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Smale flows on $\mathbb{S}^2 \times \mathbb{S}^1$

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Abstract. In this paper, we use abstract Lyapunov graphs as a combinatorial tool to obtain a complete classification of Smale flows on $\mathbb{S}^2 \times \mathbb{S}^1$. This classification gives necessary and sufficient conditions that must be satisfied by an (abstract) Lyapunov graph in order for it to be associated to a Smale flow on $\mathbb{S}^2 \times \mathbb{S}^1$.

1. *Introduction*

In this paper, we investigate the relationship between the topology and dynamics of Smale flows on $\mathbb{S}^2 \times \mathbb{S}^1$. A strategy that has been very successful in analyzing smooth flows on manifolds is firstly to consider the local dynamics described by the chain-recurrent behavior on the basic sets, and secondly to examine how these basic sets fit together.

Our approach will be qualitative in nature and make use of the methods introduced by Franks [**10**], and further developed by de Rezende [**7**] in their study of Smale flows on $\mathbb{S}^3$ and by Yu [**13**] in his study of non-singular Smale flows on $\mathbb{S}^2 \times \mathbb{S}^1$. Our main contribution here is to consider Smale flows on $\mathbb{S}^2 \times \mathbb{S}^1$ with singularities, which adds considerable difficulty to the embedding problem of the basic blocks containing the basic sets.

A qualitative analysis of a flow on a compact manifold M can be divided into two parts, namely, into chain-recurrent pieces and orbits that exhibit gradient-like behaviour among these pieces. This was very nicely captured in a theorem of Conley's [**5**] that proves the existence of continuous Lyapunov functions for continuous flows $\phi_t : M \to M$. Results of Wilson [**12**] assert that it is possible to choose a smooth Lyapunov function $f : M \to \mathbb{R}$.

Definition 1.1. If $\phi_t : M \to M$ is a flow and $\epsilon > 0$, we say there is an ϵ-chain from x to y provided that there exist points $x_1 = x, x_2, \ldots, x_n = y$ and real numbers $t(i) > 1$ such that

$$d(\phi_{t(i)}(x_i), x_{i+1}) < \epsilon$$

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for all $1 \leq i < n$. A point x is called *chain-recurrent* if for any $\epsilon > 0$ there is an ϵ-chain from x to x. The set of chain-recurrent points $\mathfrak{R}$ is called the *chain-recurrent set*.

It is easy to see that the chain-recurrent set is compact and invariant under the flow.

Definition 1.2. If $\phi_t : M \to M$ is a smooth flow, then a smooth function $f : M \to \mathbb{R}$ will be called a *Lyapunov function* provided that:

(1) $d/dt(f(\phi_t x)) < 0$ if it is not in the chain-recurrent set $\mathfrak{R}$;
(2) if $x, y \in \mathfrak{R}$ then $f(x) = f(y)$ if and only if for each $\epsilon > 0$ there are ϵ-chains from x to y and from y to x.

The qualitative study of flows will first be local in nature, that is, it comprises the analysis of the dynamics of the chain-recurrent pieces. In this work, for example, dynamical-topological invariants associated to the basic sets will characterize the possible basic blocks that contain them as maximal invariant sets. The second part constitutes a global characterization of how these pieces fit together to form the manifold and realize a smooth flow on it. The question of how the collection of chain-recurrent pieces fits together to form $\mathbb{S}^2 \times \mathbb{S}^1$ is, in fact, an embedding problem. In this work, as in [**7**] and [**13**], we make use of a Lyapunov graph as a combinatorial tool which gives a global picture of the disposition of the basic sets within the manifold.

For a Smale flow ϕ_t on a compact 3-manifold M with Lyapunov function $f : M \to \mathbb{R}$, define the following equivalence relation on M: $x \sim_f y$ if and only if x and y belong to the same connected component of a level set of f. This determines a graph $M/\sim_f$ which we say is a *Lyapunov graph associated to the Lyapunov function* f *and the flow* ϕ_t if the vertices are labeled with a chain-recurrent flow on a compact set and the edges with the genera of the level surfaces given by $f^{-1}(c)$ where c is a regular value.

We should point out that this definition differs from those in [**10**] and [**13**] since the dynamics under consideration therein, being non-singular, forced the genera to be equal to one in the first case and less than or equal to one in the second case. The presence of singularities, as is the case here, permits level surfaces of all genera. As a guiding principle, the more complicated the flow or if the phase space is higher-dimensional, the more labeling with topological invariants of the associated Lyapunov graph may be required in order to obtain meaningful characterization results.

The *cycle rank* of a graph is the maximum number of edges that can be removed without disconnecting the graph and is denoted by $\beta(L)$. The *indegree* of a vertex u in L is the number of incoming edges incident to u and the *outdegree* of u is the number of outgoing edges incident to u.

Definition 1.3. An *abstract Lyapunov graph* is a finite, connected, oriented graph L such that:

(1) L has no oriented cycles;
(2) each vertex is labeled with a chain-recurrent flow on a compact space;
(3) each edge is labeled with a non-negative number g, which we refer to as the weight on the edge.

The following result is the main theorem in this paper which characterizes Smale flows on $\mathbb{S}^2 \times \mathbb{S}^1$.

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THEOREM 1.1. *Let L be an abstract Lyapunov graph. L is associated to a Lyapunov function and a Smale flow ϕ_t on $\mathbb{S}^2 \times \mathbb{S}^1$ if and only if the following statements hold.*

(1) *The underlying graph L is an oriented graph with exactly one edge attached to each sink or source vertex. Moreover, the sink (source) vertex is labeled with an index 0 (index 3) singularity or an attracting (repelling) periodic orbit.*

(2) *If a vertex of indegree e^+ and outdegree e^- is labeled with a singularity of index 2 (index 1), then $1 \le e^+ \le 2$ and $e^- = 1$ ($1 \le e^- \le 2$ and $e^+ = 1$).*

(3) *If a vertex v of indegree $e^+ > 0$ and outdegree $e^- > 0$ is labeled with a suspension of a subshift of finite type associated to a non-negative integer matrix $A_{m\times m}$ with $k = \dim \ker((I - B) : F_2^m \to F_2^m)$, $F_2 = \mathbb{Z}/2$, $b_{ij} = a_{ij} \bmod 2$ and $g_i^+(g_j^-)$ are the weights on the incoming (outgoing) edges incident to the vertex v, where $G^- = \sum_{i=1}^{e^-} g_i^-$ and $G^+ = \sum_{i=1}^{e^+} g_i^+$, then there are two cases to consider: $\beta(L) = 0$ and $\beta(L) = 1$.*

   (a) *If $\beta(L) = 1$ then*

$$k + 1 - G^- \le e^+ \le k + 1 \quad \text{and} \quad k + 1 - G^+ \le e^- \le k + 1. \tag{1}$$

   (b) *If $\beta(L) = 0$ we have the following.*

      (i) *There exists at most one vertex labeled with a suspension of a subshift of finite type with*

$$k - G^- = e^+ \quad \text{and} \quad k - G^+ = e^-. \tag{2}$$

      Any other vertex labeled with a suspension of a subshift of finite type satisfies the inequalities in (1).

      (ii) *If no vertex on L satisfies the equality in (2) then there must be an edge in L with non-zero weight.*

(4) *All vertices must satisfy the Poincaré–Hopf condition, i.e. for a vertex labeled with a singularity of index r, the condition is*

$$(-1)^r = e^+ - e^- - \sum g_j^+ + \sum g_i^-, \tag{3}$$

and for a vertex labeled with a suspension of a subshift of finite type or a periodic orbit, the condition is

$$0 = e^+ - e^- - \sum g_j^+ + \sum g_i^-. \tag{4}$$

The proof of the necessity of the conditions in Theorem 1.1 follows from Propositions 5.1–5.4. The proof of the sufficiency of the conditions in Theorem 1.1 follows from Proposition 6.4.

In §2 we introduce background material. In §3 we provide some topological properties of embedded surfaces in $\mathbb{S}^2 \times \mathbb{S}^1$. We introduce the notion of 3-manifolds of handlebody type. In §4 we study the relationship between the graph and manifolds of handlebody type. In §§5 and 6 we state and prove propositions which comprise the foundation of the proof of the main result, Theorem 1.1. Throughout this paper, we consider homology with $F_2 = \mathbb{Z}_2$ coefficients.

2. *Background and definitions*

In this section, we provide background material, basic definitions and theorems essential to the understanding of our work.

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2.1. *Smale flows.* A compact invariant set Λ for a smooth flow $\phi_t : M \to M$ is said to have a *hyperbolic structure* provided that the tangent bundle of M restricted to Λ can be written as the Whitney sum of three subbundles $E^u + E^s + E^c$, each being invariant under $D\phi_t$ for all t, in such a way that E^c is spanned by the vector field tangent to the flow and there are constants $C, \alpha > 0$ satisfying

$$\|D\phi_t(v)\| \leq Ce^{-\alpha t}\|v\| \quad \text{for } v \in E^s, t \geq 0,$$

and

$$\|D\phi_t(v)\| \geq C^{-1}e^{\alpha t}\|v\| \quad \text{for } v \in E^u, t \geq 0.$$

An important consequence of the hyperbolicity is that $\mathfrak{R}$ is decomposed into finitely many irreducible pieces [**11**].

THEOREM 2.1. *Suppose that the chain-recurrent set* $\mathfrak{R}$ *of a flow on a compact manifold* $\phi_t : M \to M$ *has a hyperbolic structure. Then* $\mathfrak{R}$ *is a finite disjoint union of compact invariant sets* $\Lambda_1, \ldots, \Lambda_n$ *and each* Λ_i *contains a point whose forward orbit is dense in* Λ_i.

The sets Λ_i are called *basic sets* of the flow and are precisely the chain-transitive pieces of $\mathfrak{R}$, which are not separated by Lyapunov function.

If Λ is a compact invariant hyperbolic set for a flow then each orbit in Λ has a *stable* and *unstable manifold*. These are defined for $x \in \Lambda$ as

$$W^s(x) = \{y \in M \mid \text{for some } r \in \mathbb{R},\ d(\phi_t(y), \phi_{t+r}(x)) \to 0 \text{ as } t \to \infty\}$$

and

$$W^u(x) = \{y \in M \mid \text{for some } r \in \mathbb{R},\ d(\phi_t(y), \phi_{t+r}(x)) \to 0 \text{ as } t \to -\infty\}.$$

Definition 2.1. A flow $\phi_t : M \to M$ with hyperbolic chain-recurrent set $\mathfrak{R}$ is said to satisfy the *transversality condition* provided that, for each $x, y \in \mathfrak{R}$, the manifolds $W^s(x)$ and $W^u(y)$ intersect transversally.

Definition 2.2. A smooth flow $\phi_t : M \to M$ on a compact manifold is called a *Smale flow* provided:

(1) its chain-recurrent set $\mathfrak{R}$ has a hyperbolic structure and $\dim \mathfrak{R} \leq 1$;
(2) it satisfies the transversality condition.

Bowen [**3**] gives a complete dynamical description of the basic sets of Smale flows.

THEOREM 2.2. *Let* ϕ_t *be a flow with hyperbolic chain-recurrent set and* Λ *a basic set of dimension one. Then* ϕ_t *restricted to* Λ *is topologically conjugate to the suspension of a subshift of finite type associated to an irreducible matrix.*

Hence, the chain-recurrent set of a Smale flow is made up of singularities, periodic orbits and suspensions of subshifts of finite type.

Definition 2.3. If A and B are non-negative integer matrices they are *flow equivalent* provided the suspensions of the subshifts of finite type $\sigma(A)$ and $\sigma(B)$ are topologically equivalent.

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Also, we will need to define the genus of a manifold and use the following result which characterizes it.

Definition 2.4. Let M be a smooth, compact, connected n-manifold with boundary. The *genus of manifold* M, $g(M)$, is the maximal number of mutually disjoint, smooth, compact, connected, two-sided codimension-one submanifolds that do not disconnect M.

The following result gives the relation between the cycle rank of L and the genus of the fundamental group of M.

PROPOSITION 2.1. *Let M be a connected, closed, smooth n-manifold. Let ϕ_t be a smooth flow on M with associated Lyapunov function f. Let L be a Lyapunov graph associated to f. Then*

$$\beta(L) \leq g(M).$$

See [**6**] for the proof.

Let ϕ_t be a smooth flow on $\mathbb{S}^2 \times \mathbb{S}^1$ with L its associated Lyapunov graph. Since $g(\mathbb{S}^2 \times \mathbb{S}^1) = 1$ by Proposition 2.1,

$$\beta(L) \leq 1.$$

Thus, Lyapunov graphs associated with smooth flows, in particular Smale flows, on $\mathbb{S}^2 \times \mathbb{S}^1$ have at most one cycle.

2.2. *Filtration and homology.* The existence of a smooth Lyapunov function for a flow ϕ_t implies the existence of a filtration associated to it.

Definition 2.5. If $\phi_t : M \to M$ is a flow with hyperbolic chain-recurrent set with basic sets $\{\Lambda_i\}$ $(i = 1, \ldots, n)$, a *filtration* associated to ϕ_t is a collection of submanifolds $M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that:

(1) $\phi_t(M_i) \subset \operatorname{int} M_i$, for each $t > 0$;

(2) $\Lambda_i = \bigcap_{t=-\infty}^{\infty} \phi_t(M_i - M_{i-1})$.

The following is a result due to Bowen and Franks [**4**].

THEOREM 2.3. *Let ϕ_t be a Smale flow and let M_i, $i = 1, \ldots, n$, be a filtration associated to ϕ_t. Suppose that $\Lambda_i = \bigcap_{t=-\infty}^{\infty} \phi_t(M_i - M_{i-1})$ is a basic set of index k labeled with a structure matrix $A_{m\times m}$. Then*

$$H_s(M_i, M_{i-1}; F_2) = 0 \quad \text{if } s \neq k, k+1;$$

$$H_k(M_i, M_{i-1}; F_2) \cong F_2^m/(I - B)F_2^m;$$

$$H_{k+1}(M_i, M_{i-1}; F_2) \cong \ker((I - B) \text{ on } F_2^m),$$

where $B = A \bmod 2$.

See [**9**] for the proof.

The following theorem in [**7**], for Smale flows on S^3, completely classifies Lyapunov graphs for flows on S^3.

