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Informal research statement

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Abstract. The editors are reproducing here an informal research statement that Dan wrote in one of his last years at the University of Maryland. It provides a rare insight into how he viewed his own work and as usual he did an excellent job in setting forth the main points in a clear way. The summary was retyped, lightly edited to correct typos, and provided with references by Ayşe Şahin. One of the editors (B.W.) added a brief appendix to cover some of the highlights of Dan's work during the last years of his life. Following the research summary is a complete list of Dan's publications as well as a second list for the other references cited.

1. *Introduction*

This is an 'informal' research statement for two reasons: first, I will speak in the first person and be rather casual in my discussions; second, I will not be complete in giving credit for many of the ideas I will talk about and will not offer a bibliography†. This latter is not because I do not want to, but rather that this is a first draft. I will work to give correct attribution as the document matures. I will cite my work by its number in my CV‡. I will not discuss all the work I have done but rather limit myself to a core set of areas that can be discussed in a relatively brief and cohesive manner.

My area of study is measurable dynamics, what is usually called 'ergodic theory'. This is a central branch of dynamical systems with broad connections to smooth and low-dimensional dynamics, symbolic dynamics, topological dynamics, you name it, and to other branches of mathematics, functional analysis, geometry, combinatorics, number theory, you name it. The central assumption of dynamics is that one has a phase space and some group or semi-group of self-maps of that space that play the role of describing time evolution of the phase space.

Although the dynamics may have come from other considerations, one assumes here that the phase space is a standard probability space $(X, \mathcal{F}, \mu)$ and the self-maps $\{T_g\}_{g\in G}$ are measure preserving. A standard probability measure is always a convex combination of an atomic measure and a continuous measure. Usually, issues related to atoms are simple and the real issues concern continuous probability measures. In measurable dynamics one seeks to understand almost sure behavior. Behavior on sets of measure zero is discounted. It is quite important now to note that, up to sets of measure zero, there is only one non-atomic standard probability space. That is to say, if $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ are two such,

† References have been included.

‡ These references are now numbered according to the complete bibliography of Rudolph's work, which appears at the end of the article.

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then there are subsets $X_0 \subset X$ and $Y_0 \subset Y$, both of full measure, and a measure preserving bijection $\phi : X_0 \to Y_0$. Thus, all of measurable dynamics on standard probability spaces (excepting atomic behavior) happens on the same space. One can choose whatever is most convenient for this space, the unit interval, a Cantor set, a torus, whatever compact metrizable space one wants.

As is general in dynamics, one is interested here in observing 'orbits' of the system $\mathcal{O}(x) = \{T_g(x)\}_{g \in G}$. As written, this is just a set. Notice, though, that any structure coming from G can be viewed now as a structure on $\mathcal{O}(x)$ so long as the map $g \to T_g(x)$ is one-to-one, i.e. the action is free. For example, if the group is $\mathbb{Z}$, then each free orbit is well ordered and if G is a topological group, each free orbit inherits this topology. I am most interested in the classical situations where each orbit can be exhausted by compact subsets with small boundary. Such groups (or more generally orbit relations) are called amenable. Classical 'ergodic theorems' involve calculating average values of functions or densities of occurrences of sets over such large compact regions of orbits and allowing the size of the regions to grow. Smallness of the boundary in this case tells one that such averages over large compact sets are close to invariant.

To organize the description of my work, I will break it into a collection of overlapping areas that cover most of what I have done.

(1) Representations of $\mathbb{R}^n$-actions.
(2) Joinings.
(3) Equivalence relations and theorems.
(4) Counterexamples and rigid examples.
(5) Isometric extensions.
(6) Endomorphisms and reverse filtrations.
(7) Ergodic theory of amenable group actions.

2. *Representations of $\mathbb{R}^n$-actions*

The standard approach to discretizing actions of $\mathbb{R}$ is to construct a Poincaré section and a return map. In measure preserving dynamics, this is the Ambrose–Kakutani theorem. It says that any action of $\mathbb{R}$ can be written as a flow under a function. To build such a flow, one takes a probability space $(X, \mathcal{F}, \mu)$, a measure preserving map $T : X \to X$, and a positive function $f \in L^1(\mu)$. The flow acts on the set in $X \times \mathbb{R}^+$ under the graph of f. The flow moves (x, t) vertically until it hits the graph of f at which time it returns to the base at $(T(x), 0)$. The Ambrose–Kakutani theorem says that any finite measure preserving flow can be represented by such a picture. This discretizes the flow in that T is a discrete action. What is not discrete though is the function f. The first significant result I ever obtained [**Rud76a**] was to show that for any values α and β in $\mathbb{R}^+$ with $(\alpha/\beta) \notin \mathbb{Q}$ the function f could be chosen to take on only the values α and β. Now the picture really is discretized in that f also is a discrete-valued function. It is easily seen that one cannot do better. Krengel showed that one could also specify $\int f$ to be any value strictly between α and β [**17**].

