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Basic series corresponding to a class of hypergeometric polynomials†

BY JERRY L. FIELDS AND JET WIMP
Midwest Research Institute, Kansas City, Missouri

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1. *Introduction.* Let $g_n(z)$ be a polynomial of degree n in z. Then there exist constants $\Pi_{k,n}$ such that

$$z^k = \sum_{n=0}^{\infty} \Pi_{k,n} g_n(z). \tag{1·1}$$

If $f(z)$ is analytic in a neighbourhood of the origin,

$$f(\lambda z) = \sum_{k=0}^{\infty} \xi_k \lambda^k z^k, \quad \xi_k = f^{(k)}(0)/k!, \tag{1·2}$$

and formal substitution of (1·1) into (1·2) yields‡

$$f(\lambda z) = \sum_{n=0}^{\infty} C_n g_n(z), \tag{1·3}$$

$$C_n = \sum_{k=0}^{\infty} \Pi_{k,n} \xi_k \lambda^k. \tag{1·4}$$

Equation (1·3) is called the basic series for $f(\lambda z)$ corresponding to the set $\{g_n(z)\}$. For a general treatment of basic series see (1).

In this paper we study basic series for the case where $g_n(z)$ is the hypergeometric polynomial ((2), §4·1)

$$\left.\begin{aligned} G_n(z,\gamma) &= {}_{p+2}F_q\left(\begin{matrix} -n, n+\gamma, a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\middle| z\right), \\ &= \sum_{k=0}^{n} \frac{(-n)_k (n+\gamma)_k \prod_{i=1}^{q} (a_i)_k}{\prod_{i=1}^{q} (b_i)_k} \frac{z^k}{k!}, \end{aligned}\right\} \tag{1·5}$$

where

$$(\sigma)_\mu = \frac{\Gamma(\sigma+\mu)}{\Gamma(\sigma)}. \tag{1·6}$$

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‡ Note that if $\Pi_{k,n} = O(1)$ uniformly in n as $k \to \infty$, convergence of C_n is assured if $|\lambda| <$ radius of convergence of $f(z)$.

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We will use the contracted notation

$$G_n(z,\gamma) = {}_{p+2}F_q\left(\left.\begin{matrix}-n, n+\gamma, a_p \\ b_q\end{matrix}\right| z\right) = \sum_{k=0}^{n} \frac{(-n)_k (n+\gamma)_k (a_p)_k}{(b_q)_k k!} z^k. \tag{1·7}$$

Thus $(a_p)_k$ is to be interpreted as $\prod_{i=1}^{p} (a_i)_k$ and similarly for $(b_q)_k$. To insure that $G_n(z,\gamma)$ is well defined and is a polynomial of order n, we assume that none of the quantities $\gamma+1$, a_i and b_j is a negative integer or zero. We also assume no a_i is equal to any b_j.

By the following limit procedure (called a confluence with respect to γ) we obtain

$$\begin{aligned}\lim_{\gamma\to\infty} G_n\left(\frac{z}{\gamma}, \gamma\right) &= \lim_{\gamma\to\infty} {}_{p+2}F_q\left(\left.\begin{matrix}-n, n+\gamma, a_p \\ b_q\end{matrix}\right| \frac{z}{\gamma}\right) \\ &= {}_{p+1}F_q\left(\left.\begin{matrix}-n, a_p \\ b_q\end{matrix}\right| z\right) = G_n(z). \end{aligned} \tag{1·8}$$

Hence, $\{G_n(z,\gamma)\}$ and its confluent form $\{G_n(z)\}$ contain the orthogonal polynomials of Jacobi, Laguerre, Hermite and Bessel.

Henceforth we use the notation

$$\left.\begin{aligned} z^k &= \sum_{n=0}^{\infty} \pi_{k,n}(\gamma)\, G_n(z,\gamma), \\ f(\lambda z) &= \sum_{n=0}^{\infty} C_n(\lambda,\gamma)\, G_n(z,\gamma), \\ C_n(\lambda,\gamma) &= \sum_{k=0}^{\infty} \pi_{k,n}(\gamma)\, \xi_k \lambda^k, \end{aligned}\right\} \tag{1·9}$$

and

$$\left.\begin{aligned} z^k &= \sum_{n=0}^{\infty} \pi_{k\ n} G_n(z), \\ f(\lambda z) &= \sum_{n=0}^{\infty} C_n(\lambda)\, G_n(z), \\ C_n(\lambda) &= \sum_{k=0}^{\infty} \pi_{k\ n} \xi_k \lambda^k, \end{aligned}\right\} \tag{1·10}$$

where $\xi_k = f^{(k)}(0)/k!$.