THEOREM 2.4. *Let L be an abstract Lyapunov graph. L is associated to a Smale flow ϕ_t on $\mathbb{S}^3$ if only if the following conditions hold.*

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(1) *The underlying graph L is a tree with exactly one edge attached to each sink or source vertex. Moreover, the sink (source) vertex is labeled with an index* 0 *(index* 3*) singularity or an attracting (repelling) periodic orbit.*

(2) *If a vertex is labeled with a singularity of index* 2 *(index* 1*) then* $1 \le e^+ \le 2$ *and* $e^- = 1$ *(*$e^+ = 1$ *and* $1 \le e^- \le 2$*). If a vertex is labeled with a suspension of a subshift of finite type and* $A_{m\times m}$ *is the non-negative integer matrix representing this subshift, then*

$$e^+,\ e^- > 0,$$

$$k + 1 - G^- \le e^+ \le k + 1 \quad \text{with } G^- = \sum_{i=1}^{e^-} g_i^-,$$

$$k + 1 - G^+ \le e^- \le k + 1 \quad \text{with } G^+ = \sum_{j=1}^{e^+} g_j^+,$$

where $k = \dim \ker((I - B) : F_2^m \to F_2^m)$ *and* $b_{ij} = a_{ij} \bmod 2$*,* $e^+(e^-)$ *is the number of incoming (outgoing) edges incident to the vertex and* $g_j^+(g_i^-)$ *is the weight on an incoming (outgoing) edge incident to the vertex.*

(3) *All vertices satisfy the Poincaré–Hopf condition (3) and (4).*

3. *Surfaces embedded in* $\mathbb{S}^2 \times \mathbb{S}^1$

In our work it is important to have a characterization of the generators of the two-dimensional homology of $\mathbb{S}^2 \times \mathbb{S}^1$.

LEMMA 3.1. *Let S be an orientable closed connected surface in* $\mathbb{S}^2 \times \mathbb{S}^1$ *such that S is non-separating and* $i : S \to \mathbb{S}^2 \times \mathbb{S}^1$ *is the inclusion map. Then the induced homomorphism*

$$i_* : H_2(S) \to H_2(\mathbb{S}^2 \times \mathbb{S}^1)$$

is an isomorphism.

Proof. Since S is non-separating, there are two points A and B on a tubular neighborhood of S which can be connected by a path that intersects S at one single point C. This path can be extended to a loop which intersects S at C. It follows that the one-dimensional homology class $[\gamma]$ and the two-dimensional homology class $[S]$ have a non-zero intersection number. Therefore, both $[\gamma]$ and $[S]$ are non-trivial elements in the homology of $\mathbb{S}^2 \times \mathbb{S}^1$. Since $H_2(\mathbb{S}^2 \times \mathbb{S}^1) \cong F_2$, we have that i_* is an isomorphism. □

Definition 3.1. Let S be a closed surface in $\mathbb{S}^2 \times \mathbb{S}^1$ and let $i : S \to \mathbb{S}^2 \times \mathbb{S}^1$ be the inclusion embedding. Let

$$i_* : \pi_1(S) \to \pi_1(\mathbb{S}^2 \times \mathbb{S}^1)$$

be the homomorphism induced by i. We say that S is π_1-*trivial* in $\mathbb{S}^2 \times \mathbb{S}^1$ if i_* is trivial and π_1-*non-trivial* otherwise.

The study of embedded surfaces in $\mathbb{S}^2 \times \mathbb{S}^1$ is essential to our work. Some of these embeddings have been previously studied, such as the embedding of the sphere $\mathbb{S}^2$ in [**2**]

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and the embedding of the torus in [**13**]. In this section we present other embeddings that are relevant for our work. The following result of the embedding of $\mathbb{S}^2$ in $\mathbb{S}^2 \times \mathbb{S}^1$ is a well-known result with a nice proof in [**1**].

PROPOSITION 3.1. *Let S be a differential embedding of the 2-sphere in $\mathbb{S}^2 \times \mathbb{S}^1$. Then:*
(1) *either S bounds a 3-ball;*
(2) *or S is homotopic to a fiber $S^2 \times \{t\}$, with $t \in S^1$.*

Now let ϕ_t be a Smale flow on $\mathbb{S}^2 \times \mathbb{S}^1$ with Lyapunov function $f : \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{R}$ and let L be the Lyapunov graph associated to f. Let $c \in \mathbb{R}$ be a regular value. then $f^{-1}(c)$ is the disjoint collection of orientable, closed connected surfaces. Let $\mathcal{T}_g \subset f^{-1}(c)$ be a connected, closed surface of genus g. Then by the topology of $\mathbb{S}^2 \times \mathbb{S}^1$, $\mathcal{T}_g$ can be a non-separating surface in $\mathbb{S}^2 \times \mathbb{S}^1$.

If $\mathcal{T}_g$ is non-separating, there exists another connected component of the regular level set $\mathcal{T}_{\widehat{g}}$ non-parallel to $\mathcal{T}_g$, such that $\mathbb{S}^2 \times \mathbb{S}^1 - (\mathcal{T}_g \sqcup \mathcal{T}_{\widehat{g}})$ has two components. We denote their closures by M_1 and M_2. Therefore, M_1 and M_2 satisfy

$$M_1 \cup M_2 = \mathbb{S}^2 \times \mathbb{S}^1 \quad \text{and} \quad M_1 \cap M_2 = \mathcal{T}_g \sqcup \mathcal{T}_{\widehat{g}}.$$

Also, $\partial M_1 = \partial M_2 = \mathcal{T}_g \sqcup \mathcal{T}_{\widehat{g}}$. The following lemma gives us homological information on M_1 and M_2.

LEMMA 3.2. *M_1 and M_2 as defined above satisfy:*
(1) $H_2(M_i) \cong F_2$;
(2) $\dim H_1(M_i) = \widehat{g} + g$.

Proof. We consider the Mayer–Vietoris exact sequence of the pair (M_1, M_2),

$$\begin{aligned} 0 \to H_3(M_1) \oplus H_3(M_2) \to H_3(\mathbb{S}^2 \times \mathbb{S}^1) \to H_2(M_1 \cap M_2) \\ \to H_2(M_1) \oplus H_2(M_2) \xrightarrow{\alpha} H_2(\mathbb{S}^2 \times \mathbb{S}^1) \to \cdots . \end{aligned}$$

Since M_1 and M_2 are compact manifolds with boundary, $H_3(M_1) = 0 = H_3(M_2)$. On the other hand, $H_3(\mathbb{S}^2 \times \mathbb{S}^1) \cong F_2 \cong H_2(\mathbb{S}^2 \times \mathbb{S}^1)$ and as $M_1 \cap M_2$ has two components, $H_2(M_1 \cap M_2) \cong F_2^2$. Therefore,

$$\dim H_2(M_1) + \dim H_2(M_2) \leq 2. \tag{5}$$

On the other hand, we consider the exact sequence of the pair $(M_i, \partial M_i)$,

$$0 \to H_3(M_i, \partial M_i) \to H_2(\partial M_i) \to H_2(M_i) \to H_2(M_i, \partial M_i) \to \cdots . \tag{6}$$

Since M_i is a 3-manifold with boundary, $H_3(M_i, \partial M_i) \cong F_2$. Also ∂M_i has two components, and hence $H_2(\partial M_i) \cong F_2^2$. Therefore, by exactness,

$$\dim H_2(M_i) \geq 1. \tag{7}$$

By (5) and (7), we conclude that

$$H_2(M_i) \cong F_2.$$

So we obtain the following exact sequence from (6):

$$0 \to H_2(M_i, \partial M_i) \to H_1(\partial M_i) \to H_1(M_i) \to H_1(M_i, \partial M_i) \to \widetilde{H}_0(\partial M_i) \to 0.$$

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Therefore, by exactness,

$$\begin{aligned}\dim H_1(M_i) = &\dim H_1(\partial M_i) - \dim H_2(M_i, \partial M_i) \\ &+ \dim H_1(M_i, \partial M_i) - \dim \widetilde{H}_0(\partial M_i). \end{aligned} \tag{8}$$

At this point, we need to consider the following proposition, which follows from the universal coefficient theorem.

PROPOSITION 3.2. *Let* (X, A) *be a pair of topological spaces, and let* F *be a field. There is a natural isomorphism*

$$\beta : H^n(X, A; F) \to \mathrm{Hom}_F(H_n(X, A; F), F) \cong H_n(X, A; F)^*.$$

Since $\partial M_i \cong \mathcal{T}_g \sqcup \mathcal{T}_{\hat{g}}$, we have $\dim H_1(\partial M_i) = 2g + 2\hat{g}$ and $\widetilde{H}_0(\partial M_i) \cong F_2$. On the other hand, by Proposition 3.2 and by the Poincaré duality,

$$H_1(M_i, \partial M_i) \cong (H_2(M_i))^* \quad \text{and} \quad H_2(M_i, \partial M_i) \cong (H_1(M_i))^*.$$

In (8) we have

$$\dim H_1(M_i) = g + \hat{g}. \tag{9}$$

□

If $\mathcal{T}_g$ is separating, this embedding splits $\mathbb{S}^2 \times \mathbb{S}^1$ into two submanifolds M_1 and M_2 such that

$$\mathbb{S}^2 \times \mathbb{S}^1 = M_1 \cup M_2 \quad \text{and} \quad M_1 \cap M_2 = \mathcal{T}_g.$$

LEMMA 3.3. *Let* M_1 *and* M_2 *be defined as above. Then:*
(1) $\dim H_2(M_1) + \dim H_2(M_2) \leq 1$*;*
(2) *if* $H_2(M_1) = 0$*, then* $\dim H_1(M) = g$*;*
(3) *if* $H_2(M_1) \cong F_2$*, then* $\dim H_1(M) = g + 1$.
Analogous results hold for M_2.

Proof. Consider the Mayer–Vietoris exact sequence of M_1 and M_2:

$$\begin{aligned}0 \to H_3(M_1) \oplus H_3(M_2) \to H_3(\mathbb{S}^2 \times \mathbb{S}^1) \to H_2(T_g^2) \\ \to H_2(M_1) \oplus H_2(M_2) \to H_2(\mathbb{S}^2 \times \mathbb{S}^1) \to \cdots. \end{aligned} \tag{10}$$

Since M_1 and M_2 are compact manifolds with boundary, $H_3(M_1) = 0 = H_3(M_2)$. Also, $M_1 \cap M_2$ has one component, hence $H_2(M_1 \cap M_2) \cong F_2$. Therefore, by the exactness of sequence (10),

$$\dim H_2(M_1) + \dim H_2(M_2) \leq 1. \tag{11}$$

Hence, there are the following possibilities:
(a) either $H_2(M_1) = 0$ and $H_2(M_2) = 0$;
(b) or $H_2(M_1) \cong F_2$ and $H_2(M_2) = 0$;
(c) or $H_2(M_1) = 0$ and $H_2(M_2) \cong F_2$.
Now consider the exact sequence of the pair $(M_i, \partial M_i)$,

$$0 \to H_3(M_i, \partial M_i) \to H_2(\partial M_i) \to H_2(M_i) \to H_2(M_i, \partial M_i) \to \cdots. \tag{12}$$

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Since M_i is a 3-manifold with boundary, $H_3(M_i, \partial M_i) \cong F_2$. Also, ∂M_i has one component, hence $H_2(\partial M_i) \cong F_2$. Therefore, $H_3(M_i, \partial M_i)$ is isomorphic to $H_2(\partial M_i)$. Thus, we can write the exact sequence (12) as

$$0 \to H_2(M_i) \to H_2(M_i, \partial M_i) \to H_1(\partial M_i) \to H_1(M_i) \to H_1(M_i, \partial M_i) \to 0$$

which implies that

$$2g = \dim H_2(M_i, \partial M_i) - \dim H_2(M_i) + \dim H_1(M_i) - \dim H_1(M_i, \partial M_i). \quad (13)$$

By Proposition 3.2 and by Poincaré duality,

$$H_1(M_i, \partial M_i) \cong (H_2(M_i))^* \cong (H_2(M_i)),$$
$$H_2(M_i, \partial M_i) \cong (H_1(M_i))^* \cong (H_1(M_i)).$$

Hence, equation (13) can be written as

$$g = \dim H_1(M_i) - \dim H_2(M_i). \quad (14)$$

We conclude the proof with a case analysis.

Case (*a*). Suppose that $H_2(M_1) = 0$ and $H_2(M_2) = 0$.

Hence, by equation (14), for $i = 1, 2$,

$$g = \dim H_1(M_i).$$

Case (*b*). Suppose that $H_2(M_1) \cong F_2$ and $H_2(M_2) = 0$.

(i) By the previous case, if $H_2(M_2) = 0$ then $H_1(M_2) \cong F_2^g$, hence, $\dim H_1(M_2) = g$.

(ii) In the case $H_2(M_1) \cong F_2$, equation (14) is written as

$$g = \dim H_1(M_1) - 1,$$

which implies that $\dim H_1(M_1) = g + 1$.

Case (*c*). Suppose that $H_2(M_1) = 0$ and $H_2(M_2) \cong F_2$.

In a similar fashion to the previous case we have that:

(i) if $H_2(M_1) = 0$, then $\dim H_1(M_1) = g$;

(ii) if $H_2(M_2) \cong F_2$, then $\dim H_1(M_2) = g + 1$.

□

Definition 3.2. A manifold M^3 is of *handlebody type* whenever ∂M^3 is homeomorphic to a closed and orientable surface of genus g, for some $g \geq 0$, and satisfies

$$H_2(M^3) = 0 \quad \text{and} \quad \dim H_1(M^3) = g.$$

The positive number g is defined as the genus of the manifold of handlebody type.

Remark 3.1. By inequality (11) and Lemma 3.3, at least one of M_1 and M_2 is of handlebody type.

LEMMA 3.4. *With the above notation, M_2 is not a manifold of handlebody type if and only if*

$$H_2(M_2) \cong F_2.$$

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Proof. ($\Rightarrow$) By inequality (11) we have that $\dim H_2(M_2) \leq 1$, hence we have two possibilities. If $H_2(M_2) = 0$, by case (b) in the proof of Lemma 3.3, it follows that $\dim H_1(M_2) = g$. Hence, M_2 is of handlebody type, contradicting the hypothesis. Therefore, $H_2(M_1) \cong F_2$.

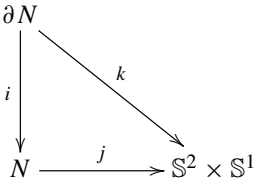
($\Leftarrow$) Since $H_2(M_2) \cong F_2$, by definition M_2 is not of handlebody type. □

LEMMA 3.5. *Let N be a 3-submanifold of $\mathbb{S}^2 \times \mathbb{S}^1$, such that N is of handlebody type and ∂N is a π_1-trivial surface in $\mathbb{S}^2 \times \mathbb{S}^1$. Then the inclusion $j : N \to \mathbb{S}^2 \times \mathbb{S}^1$ induces a homomorphism*

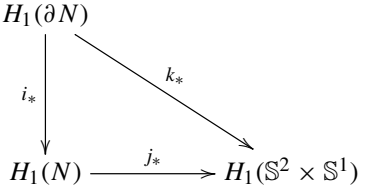
$$j_* : H_1(N) \to H_1(\mathbb{S}^2 \times \mathbb{S}^1)$$

which is trivial.