This result settled a problem of Sinai, who asked whether a K-flow, one of completely positive entropy, was exact in that there existed a σ-field $\mathcal{F}_0$ so that $\bigcup_{t>0} T_t(\mathcal{F}_0)$ nested up

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to $\mathcal{F}$ and $\bigcap_{t<0} T_t(\mathcal{F}_0)$ nested down to the trivial algebra. The answer is yes and it comes out of the construction.

One can ask if a similar construction works in $\mathbb{R}^n$, $n > 1$ and the answer is yes [**Rud88b, Rud89**]. At the level of Ambrose–Kakutani, Katok has shown that one can lay out a section on an orbit of an action of $\mathbb{R}^n$ that looks like the knots in a fishnet, cutting the space into approximate squares [**15**]. Notice that this says that the orbits can be measurably tiled by approximate squares. The two-valued theorem above can be viewed as showing that orbits of free $\mathbb{R}$ actions can be measurably tiled by intervals of just lengths α and β. This indicates where one can go for a new representation in $\mathbb{R}^n$. Consider a collection of tiles of the form $\otimes[0, x_i)$, where x_i is either α_i or β_i and these positive values are irrationally related. This is a collection of 2^n boxes. One can ask if orbits of an $\mathbb{R}^n$ action can be measurably tiled by this collection of tiles. The answer is yes, but this is still not enough to discretize the picture, as these tiles can meet in a continuum of ways [**Rud88b**]. So, suppose you also ask that as you move from one tile to another across a boundary of a tile orthogonal to direction i, that the new tile is translated from the one you are in by a vector whose jth coordinate is one of $\pm(\alpha_j - \beta_j)$ or 0. It is useful here to regard α_j and β_j as close together, so that these values are all small. One finds that any action can be tiled with this 'corner condition'. Moreover, one finds that the tiles of each orbit are canonically labeled by $\mathbb{Z}^n$, and that the shape of the tiles and the corner condition give a $\mathbb{Z}^n$ subshift of finite type and that this subshift is independent of the actual values α_i and β_i. A wonderful open question is to find the topological entropy of this subshift.

3. *Joinings*

A coupling of k probability spaces $(X_i, \mathcal{F}_i, \mu_i)$, $i = 1, \ldots, k$, is a measure $\hat{\mu}$ on $\otimes X_i$ which has μ_i as its ith marginal. This is an old idea in probability theory. Now suppose we have a measure preserving action T_i on each X_i. A joining of these systems is a coupling that is $\otimes T_i$ invariant. I believe this notion originated with Mackey or Furstenberg. The space of joinings is a weak* compact and convex space. If the T_i are ergodic, then the ergodic joinings are the extreme points. Joinings are a tool and I want to indicate a few of their uses. Furstenberg defined 'disjointness' of two systems to mean that the only joining of the two was product measure. The weakly mixing systems can be described as those disjoint from all maps of pure point spectrum. The K-systems can be described as those systems disjoint from all systems of zero entropy. These two characterizations make these two properties particularly friendly to work with.

Conjugacies of systems and more generally the existence of conjugate factors show up in the structure of the joinings of the systems. One can always support a joining on the graph of a conjugacy and, if the systems have a common factor, then one can construct the 'relatively independent joining' over the common factor. This material forms Chapter 6 of my book, *Fundamentals of Measurable Dynamics* [**Rud90a**].

Let me describe three ways in which I have used joinings. First, I have considered self-joinings and more generally joinings of powers of a common transformation. A single transformation is never disjoint from itself. One can always support a joining on the diagonal or, more generally, on the graph of any power of T. Any finite number of copies of T could be joined on the graph of some map $I \times T^{j_1} \cdots \times T^{j_k}$ and of course one could

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take a direct product of such 'off-diagonal' measures. A transformation for which these are the only self-joinings available is said to be of 'minimal self-joinings'. I was able to construct a map with this property [**Rud79b**]. Later I will talk about how such an example can be used.

A much larger use of joinings builds on an observation of Burton and Rothstein [**2**]. What they did was to recast Ornstein's isomorphism theorem for Bernoulli shifts into the language of joinings. In the end, what they showed was that if T is a Bernoulli shift of entropy h and S is an ergodic action of entropy $\geq h$, then, in the space of joinings of T and S, those joinings supported on the graph of a factor map from S to T are a residual set, a dense G_δ. This gives us Sinai's theorem, that such a factor map exists. But, by showing that this set is residual in a Polish space, the isomorphism theorem now comes for free. Suppose S is also a Bernoulli shift of entropy h. Then those joinings which are supported on the graphs of conjugacies will simply be the intersection of two residual sets of joinings, and hence be residual. In seeking to build generalizations of Ornstein's theorem to other equivalence relations weaker than conjugacy, I have always sought to cast them as the existence of a certain residual set of joinings of some kind. This idea will appear in the next section.

