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Sofic entropy of Gaussian actions

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Abstract. Associated to any orthogonal representation of a countable discrete group, is a probability measure-preserving action called the Gaussian action. Using the Polish model formalism we developed before, we compute the entropy (in the sense of Bowen [*J. Amer. Math. Soc.* **23** (2010) 217–245], Kerr and Li [*Invent. Math.* **186** (2011) 501–558]) of Gaussian actions when the group is sofic. Computation of entropy for Gaussian actions has only been done when the acting group is abelian and thus our results are new, even in the amenable case. Fundamental to our approach are methods of non-commutative harmonic analysis and C^*-algebras which replace the Fourier analysis used in the abelian case.

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1. *Introduction*

This paper is concerned with giving new computations for sofic entropy, specifically in computing entropy of Gaussian actions. Entropy for actions of $\mathbb{Z}$ is classical and goes back to work of Kolmogorov and Sinaĭ. Entropy roughly measures how chaotic the action of $\mathbb{Z}$ is. Kieffer, in [**20**], showed that one can generalize entropy to actions of amenable groups. An amenable group is a group which has a sequence of non-empty finite sets which are almost invariant under translation by elements of the group. Abelian groups, nilpotent groups and solvable groups are amenable, but the free group on r letters is not if $r \geq 2$. Entropy for amenable groups has been studied by many people and is a useful invariant in ergodic theory: it can be computed in many cases and positivity of entropy implies interesting structural properties.

Fundamental examples in [**24**] led many to believe that there cannot be a good entropy theory beyond the realm of amenable groups. In groundbreaking and seminal work,

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Bowen, in [**1**], defined a notion of entropy for actions of sofic groups. The class of sofic groups is considerably larger than that of amenable groups: it contains all amenable groups, all residually finite groups, all linear groups and is closed under free products with amalgamation over amenable subgroups (see [**10–12**, **25**, **28**]). Since the subject is fairly young, not as much is known about sofic entropy as entropy for actions of amenable groups, but some structure is beginning to emerge. It can be calculated for some interesting examples such as Bernoulli shifts (see [**1**, **19**]) as well as algebraic actions (see [**2**, **5**, **15**, **19**]). Additionally, recent work in [**17**, **22**] shows that one can deduce structural properties of an action from assumptions of positive sofic entropy.

Our goal in this paper is to add to the list of computations of sofic entropy by computing the entropy of *Gaussian actions*. The Gaussian action construction is a way to associate, in a functorial way, to any orthogonal representation $\rho : \Gamma \to \mathcal{O}(\mathcal{H})$, a probability measure-preserving action called the Gaussian action. We refer the reader to §4.1 for the precise definition. For intuition, we mention that if $\mathcal{H}$ is finite-dimensional, then this action is just the induced action of Γ on $\mathcal{H}$ with the Gaussian measure. The Gaussian action is a natural generalization of this construction to the case of infinite-dimensional representations.

Recall that the left-regular representation $\lambda_{\mathbb{R}} : \Gamma \to \ell^2(\Gamma, \mathbb{R})$ is given by

$$(\lambda_{\mathbb{R}}(g)\xi)(h) = \xi(g^{-1}h) \quad \text{for } \xi \in \ell^2(\Gamma, \mathbb{R}),\ g, h \in \Gamma.$$

It is known that the Gaussian action associated to the left-regular representation $\Gamma \curvearrowright \ell^2(\Gamma, \mathbb{R})$ is the Bernoulli action with base $(\mathbb{R}, \nu)$, where ν is the Gaussian measure. Thus Gaussian actions are a class of actions which are similar to the class of Bernoulli shifts.

To the best of our knowledge, entropy for Gaussian actions has only been computed when the acting group is abelian (see [**9**, **21**]). To state our result, we need to introduce the following decomposition of representations. We define singularity of orthogonal representations exactly as in the unitary case. Given orthogonal representations ρ_1, ρ_2 of Γ, we say that ρ_1 is singular with respect to ρ_2, and write $\rho_1 \perp \rho_2$, if no non-zero subrepresentation of ρ_1 embeds into ρ_2. We say that ρ_1 is *absolutely continuous with respect to* ρ_2, and write $\rho_1 \ll \rho_2$, if ρ_1 embeds into $\rho_2^{\oplus\infty}$. For general pairs of orthogonal representations ρ_1, ρ_2 we can always write

$$\rho_1 = \rho_{1,a} \oplus \rho_{1,s},$$

where $\rho_{1,s}$, ρ_1 are singular and $\rho_{1,a} \ll \rho_2$. If Γ is abelian, this reduces to the Lebesgue decomposition in measure theory. We thus regard this as a non-commutative analog of the Lebesgue decomposition.

THEOREM 1.1. *Let Γ be a countable discrete sofic group with sofic approximation Σ. Let $\rho : \Gamma \to \mathcal{O}(\mathcal{H})$ be an orthogonal representation with $\mathcal{H}$ separable. Let $\Gamma \curvearrowright (X_\rho, \mu_\rho)$ be the induced Gaussian action; write $\rho = \rho_a \oplus \rho_s$ where ρ_s and $\lambda_{\Gamma,\mathbb{R}}$ are singular and $\rho_a \ll \lambda_{\Gamma,\mathbb{R}}$. Then*

$$h_{\Sigma,\mu_\rho}(X_\rho, \Gamma) = \begin{cases} -\infty & \text{if } h_{\Sigma,\mu_{\rho_s}}(X_{\rho_s}, \Gamma) = -\infty, \\ 0 & \text{if } \rho_a = 0 \text{ and } h_{\Sigma,\mu_{\rho_s}}(X_{\rho_s}, \Gamma) \neq -\infty, \\ \infty & \text{if } \rho_a \neq 0 \text{ and } h_{\Sigma,\mu_{\rho_s}}(X_{\rho_s}, \Gamma) \neq -\infty. \end{cases}$$

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Since it appears to be new, we specifically mention the amenable case.