Proof. Consider the diagram



where i, j and k are inclusions. These functions induce the diagram



Since ∂N is π_1-trivial, it follows that k_* is trivial. Thus, in order to prove that j_* is trivial, it is only necessary to prove that i_* is surjective. Now consider the exact homology sequence of the pair $(N, \partial N)$,

$$0 \to H_2(N, \partial N) \to H_1(\partial N) \xrightarrow{i_*} H_1(N) \to H_1(N, \partial N) \to \cdots .$$

By Proposition 3.2 and the Poincaré duality,

$$H_1(N, \partial N) \cong (H_2(N))^*.$$

On the other hand, $H_2(N) = 0$. Hence the lemma follows. □

LEMMA 3.6. *Let $\mathcal{T}_g$ be a separating connected component of a regular level set associated to some Smale flow ϕ_t on $\mathbb{S}^2 \times \mathbb{S}^1$. Let M_1 and M_2 be the closure of the two connected*

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components of $\mathbb{S}^2 \times \mathbb{S}^1 - \mathcal{T}_g$. *Hence:*

(1) *if* $\mathcal{T}_g$ *is* π_1*-trivial in* $\mathbb{S}^2 \times \mathbb{S}^1$, *then either* M_1 *or* M_2 *is of handlebody type;*

(2) *if* $\mathcal{T}_g$ *is* π_1*-non-trivial in* $\mathbb{S}^2 \times \mathbb{S}^1$, *then* M_1 *and* M_2 *are of handlebody type.*

Proof.

(1) It follows by Remark 3.1 that at least one of the submanifolds M_1 and M_2 is of handlebody type. Suppose that M_1 is a manifold of handlebody type. By Lefschetz duality, that

$$H_1(\mathbb{S}^2 \times \mathbb{S}^1, M_1) \cong H_2(M_2). \tag{15}$$

Now consider the homology exact sequence of the pair $(\mathbb{S}^2 \times \mathbb{S}^1, M_1)$,

$$\to H_2(\mathbb{S}^2 \times \mathbb{S}^1, M_1) \to H_1(M_1) \xrightarrow{\eta} H_1(\mathbb{S}^2 \times \mathbb{S}^1) \to H_1(\mathbb{S}^2 \times \mathbb{S}^1, M_1) \to 0.$$

Since $\mathcal{T}_g$ is π_1-trivial in $\mathbb{S}^2 \times \mathbb{S}^1$, by Lemma 3.5, we have that η is equal to zero. Therefore, $H_1(\mathbb{S}^2 \times \mathbb{S}^1) \cong H_1(\mathbb{S}^2 \times \mathbb{S}^1, M_1)$ and by (15) we have that

$$H_2(M_2) \cong F_2.$$

The result now follows by Lemma 3.4.

(2) If $\mathcal{T}_g$ is π_1-non-trivial in $\mathbb{S}^2 \times \mathbb{S}^1$ there exists a simple closed curve β in $\mathcal{T}_g$ such that $[\beta] \in \pi_1(\mathbb{S}^2 \times \mathbb{S}^1)$ is non-trivial. Now consider the Mayer–Vietoris exact homology sequence

$$0 \to H_2(M_1) \oplus H_2(M_2) \to H_2(\mathbb{S}^2 \times \mathbb{S}^1) \xrightarrow{\Delta} H_1(X \cap Y) \to \cdots . \tag{16}$$

We have that Δ is non-zero, otherwise we would have that the image of Δ in $H_1(\mathcal{T}_g)$ is homologous to a curve γ that separates $\mathcal{T}_g$ and the intersection number of β and γ would be zero, since $[\gamma] = 0$ in $\mathcal{T}_g$. However, since this intersection is $n \neq 0$, we have a contradiction and hence Δ is not zero. Thus, by (16) we have that $H_2(M_1)$ and $H_2(M_2)$ are zero. Thus M_1 and M_2 are of handlebody type. □

4. *Manifolds of handlebody type and Lyapunov graphs*

In this section we study the relation between Lyapunov graphs and manifolds of handlebody type. Let ϕ_t be a Smale flow on $\mathbb{S}^2 \times \mathbb{S}^1$ with Lyapunov function f and let L be the associated Lyapunov graph. Suppose that L is a tree. Choose a regular level set which is a separating surface $\mathcal{T}_g$ which splits $\mathbb{S}^2 \times \mathbb{S}^1$ in two 3-submanifolds, $\mathcal{M}^-_{\mathcal{T}_g}$ and $\mathcal{M}^+_{\mathcal{T}_g}$, and such that the flow is transversal and outward-going on $\mathcal{M}^-_{\mathcal{T}_g}$ and inward-coming on $\mathcal{M}^+_{\mathcal{T}_g}$.

Remark 4.1. By Lemma 3.6:

- if $\mathcal{T}_g$ is π_1-trivial, then either $\mathcal{M}^+_{\mathcal{T}_g}$ or $\mathcal{M}^-_{\mathcal{T}_g}$ is of handlebody type;
- if $\mathcal{T}_g$ is π_1-non-trivial, then $\mathcal{M}^+_{\mathcal{T}_g}$ and $\mathcal{M}^-_{\mathcal{T}_g}$ are of handlebody type;
- if we cut $\mathbb{S}^2 \times \mathbb{S}^1$ along a separating regular level set, this corresponds to disconnecting the graph L into two subgraphs with dangling edges that represent each one of the manifolds $\mathcal{M}^+_{\mathcal{T}_g}$ and $\mathcal{M}^-_{\mathcal{T}_g}$.

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PROPOSITION 4.1. *Let u be a vertex in L and let e be an incoming edge incident to u. Suppose that there exists a vertex v labeled with a basic set Λ such that Λ is contained in $\mathcal{M}^-_{\mathcal{T}_g}$. Then there exists an incoming (outgoing) edge with weight $\widehat{g}$ of v such that $\mathcal{M}^+_{\mathcal{T}_{\widehat{g}}}$ ($\mathcal{M}^-_{\mathcal{T}_{\widehat{g}}}$) is contained in $\mathcal{M}^-_{\mathcal{T}_g}$.*

Proof. Since L is a tree, there exists an incoming or outgoing edge e_1 (with weight $\widehat{g}$) of v such that cutting along this edge separates u and v into distinct subgraphs. First suppose that e_1 is an incoming edge incident to v. Hence, if we cut $\mathbb{S}^2 \times \mathbb{S}^1$ along $\mathcal{T}_{\widehat{g}}$ then $\mathcal{M}^+_{\mathcal{T}_{\widehat{g}}} \subset \mathcal{M}^-_{\mathcal{T}_g}$. On the other hand, if e_1 is an outgoing edge incident to v then $\mathcal{M}^-_{\mathcal{T}_{\widehat{g}}} \subset \mathcal{M}^-_{\mathcal{T}_g}$. □

We have a similar result if the edge e incident to u is outgoing. In what follows other interesting properties of L are determined. Given a vertex v of the graph L labeled with a basic set Λ_v, each edge point represents a regular level set which is a surface of genus g_i^+ (g_j^-) embedded in $\mathbb{S}^2 \times \mathbb{S}^1$ denoted by $\mathcal{T}_{g_i^+}$ ($\mathcal{T}_{g_j^-}$). The incoming (outgoing) edges are labeled with weights corresponding to g_i^+ (g_i^-).

COROLLARY 4.1. *Let v be a vertex on L and with the notation above.*

(1) *For $\alpha \in \{1, \ldots, e_v^-\}$ fixed,*

$$\mathcal{M}^-_{\mathcal{T}_{g_k^+}} \subset \mathcal{M}^-_{\mathcal{T}_{g_\alpha^-}} \quad \text{and} \quad \mathcal{M}^+_{\mathcal{T}_{g_j^-}} \subset \mathcal{M}^-_{\mathcal{T}_{g_\alpha^-}}$$

with $k \in \{1, \ldots, e^+\}$ and $j \in \{1, \ldots, e^-\} - \{\alpha\}$.

(2) *For $\beta \in \{1, \ldots, e^+\}$ fixed,*

$$\mathcal{M}^-_{\mathcal{T}_{g_k^+}} \subset \mathcal{M}^+_{\mathcal{T}_{g_\beta^+}} \quad \text{and} \quad \mathcal{M}^+_{\mathcal{T}_{g_j^-}} \subset \mathcal{M}^+_{\mathcal{T}_{g_\beta^+}}$$

with $k \in \{1, \ldots, e^+\} - \{\beta\}$ and $j \in \{1, \ldots, e^-\}$.

Proof.

(1) Since $\mathcal{T}_{g_\alpha^-}$ corresponds to an outgoing edge incident to v, let u be the vertex on which the edge is incoming. Note that each $\mathcal{T}_{g_k^+}$ corresponds to an outgoing edge incident to vertices v_k that are incoming and incident to v. Hence, by Lemma 3.6 it follows that

$$\mathcal{M}^-_{\mathcal{T}_{g_k^+}} \subset \mathcal{M}^-_{\mathcal{T}_{g_\alpha^-}}.$$

Analogously, each $\mathcal{T}_{g_j^-}$ corresponds to an outgoing edge incident to vertices v_j and by Lemma 3.6 it follows that

$$\mathcal{M}^+_{\mathcal{T}_{g_j^-}} \subset \mathcal{M}^-_{\mathcal{T}_{g_\alpha^-}}.$$

(2) To prove this, it suffices to reverse the orientation on the graph. □

COROLLARY 4.2. *Let v be a vertex of L and with the notation above.*

(1) *If $\mathcal{M}^-_{\mathcal{T}_{g_\alpha^-}}$ is of handlebody type for $\alpha \in \{1, \ldots, e^-\}$, then*

$$\mathcal{M}^-_{\mathcal{T}_{g_k^+}} \quad \text{and} \quad \mathcal{M}^+_{\mathcal{T}_{g_j^-}}$$

are of handlebody type for $k \in \{1, \ldots, e^+\}$ and $j \in \{1, \ldots, e^-\} - \{\alpha\}$.

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(2) *If* $\mathcal{M}^+_{\mathcal{T}_{g^+_\beta}}$ *is of handlebody type for* $\beta \in \{1, \ldots, e^+\}$, *then*

$$\mathcal{M}^-_{\mathcal{T}_{g^+_k}} \quad \text{and} \quad \mathcal{M}^+_{\mathcal{T}_{g^-_j}}$$

are of handlebody type for $k \in \{1, \ldots, e^+\} - \{\beta\}$ *and* $j \in \{1, \ldots, e^-\}$.

Proof. This follows directly from Remark 4.1, Corollary 4.1 and the fact that a submanifold with a surface boundary of a manifold of handlebody type is of handlebody type. □

5. *Necessary conditions on Lyapunov graphs associated to Smale flows*

In this section we will prove a series of propositions that will comprise the proof of the necessity of the conditions stated in the main theorem of this work, Theorem 1.1.

Let ϕ_t be a Smale flow on $\mathbb{S}^2 \times \mathbb{S}^1$ with Lyapunov function $f : \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{R}$. Let Λ be a basic set of ϕ_t and $f(\Lambda) = c$. Choose $\epsilon > 0$ sufficiently small so that c is the only critical value in $[c - \epsilon, c + \epsilon]$. Let $X_0 = f^{-1}(-\infty, c + \epsilon]$ and $Z_0 = f^{-1}(-\infty, c - \epsilon]$.

5.1. *Lyapunov graph with a cycle.*

Let U be the closure of the component of $X_0 - Z_0$ which contains Λ. Then U is a neighborhood of Λ whose boundary consists of closed orientable surfaces to which the flow is transverse. The flow enters e^+ of these surfaces and exits the remaining e^-.

PROPOSITION 5.1. *Let L be a Lyapunov graph associated to a Smale flow ϕ_t and a Lyapunov function on $\mathbb{S}^2 \times \mathbb{S}^1$, such that $\beta(L) = 1$. Let v be a vertex of L on the cycle, labeled with a suspension of a subshift of finite type, and $A_{m\times m}$ be the non-negative integer matrix representing this subshift. Let* $k = \dim \ker \overline{I - A} : F_2^m \to F_2^m$ *where* $\overline{I - A}$ *is the* mod 2 *reduction of* $I - A$.

If e^+ and e^- are respectively the indegree and outdegree of v, then

$$e^+, e^- > 0,$$
$$e^+ \le k + 1,$$
$$e^- \le k + 1,$$
$$k + 1 - G^- \le e^+ \quad \text{with } G^- = \sum_{i=1}^{e^-} g_i^-,$$
$$k + 1 - G^+ \le e^- \quad \text{with } G^+ = \sum_{j=1}^{e^+} g_j^+,$$

where the g_i^+ (g_j^-) are the weights on the incoming (outgoing) edges incident to the vertex v.

Proof. Suppose that Λ is the basic set of ϕ_t corresponding to v. First of all, suppose that both edges of the cycle are incoming edges incident to the vertex v, as shown in Figure 1. Let Z be the union of the components of $\mathbb{S}^2 \times \mathbb{S}^1 - (\partial U \cap Z_0)$ which do not contain Λ.

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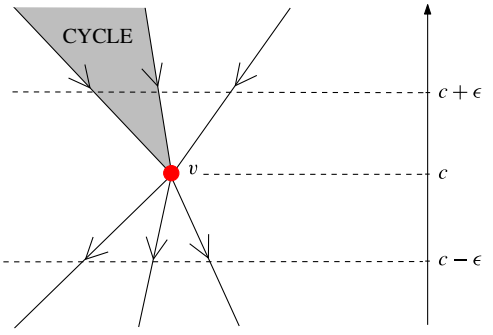


FIGURE 1. v is a vertex in the cycle.