One can also use joinings to attack non-standard ergodic theorems. Bourgain, Furstenberg, Katznelson, and Ornstein have proven the following theorem [**1**].

THEOREM 3.1. [**1**] *Suppose T is an ergodic measure preserving action and $f \in L^2(\mu)$. Then there is a subset X_f of measure 1 in X so that for any other ergodic measure preserving action S and function $g \in L^2(\mu)$, for all $x \in X_f$ and almost every $y \in Y$ the averages*

$$\frac{1}{n}\sum_{i=0}^{n-1} f(T(x))g(S(y))$$

converge.

This is usually referred to as the 'return times theorem', because if one takes f to be the characteristic function of a set then one is averaging g over the return times to the set and the theorem says for almost every x this subsequence is universal for the ergodic theorem. I was able to obtain this result using joinings [**Rud94**] and was also able to obtain an inductive generalization [**Rud98b**].

THEOREM 3.2. *For all $n \in \mathbb{N}$ and lists of measure preserving ergodic actions $T_1, \ldots, T_n$ and functions $f_i \in L^\infty(\mu_i)$, there are subsets*

$$X_{f_1} \subset X_1,\ X_{f_1,f_2} \subset X_2, \ldots, X_{f_1,\ldots,f_n} \subset X_n$$

all of full measure so that for any further action T_{n+1} and function $f_{n+1} \in L^\infty(\mu_{n+1})$ for all $x_i \in X_{f_1,\ldots,f_i}$, $i = 1, \ldots, n$, and almost every $x_{n+1} \in X_{n+1}$, the averages

$$\frac{1}{n}\sum_{i=0}^{n-1}\prod_{j=1}^{n+1} f_j(T_j^i(x_j))$$

converge.

To explain what is happening here, if one uses product measure $\hat{\mu} = \mu_1 \times \mu_2 \times \cdots \times \mu_{n+1}$, then the Birkhoff theorem tells us that for $\hat{\mu}$-almost every $(x_1, \ldots, x_{n+1})$ these

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averages converge. What this result does is to show that the good set for this ergodic theorem has the inductive form described above. Joinings make this result approachable.

4. *Equivalence relations and theorems*

I mentioned Ornstein's isomorphism theorem for Bernoulli shifts above. A completely parallel theory was built up in the late 1970s by Feldman, Ornstein, Weiss, Katok, and myself [**ORW82**] around the notion of Kakutani equivalence. This concept goes back to the Ambrose–Kakutani theorem mentioned earlier. A flow will have many different representations as a flow under a function. The Poincaré maps of different sections, though, will always be related in a very simple way. One can always find subsets A and B of the two cross-sections, respectively, so that the induced maps on these subsets are conjugate. This is the classical notion of 'Kakutani equivalence'. Kakutani had shown this to be an equivalence relation. What Ornstein and Weiss showed was that one could develop a theory completely analogous to the isomorphism theory that characterized those actions Kakutani equivalent to a Bernoulli shift. This theory also has a zero-entropy case, first discussed by Katok [**14**], that characterizes those systems Kakutani equivalent to an isometry of a compact metric space. Where this theory appears to differ from the isomorphism theory is that entropy is not an invariant of Kakutani equivalence. If one asks just a bit more, that the subsets A and B have the same measure, one obtains a refinement, called 'even Kakutani equivalence', and this equivalence relation does indeed preserve entropy. In the late 1950s an even more striking 'equivalence theorem' had been proven by Dye. He showed that all free ergodic measure preserving actions on a standard space are orbit equivalent [**5**]. That is to say, there is a measure preserving bijection, defined almost everywhere, taking the orbits of one action of $\mathbb{Z}$ to the other. In the late 1970s Feldman suggested that all these might be special cases of one grand theorem. The role of entropy seemed particularly unclear though, as two of the relations preserved entropy but the third did not.