In § 2 our main result determines (1·9) and (1·10), in § 3 we examine the case where $\{G_n(z,\gamma)\}$ are the Jacobi polynomials, and in § 4 we apply our results to obtain economic polynomial representations of several higher transcendental functions.

2. *The determination of $\pi_{k,n}(\gamma)$ and $\pi_{k,n}$.*

THEOREM. *Let none of the quantities $\gamma+1$, a_i and b_j be a negative integer or zero. Then*

$$\pi_{k,n}(\gamma) = \frac{(b_q)_k (-k)_n (2n+\gamma)}{(a_p)_k (n+\gamma+1)_k (n+\gamma)\, n!}, \tag{2·1}$$

and hence

$$C_n(\lambda,\gamma) = \frac{(b_q)_n (-\lambda)^n}{(a_p)_n (n+\gamma)_n} \sum_{k=0}^{\infty} \frac{(n+b_q)_k (n+1)_k \xi_{k+n} \lambda^k}{(n+a_p)_k (2n+\gamma+1)_k k!}. \tag{2·2}$$

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We prove (2·1) by induction on p and q using the Laplace transform techniques discussed in (3). Notice that by (1·9), (2·1) is equivalent to

$$z^k = \frac{(b_q)_k}{(a_p)_k} \sum_{n=0}^{k} \frac{(-k)_n (2n+\gamma)}{(n+\gamma+1)_k (n+\gamma)\, n!}\, {}_{p+2}F_q\left(\begin{matrix} -n, n+\gamma, a_p \\ b_q \end{matrix}\middle| z\right). \tag{2·3}$$

First we prove (2·3) for the case $p = q = 0$. It is known classically ((2), § 10·20) or deducible from ((3), (2·5)) that

$$z^k = (\alpha+1)_k \sum_{n=0}^{k} \frac{(-k)_n (2n+\alpha+\beta+1)\, P_n^{(\alpha,\beta)}(1-2z)}{(\alpha+1)_n (n+\alpha+\beta+2)_k (n+\alpha+\beta+1)}, \tag{2·4}$$

where

$$P_n^{(\alpha,\beta)}(x) = \frac{(-)^n (\beta+1)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \beta+1 \end{matrix}\middle| \frac{1+x}{2}\right). \tag{2·5}$$

Making use of the property

$$P_n^{(\alpha,\beta)}(-x) = (-)^n P_n^{(\beta,\alpha)}(x), \tag{2·6}$$

interchanging α and β, and letting $\gamma = \alpha+\beta+1$, $b_1 = \beta+1$, we arrive at (2·3) for $p = 0$, $q = 1$. The case $p = q = 0$ follows simply from this case if we replace z by zb_1, divide both sides of (2·3) by b_1^k and let $b_1 \to \infty$.

Now assume (2·3) true for p and q. Multiplying both sides of (2·3) by $z^{\sigma-1}$, and taking the Laplace transform of both sides with respect to z, we obtain, see (4), p. 219, (17),

$$\begin{aligned} \int_0^\infty e^{-\lambda z} z^{\sigma-1+k}\, dz &= \frac{\Gamma(\sigma+k)}{\lambda^{\sigma+k}} \\ &= \frac{\Gamma(\sigma)\,(b_q)_k}{\lambda^\sigma (a_p)_k} \sum_{n=0}^{k} \frac{(-k)_n (2n+\gamma)}{(n+\gamma+1)_k (n+\gamma)\, n!}\, {}_{p+3}F_q\left(\begin{matrix} -n, n+\gamma, a_p, \sigma \\ b_q \end{matrix}\middle| \frac{1}{\lambda}\right). \end{aligned} \tag{2·7}$$

Replacing $1/\lambda$ by z completes the induction with respect to p. The induction with respect to q is effected by multiplying both sides of (2·3) by z^σ, letting $z = 1/\lambda$ and applying the inverse-Laplace transform, see (4), p. 297, (1). Note that for convergence of the integral in (2·7), the hypothesis $\Re(\sigma) > 0$ is necessary. By an appeal to the analytic continuation of (2·3), we may relax this restriction.