By Lemma 3.6, each component of Z is a manifold of handlebody type. Now we write $X = Z \cup U$ and $Y = \mathbb{S}^2 \times \mathbb{S}^1 - X$. Suppose that Y_1 is a component of Y such that it corresponds to the cycle on the graph. Therefore, by Lemma 3.2,

$$\dim H_1(Y_1) = g_1^+ + g_2^+ \quad \text{and} \quad \dim H_2(Y_1) = 1, \tag{17}$$

and any other component of Y is of handlebody type. On the other hand, a result (9.11) of [**9**] says that $H_2(X, Z)$ has dimension k. We consider the exact homology sequence of the pair (X, Z),

$$H_3(X, Z) \to H_2(Z) \to H_2(X) \to H_2(X, Z) \to H_1(Z) \xrightarrow{\alpha} H_1(X) \to \cdots . \tag{18}$$

Since $H_3(X, Y) = 0$ and $H_2(Z) = 0$, this implies that

$$\dim H_2(X) \leq \dim H_2(X, Z) = k. \tag{19}$$

On other hand, we consider the Mayer–Vietoris exact homology sequence,

$$\begin{aligned} H_3(X) \oplus H_3(Y) \to H_3(\mathbb{S}^2 \times \mathbb{S}^1) \to H_2(X \cap Y) \\ \to H_2(X) \oplus H_2(Y) \xrightarrow{\beta} H_2(\mathbb{S}^2 \times \mathbb{S}^1) \to \cdots . \end{aligned} \tag{20}$$

Both X and Y are compact 3-manifolds with boundary, so $H_3(X) = 0 = H_3(Y)$. Also, $X \cap Y$ is composed of e^+ closed orientable surfaces, so $H_2(X \cap Y) \cong F_2^{e^+}$. On the other hand, we know that $H_3(\mathbb{S}^2 \times \mathbb{S}^1) \cong F_2 \cong H_2(\mathbb{S}^2 \times \mathbb{S}^1)$ contains a non-separating regular level set. By Lemma 3.1, it follows that β is surjective. Therefore,

$$\dim H_2(X) = e^+ - 1.$$

By inequality (19),

$$e^+ \leq k + 1.$$

If $a = \dim \operatorname{Im} \alpha$, from the exact sequence (18), we have

$$k = e^+ - 1 + G^- - a.$$

Hence,

$$k + 1 - G^- \leq e^+. \tag{21}$$

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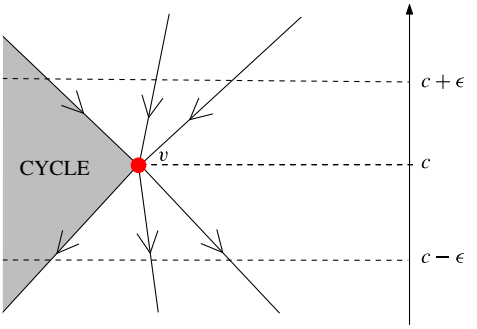


FIGURE 2. v is a vertex in the cycle.

On the other hand, U satisfies the Poincaré–Hopf condition (4) and thus

$$e^+ + G^- = e^- + G^+.$$

By inequality (21),

$$k + 1 - G^+ \leq e^-.$$

Now for the last inequality, we need to consider the reverse flow which switches the roles of e^+ and e^- and transforms A to A^t so k is unchanged.

We write $W = U \cup Y$ and in this case the flow exits U through Y. We consider the exact homology sequence of the pair (W, Y),

$$\to H_3(W, Y) \to H_2(Y) \to H_2(W) \to H_2(W, Y) \to \cdots . \tag{22}$$

By Theorem 2.3, we have $H_3(W, Y) = 0$ and $H_2(Y) \cong F_2$. Therefore, by sequence (22),

$$\dim H_2(W) \leq k + 1. \tag{23}$$

Now consider the Mayer–Vietoris exact homology sequence

$$\begin{aligned} \to H_3(W) \oplus H_3(Z) \to H_3(\mathbb{S}^2 \times \mathbb{S}^1) \to H_2(W \cap Z) \\ \to H_2(W) \oplus H_2(Z) \xrightarrow{\gamma} H_2(\mathbb{S}^2 \times \mathbb{S}^1) \to \cdots . \end{aligned} \tag{24}$$

Both W and Z are compact 3-manifolds with boundary, so $H_3(W) = 0 = H_3(Z)$. $W \cap Z$ is composed of e^- closed orientable surfaces, hence $H_2(X \cap Y) \cong F_2^{e^-}$. Since the submanifold W contains a non-separating regular level set by Lemma 3.1, γ is surjective. Therefore, $\dim H_2(W) = e^-$ and, by inequality (23),

$$e^- \leq k + 1.$$

The proof is complete in this case.

Now suppose that an edge on the cycle is incoming and incident to v and another edge is outgoing and incident to v, as shown in Figure 2. We have that $\partial U \cap Z_0$ is a union of surfaces and only one of these is non-separating and corresponds to the outgoing edge incident to v on the cycle. We call Z_1 the connected component of Z_0 that contains the aforementioned non-separating surface.

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Let Z be the union of Z_1 with the components of $\mathbb{S}^2 \times \mathbb{S}^1 - (\partial U \cap (Z_0 - Z_1))$ which do not contain Λ. By Lemma 3.6, each component of $Z - Z_1$ is a manifold of handlebody type. Now we define $X = Z \cup U$ and $Y = \mathbb{S}^2 \times \mathbb{S}^1 - X$.

We call Y_1 the connected component of Y corresponding to the cycle. By Lemma 3.6 we have that each component of $Y - Y_1$ is a manifold of handlebody type. Note that $Z \cap Y$ is a closed orientable surface of genus g. Moreover, by Lemma 3.1 we have $H_2(Z) \cong F_2$ and $\dim H_1(Z) = \sum g_i^- + g$. With this notation, the result (9.11) of [**9**] says that $H_2(X, Z)$ has dimension k.

Now consider the exact homology sequence of the pair (X, Z),

$$\to H_3(X, Z) \to H_2(Z) \to H_2(X) \to H_2(X, Z) \to H_1(Z) \xrightarrow{\alpha} H_1(X) \to \cdots . \quad (25)$$

Since $H_3(X, Y) = 0$ and $H_2(Z) \cong F_2$, this implies

$$\dim H_2(X) \leq k + 1. \quad (26)$$

Consider the Mayer–Vietoris exact homology sequence

$$0 \to H_3(\mathbb{S}^2 \times \mathbb{S}^1) \to H_2(X \cap Y) \to H_2(X) \oplus H_2(Y) \xrightarrow{\beta} H_2(\mathbb{S}^2 \times \mathbb{S}^1) \to \cdots . \quad (27)$$

Since $X \cap Y$ is composed of $e^+ + 1$ closed orientable surfaces, $H_2(X \cap Y) \cong F_2^{e^+ + 1}$. In addition, we know that $H_3(\mathbb{S}^2 \times \mathbb{S}^1) \cong F_2 \cong H_2(\mathbb{S}^2 \times \mathbb{S}^1)$. Furthermore, there exists a non-separating regular level set contained in Y. By Lemma 3.1, β is surjective. Therefore,

$$\dim H_2(X) = e^+.$$

By inequality (26),

$$e^+ \leq k + 1.$$

On the other hand, as β in (27) is surjective,

$$\begin{aligned} 0 \to H_1(X \cap Y) \to\ & H_1(X) \oplus H_1(Y) \to H_1(\mathbb{S}^2 \times \mathbb{S}^1) \to H_0(X \cap Y) \\ & \to H_0(X) \oplus H_0(Y) \to H_0(\mathbb{S}^2 \times \mathbb{S}^1) \to 0. \end{aligned}$$

Since $\dim H_1(X \cap Y) = 2G^+ + 2g$, $\dim H_1(Y) = G^+ + g$, $\dim H_0(X \cap Y) = e^+ + 1$ and $\dim H_0(Y) = e^+$,

$$\dim H_1(X) = G^+ + g.$$

Since $\partial X \cap \partial Z$ is a genus g surface, $H_1(X)$ and $H_1(Z)$ both have a F_2^g summand that comes from this surface. Then $\dim \operatorname{Im} \alpha \geq g$. Also, if $a = \dim \operatorname{Im} \alpha$, from the exact sequence (25), we have

$$k = e^+ - 1 + \sum g_i^- + g - a$$

as $g - a \leq 0$, hence

$$k + 1 - G^- \leq e^+. \quad (28)$$

Moreover, U satisfies the Poincaré–Hopf condition (4) and thus

$$e^+ + G^- = e^- + G^+.$$

By inequality (28),

$$k + 1 - G^+ \leq e^-.$$

Now, for the last inequality, we consider the reverse flow which switches the roles of e^+ and e^- and transforms A to A^t so that k remains unchanged. □

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Note that by Lemma 3.6 every other component of the chain-recurrent set that corresponds to a vertex which is not on the cycle is inside a manifold of handlebody type and all separating regular level sets are π_1-trivial. In this context we have the following proposition.

PROPOSITION 5.2. *Let L be a Lyapunov graph associated to a Smale flow ϕ_t and a Lyapunov function on $\mathbb{S}^2 \times \mathbb{S}^1$. Let v be a vertex of L labeled with a suspension of a subshift of finite type. $A_{m\times m}$ is the non-negative integer matrix representing this subshift and $k = \dim \ker \overline{I - A} : F_2^m \to F_2^m$ where $\overline{I - A}$ is the* mod 2 *reduction of $I - A$. Let e^+ and e^- be respectively the indegree and outdegree of v. If v represents a basic set, which is contained in some manifold of handlebody type M in $\mathbb{S}^2 \times \mathbb{S}^1$, then*

$$e^+, e^- > 0,$$
$$e^+ \le k + 1,$$
$$e^- \le k + 1,$$
$$k + 1 - G^- \le e^+ \quad \text{with } G^- = \sum_{i=1}^{e^-} g_i^-,$$
$$k + 1 - G^+ \le e^- \quad \text{with } G^+ = \sum_{j=1}^{e^+} g_j^+,$$

where the g_i^+ (g_j^-) are the weights on the incoming (outgoing) edges incident to the vertex v.

Proof. By Remark 3.1, we can suppose that the component Y_1 is not a manifold of handlebody type. Then by Corollary 4.2 the other components of Y and all components of Z are manifolds of handlebody type. In other words, $H_2(Y) \cong F_2$, $H_2(Z) = 0$ and $\dim H_1(Z) = G^-$.

Consider the exact homology sequence of the pair (X, Z),

$$\to H_3(X, Z) \to H_2(Z) \to H_2(X) \to H_2(X, Z) \to H_1(Z) \xrightarrow{\alpha} H_1(X) \to \cdots . \quad (29)$$

Since $H_3(X, Z) = 0$ and $H_2(Z) = 0$, this implies that

$$\dim H_2(X) \le \dim H_2(X, Z) = k. \quad (30)$$

Now consider the exact homology reduced sequence of the pair $(\mathbb{S}^2 \times \mathbb{S}^1, Y)$,

$$\to \widetilde{H}_1(Y) \to \widetilde{H}_1(\mathbb{S}^2 \times \mathbb{S}^1) \to \widetilde{H}_1(\mathbb{S}^2 \times \mathbb{S}^1, Y) \to \widetilde{H}_0(Y) \to \widetilde{H}_0(\mathbb{S}^2 \times \mathbb{S}^1).$$

Since $\widetilde{H}_0(\mathbb{S}^2 \times \mathbb{S}^1) = 0$ and $\dim(\widetilde{H}_0(Y)) = e^+ - 1$, we conclude that

$$\dim \widetilde{H}_1(\mathbb{S}^2 \times \mathbb{S}^1, Y) \ge e^+ - 1.$$

Moreover, by Lefschetz duality $H_1(\mathbb{S}^2 \times \mathbb{S}^1, Y) \cong H_2(\mathbb{S}^2 \times \mathbb{S}^1 - Y, \mathbb{S}^2 \times \mathbb{S}^1 - \mathbb{S}^2 \times \mathbb{S}^1) \cong H_2(X)$. Therefore,

$$\dim H_2(X) \ge e^+ - 1 \quad (31)$$

and, from inequality (30),

$$e^+ \le k + 1.$$

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Now we consider the Mayer–Vietoris exact homology sequence

$$0 \to H_3(\mathbb{S}^2 \times \mathbb{S}^1) \to H_2(X \cap Y) \to H_2(X) \oplus H_2(Y) \to H_2(\mathbb{S}^2 \times \mathbb{S}^1) \to \cdots .$$

Since $X \cap Y$ is composed of e^+ closed orientable surfaces, we have that $H_2(X \cap Y) \cong F_2^{e^+}$. Moreover, we know that $H_3(\mathbb{S}^2 \times \mathbb{S}^1) \cong F_2 \cong H_2(\mathbb{S}^2 \times \mathbb{S}^1)$. Therefore,

$$\dim H_2(X) \leq e^+ - 1. \tag{32}$$

By inequalities (31) and (32), we conclude that

$$\dim H_2(X) = e^+ - 1.$$

If $a = \dim \operatorname{Im} \alpha$, from exact sequence (29), we have that

$$k = \dim H_2(X) + G^- - a.$$

Hence,

$$k + 1 - G^- \leq e^+. \tag{33}$$

On the other hand, U satisfies the Poincaré–Hopf condition (4) and thus

$$e^+ + G^- = e^- + G^+.$$

Thus,

$$k + 1 - G^+ \leq e^-.$$

Now for the last inequality, we need to consider the reverse flow which switches the roles of e^+ and e^- and transforms A to A^t so that k is unchanged. In this case, the flow enters U through Z and exits through Y. We write $W = Y \cup U$ and we consider the exact homology sequence of the pair (W, Y),

$$\to H_3(W, Y) \to H_2(Y) \to H_2(W) \to H_2(W, Y) \to \cdots . \tag{34}$$

By Theorem 2.3, we have $H_3(W, Y) = 0$ and $H_2(Y) \cong F_2$. Therefore, by sequence (34),

$$\dim H_2(W) \leq k + 1 \quad \text{and} \quad \dim H_2(W) \geq 1. \tag{35}$$

Now consider the exact sequence of the pair $(\mathbb{S}^2 \times \mathbb{S}^1, Z)$, and by Lemma 3.5 we have that $\dim H_2(W) = e^-$. It follows from (35) that

$$e^- \leq k + 1. \qquad \square$$

5.2. *Lyapunov graph without cycles.* We continue to use the notation established at the beginning of this section. Let U be the closure of the component of $X_0 - Z_0$ which contains Λ. Then U is a neighborhood of Λ whose boundary consists of closed orientable surfaces. Also, the flow is transverse to the boundary of U. The flow enters e^+ of these surfaces and exits the remaining e^- surfaces. Let Z be the union of the components of $\mathbb{S}^2 \times \mathbb{S}^1 - (\partial U \cap Z_0)$ which do not contain Λ. Now define $X = Z \cup U$ and $Y = \mathbb{S}^2 \times \mathbb{S}^1 - X$. With this notation, result (9.11) in [**9**] says that $H_2(X, Z)$ has dimension k.

First of all, suppose that some component of ∂U^- is a π_1-non-trivial regular level set. Furthermore, we can suppose that ∂Z_1 is also a π_1-non-trivial regular level set. By Lemma 3.6 and Corollary 4.2, we have that all components of Z and all components of Y are manifolds of handlebody type.