In a series of major works [**KR97, KR02, Rud85b**], we were able to show that this is indeed the case. I want to spend some time on the final form this takes in *Restricted Orbit Equivalence for Actions of Discrete Amenable Groups*, joint with my former student Janet Kammeyer. For this theory the group acting is any discrete and amenable group. Ornstein and Weiss had shown that one has an entropy theory and that the isomorphism theory holds in this category [**23**]. Connes, Feldman, and Weiss [**3**] have characterized measure preserving actions of countable discrete amenable groups as precisely those that are orbit equivalent to $\mathbb{Z}$-actions, a broad strengthening of Dye's theorem. What we set out to do was to construct a perturbation theory for a general notion of 'restricted orbit equivalence'. Here is the idea. Fix a measure space $(X, \mathcal{F}, \mu)$ and some free and ergodic action of the countable discrete and amenable group G, called $T = \{T_g\}$. Now consider perturbing the orbits of T by an element ϕ of the full group of T. This full group Γ consists of all measure preserving bijections with $(x, \phi(x))$ in the orbit relation. The perturbation is simply $\phi T_g \phi^{-1}$. One sets up some axioms for what we call the 'size' $m(T, \phi)$ of this perturbation. The most important thing these axioms require is that the 'size' $m(\phi_1 T_g \phi_1^{-1}, \phi_2 \phi_1^{-1})$ of the perturbation from $\phi_1 T_g \phi_1^{-1}$ to $\phi_2 T_g \phi_2^{-1}$ should be a pseudometric $m_T(\phi_1, \phi_2)$ on the full group. In addition, some basic continuity facts are required, which I will not spell out.

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Abstractly, what one now does is to complete the full group with respect to this pseudometric obtaining a complete metric space. One finds that limits of Cauchy sequences of perturbations correspond to actual actions of G on the orbits of T. If S is such an action, one subtle aspect of the theory is that it may not now be possible to get back to T via a Cauchy sequence of perturbations of S. None-the-less, if one takes those S reachable by perturbations of T and which in turn can be perturbed back to T, one gets a residual subset of the completion, hence a Polish space. Further, this relation between actions T and S is an equivalence relation. We call such a relation a 'restricted orbit equivalence' or m-equivalence to specify the size involved.

One now proves some broad facts. I will state two. First, one can define an entropy that will be an invariant of a restricted orbit equivalence just by taking the infimum of the entropies in an equivalence class. We call this m-entropy. One can show that a restricted orbit equivalence must be of one of two types. Either the m-entropy is the classical entropy, or it is universally zero. In the latter case, in each equivalence class the actions of zero entropy are residual. We call the first type of size or equivalence relation 'entropy preserving' and the second type 'entropy free'. This explains the difference between Dye's theorem and the isomorphism theorem, as both conjugacy and orbit equivalence are restricted orbit equivalences but one is entropy preserving and the other is entropy free. Second, one also shows that for each size m there are distinguished classes, called m-finitely determined, and that in these classes m-entropy is a complete invariant of m-equivalence. We refer to this as the equivalence theorem. This result now subsumes Ornstein's theorem, the Kakutani equivalence theorem, and Dye's theorem.

Let me mention two further examples of restricted orbit equivalences. Earlier we indicated that any flow could be represented as a flow under a two-valued function. Fix the values to be 1 and $1 + \alpha$, where α is a positive irrational. Now say two actions are α-related if they arise as sections of a common flow, with height functions taking on only these two values and for which the integrals agree. It is not clear that this is an equivalence relation, but it is, and in fact it can be described as a restricted orbit equivalence [**FdJR94**]. Notice that it refines even Kakutani equivalence. We showed further that in the finitely determined classes it cuts an even Kakutani class into countably many α-equivalence classes. For other classes this is not true [**FR92**]. My former student Ayşe Şahin showed in her thesis that this work extended to the tiling representations of $\mathbb{R}^2$ but it remains unknown if the work extends to dimensions three or higher [**25**]. To me the most fascinating example of a restricted orbit equivalence is equality of entropy. It is evident that equality of entropy is an equivalence relation. I recently showed that it could be obtained as a restricted orbit equivalence [**Rud05**].

I will end this section by referring back to the discussion of joinings. What we do to prove the equivalence theorem is to define a space of m-joinings, a generalization of the original idea. An m-joining is a common cover for the two spaces with lifts of the two actions sitting as m-equivalent actions. We show how to topologize the space of m-joinings as a Polish space and it is here that the equivalence theorem is proven. Of course we show that for m-finitely determined actions of equal m-entropy, the m-joinings supported on graphs of m-equivalences form a residual set. Without the m-joining machinery the bookkeeping of the theorem would, I believe, be quite unmanageable.

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5. *Counterexamples and rigid examples*

Examples and applications are the heart of dynamics. In measurable dynamics there are a number of standard constructive methods for the building of examples. The one with the strongest track record goes by the name of 'cutting and stacking'. Here one constructs a map on the unit interval that is piecewise linear by successively cutting and restacking a stack of intervals. It really is very similar to the construction of a substitution or of an adic transformation in the sense of Vershik. The advantage is that it is completely general; any measure preserving system will come from such a construction. A second constructive method is through skew products or twisted products, most especially with isometries. Here one starts with an action S on a space Y and a function T mapping Y measurably to the measure preserving maps on some other space X. Now set $\hat{S}(y, x) = (S(y), T(y)(x))$.