If in (2·3) we replace z by z/γ and let $\gamma \to \infty$, we arrive at the following

COROLLARY. *Let neither a_i nor b_j be a negative integer or zero. Then*

$$\pi_{k,n} = \frac{(b_q)_k (-k)_n}{(a_p)_k\, n!}, \tag{2·8}$$

and hence

$$C_n(\lambda) = \frac{(b_q)_n (-\lambda)^n}{(a_p)_n} \sum_{k=0}^{\infty} \frac{(n+b_q)_k (n+1)_k\, \xi_{k+n} \lambda^k}{(n+a_p)_k\, k!}. \tag{2·9}$$

3. *The shifted Jacobi polynomials.* The important case $p = 0$, $q = 1$, $\gamma = 1+\alpha+\beta$, $b_1 = 1+\beta$ of (2·2) and (1·9) yields†

$$\left.\begin{aligned} f(\lambda z) &= \sum_{n=0}^{\infty} S_n(\lambda)\, R_n^{(\alpha,\beta)}(z), \\ S_n(\lambda) &= \frac{n!\,\lambda^n}{(n+\alpha+\beta+1)_n} \sum_{k=0}^{\infty} \frac{(n+\beta+1)_k (n+1)_k}{(2n+\alpha+\beta+2)_k}\, \xi_{k+n} \frac{\lambda^k}{k!}, \end{aligned}\right\} \tag{3·1}$$

$$R_n^{(\alpha,\beta)}(z) = P_n^{(\alpha,\beta)}(2z-1). \tag{3·2}$$

† For $m = 0$, $(m+\alpha+\beta+1)_m = 1$, regardless of the values of α and β. This also applies to (3·5).

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$R_n^{(\alpha,\beta)}(z)$ is called the shifted Jacobi polynomial and is interpolatory for z real, $0 \leqslant z \leqslant 1$. Using (3·1) with $\beta = \mp \frac{1}{2}$ together with the transformations ((5), 4·1·5)

$$R_n^{(\alpha,-\frac{1}{2})}(z^2) = \frac{(n+1)_n}{(n+\alpha+1)_n} P_{2n}^{(\alpha,\alpha)}(z), \tag{3·3}$$

$$zR_n^{(\alpha,\frac{1}{2})}(z^2) = \frac{(2n+1)(n+1)_n}{(2n+\alpha+1)(n+\alpha+1)_n} P_{2n+1}^{(\alpha,\alpha)}(z), \tag{3·4}$$

we can write

$$\left.\begin{aligned} f(\lambda z^2) &= \sum_{n=0}^{\infty} E_n(\lambda) P_{2n}^{(\alpha,\alpha)}(z), \\ E_n(\lambda) &= \frac{(2n)!(4\lambda)^n}{(2n+2\alpha+1)_{2n}} \sum_{k=0}^{\infty} \frac{(n+\frac{1}{2})_k (n+1)_k \xi_{k+n} \lambda^k}{(2n+\alpha+\frac{3}{2})_k k!}, \end{aligned}\right\} \tag{3·5}$$

$$\left.\begin{aligned} zf(\lambda z^2) &= \sum_{n=0}^{\infty} O_n(\lambda) P_{2n+1}^{(\alpha,\alpha)}(z), \\ O_n(\lambda) &= \frac{2(2n+1)!(4\lambda)^n}{(2n+2\alpha+2)_{2n+1}} \sum_{k=0}^{\infty} \frac{(n+\frac{3}{2})_k (n+1)_k \xi_{k+n} \lambda^k}{(2n+\alpha+\frac{5}{2})_k k!}. \end{aligned}\right\} \tag{3·6}$$

One special case of the foregoing is of particular interest. If $\alpha = \beta = -\frac{1}{2}$, $P_n^{(\alpha,\beta)}(z)$ is essentially the classical Chebyshev polynomial $T_n(z)$ which has proved so useful in approximation theory (6).

$$\left.\begin{aligned} T_n(z) &= \frac{n!}{(\frac{1}{2})_n} P_n^{(-\frac{1}{2},-\frac{1}{2})}(z), \\ &= T_n^*\left(\frac{1+z}{2}\right). \end{aligned}\right\} \tag{3·7}$$

$T_n^*(z)$ is called the shifted Chebyshev polynomial. Combining (3·5) with the quadratic transformation (6)

$$T_n^*(z^2) = T_{2n}(z), \tag{3·8}$$

one obtains

$$\left.\begin{aligned} f(\lambda z) &= \sum_{n=0}^{\infty} h_n(\lambda) T_n^*(z), \\ h_0(\lambda) &= \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k}{k!} \xi_k \lambda^k, \\ h_n(\lambda) &= 2(\tfrac{1}{4}\lambda)^n \sum_{k=0}^{\infty} \frac{(2n+k+1)_k}{k!} \xi_{k+n} (\tfrac{1}{4}\lambda)^k \quad (n \geqslant 1). \end{aligned}\right\} \tag{3·9}$$