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PROPOSITION 5.3. *Let L be a Lyapunov graph associated to a Smale flow ϕ_t and a Lyapunov function on $\mathbb{S}^2 \times \mathbb{S}^1$, such that L is a tree. Furthermore, suppose there exists a π_1-non-trivial regular level set. Let v be a vertex of L labeled with a suspension of a subshift of finite type and $A_{m\times m}$ be the non-negative integer matrix representing this subshift. Let $k = \dim \ker \overline{I - A} : F_2^m \to F_2^m$ where $\overline{I - A}$ is the* mod 2 *reduction of $I - A$. If e^+ and e^- are respectively the indegree and outdegree of v, then*

$$\begin{gathered} e^+, e^- > 0, \\ e^+ \le k + 1, \\ e^- \le k + 1, \\ k + 1 - G^- \le e^+ \quad \text{with } G^- = \sum_{i=1}^{e^-} g_i^-, \\ k + 1 - G^+ \le e^- \quad \text{with } G^+ = \sum_{j=1}^{e^+} g_j^+, \end{gathered}$$

where the g_i^+ (g_j^-) are the weights on the incoming (outgoing) edges incident to the vertex v.

Proof. Suppose that Λ is the basic set of ϕ_t corresponding to v. Consider the exact homology sequence of the pair (X, Z),

$$\to H_3(X, Z) \to H_2(Z) \to H_2(X) \to H_2(X, Z) \to H_1(Z) \xrightarrow{\alpha} H_1(X) \to \cdots . \quad (36)$$

Since $H_3(X, Y) = 0$ and $H_2(Z) = 0$, this implies that

$$\dim H_2(X) \le k. \quad (37)$$

Now consider the exact homology reduced sequence of the pair $(\mathbb{S}^2 \times \mathbb{S}^1, Y)$,

$$\to \widetilde{H}_1(Y) \to \widetilde{H}_1(\mathbb{S}^2 \times \mathbb{S}^1) \to \widetilde{H}_1(\mathbb{S}^2 \times \mathbb{S}^1, Y) \to \widetilde{H}_0(Y) \to \widetilde{H}_0(\mathbb{S}^2 \times \mathbb{S}^1) \to 0. \quad (38)$$

Since $\widetilde{H}_0(\mathbb{S}^2 \times \mathbb{S}^1) = 0$ and $\dim(\widetilde{H}_0(Y)) = e^+ - 1$, we conclude that

$$\dim \widetilde{H}_1(\mathbb{S}^2 \times \mathbb{S}^1, Y) \ge e^+ - 1.$$

Moreover, by Lefschetz duality, $H_1(\mathbb{S}^2 \times \mathbb{S}^1, Y) \cong H_2(\mathbb{S}^2 \times \mathbb{S}^1 - Y, \mathbb{S}^2 \times \mathbb{S}^1 - \mathbb{S}^2 \times \mathbb{S}^1) \cong H_2(X)$. Therefore, $\dim H_2(X) \ge e^+ - 1$ and, from inequality (37),

$$e^+ \le k + 1.$$

Also, from sequence (38),

$$\dim H_2(X) \le e^+ - 1 + \dim H_1(\mathbb{S}^2 \times \mathbb{S}^1) = e^+. \quad (39)$$

If $a = \dim \operatorname{Im} \alpha$, from the exact sequence (36), we have

$$k = \dim H_2(X) + G^- - a. \quad (40)$$

The hypothesis that there exists a π_1-non-trivial regular level set in Z implies that $\operatorname{Im} \alpha \neq 0$. Therefore, from inequality (40) we can conclude that

$$k < \dim H_2(X) + G^-. \quad (41)$$

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Now, by inequality (39), we have that

$$k + 1 - G^- \leq e^+.$$

On the other hand, U satisfies the Poincaré–Hopf condition (4) and thus

$$e^+ + G^- = e^- + G^+.$$

Hence,

$$k + 1 - G^+ \leq e^-.$$

Now for the last inequality, we need to consider the reverse flow which switches the roles of e^+ and e^- and transforms A to A^t so that k is unchanged.

For the general case, we can suppose that a π_1-non-trivial regular level set is contained in some component of Z. Then it suffices to repeat the arguments in the above proof. □

Now we suppose that each connected component of a regular level set is π_1-trivial in $\mathbb{S}^2 \times \mathbb{S}^1$.

Let $k = \dim \ker \overline{I - A} : F_2^m \to F_2^m$ where $\overline{I - A}$ is the mod 2 reduction of $I - A$, and let e^+ (e^-) be the indegree (outdegree) of v. With X, Y and Z as defined at the beginning of §5.2, we have the following lemma.

PROPOSITION 5.4. *Let v be a vertex of L labeled with a suspension of a subshift of finite type and $A_{m\times m}$ be the non-negative integer matrix representing this subshift. Suppose that all components of Y and Z are manifolds of handlebody type. Then*

$$e^+, e^- > 0,$$
$$e^+ \leq k,$$
$$e^- \leq k,$$
$$k - G^- \leq e^+ \quad \text{with } G^- = \sum_{i=1}^{e^-} g_i^-,$$
$$k - G^+ \leq e^- \quad \text{with } G^+ = \sum_{j=1}^{e^+} g_j^+,$$

where the g_i^+ (g_j^-) are the weights on the incoming (outgoing) edges incident to v.

Proof. Since Z and Y are disjoint unions of manifolds of handlebody type, we have that $H_2(Z) = 0 = H_2(Y)$, $\dim H_1(Z) = G^-$ and $\dim H_1(Y) = G^+$. Now consider the exact homology sequence of the pair (X, Z),

$$\to H_3(X, Z) \to H_2(Z) \to H_2(X) \to H_2(X, Z) \to H_1(Z) \xrightarrow{\alpha} H_1(X) \to \cdots. \quad (42)$$

Since $H_3(X, Z) = 0$ and $H_2(Z) = 0$,

$$\dim H_2(X) \leq k. \quad (43)$$

Consider the reduced exact homology sequence of the pair $(\mathbb{S}^2 \times \mathbb{S}^1, Y)$,

$$\to H_1(Y) \xrightarrow{\beta} H_1(\mathbb{S}^2 \times \mathbb{S}^1) \to H_1(\mathbb{S}^2 \times \mathbb{S}^1, Y) \to \widetilde{H}_0(Y) \to 0. \quad (44)$$

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By Lemma 3.5 we have that β is trivial and $\dim \widetilde{H}_0(Y) = e^- + 1$. Thus, by sequence (44),

$$\dim \widetilde{H}_1(\mathbb{S}^2 \times \mathbb{S}^1, Y) = e^+.$$

Using the Lefschetz duality, $H_1(\mathbb{S}^2 \times \mathbb{S}^1, Y) \cong H_2(\mathbb{S}^2 \times \mathbb{S}^1 - Y, \mathbb{S}^2 \times \mathbb{S}^1 - \mathbb{S}^2 \times \mathbb{S}^1) \cong H_2(X)$. Therefore, $\dim H_2(X) = e^+$ and, from equality (43),

$$e^+ \leq k.$$

On the other hand, if $a = \dim \operatorname{Im} \alpha$ then, from sequence (42),

$$k = \dim H_2(X) + \dim H_1(Z) - a.$$

Therefore,

$$k - G^- \leq e^+.$$

Furthermore, U satisfies the Poincaré–Hopf condition (4) and thus

$$e^+ + G^- = e^- + G^+.$$

Hence,

$$k - G^+ \leq e^-.$$

Now, for the last inequality, we need to consider the reverse flow which switches the roles of e^+ and e^- and transforms A to A^t so that k is unchanged. □

Proof of the necessity of the conditions of Theorem 1.1.

(1) Items (1) and (2) follow from Corollary 3.1 in [**8**].
(2) Item (4) follows by Theorem 4.7 in [**6**].
(3) Item (3) follows from Proposition 2.1.
(4) Item 3(a) follows from Propositions 5.1 and 5.2.
(5) Item 3(b)(i) follows from Propositions 5.2–5.4.
(6) Item 3(b)(ii) is proved as follows. If there exists a regular level set $\mathcal{T}_g$ which is π_1-non-trivial, then clearly $g \neq 0$. Otherwise, if each regular level set is π_1-trivial, by the use of Lemma 3.6 on the edges of the graph we obtain that there exists a vertex v_0 labeled with a suspension of a subshift of finite type associated to the non-negative integer matrix $A_{m\times m}$ such that

$$e^+, e^- > 0,$$
$$e^+ \leq k,$$
$$e^- \leq k,$$
$$k - G^- \leq e^+ \quad \text{with } G^- = \sum_{i=1}^{e^-} g_i^-,$$
$$k - G^+ \leq e^- n \quad \text{with } G^+ = \sum_{j=1}^{e^+} g_j^+,$$

where $k = \dim \ker((I - B) : F_2^m \to F_2^m)$, $F_2 = \mathbb{Z}/2$, $b_{ij} = a_{ij} \bmod 2$ and $g_i^+ (g_j^-)$ are the weights on the incoming (outgoing) edges incident to the vertex v_0. By hypothesis,

$$k - G^- < k + 1 - G^- \leq e^- \leq k$$

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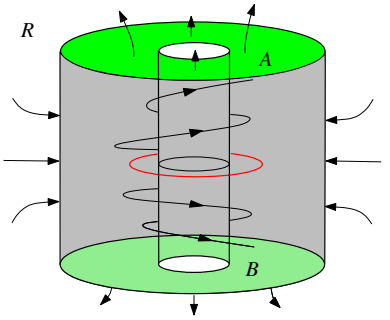


FIGURE 3. Round handle.

so $k - G^- < k$, that is, $G^- > 0$, thus there is at least one $g_j^- > 0$. Similarly, there is at least one $g_k^+ > 0$.

6. *Sufficient conditions on abstract Lyapunov graphs associated to Smale flows*

In this section we prove propositions that will combine in order to form the proof of the sufficiency of the conditions stated in the main theorem of this work, Theorem 1.1. In [**7**] the construction of basic blocks for singularities, periodic orbits and subshifts of finite type that satisfy the conditions in Theorem 2.4 is presented, and we refer the reader to that paper. In what follows, we present a construction of a basic block for a vertex v that satisfies the condition of Proposition 5.4. For that we need some definitions.

Definition 6.1. Let C_1 and C_2 be solid concentric cylinders with $C_2 \subset C_1$. A *round handle* R is a 3-manifold homeomorphic to $C_1 - C_2$ containing a saddle-type periodic orbit of period equal to one and with a flow defined on R as follows: the flow enters two disjoint boundary components of R homeomorphic to annuli and exits two other disjoint boundary components also homeomorphic to annuli. See Figure 3.

Definition 6.2. A one-handle H_i is homeomorphic to a strip $D^1 \times D^1$ and is transversal to the flow. A one-handle H_s in a round handle R_s will be chosen so that $H_s \times \mathbb{S}^1 = R_s$. A one-handle H_t in a nilpotent handle N_t will be chosen so that $H_t \times D^1 = N_t$.

Definition 6.3. A *basic block* for a one-dimensional set Λ of a Smale flow ϕ_t on $\mathbb{S}^2 \times \mathbb{S}^1$ associated to a Lyapunov function $f : \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{R}$ with $f(\Lambda) = c$ is the component of $X = f^{-1}([c - \epsilon, c - \epsilon])$ which contains Λ, where $\epsilon > 0$ is chosen so that X contains no other basic set and:

(1) there exist (not necessarily connected) codimension-one submanifolds with boundary U and V in X with $U \subset V$ everywhere transversal to the flow;
(2) the first return map $\mu : U \to \operatorname{int} V$ is a well defined smooth map and there is a hyperbolic handle set $H \subset U$ with every orbit of Λ intersecting H and every $h_i \subset H$ intersecting Λ;
(3) if $x \in H$ but $r(x) \notin H$ then $\phi_t(x) \cap H = \emptyset$ for all $t > 0$ and $f(\phi_{t_0}(x)) = c - \epsilon$ for some $t_0 > 0$; likewise, if $x \in H$ but $r^{-1}(x) \notin H$ then $\phi_t(x) \cap H = \emptyset$ for all $t < 0$ and $f(\phi_{t_0}(x)) = c + \epsilon$ for some $t_0 < 0$.

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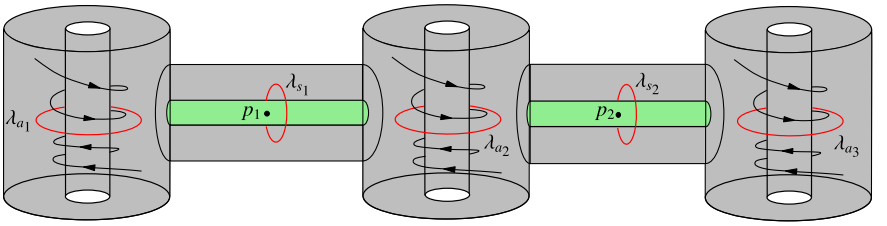


FIGURE 4. H_3 is a handlebody of genus 3.

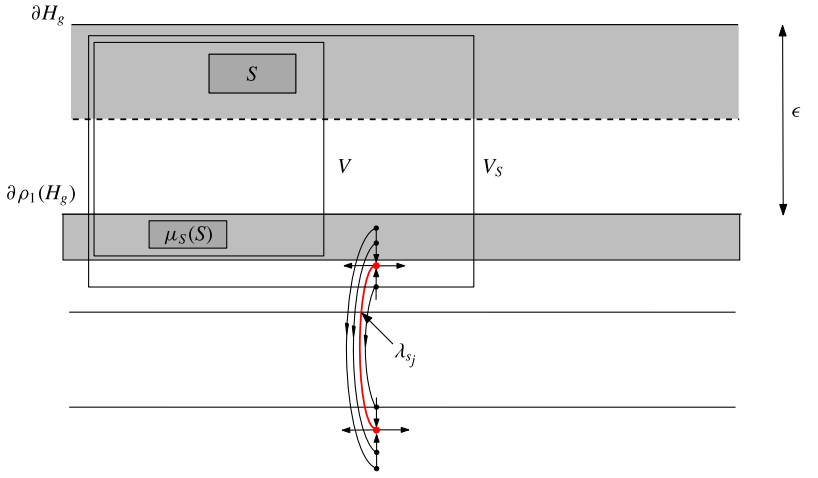


FIGURE 5. The rectangle S is the interior of $\partial H_g \times I$.