Let me describe a few constructive results. In my thesis, using cutting and stacking and isometric extensions, I constructed two non-isomorphic K-systems with isomorphic squares [**Rud76b**]. Later I showed how to construct two non-isomorphic K-systems all of whose powers beyond 1 were isomorphic [**Rud77**]. In my thesis I also constructed two $\mathbb{R}$ actions which were isomorphic at a dense collection of times but not at all times, but I did not publish the result.

Referring back to the discussion of joinings, I constructed an example [**Rud79b**] with minimal self-joinings, that is to say for which all joinings of powers of T must be products of off-diagonals. The point of this example was that using it one could relatively painlessly construct counterexamples to a wide variety of questions. For example, Furstenberg had asked whether two actions with no common factors must be disjoint. The answer is no and it is quite easy to get that from minimal self-joinings.

The T, T^{-1} map is a skew product defined over a base S, the shift map on coin-tossing random variables. The map T can be anything, but is most interesting when it is of positive entropy. The map on the second coordinate is set to be T when the base point y is a head and T^{-1} when it is a tail. Kalikow, in brilliant work, showed that this system was not Kakutani equivalent to a Bernoulli shift [**12**]. This gave the first natural example of a system that was K and not Bernoulli. Borrowing heavily from Kalikow, I showed [**Rud88a**] that this result could be broadly generalized. Rather than describe the general criteria, let me give one example. Take for S a hyperbolic toral automorphism on a torus given as $\{(\theta, \eta) : 0\theta, \eta < 1\}$. Now suppose T_t is an ergodic $\mathbb{R}$ action of positive entropy, say a geodesic flow on a compact hyperbolic manifold. Now define

$$\hat{S}((\theta, \eta), y) = (S(\theta, \eta), T_{\sin\theta}(y)).$$

This is a real analytic action and is K and not Bernoulli. I will say more about T, T^{-1} maps when I talk about endomorphisms.

Furstenberg had shown back in the 1970s that for x irrational, the values $2^i 3^j x \mod 1$ are dense in the unit interval. He had conjectured that perhaps they must be uniformly distributed. Note that this is distinctly different from what happens for a single action $2^i x$, where orbits may have very interesting fractal closures. Uniform distribution is almost equivalent to showing that the only non-atomic invariant measure for the $\mathbb{N}^2$ action given by $x \to 2^i 3^j x$ is Lebesgue measure. I was able to prove the following theorem [**Rud90b**].

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THEOREM 5.1. *Suppose p and q are relatively prime natural numbers and μ is an invariant measure for both $x \to px \mod 1$ and $x \to qx \mod 1$. Suppose further that for one of these maps μ has positive entropy and the joint $\mathbb{N}^2$ action is ergodic. It follows that μ must be Lebesgue measure.*

My former student Aimee Johnson generalized this to multiplicatively independent integers ($p^i \neq q^j$ for any i, j) in her thesis [**11**]. Various new proofs and deep strengthenings exist to this result but to date no one has cracked the zero-entropy wall for this problem. With Johnson [**JR92**], we applied our work to show that if f and g are two commuting C^k actions on the circle, $k > 1$, one of which is p to 1 and the other q to 1, p and q multiplicatively independent, then the pair are C^k conjugate to $\times p$ and $\times q$. This level of smoothness of the conjugacy need not hold for one map alone.

One very important aspect of Ornstein's isomorphism theory is that it gives one a tool, in the form of the 'very weakly Bernoulli' condition, for verifying whether or not a system is conjugate to a Bernoulli shift. One significant early application was to show that the geodesic flow on a compact or finite-volume negatively curved manifold was Bernoulli. The invariant measure here of course is Lebesgue measure. Patterson and Sullivan had shown that for a certain class of infinite-volume hyperbolic manifolds, the 'geometrically finite' ones, there would be an invariant compact set that carried an invariant measure (the Patterson–Sullivan measure) for the geodesic flow [**24, 26**]. I was able to show [**Rud82a**] that once again the geodesic flow was Bernoulli. The idea is to use hyperbolicity, the existence of expanding and contracting foliations, much as is done for the finite-volume case. The expanding and contracting foliations are horospheres; these are geometrically flat. There is no action on these horospheres though and the measure, when diffused onto these leaves, is highly singular. What I did though was to prove that an ergodic theorem holds on these leaves. If μ_x is the leaf measure centered at x (unique up to a multiplier), then, for any bounded measurable function f, one can calculate an average value over a ball $B_r(x)$:

$$\frac{\int_{B_r(x)} f(x)\, d\mu_x}{\mu_x(B_r(x))}.$$

I showed that, at least along a subsequence, this converges to the integral of f. That is to say, even though the measure is not invariant, or even non-singular, for a horospherical flow (in fact there is no horospherical flow) there are ergodic averages that converge to a constant almost everywhere. This ergodicity of the horospheres is essentially equivalent to Bernoullicity of the geodesic flow. My former student Florence Newberger generalized this to complex hyperbolic and quaternionic manifolds using the corresponding invariant measure, now called the Bowen–Margulis measure [**21**]. Later I will say more about this idea of pushing ergodic theory into this realm of singular measures†.