If R is the radius of convergence of $f(z)$, then $S_n(\lambda)$, $E_n(\lambda)$, $O_n(\lambda)$ and $h_n(\lambda)$ converge for $|\lambda| < R$. Furthermore, $h_n(\lambda)$ is numerically small compared to the corresponding term of the Taylor series for $f(z)$; e.g. if $R > 1$, then $h_n(1)$ is of order $2^{1-2n} \cdot f^{(n)}(0)/n!$.

In § 4 we demonstrate the usefulness of the above basic series by obtaining rapidly convergent approximations to some transcendental functions.

4. *Applications.* Our first example is the incomplete gamma function

$$\gamma(a+1,\lambda z) = \int_0^{\lambda z} e^{-t} t^a dt = \frac{(\lambda z)^{a+1}}{(a+1)} {}_1F_1\left(\begin{matrix} a+1 \\ a+2 \end{matrix}\middle| -\lambda z\right) \quad (a > -1). \tag{4·1}$$

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In (3·1), let $\alpha = \beta = a$ and

$$\left.\begin{aligned} f(\lambda z) &= \lambda z\, {}_1F_1\left(\begin{matrix} a+1 \\ a+2 \end{matrix}\middle| -\lambda z\right), \\ \xi_k &= \frac{(a+1)\,\Gamma(a+k)\,(-)^{k-1}}{\Gamma(a+k+1)\,\Gamma(k)}. \end{aligned}\right\} \tag{4·2}$$

Then we find that

$$S_n(\lambda) = \frac{n(a+1)\,(-)^{n+1}\lambda^n}{(n+a)\,(n+2a+1)_n}\, {}_2F_2\left(\begin{matrix} n+1, n+a \\ n, 2n+2a+2 \end{matrix}\middle| -\lambda\right). \tag{4·3}$$

Now the equation

$${}_2F_2\left(\begin{matrix} n+1, n+a \\ n, 2n+2a+2 \end{matrix}\middle| -\lambda\right) = A\, {}_1F_1\left(\begin{matrix} n+a \\ 2n+2a \end{matrix}\middle| -\lambda\right) + (B+C\lambda)\, {}_1F_1\left(\begin{matrix} n+a+1 \\ 2n+2a+2 \end{matrix}\middle| -\lambda\right) \tag{4·4}$$

may be solved for A, B and C to yield

$$A = -\frac{a(2n+2a+1)}{n(n+a+1)}, \quad B = 1+\frac{a(2n+2a+1)}{n(n+a+1)}, \quad C = \frac{-a}{2n(n+a+1)}. \tag{4·5}$$

Converting the confluent hypergeometric functions in (4·4) into Bessel functions by (2), §7·2·2, (12) and referring to (4·1), we have the expansion

$$\begin{aligned} \gamma(a+1, \lambda z) = z^a(\pi\lambda)^{\frac{1}{2}} e^{-\frac{1}{2}\lambda} \sum_{n=0}^{\infty} \frac{(-)^n\,(n+a+1)_a\,(n+a+\frac{1}{2})}{(n+a)\,(n+a+1)} \\ \times\left\{aI_{n+a-\frac{1}{2}}(\tfrac{1}{2}\lambda) + \left[a - \frac{2(n+a)\,(n+2a+1)}{\lambda}\right] I_{n+a+\frac{1}{2}}(\tfrac{1}{2}\lambda)\right\} R_n^{(a,a)}(z) \quad (a > -1). \end{aligned} \tag{4·6}$$

If $z = 1$ in (4·6), the result agrees with an expansion given by Buchholz ((7), p. 130).