In the following construction, we build specific Smale flows on handlebodies which are transverse to its boundary. This construction will be very important for the next subsection. There exists a Smale flow on a handlebody H_g of genus g, with g attracting periodic orbits λ_{a_i}, $g-1$ saddle periodic orbits λ_{s_i} and $g-1$ repelling singularities p_i. H_g is obtained by gluing $g-1$ round handles to g solid tori as shown in Figure 4. The flow on H_g is induced by the periodic orbits and singularities and will be denoted by ρ_t. Since the flow is transversal and points inward on ∂H_g, we have

$$\epsilon = d(\partial H_g, \rho_1(\partial H_g)) > 0.$$

PROPOSITION 6.1. *Let S be a surface homeomorphic to $D^1 \times D^1$ embedded in a tubular $\epsilon/2$-neighborhood of the boundary of handlebody, such that the flow described above is transversal to S. Then there exists a neighborhood V of S such that the first return map, $\mu : S \to V$, is smooth.*

Proof. Suppose that S is embedded in some round handle R_j with periodic orbit λ_{s_j}. Hence, we can extend S to a surface V_S, which is homeomorphic to $D^1 \times D^1$ and has transversal intersection with λ_{s_j}, as shown in Figure 5.

Therefore, V_S is a cross section of λ_{s_j}, so the first return map μ_S is defined for V_S and is smooth. Now if necessary we can extend V_S, such that V_S contains $\mu_S(S)$. On the

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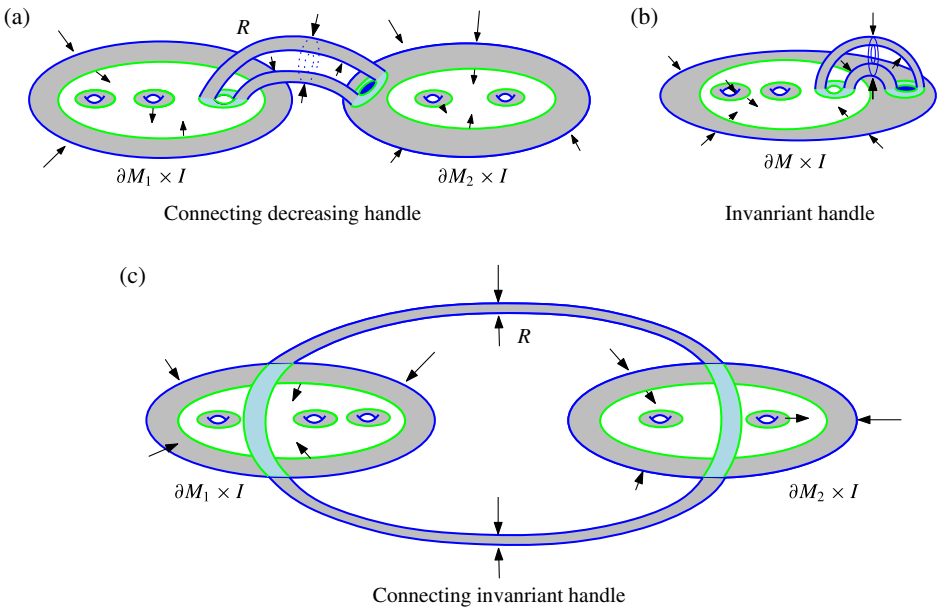


FIGURE 6. Distinct gluings of round handles.

other hand, since λ_{s_j} has an index 1 periodic orbit, we have that $\mu_S(S)$ does not intersect λ_{s_j}. Hence, we can find a surface $V \subset V_S$ such that V contains S and $\mu_S(S)$ and does not intersect with λ_{s_j}. See Figure 5. Now suppose that S is contained in some neighborhood of an attracting periodic orbit λ_{a_i}; we proceed in a similar way. For the general case, we can choose a smaller neighborhood contained in S and argue as above. □

Given a handlebody M, we consider a flow on M as described above, and the restriction of this flow to the tubular neighborhood of ∂M. Note that the tubular neighborhood of ∂M is homeomorphic to the collaring of the boundary, $\partial M \times I$, where I is a closed interval. For this reason we refer to the tubular neighborhood of ∂M as the collaring of ∂M. In [**7**] there is a description of distinct ways of gluing a round handle R to a collaring of ∂M.

In [**7**] three types of gluing of round handles to a collaring of ∂M are presented. See Figure 6. In the next subsection we construct a special round handle that cannot be realized in $\mathbb{S}^3$. Subsequently, we make use of this special round handle to construct special basic blocks which consequently are also not realizable in $\mathbb{S}^3$.

6.1. *Special round handle in* $\mathbb{S}^2 \times \mathbb{S}^1$. Due to the topology of $\mathbb{S}^2 \times \mathbb{S}^1$ there is another case to consider. In order to describe it, we build a special round handle R_1 in $\mathbb{S}^2 \times \mathbb{S}^1$. Start with a non-separating sphere $\mathbb{S}^2$ and remove two disjoint disks D_1 and D_2. Now consider a product of $\mathbb{S}^2 - (D_1 \sqcup D_2)$ with an interval I. This manifold is a 3-manifold, R_1, such that the boundary ∂R_1 is composed by four annuli. We can put a saddle periodic orbit inside of R_1, as shown in Figure 7.

Let N be a compact, connected 3-manifold with non-empty and connected boundary which is inside a 3-ball in $\mathbb{S}^2 \times \mathbb{S}^1$. In the boundary of N, ∂N, consider two disjoint

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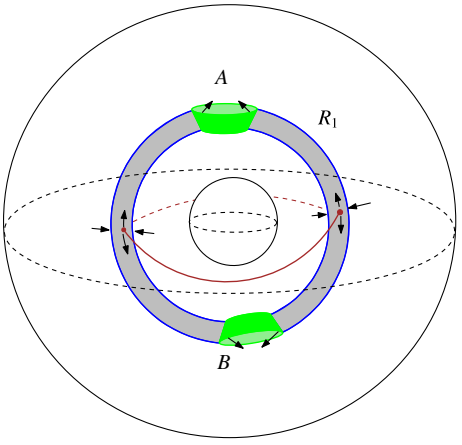


FIGURE 7. Special type of round handle in $\mathbb{S}^2 \times \mathbb{S}^1$.

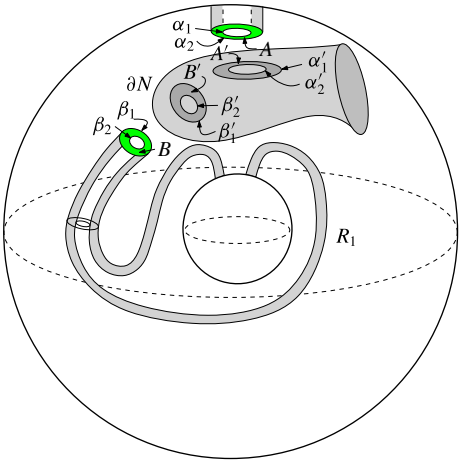


FIGURE 8. Special gluing of a round handle in $\mathbb{S}^2 \times \mathbb{S}^1$.

disks D_1 and D_2 and inside each one of them other smaller disks $\overline{D}_1 \subset D_1$ and $\overline{D}_2 \subset D_2$ such that the annuli $A_1 = D_1 - \overline{D}_1$ and $B_1 = D_2 - \overline{D}_2$ have boundaries $\alpha'_1 = \partial D_1$, $\alpha'_2 = \partial \overline{D}_1$, $\beta'_1 = \overline{D}_2$ and $\beta'_2 = \partial \overline{D}_2$. A round handle R is homeomorphic to $\mathbb{S}^1 \times [a, b] \times [c, d]$. Consider the circles $\beta_1 = \mathbb{S}^1 \times \{a\} \times \{c\}$, $\beta_2 = \mathbb{S}^1 \times \{b\} \times \{c\}$, $\alpha_1 = \mathbb{S}^1 \times \{a\} \times \{d\}$ and $\alpha_2 = \mathbb{S}^1 \times \{b\} \times \{d\}$. Also consider the annuli $A = \mathbb{S}^1 \times [a, b] \times \{d\}$, $B = \mathbb{S}^1 \times [a, b] \times \{c\}$, $E = \mathbb{S}^1 \times \{a\} \times [c, d]$ and $F = \mathbb{S}^1 \times \{b\} \times [c, d]$. With this decomposition we now describe the special gluing of R_1 to the 3-manifold N. The annulus A will be glued to A_1 and B to B_1 in such a way that α_1 is identified to α'_2 and α_2 is identified to α'_1. Also, β_1 is identified to β'_1 and β_2 is identified to β'_2. See Figure 8.

LEMMA 6.1. *With the notation above, $\partial(N \cup R_1)$ is homeomorphic to ∂N.*

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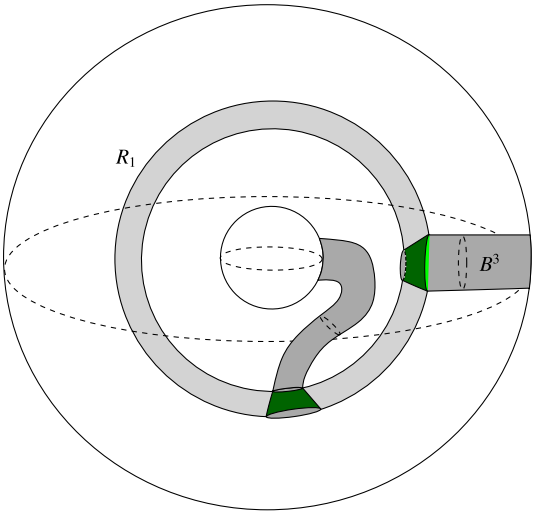


FIGURE 9. The special round handle R_1 glues to B^3.

Proof. We have that

$$\partial N = \overline{D}_2 \cup_{\beta_2'} B_1 \cup_{\beta_1'} C \cup_{\alpha_1'} A_1 \cup_{\alpha_2'} \overline{D}_1$$

where $C = \overline{\partial N - (D_1 \cup D_2)}$. On the other hand, we have

$$\partial(N \cup R_1) = \overline{D}_1 \cup_{\alpha_1=\alpha_1'} E \cup_{\beta_1=\beta_1'} C \cup_{\alpha_2=\alpha_2'} F \cup_{\beta_2=\beta_2'} \overline{D}_2.$$

Note that both $\overline{D}_1 \cup E$ and $F \cup \overline{D}_2$ are disks. Hence, ∂N is homeomorphic to $\partial(N \cup R_1)$. □

COROLLARY 6.1. *Let N be a handlebody of genus g embedded in a 3-ball in $\mathbb{S}^2 \times \mathbb{S}^1$. Let*

$$X = N \cup R_1$$

as described above. Then $\mathbb{S}^2 \times \mathbb{S}^1 - X$ is homeomorphic to a handlebody of genus g.

Proof. Recalling the construction of R_1, we have that

$$R_1 = \overline{\mathbb{S}^2 - (D_1^2 \cup D_2^2)} \times I,$$

where $\mathbb{S}^2$ is a non-separating sphere. The annuli $\partial D_1^2 \times I$ and $\partial D_2^2 \times I$ are the gluing regions. For the case of a genus zero handlebody, a 3-disk should be glued to the above annuli using two 2-disks on the boundary of the 3-disk, as shown in the Figure 9.

Note that the final manifold is isotopic to a tubular neighborhood of $\mathbb{S}_a^2 \vee \mathbb{S}_b^1$, where $\mathbb{S}_a^2$ is a fiber of $\mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^1$ and $\mathbb{S}_b^1$ is a fiber of $\mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2$. Thus, the complement in $\mathbb{S}^2 \times \mathbb{S}^1$ is homeomorphic to B^3. In the general case of a handlebody of any genus, it is enough to observe that when we glue a handle in B^3, we increase the genus of the complement. This completes the proof. □

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6.2. *Construction of special basic block in* $\mathbb{S}^2 \times \mathbb{S}^1$. Given $A_{m\times m}$, a non-negative irreducible integer matrix which is not a permutation matrix, our aim is to construct a block for the suspension of $\sigma(A)$ so that it satisfies the conditions in Proposition 5.4.

Now suppose that $A_{m\times m}$ is a matrix with k ones on the diagonal and all other entries equal to zero. Then we have the following proposition.

PROPOSITION 6.2. *Let v be a vertex of L labeled with the suspension of a subshift of finite type $\sigma(A)$, where $A_{m\times m}$ is a matrix with k ones on the diagonal and other entries equal to zero. Let e^+ and e^- be the indegree and outdegree of v, and $\{g_i^+\}$ and $\{g_j^-\}$ the weights on the incoming and outgoing edges incident to v, respectively. Suppose that*

$$k - \sum g_i^- = e^+ \quad \text{and} \quad k - \sum g_i^+ = e^-.$$

Then there exists a basic block X in $\mathbb{S}^2 \times \mathbb{S}^1$ for the suspension of $\sigma(A)$ with e^+ (e^-) entering (exiting) boundary components each being a 2-manifold of genus g_i^+ (g_j^-). Furthermore, $\mathbb{S}^2 \times \mathbb{S}^1 - X$ is homeomorphic to $H_1^{g_1^+} \sqcup \cdots \sqcup H_{e^+}^{g_{e^+}^+} \sqcup H_1^{g_1^-} \sqcup \cdots \sqcup H_{e^-}^{g_{e^-}^-}$.

Proof. The dynamics associated to the suspension of $\sigma(A)$ is simple. It consists of a basic set with k periodic orbits. We can rewrite the hypothesis as

$$(k-1) + 1 - \sum g_i^- = e^+,$$
$$(k-1) + 1 - \sum g_i^+ = e^-.$$

By [**7**, Proposition 8.2] there exists a way of gluing $k-1$ round handles to e^- collarings of $\partial M_1, \ldots, \partial M_{e^-}$, where each handlebody M_i has genus g_i^-, such that the resulting 3-manifold X_1 is a basic block which contains $k-1$ periodic orbits. Also, $\partial X_1^+ (\partial X_1^-)$ is composed of $e^+(e^-)$ closed surfaces of genus $g_i^+(g_j^-)$ respectively, and the complement in $\mathbb{S}^3$ is homeomorphic to $H_1^{g_1^+} \sqcup \cdots \sqcup H_{e^+}^{g_{e^+}^+} \sqcup H_1^{g_1^-} \sqcup \cdots \sqcup H_{e^-}^{g_{e^-}^-}$. Recall that, since X_1 is a compact 3-manifold in $\mathbb{S}^3$, we can embed X_1 in $\mathbb{S}^2 \times \mathbb{S}^1$. For simplicity we continue to denote this embedding by X_1. Now glue X_1 to R_1 in order to construct a new basic block $X_2 = X_1 \cup R_1$ with k periodic orbits. By Lemma 6.1 and Corollary 6.1, X_2 satisfies $\partial X_2 \approx \partial X_1^+ \cup \partial X_1^-$ and $\mathbb{S}^2 \times \mathbb{S}^1 - X_2$ is homeomorphic to $H_1^{g_1^+} \sqcup \cdots \sqcup H_{e^+}^{g_{e^+}^+} \sqcup H_1^{g_1^-} \sqcup \cdots \sqcup H_{e^-}^{g_{e^-}^-}$. At this point, a basic set with k periodic orbits corresponding to the k ones on the diagonal of A has been constructed. Also, $m-k$ nilpotent handles must be glued to X_2. Recall that these handles are homeomorphic to $D^1 \times D^1 \times D^1$ and contain no recurrent points. These handles will correspond to the $m-k$ zeros on the diagonal of A. Finally, denote X as X_2 glued with $m-k$ nilpotent handles. Recall that these nilpotent handles do not modify ∂X_2. □

For the general case, we can construct a basic block for the suspension of $\sigma(N)$ where N is a matrix that is flow equivalent to A. By definition, this implies that the suspensions of $\sigma(A)$ and $\sigma(N)$ are topologically equivalent. An essential proposition at this point is due to Franks [**10**].