6. *Isometric extensions*

Earlier we saw what a skew extension looks like: $\hat{S}(x, y) = (S(x), T(x)(y))$. If Y is a compact metrizable homogeneous space and the maps $T(x)$ are all isometries of Y, then

† Eric Holt was the last of Rudolph's students who finished his PhD research before 2010. In his dissertation he used Rudolph's technique of diffusing a measure onto orbits of an action to prove a ratio ergodic theorem. Partial results from his thesis appear in [**10**].

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we call this an 'isometric extension'. One puts Haar measure on Y of course. Notice that the simplest such example would be a direct product. In this case $\hat{S}$ would have an isometric factor and hence some point spectrum. If S were weakly mixing, i.e. had no point spectrum, but $\hat{S}$ was not weakly mixing then one can argue that, at least partially, the extension must be a direct product. What interested me was what happened at the other extreme, when the extension was also weakly mixing. In the end, what one learns is that all mixing properties that S might have will then automatically lift to $\hat{S}$. Parry had shown this for the K-property. That is to say, if S is K and $\hat{S}$ is weakly mixing, then $\hat{S}$ will also be K. I used joinings in a rather simple argument to show that if S was mixing or, more generally, k-fold mixing, and $\hat{S}$ was weakly mixing, then $\hat{S}$ must also be k-fold mixing [**Rud85a**]. In a far more technical argument, I was able to show that the Bernoulli property would also lift [**Rud78a, Rud78d, Rud78c, Rud83**]. That is to say, if S was Bernoulli and $\hat{S}$ was weakly mixing, then in fact $\hat{S}$ must be Bernoulli.

With del Junco [**dJR96**], we showed that if S was loosely Bernoulli (Kakutani equivalent to a Bernoulli shift or isometry) and $\hat{S}$ was just ergodic, then $\hat{S}$ would again have to be loosely Bernoulli. This is a very pleasing result to me in that it uses in an essential way the fact that the Kakutani equivalences of the restricted orbit equivalence theory are a residual subset of the Kakutani joinings.

These results were for actions of $\mathbb{Z}$. Kammeyer, in her thesis, showed that for $\mathbb{Z}^d$ actions the Bernoulli property again lifts to weakly mixing isometric extensions. The question for general discrete and amenable groups seems to be quite difficult.

7. *Endomorphisms and reverse filtrations*

To this point virtually all the work described is about group actions. I now want to discuss a natural class of actions of the semigroup $\mathbb{N}$ (I put 0 in $\mathbb{N}$). Notice that for a measure preserving endomorphism T, the σ-fields $T^{-i}(\mathcal{F})$ are nested and decrease in i. Such a nested sequence of σ-fields one calls a 'reverse filtration'. If one had a conjugacy between two endomorphisms it would automatically preserve this reverse filtration. This tells us two things, first the reverse filtration is an invariant and second there is a weaker notion of equivalence around, that of just preserving the reverse filtration.

My work has exclusively concerned finite to one endomorphisms. This means that each $\mathcal{F}_i$ will have finite atomic fibers over $\mathcal{F}_{i+1}$. These atomic fiber measures will give various masses to atoms and the masses that occur will be an invariant of conjugacy of the filtrations. Most work on this subject, beginning with Vershik, concerns the case where this invariant is trivial, i.e. where each fiber has p points of equal mass $1/p$. This is called a 'uniformly p to 1' filtration. The 'standard' ones are the uniform 1-sided Bernoulli p-shifts. The first result I contributed to in this area is in the thesis of my former student Deborah Heicklen. Notice that one can measurably put on each p-point fiber a cyclic permutation of order p. Conjugating by the shift map builds up an action of $\mathbb{Z}_p^{\mathbb{N}}$. A conjugacy between two such reverse filtrations will give an orbit equivalence between the two $\mathbb{Z}_p^{\mathbb{N}}$ actions. Heicklen showed that this was a restricted orbit equivalence as described earlier and that the m-finitely determined class was the one containing the standard example. Hence, one obtained a characterization of those reverse filtrations conjugate to the standard one [**7**]. Hoffman and Heicklen followed this up by showing

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that for T of positive entropy, the T, T^{-1}-endomorphism (where the base coin flipping sequence is one sided) is not standard [**8**]. Working with them, I showed [**HHR00**] that in fact the entropy of T is an invariant of conjugacy of the filtrations.

