2 d\mu(\theta) = 2 \sum_{1 \leqslant m < n \leqslant N} \hat{\mu}(k(\lambda_n - \lambda_m))$$
$$\leqslant K_1 \sum_{1 \leqslant m < n \leqslant N} \frac{1}{\{\log 2(n-m)\}^{5+\delta}}$$
$$\leqslant K_1 . N \sum_{n=2}^{2N} \frac{1}{(\log n)^{5+\delta}}$$

using (1) and (2), where K_1 depends only on μ and c. The last sum is dominated by

$$64 + \sum_{n}^{2N} \frac{1}{(\log x)^{5+\delta}} \leqslant \frac{CN}{(\log N)^{5+\delta}}$$

where C is a numerical constant. The last inequality may be proved by dominating the integrand with a multiple of

$$\frac{d}{dx}\left(\frac{x}{(\log x)^{5+\delta}}\right) = \frac{(\log x)^{5+\delta} - (5+\delta)(\log x)^{4+\delta}}{(\log x)^{10+2\delta}}$$

for $x \geqslant x_0$ say. Thus for $k = 1, 2, \ldots, N = 1, 2, \ldots$

$$\int_R \left| \sum_{n=1}^{N} e^{2\pi i k \lambda_n \theta} \right|^2 d\mu(\theta) \leqslant \frac{K_2 . N^2}{(\log N)^{5+\delta}}, \tag{11}$$

where K_2 depends only on μ and c. Substituting this into (10),

$$\int_R N^2 D^2(N, \theta)\, d\mu(\theta) \leqslant K^2 \left(N\mu(R) + 2\sqrt{N} \sum_{h=1}^{[\sqrt{N}]} \frac{\mu^{\frac{1}{2}}(R)}{h} K^{\frac{1}{2}} \frac{N}{(\log N)^{\frac{1}{2}(5+\delta)}} \right)$$
$$+ K^2 \sum_{h=1}^{[\sqrt{N}]} \sum_{k=1}^{[\sqrt{N}]} \frac{1}{hk} \frac{N^2}{(\log N)^{5+\delta}}$$
$$\leqslant K^2 N\mu(R) + K_3 N^{\frac{3}{4}} (\log N)^{-\frac{1}{2}(3+\delta)} + K^2 (\log N)^{-(3+\delta)}$$
$$\leqslant K_0 N^2 (\log N)^{-(3+\delta)}$$

as asserted, where K_3, K_0 depend only on μ and c.

Proof of theorem 4·1. Let M be any integer $\geqslant 1$. The sequence $(\lambda_{M+n})_{n=1}^{\infty}$ satisfies the hypotheses of Lemma 3. The discrepancy of $\lambda_{M+1}\theta, \lambda_{M+2}\theta, \ldots$ being written $D(M, N, \theta)$, the function $F(M, N, \theta) = ND(M, N, \theta)$ satisfies (5), as is easily verified. Thus Lemma 2 is applicable and in this case (7) reads

$$D(0, N, \theta) = o(1)$$

for almost all θ with respect to μ, which proves Theorem 4·1.

Remarks. (*a*) It is not difficult to see that what is required of $\hat{\mu}$ is precisely

$$\sum_{n=1}^{N} |\hat{\mu}(dn)| = O\left(\frac{N}{(\log N)^{5+\delta}}\right),$$

each $d > 0$ which certainly follows from the hypothesis of Theorem 3·5 (see the argument just below).

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(*b*) We could have obtained Theorem 4·1 for a compact connected group G instead of R (e.g. $G = T$). The hypothesis would be that, ordering the elements of $\hat{G}$ for which $\hat{\mu}(\lambda) \neq 0$ in any way as $\gamma_1, \gamma_2, \ldots$ we have

$$\sum_{n=1}^{N} |\hat{\mu}(\gamma_n)| = 0\left(\frac{N}{(\log N)^{5+\delta}}\right) \quad (\delta > 0) \tag{12}$$

and the conclusion that $\mu(E(\Lambda)) = 0$. We leave the details to the reader. The conclusion can be drawn in particular if $\hat{\mu} \in L^p(\hat{G})$, $0 < p < \infty$, for then by Hölder's inequality

$$\sum_{n=1}^{N} |\hat{\mu}(\gamma_n)| \leqslant \|\hat{\mu}\|_p \left(\sum_{n=1}^{N} 1^{p'}\right)^{1/p'} = O(N^{1/p'}) \quad (p' > 1).$$

So Theorem 3·1 is a corollary. However, the simple direct proof that $\mu(E(\Lambda)) = 0$ covers many more groups, and (12) seems rather artificial on a group.

(*c*) Theorem 4·1 tells us in particular that if E is a compact subset of R, and E supports a measure $\mu \geqslant 0$ satisfying (2), then uncountably many elements of E are non-normal numbers for a $(\lambda_n)_{n=1}$ satisfying (1).

REFERENCES

(1) Baker, R. C. A Diophantine problem on groups, I. To appear. *Trans. Amer. Math. Soc.*

(2) Baker, R. C. A Diophantine problem on groups, II. To appear. *Proc. Lond. Math. Soc.*

(3) Erdös, P. and Koksma, J. F. On the uniform distribution of sequences $(f(n, \theta))$. *Proc. Kon. Ned. Akad. van Wetensch.* **52** (1949), 851–854.

(4) Gál, I. S. and Koksma, J. F. Sur l'ordre de grandeur des fonctions sommables. *C.R. Acad. Sci. (Paris)* **227** (1948), 1321–1323.

(5) Hewitt, E. and Zuckerman, H. S. Continuous measures with absolutely continuous convolution squares. *Proc. Camb. Phil. Soc.* **62** (1966), 339–420. Corrigendum, *ibid.* **63** (1967), 367–368.

(6) Hewitt, E. and Ross, K. A. *Abstract harmonic analysis I* (Springer-Verlag, 1963).

(7) Kahane, J. P. Sur les mauvaises repartitions modulo 1. *Ann. Inst. Fourier (Grenoble)*, **14** (1965), 519–526.

(8) Kahane, J. P. and Salem, R. Distribution modulo one and sets of uniqueness. *Bull. Amer. Math. Soc.* **70** (1964), 259–261.

(9) Rudin, W. *Fourier analysis on groups.* (Interscience, 1962.)

(10) Salem, R. Uniform distribution and capacity of sets. *Communications Sem. Math. Univ. Lund. Supplement*, 1952, pp. 193–195.

(11) Wiener, N. and Wintner, A. Fourier–Stieltjes transforms and singular infinite convolution. *Amer. J. Math.* **60** (1938), 513–522.

(12) Zygmund, A. *Trigonometric series.* Second edition (Cambridge, 1959).

https://doi.org/10.1017/S0305004100049641 Published online by Cambridge University Press