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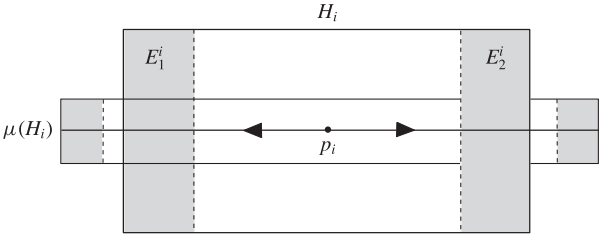


FIGURE 10. p_i is the center of the one-handle H_i.

PROPOSITION 6.3. *If A is a non-negative irreducible integer matrix which is not a permutation matrix then, given an integer $M > 0$, there is a matrix which is flow equivalent to N and which has every entry greater than M and non-diagonal entries even. The size of N depends only on A and not on M.*

By Proposition 6.3, the matrix $A_{m\times m}$ is flow equivalent to some matrix $N_{n\times n}$ which has all non-diagonal entries even,

$$\dim\ \ker((I - \overline{N} : F_2^n \to F_2^n)) = \dim\ \ker((I - \overline{A} : F_2^n \to F_2^n)) = k,$$

where $\overline{N}$ denotes the mod 2 reduction of N. By Proposition 6.2 we are able to construct a basic block X for the suspension of $\sigma(\overline{N})$. Our goal is to construct a basic block for the suspension of $\sigma(N)$. In order to achieve this, we wish to maintain the block X while modifying the flow within it. We will use the notion of one-handles H_i within round handles R_i and nilpotent handles N_t.

Let x be a point of $D^1 \times p \subset H_i$ where $p \in D^1$. The interval $D^1 \times p$ will be denoted by $W_i^u(x)$. Similarly, let x be a point of $q \times D^1$. The interval $q \times D^1$ will be denoted by $W_i^s(x)$. The reason for introducing one-handles is that if H_i intersects $\mu(H_j)$ m times then the ijth entry of $\overline{N}$ increases by m, where $\mu(x)$ is the first return map for the one-handle of the round handle.

This is precisely what we want to do, namely, increase the entries of $\overline{N}$ to obtain N. For this reason we will construct a connected surface U, such that U contains all one-handles and the flow is transversal to U.

For each one-handle H_i, we consider two disjoint subsets E_1^i and E_2^i contained in H_i as shown in Figure 10, homeomorphic to $D^1 \times D^1$ such that

$$\mu(E_1^i \sqcup E_2^i) \cap H_i = \emptyset,$$

and we will call E_1^i and E_2^i the *set of ends* of H_i. See Figure 10.

Thus, our aim is the construction of a surface $U \subset X$ such that U contains all one-handles H_i and all nilpotent handles N_j.

On the other hand, in order to connect the one-handles we consider disjoint rectangles embedded in the union of collaring of ∂M_i for $i \in \{1, \ldots, e^-\}$ and consider the flow transversal to the collaring as described in Proposition 6.1. We will use these rectangles to connect the ends of the one-handles associated to the collaring.

First fix a collaring of a handlebody $\partial M_i \times I$ for $i \in \{1, \ldots, e^-\}$. Now consider the round handles $R_{j_1}, \ldots, R_{j_l}, \ldots, R_{j_{n(i)}}$ which glue to $\partial M_i \times I$. Without loss of

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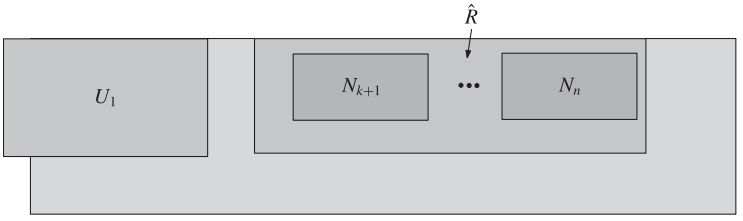


FIGURE 11. $\hat{R}$ contains all nilpotent one-handles.

generality, we can suppose that $R_{j_1}, \ldots, R_{j_l}$ are round handles that have only one annulus which glues to $\partial M_i \times I$, and the remaining ones are invariant round handles with two annuli which glue to $\partial M_i \times I$. The special round handle R_1 shall be considered as if R_1 has one annulus glued to $\partial M_1 \times I$. Now we connect the ends of the one-handles $H_{j_1}, \ldots, H_{j_l}$ with a rectangle embedded in the collaring of ∂M_i as described above. Thus, we can find a connected surface U_1^i such that U_1^i contains all one-handles $H_{j_1}, \ldots, H_{j_l}$ of the round handles $R_{j_1}, \ldots, R_{j_l}$ glued to $\partial M_i \times I$. For the invariant round handles, we glue a rectangle to $E_1^{j_{l+1}}$ and another rectangle to $E_2^{j_{n(i)}}$. Now we connect $H_{j_{l+1}}$ with $H_{j_{l+2}}$ by gluing a rectangle joining $E_2^{j_{l+1}}$ and $E_1^{j_{l+2}}$. We do this successively until the last pair $H_{j_{n(i)-1}}$ and $H_{j_{n(i)}}$ have been joined. Thus, we can find a connected surface U_2^i such that U_2^i contains all one-handles $H_{j_{l+1}}, \ldots, H_{j_{n(i)}}$. Now we connect U_1^i and U_2^i with a rectangle to obtain a connected surface U^i. Note that the flow is transversal to U^i.

In this fashion, we construct a collection $\mathcal{U} = \{ U_i \mid 1 \le i \le e^- \}$. Note that $U^i \cap U^j$ is composed by the one-handles that connect the collaring of ∂M_i with the collaring of ∂M_j. Define

$$U_1 = \bigcup_{i=1}^{e^-} U^i.$$

The surface U_1 contains all one-handles associated to $R_1, \ldots, R_k$. Extend U_1 by a rectangle $\hat{R}$, which contains all nilpotent one-handles. See Figure 11.

Thus, by Proposition 6.1 we can find a surface U that contains U_1 and $\hat{R}$. Actually, we find a surface V, such that $U \subset V$ and the first return map $\mu : U \to V$ is smooth. By construction,

$$\overline{U} = (U - \mu(U)) \cup \bigcup_{i=1}^{e^-} \mu(E_1^i \sqcup E_2^i)$$

is connected.

LEMMA 6.2. *Let U_0 be a small neighborhood of U in V as defined above. There exists an isotopy $\Psi_t : V \to V$ supported on the interior of U_0 such that Ψ_0 is the identity. Also, $\mu_1 = \Psi_1 \circ \mu : U \to V$ is such that its suspension flow with induced flow φ_t has a chain-recurrent set which is topologically equivalent to the suspension of a subshift of finite type with matrix $A_{m \times m}$.*

Proof. We have that $\overline{U}$ is connected. Hence, for $p \in \text{int}\,(\mu(W_i^u(p_i)) \cap (\partial M_i \times I))$ and $q \in \text{int}\, H_j$ such that $W_j^u(q)$ and $\mu(H_\alpha)$ are disjoint for all $\alpha \in \{1, \ldots, e^-\}$, there exists

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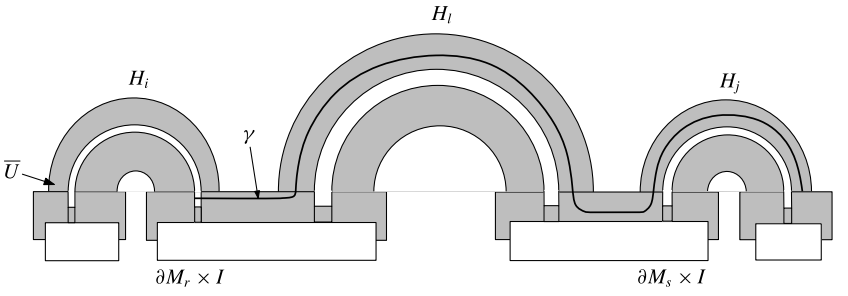


FIGURE 12. H_l is a connecting handle.

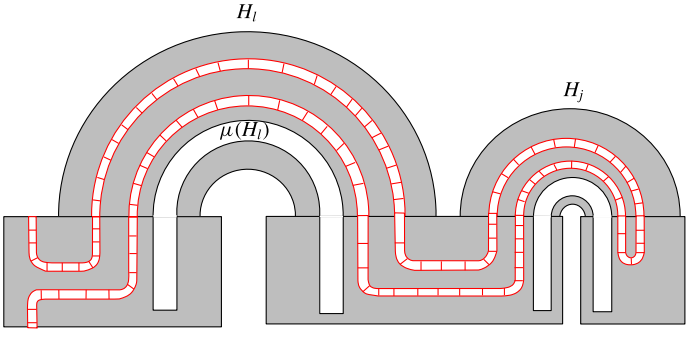


FIGURE 13. Deformation of $\phi_1(H_i)$.

a curve $\gamma : [0, 3] \to \operatorname{int} \overline{U}$ such that $\gamma(1) = p$ and $\gamma(2) = q$. Also, the intersection of $\mu(H_i)$ and $\gamma([0, 3])$ is connected and equal to $\mu(W_i^s(\mu^{-1}(p)))$ and its intersection with H_j lies in $W_j^u(q)$. For the other H_l with $l \in \{1, \ldots, n\} - \{i, j\}$ and $\gamma([0, 3]) \cap H_l \neq \emptyset$, γ is transversal to $W_l^u(p_l)$. See Figure 12.

Thus, there is an isotopy supported in a tubular neighborhood of $\gamma([0, 1])$ which pushes a small interval of $\mu(W_i^u(p_i))$ along the curve γ until it intersects $W_j^s(p_j)$ at two points. It is possible that this process causes intersection with other $W_l^s(p_l)$, but always an even number of them. See Figure 13.

Note that this isotopy preserves the hyperbolicity of the one-handles with respect to the first return map ϕ_1. In other words, if $\mu(W_i^u(x)) \cap W_j^u(y) \neq \emptyset$ then $W_j^u(y) \subset \mu(W_i^u(x))$, and similarly if $\mu(W_i^s(x)) \cap W_j^s(y) \neq \emptyset$ then $\mu(W_i^s(x)) \subset W_j^s(y)$ for all $x \in H_i$ and $y \in H_j$.

Hence, to achieve the geometric intersection matrix N, which is flow equivalent to A, we argue as in [**10**]. First we need to find a pairwise disjoint family of embedded curves $\gamma_{ij} : [0, 3] \to \overline{U}$ with the properties described above for $i, j \in \{1, \ldots, n\}$. The existence of this family $\{\gamma_{ij}\}$ with these properties is easy to see except for the fact that they are disjoint. Now, by using an isotopy supported on a neighborhood of γ_{11}, we push off all other curves. We do this for γ'_{12} and in the same way for all the remaining curves.

By Proposition 6.3 we can, if necessary, replace the matrix N with another matrix which is flow equivalent to it, congruent mod 2, and has every entry as large as we want and, in

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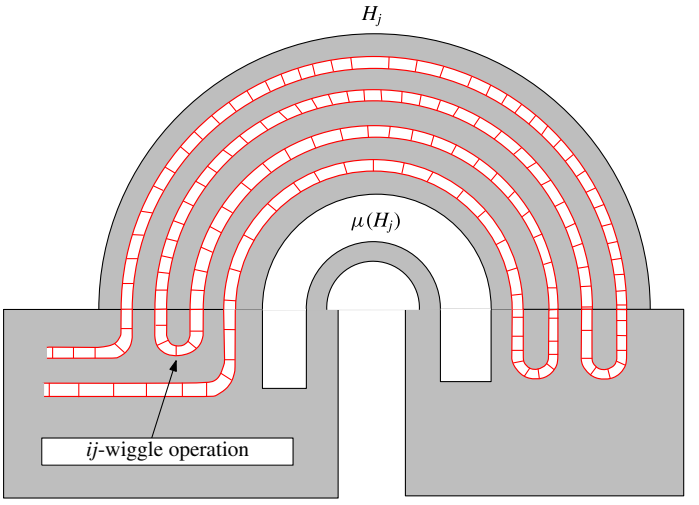


FIGURE 14. ij-wiggle operation.

particular, larger than the corresponding entry of A. We can suppose that this has been done and for simplicity of notation continue to call this matrix N. Now for each γ_{ij} we can increase the intersection points of $W^s_j(p_j)$ with the image of $W^u_i(p_i)$ if we push back the curve through H_j again. See Figure 14.

This is referred to as the ij-wiggle operation. If this operation is repeated r times, $2r+1$ intersection points are added. Do this with $r = 1/2[(A)_{ij} - (N)_{ij})]$ so that the ijth entry of the matrix $\overline{N}$ agrees with $(N)_{ij}$. Repeat this process in order to get to the matrix N. Now we perform an isotopy to change μ so that the final map has the desired property. □

PROPOSITION 6.4. *Let $A_{m\times m}$ be a non-negative integer matrix with $k = \dim\ker(\overline{(I-A)}F_2^m \to F_2^m$, where $\overline{A}$ is the* mod 2 *reduction of A. Suppose also that $e^+, e^-, g_j^+, g_i^- \in \mathbb{N}$, with $i = 1,\dots,e^+$ and $j = 1,\dots,e^-$, are positive integers satisfying:*

(1) $e^- - e^+ = \sum g_i^- - \sum g_j^+$;
(2) $k - \sum g_i^- = e^+$; *and*
(3) $k - \sum g_i^- = e^+$.