My most significant work in this area is joint with Hoffman [**HR02b**], showing that for uniformly p to 1 endomorphisms of entropy $\log p$, one could develop a full Ornstein theory of isomorphism. This does not come under the restricted orbit equivalence umbrella (yet) as the action is not of a group. One gives a 'finitely determined' notion, called *tree*-finitely determined, and shows it to be a conjugacy invariant. One proceeds through a theory of joinings. In this case one cannot use conventional joinings but must consider 'one-sided' joinings which will lead to conjugacies of the endomorphisms. Without defining this idea, the set of one-sided joinings is a compact convex space with the ergodic one-sided joinings as extreme points. One shows the precise analog of the Burton–Rothstein fact, that if S is tree finitely-determined and T is some uniformly p to 1 endomorphism, then the one-sided joinings supported on graphs of factor maps from T to S are a residual set. The isomorphism of any two tree finitely-determined actions now follows. Heicklen and Hoffman found a very nice application of this result [**9**]. If f is a rational map of the Riemann sphere of degree p, then on its Julia set it will be p to 1. One can start from one point on the Julia set and diffuse out a measure by putting uniform point masses at the successive pre-images of this one point. These measures converge to a natural invariant measure on the Julia set, making the action uniformly p to 1. Mane had shown that for a sufficiently high power f^j this action would be conjugate to a Bernoulli action. Our characterization allowed Heicklen and Hoffman to show that f itself is conjugate to a one-sided p-shift.

Recently, with my former student Chao-Hui Lin [**LR04**], we have embarked on an attempt to understand the theory of cross-sections, Poincaré maps, and Kakutani equivalence for endomorphic flows. The construction of sections and Poincaré maps for endomorphic flows was carried out by Kubo in 1969 [**18**]. Conjugacy of induced maps is no longer an equivalence relation for endomorphisms and no longer is equivalent to the maps arising as sections of a common flow. The classical Kakutani arguments are just not available if the maps are not invertible. We show though that if you replace conjugacy by 'shift equivalence' then most of the classical theory can be regained. We say two endomorphisms T and S are shift equivalent if there exist two factor maps $U : X \to Y$ and $V : Y \to X$, $UT = SU$, and $TV = VS$ and moreover $UV = T^j$ for some j. This latter condition is just asking that $UV(x)$ should lie on the same orbit as x. If T or S were an automorphism, then shift equivalence implies conjugacy but not for endomorphisms. If one says two actions are 'Kakutani shift equivalent' if they induce shift-equivalent actions on subsets, the classical arguments of Kakutani reappear†.

Much remains to be done in the study of endomorphisms. What one finds oneself looking at, instead of an orbit as a copy of a group, is a tree of inverse images with weights on the nodes coming from conditional expectations. The arguments, though, now have much the same flavor as for automorphisms. One can hope to develop a perturbation theory much like that of restricted orbit equivalence to try to manage this more elaborate and very

† Rudolph's student McClendon also worked on the problem of classifying endomorphic flows. Some partial results from his thesis appear in [**20**].

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interesting structure. This leads me to discuss a much broader context for measurable dynamics. One can consider a generalized orbit as a foliation of the measure space by copies of some graph or metric space. One would want this metric space to be, at least in the large, a symmetric space, i.e. orbits of distinct points should look the same, at least after some bounded shift. Since this is ergodic theory, there should be a measure on these leaves, or rather a family of equivalent measures up to normalization. In this broad context what can one do? This is a major project that I am pursuing with Elon Lindenstrauss. Right now we are working to show that if the measure is simply recurrent on the leaves, i.e. the mass of almost every leaf is infinite, then there is a leaf-wise ratio ergodic theorem [**19**]†. More generally, one might hope to establish the restricted orbit equivalence theory in the broad context, but this is wildly conjectural.

8. *Ergodic theory of amenable group actions*

The restricted orbit equivalence theory was cast in the category of actions of countable and discrete amenable groups. The reason for this was that Ornstein and Weiss had laid out the basic ergodic theory here, most especially a theory of measurable quasi-tilings and an entropy theory [**23**]. Moreover, Connes, Feldman, and Weiss had shown that all such ergodic actions were orbit equivalent [**3**]. These broad results give one hope to establish this as the natural home for discrete ergodic theory and in particular it means a vast array of results known for $\mathbb{Z}$ could perhaps be generalized. There are two particular results that came to the fore in this direction: are there K-systems (here called 'CPE' for completely positive entropy) of countable Lebesgue spectrum and are they mixing? For $\mathbb{Z}^d$, this was settled by Kamiński and Liardet [**13**]. Thouvenot has produced an argument for the discrete rationals [**27**]. The problem, of course, is that these results for $\mathbb{Z}$ depend heavily on tail fields and the well ordering of $\mathbb{Z}$. What people were seeking were tail fields and it seemed an endless project, group by group.