Our second example, the error function, is a special case of (4·1) but here the expansion is of a different nature. Let

$$zf(\lambda z^2) = z\, {}_1F_1\left(\begin{matrix} 1 \\ \frac{3}{2} \end{matrix}\middle| a^2z^2\right). \tag{4·7}$$

Then from (3·6) with $\alpha = -\frac{1}{2}$, $\lambda = a^2$ and from (2), §9·9, (1) we find that

$$\operatorname{erf}(z) = \int_0^z e^{-t^2}dt = \sqrt{\pi}\, e^{\frac{1}{2}a^2-z^2} \sum_{n=0}^{\infty} I_{n+\frac{1}{2}}(\tfrac{1}{2}a^2)\, T_{2n+1}(z/a). \tag{4·8}$$

Let ϵ_N denote the maximum error incurred when just N terms of (4·8) are retained. Use of (2), §10·18 and (8), p. 49, no. (1) provides the estimate

$$|\epsilon_N| \leqslant \frac{\sqrt{\pi}\, e^{a^2-z^2}\, |\frac{1}{2}a|^{2N+1}}{\Gamma(N+\frac{3}{2})}\, |1+O(1/N)| \quad (a \text{ real}, \ -a \leqslant z \leqslant a). \tag{4·9}$$

If $a = z$ or $a^2 = 2z^2$ in (4·8) expansions given by Luke and Coleman (9) and Tricomi (10), respectively, result.

More generally, if f is hypergeometric in nature, the Theorem and Corollary in § 3 yield many expansions given previously by the authors (3). Also, if only the numerical values of the coefficients for the Taylor's series of $f(z)$ are known, the formulae in §§ 2 and 3 are valuable. In the accompanying table are listed coefficients for the Chebyshev

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polynomial expansions of the Riemann zeta function, the gamma function, and its reciprocal. These were obtained by applying (3·9) to the tabulations (11), (12) and (13), p. 4. For (12), the authors thank J. C. P. Miller who graciously loaned us his manuscript. The coefficients in the table are given to a sufficient number of decimals to minimize round-off error in case the expansions are rearranged in powers of z. The number in parentheses behind each entry is the power of ten by which the entry is to be multiplied.

$$\zeta(1+z) = \frac{1}{z} + \sum_{n=0}^{6} h_n T_n^*(z) + \eta(z), \quad |\eta(z)| \leqslant 10^{-10} \quad (0 < z \leqslant 1)$$

n	h_n	
0	0·61172 44860	(+0)
1	0·33864 46129	(−1)
2	−0·65024 13912	(−3)
3	−0·52464 23167	(−5)
4	0·62132 10656	(−6)
5	−0·13838 69984	(−7)
6	−0·14078 641	(−10)

$$\Gamma(z+3) = \sum_{n=0}^{10} h_n T_n^*(z) + \eta(z), \quad |\eta(z)| \leqslant 10^{-10} \quad (0 \leqslant z \leqslant 1)$$

n	h_n	
0	0·36573 87725 10	(+1)
1	0·19575 43456 67	(+1)
2	0·33829 71138 1	(+0)
3	0·42089 51276 7	(−1)
4	0·42876 50483	(−2)
5	0·36521 21696	(−3)
6	0·27400 6424	(−4)
7	0·18124 024	(−5)
8	0·10965 78	(−6)
9	0·59871 9	(−8)
10	0·30769 3	(−9)

$$\frac{1}{\Gamma(z)} = z \sum_{n=0}^{10} h_n T_n^*(z) + \eta(z), \quad |\eta(z)| \leqslant 10^{-10} \quad (0 \leqslant z \leqslant 1)$$

n	h_n	
0	0·10637 73007 8	(+1)
1	−0·49855 8735	(−2)
2	−0·64192 54364	(−1)
3	0·50657 9862	(−2)
4	0·41660 9128	(−3)
5	−0·80481 417	(−4)
6	0·29600 098	(−5)
7	0·26897 5	(−6)
8	−0·33397	(−7)
9	0·10896	(−8)
10	0·514	(−10)

For computational purposes, the expansions need not be rearranged in powers of z. Clenshaw (14), basing his work on the recursion formulae for $T_n^*(z)$, has developed a nesting procedure to utilize the coefficients in such tables without computing $T^*(z)$ or powers of z.

For the general case, $G_n(z, \gamma)$ itself can always be computed recursively. If we employ the techniques in Rainville (15), we can show that $G_n(z, \gamma)$ obeys a linear recursion relation of the form

$$\left.\begin{aligned} \sum_{j=0}^{s} (A_j + zB_j)\, G_{n-j}(z, \gamma) = 0, \\ B_0 = B_s = 0, \quad A_0 = 1, \end{aligned}\right\} \tag{4·10}$$

where $s = \max\{p+2, q+1\}$. For $s = 3$, the authors have calculated the A_j and B_j in (4·10), while for $s = 2$, the results are classical. For $s \geqslant 3$, the determination of the A_j and B_j become tedious and for such values of s no general results of the type (4·10) seem to exist in the literature. However, specializations of the authors' formulae for $s = 3$ are given in (15), p. 233 ff.

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