Then there exists a Smale flow ϕ_t on $\mathbb{S}^2\times\mathbb{S}^1$ with a basic block X such that:

(1) *the flow ϕ_t restricted to the basic set $\Lambda\subset X$ is topologically equivalent to the suspension of $\sigma(A)$;*
(2) *$\partial X^+(\partial X^-)$ has $e^+(e^-)$ components composed by surfaces of genus $g_i^+(g_j^-)$;*
(3) *$(\mathbb{S}^2\times\mathbb{S}^1) - X$ is homeomorphic to $H_1^{g_1^+}\sqcup\cdots\sqcup H_{e^+}^{g_{e^+}^+}\sqcup H_1^{g_1^-}\sqcup\cdots\sqcup H_{e^-}^{g_{e^-}^-}$.*

Proof. By Proposition 6.3, we know that $A_{m\times m}$ is flow equivalent to $N_{n\times n}$. By Proposition 6.2, we can build a basic block X_1 for $\overline{N}$, which satisfies assertions (2) and (3) of this proposition. For the first assertion, by Lemma 6.2, we have that, for U_0 chosen sufficiently small, $\mu: U_0 \to V$ is smooth and $\tau: U_0 \to \mathbb{R}$ is the smallest $t > 0$ such that

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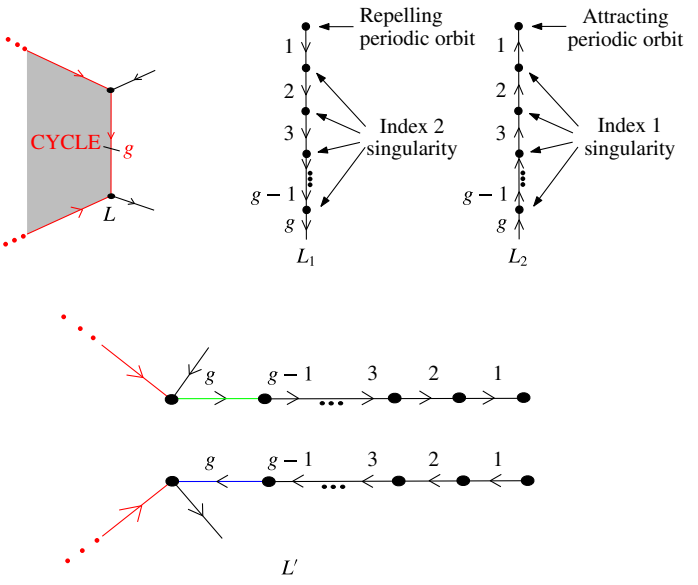


FIGURE 15. The graphs L_1 and L_2 have a dangling edge.

$\phi_t(x) = \mu(x)$. Then the partial flow on

$$Z = \{\phi_t(x) | x \in U_0,\ 0 \le t \le \tau(x)\}$$

is the suspension flow for μ. Now, by the construction, the suspension flow for μ_1 is the same as for μ since μ and μ_1 are isotopic. On the other hand, since the isotopy is supported on the interior of U_0, we have that μ and μ_1 agree near the boundary of U_0. Then near the boundary of Z the suspension flow of μ_1, φ_t and the suspension flow of μ, ϕ_t agree. Now let η_t be a flow on $\mathbb{S}^2 \times \mathbb{S}^1$ which is generated by the vector field which is tangent to the suspension flow of μ_1 on Z and tangent to ϕ_t elsewhere. Hence η_t is a Smale flow which satisfies this proposition. □

Proof of the sufficiency of the conditions of Theorem 1.1. The sufficient conditions of Theorem 1.1 will be presented in two parts, $\beta(L) = 1$ and $\beta(L) = 0$.

Case $\beta(L) = 1$. Let L be an abstract Lyapunov graph with a cycle that satisfies conditions (1), (2), (3a) and (4). Suppose that a is an edge in the cycle of L with weight g. Consider the graphs L_1 and L_2 each with a dangling edge as shown in Figure 15.

Now cut L along a and glue the graphs L_1 and L_2 by the dangling edges, as shown in Figure 15. Then a new abstract Lyapunov graph L' is obtained such that L' is a tree and each vertex satisfies the conditions in Theorem 2.4. Therefore, there exists a Smale flow ψ_t on S^3 with Lyapunov graph L' and such that L_1 and L_2 are associated to two handlebodies H_1^g and H_2^g, whose boundaries are unlinked in $\mathbb{S}^3$, as shown in Figure 16.

Now we cut the two neighborhoods $N(H_1^g)$ and $N(H_2^g)$ of H_1^g and H_2^g, respectively. Then, gluing $S^3 - (N(H_1^g) \sqcup N(H_2^g))$ along $\partial N(H_1^g)$ and $\partial N(H_2^g)$ in an appropriate manner, we obtain a Smale flow φ_t on $\mathbb{S}^2 \times \mathbb{S}^1$ with Lyapunov graph L.

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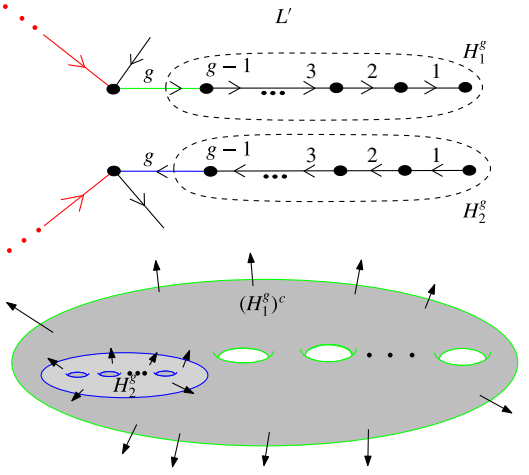


FIGURE 16. L' corresponds to a Smale flow on $\mathbb{S}^3$.

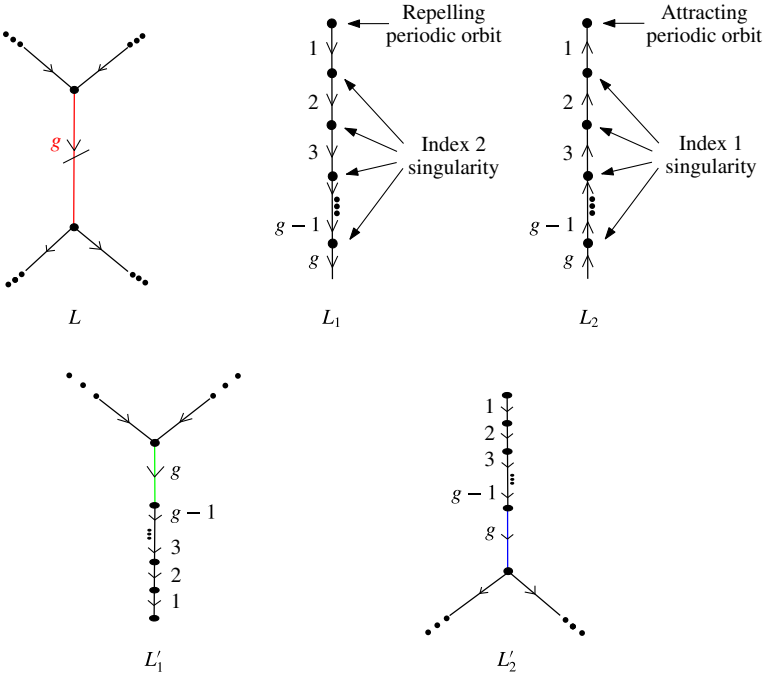


FIGURE 17. The graphs L_1 and L_2 have dangling edge.

Case $\beta(L) = 0$. In this case we will consider two possibilities: (3b)(i) and (3b)(ii).

Let L be an abstract Lyapunov tree which satisfies (1), (2), (3b)(ii) and (4).

By condition (3b)(ii), there is an edge a of L such that the weight of a is $g > 0$. Now we cut L along a and glue the graphs L_1 and L_2 by the dangling edges as shown in Figure 17.

The new graphs L_1' and L_2' are obtained. Since L_1' and L_2' satisfies the conditions of Theorem 2.4, there exist Smale flows ψ_t and ϕ_t on $\mathbb{S}^3$ such that the graph L_1 corresponds

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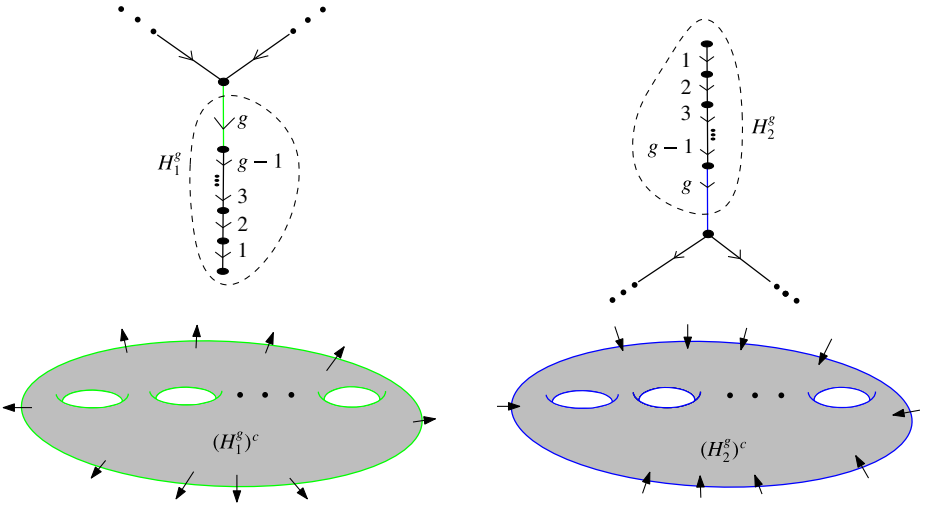


FIGURE 18. The graphs L_1' and L_2' correspond to Smale flows on $\mathbb{S}^3$.

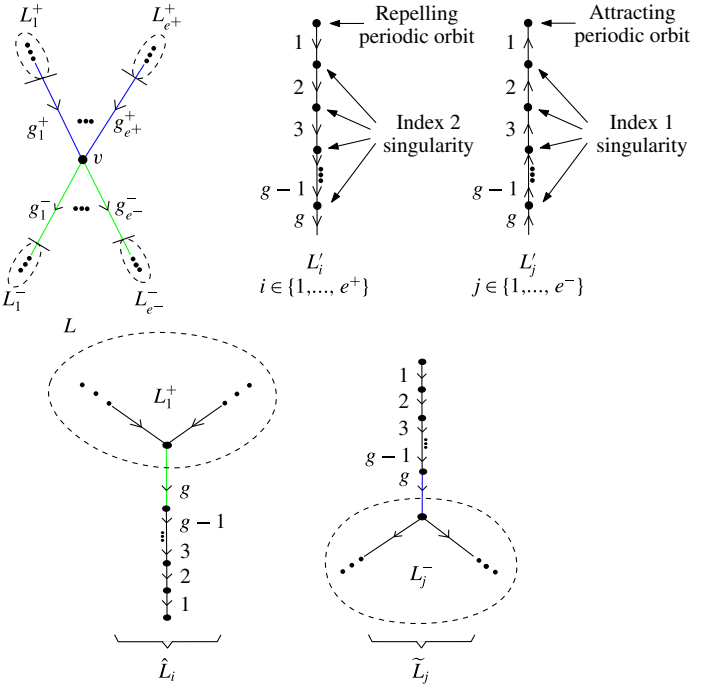


FIGURE 19. The graphs L_i' and L_j' have a dangling edge.

to a handlebody manifold H_1^g and the graph L_2 corresponds to a handlebody manifold H_2^g, as shown in Figure 18.

This implies that L_2 corresponds to a handlebody manifold H_2^g as well. Then we glue $((H_1^g)^c, \psi_t|_{(H_1^g)^c})$ and $((H_2^g)^c, \psi_t|_{(H_2^g)^c})$ in an appropriate manner along their boundaries. We obtain a Smale flow ξ_t on $\mathbb{S}^2 \times \mathbb{S}^1$ with Lyapunov graph L.

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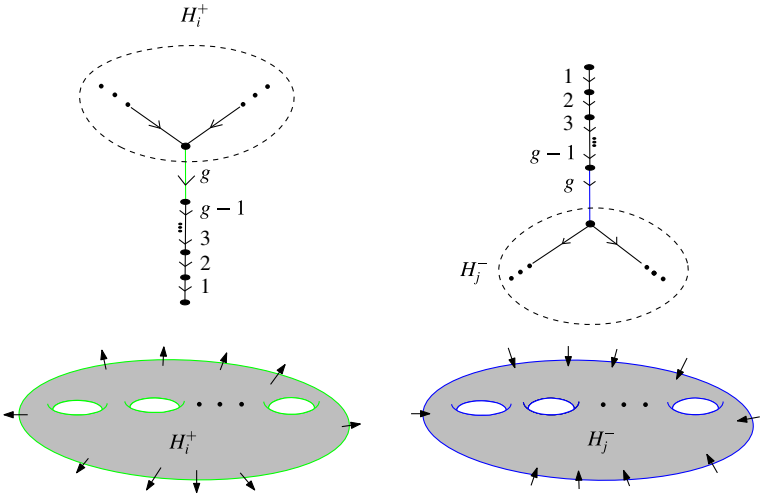


FIGURE 20. The graphs $\hat{L}_i$ and $\widetilde{L}_j$ correspond to Smale flows on $\mathbb{S}^3$.

Let L be an abstract Lyapunov tree which satisfies (1), (2), (3b)(i) and (4).

Let v be the unique vertex that satisfies the first condition of (3b)(i). By Proposition 6.4, there exist a Smale flow ϕ_t on $\mathbb{S}^2 \times \mathbb{S}^1$ and a basic block X embedded in $\mathbb{S}^2 \times \mathbb{S}^1$, associated to the vertex v. Also, $\mathbb{S}^2 \times \mathbb{S}^1 - X$ is homeomorphic to $H_1^{g_1^+} \sqcup \cdots \sqcup H_{e^+}^{g_{e^+}^+} \sqcup H_1^{g_1^-} \sqcup \cdots \sqcup H_{e^-}^{g_{e^-}^-}$. Now, if we cut L along all incoming and outgoing edges incident to the vertex v, we obtain $e^+ + e^-$ subgraphs with dangling edges which can be denoted by $L_1^+, \ldots, L_{e^+}^+$ and $L_1^-, \ldots, L_{e^-}^-$, as shown in Figure 19. We use the graphs L_i' and L_j' with dangling edges to create the new graphs $\hat{L}_1, \ldots, \hat{L}_{e^+}$ and $\widetilde{L}_1, \ldots, \widetilde{L}_{e^-}$. Then by Proposition 5.2 and Theorem 2.4, there exist Smale flows ϕ_t^i and φ_t^j on $\mathbb{S}^3$ such that L_i^+ and L_j^- correspond to handlebodies H_i^+ and H_j^- respectively, as shown in Figure 20. Then we have constructed Smale flows on $(X, \phi_t|_X)$, $(H_i^+, \phi_t^i|_{H_i^+})$ and $(H_j^-, \varphi_t^i|_{H_j^-})$ for $i \in \{1, \ldots, e^+\}$ and $j \in \{1, \ldots, e^-\}$. Finally, we glue these manifolds in an appropriate manner in order to obtain a Smale flow on $\mathbb{S}^2 \times \mathbb{S}^1$ with Lyapunov graph L.

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