What Weiss and I did [**RW00**] was to show that one could get such results out of orbit equivalence. One cannot simply say 'as the result is true for $\mathbb{Z}$ and our action is orbit equivalent to a $\mathbb{Z}$ action, the result is again true' as no property other than ergodicity is preserved by a general orbit equivalence. What we did instead was to develop a relativized theory of orbit equivalence. Suppose T has an invariant sub-σ-algebra H and the orbit equivalence is H-measurable. Then many properties of T given conditionally on H remain invariant under the orbit equivalence. The best example is the relative entropy of T over H, which we showed is invariant for an H-measurable orbit equivalence. We also showed that a uniform multiple-mixing property, equivalent to the K-property for actions of $\mathbb{Z}$, also had an H-relative version that was invariant under H-measurable orbit equivalences. This allowed us to show that CPE actions of any countable discrete amenable group action would be mixing of all orders. Let me sketch the argument, as it will indicate how the general method works. What one does is to take the direct product of the given CPE action T with an auxiliary Bernoulli action B of the same group. As B is orbit equivalent to an action U of Z, this direct product has a B measurable orbit equivalence to an extension $\hat{U}$ of U. Now as T is CPE, $T \times B$ is B-relatively CPE and so $\hat{U}$ is U-relatively CPE, hence

† Rudolph co-authored the appendix to [**19**].

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$\hat{U}$ satisfies the uniform multiple-mixing property relative to U and hence so does $T\hat{B}$ relative to B and the conclusion follows. I call the method used here the 'orbit transference method'. Dooley and Golodets have used it to show that CPE actions are of countable Lebesgue spectrum [**4**]. It is now clear that much of the theory of $\mathbb{Z}$ actions can be lifted by this method; it is just a matter of writing down the arguments. Some facts, though, appear extremely resistant. Currently the most resistant is my old theorem for $\mathbb{Z}$ actions that weakly mixing isometric extensions of mixing actions must again be mixing. One would like to say this was true for actions of general discrete amenable groups. The core issue here to using orbit transference is that the notion of mixing does not have a single good notion relative to a sub-σ-field H. I showed in [**Rud04**] that there are in fact two, a pointwise and an L^1 notion, which are distinct and natural. Until this relative mixing theory is better understood, the problem remains open.

A. *Appendix. Dan's Research During the Colorado Years (Benjamin Weiss)*

During the last years of his life Dan spent much of his time on two new areas of research, and I will limit myself in this brief appendix to his own research summary describing this work. The first area has to do with topological refinements of the classical orbit equivalence and Kakutani equivalence. A refinement of this kind was pioneered by Keane and Smorodinsky when they showed in 1979 that Bernoulli shifts with the same entropy are not only isomorphic (as had been shown by Ornstein [**22**]) but are actually finitarily isomorphic [**16**]. This means that the code from one Bernoulli shift (the source) to the other (the target) which implements the isomorphism requires only a finite amount of source data to determine the value of the target at time zero. A finitary isomorphism can be described in a more general fashion by saying that it is almost continuous, i.e. after removal of a null set it is continuous. Hamachi and Keane adapted this idea to the classical theorem of Dye, which asserts that any two ergodic actions of $\mathbb{Z}$ are orbit equivalent. They showed that the dyadic odometer is not merely orbit equivalent to the ternary odometer, but that the mapping which gives the orbit equivalence can be chosen to be almost continuous [**6**]. This initial result brought in its wake a number of generalizations to wider and wider classes of actions and in addition Dan extended this project to Kakutani equivalence [**dJRW09, DR, DR10, RR08a, RR08b, RR09a, RR09b**]. While some fairly general results have been obtained, this is still an active field of research.

The second area involves the introduction of ideas from descriptive set theory to ergodic theory. The pioneer in this area was Matt Foreman who had the idea that one should show that the isomorphism problem in ergodic theory had proven to be difficult because as a relation it is complicated in a very precise sense. The space $\mathcal{E}$ of ergodic actions of the group of integers on a standard Borel probability space is in a natural way a Polish space, and the relation of isomorphism as a subset of $\mathcal{E} \times \mathcal{E}$ is easily seen to be an analytic set. Matt had an idea as to how one could show that this set is not simply an analytic set, but the most complicated kind of analytic set there is – namely a complete analytic set. The idea was to take a canonical complete analytic set such as the set of trees with an infinite branch as a subset of the set of all countably branching trees and map the space of trees into the space of pairs of ergodic actions in such a way that the pair would be isomorphic if and only if the tree had an infinite branch. Carrying out this idea turns out to be rather

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complicated and it was only with Dan's active help that we were able to do this. While Dan took part in the final revision, it appeared too late for him to see it in print [**FRW11**].

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