Notes

105.01 On some composite Fibonacci expressions

Introduction

Let F_n denote the nth Fibonacci number, where n is a positive integer. The Fibonacci sequence $\{F_n\}_{n \geqslant 1}$ is defined by way of the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 3$, where $F_1 = F_2 = 1$.

It is in fact the case that each term of the sequence $\{F_n + 1\}_{n \geqslant 4}$ is composite. Likewise, each term of $\{F_n - 1\}_{n \geqslant 7}$ is composite. Both of these results may be deduced from Theorem 1 in [1].

In this Note we obtain results related to, if somewhat more involved than, those given in the previous paragraph. Both the Fibonacci and the Lucas sequences have roles to play with respect to deriving our results, where the Lucas sequence $\{L_n\}_{n \geqslant 1}$ possesses the same recurrence relation as the Fibonacci sequence but different initial conditions, namely $L_1 = 1$ and $L_2 = 3$.

Our first result

We start by showing that the expression

$$A(n, m) = F_{2n+m} + (-1)^n F_m$$

is composite for each pair of positive integers (n, m) except for $(1, 1)$ and all those pairs of the form $(1, m)$ for which F_{m+1} is a Fibonacci prime.

In order to do so, we make use of the following known result [2]:

$$L_k F_n = F_{n+k} + (-1)^k F_{n-k}. \tag{1}$$

On replacing each occurrence of k and each occurrence of n in this equality with n and $n + m$, respectively, we obtain

$$\begin{aligned} L_n F_{n+m} &= F_{(n+m)+n} + (-1)^n F_{(n+m)-n} \\ &= F_{2n+m} + (-1)^n F_m. \end{aligned}$$

Note that when $n \geqslant 2$, it is the case that $L_n \geqslant 3$. Also, $F_{n+m} \geqslant 2$ when $n \geqslant 2$ and $m \geqslant 1$. Therefore, $L_n F_{n+m}$ is composite when $n \geqslant 2$ and $m \geqslant 1$. From this and (1) it follows that $A(n, m)$ is composite when $n \geqslant 2$ and $m \geqslant 1$.

Finally, let us consider pairs of the form $(1, m)$. We have

$$\begin{aligned} A(1, m) &= F_{2+m} + (-1) F_m \\ &= F_{m+2} - F_m \\ &= F_{m+1}, \end{aligned}$$

from which it follows that $A(1, 1) = F_2 = 1$, noting here that 1 is considered to be neither prime nor composite. For $m \geqslant 2$, $A(1, m)$ is composite if, and only if, F_{m+1} is not a Fibonacci prime, as was required to

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be shown. Note, incidentally, that it is not known whether or not there exist infinitely many Fibonacci primes [3].

Our second result

We now show that, with the exception of those pairs of the form $(1, m)$ for which L_{m+1} is a Lucas prime, and those pairs of the form $(2, m)$ for which L_{m+2} is a Lucas prime, the expression

$$B(n, m) = F_{2n+m} + (-1)^{n+1} F_m$$

is composite for all pairs of positive integers (n, m).

This can be achieved by making use of (1) once more. On replacing each occurrence of k in this equality with $n + m$ we obtain, with the help of the result

$$F_{-k} = (-1)^{k-1} F_k$$

given in [2], that

$$\begin{aligned} L_{n+m}F_n &= F_{n+(n+m)} + (-1)^{n+m} F_{n-(n+m)} \\ &= F_{2n+m} + (-1)^{n+m} F_{-m} \\ &= F_{2n+m} + (-1)^{n+m}(-1)^{m-1} F_m \\ &= F_{2n+m} + (-1)^{n-1} F_m. \end{aligned}$$

Thus, for any pair (n, m) of positive integers such that $n \geqslant 3$ and $m \geqslant 1$, it is the case that $L_{n+m} \geqslant 7$ and $F_n \geqslant 2$, and hence that $B(n, m)$ is composite. In addition,

$$\begin{aligned} B(1, m) &= F_{2+m} + (-1)^2 F_m \\ &= F_{m+2} + F_m \\ &= L_{m+1}, \end{aligned}$$

so that $B(1, m)$ is composite if, and only if, L_{m+1} is not a Lucas prime [4].

Last of all, we have

$$\begin{aligned} B(2, m) &= F_{4+m} + (-1)^3 F_m \\ &= F_{m+4} - F_m \\ &= L_{m+2}, \end{aligned}$$

so that $B(2, m)$ is composite if, and only if, L_{m+2} is not a Lucas prime.

Note that the identities

$$F_{m+2} + F_m = L_{m+1} \qquad \text{and} \qquad F_{m+4} - F_m = L_{m+2}$$

utilised in the previous two paragraphs can easily be shown to be true by way of mathematical induction.

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105.02 Some essential properties of Farey series in one Pic(k)ture

Introduction

In this Note, which is strongly inspired by [1], some essential properties of Farey series are obtained by using Pick's theorem. The appeal of the presented approach is that all properties are derived easily from one single picture.

We consider the plane $\mathbb{R}^2$. A point (x, y) is called a *lattice point* if $x, y \in \mathbb{Z}$, and a polygon is called a *lattice polygon* if its vertices are lattice points. The beautiful theorem of Georg Pick says that a simple lattice polygon has area $i + \frac{b}{2} - 1$, where i is the number of lattice points in its interior, and b is the number of lattice points on its boundary. Since its discovery and publication by Pick in [2] it has been proved many times in a fascinating variety of ways. For a comprehensive presentation, including an overview of references, see [3].

Let n be a positive integer. The nth Farey series F_n is the finite sequence of all reduced fractions between 0 and 1, whose denominator is at most n, arranged in increasing order of magnitude. For example, F_6 is

$$\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1}.$$

Following [1] we represent a fraction $\frac{a}{b}$ by the lattice point (b, a). The slope

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