COROLLARY 1.2. *Let* Γ *be a countable discrete amenable group. Let* $\rho : \Gamma \to \mathcal{O}(\mathcal{H})$ *be an orthogonal representation and* $\Gamma \curvearrowright (X_\rho, \mu_\rho)$ *the induced Gaussian action. Write* $\rho = \rho_1 \oplus \rho_2$*, where* ρ_2 *is embeddable into* $\Gamma \curvearrowright \ell^2(\Gamma, \mathbb{R})^{\oplus\infty}$*, and* $\mathrm{Hom}_\Gamma(\rho_1, \lambda_{\Gamma,\mathbb{R}}) = \{0\}$*. Then* $h_{\mu_\rho}(X_\rho, \Gamma) = \infty$ *if and only if* $\rho_2 \neq 0$ *and, if* $\rho_2 = 0$*, then* $h_{\mu_\rho}(X_\rho, \Gamma) = 0$*.*

In [**15**], we gave a formula for entropy of a probability measure-preserving action $\Gamma \curvearrowright (X, \mu)$ when X is a Polish space, μ is a Borel probability measure and $\Gamma \curvearrowright X$ by homeomorphisms. Given an arbitrary probability measure-preserving action $\Gamma \curvearrowright (Y, \nu)$, a probability measure-preserving action $\Gamma \curvearrowright (X, \mu)$ (where X is Polish, μ is a Borel measure on X and $\Gamma \curvearrowright X$ by homeomorphisms) is called a Polish model for $\Gamma \curvearrowright (Y, \nu)$ if $\Gamma \curvearrowright (X, \mu) \cong \Gamma \curvearrowright (Y, \nu)$ as probability measure-preserving actions. Our definition of entropy in terms of a Polish model took into account the topology of X in a non-trivial way as for the definition for entropy in the presence of a compact model developed by Kerr and Li in [**19**]. Our computation for the entropy of Gaussian actions goes through Polish models. This model is associated to a family of generators for the representation, and the measure is canonically defined in terms of the representation. Although one can write down a compact model, it is unnatural and the measure is not expressed nicely in terms of the representation. We mention that entropy in the sofic case is roughly a measure of 'how many' finitary simulations a probability measure-preserving action has. The typical way to prove existence of these simulations is through a probabilistic argument. For the Gaussian action, our probabilistic argument uses Gaussian measures on finite-dimensional spaces. A consequence of our methods is that, associated to any sofic approximation of a group, we have a natural way of describing the Gaussian measure for a subrepresentation of the left-regular representation as a limit of finite-dimensional Gaussian measures.

Let us sketch how we are able to handle the case when ρ is singular with respect to $\lambda_{\mathbb{R}}$ and Γ is non-abelian. In the case $\Gamma = \mathbb{Z}$, the problem of showing that $\Gamma \curvearrowright (X_\rho, \mu_\rho)$ has zero entropy reduces to the fact that if μ, ν are two singular measures on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, then there is a $f \in C(\mathbb{T})$ with $0 \leq f \leq 1$ so that f is 'close' to one in $L^2(\mu)$ and close to zero in $L^2(\nu)$. As the representation theory of a group is captured by its universal C^*-algebra, it is natural to replace $C(\mathbb{T})$ with $C^*(\Gamma)$ in the non-abelian case. This statement about singularity of representations then becomes a statement about singularity of measures, and we can prove an analogous characterization of singularity of representations in terms of elements of $C^*(\Gamma)$. In short, we abstract the harmonic analysis present in the abelian case to non-commutative harmonic analysis using the framework of C^*-algebras.

We mention that if Γ is amenable and is an infinite conjugacy class group (that is, every non-trivial conjugacy class is infinite) then there is a easy proof that if $\rho : \Gamma \to \mathcal{O}(\mathcal{H})$ embeds into $\lambda_{\Gamma,\mathbb{R}}^{\oplus\infty}$, then $h_{\mu_\rho}(X_\rho, \Gamma) = \infty$. Namely, one can show, in this case, that there is an n so that $\rho^{\oplus n}$ contains $\lambda_{\Gamma,\mathbb{R}}$ (this is a consequence of the theory of II_1 factors). In this case, (X_ρ, μ_ρ) will factor onto (X_λ, μ_λ) which is isomorphic to $\Gamma \curvearrowright (\mathbb{R}, \eta)^\Gamma$, where η is the Gaussian measure. Since entropy for *amenable* groups decreases under factor maps, the proof is complete. This proof fails disastrously in the non-amenable case. It is *very* far from true that sofic entropy decreases under factor maps for non-amenable groups. In fact, it can be shown that, for *every* non-amenable group, there is a $\alpha > 0$ so that if

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(X, μ) is a measure space with $H(X, \mu) \geq \alpha$, then $\Gamma \curvearrowright (X, \mu)^\Gamma$ factors onto *every* non-trivial Bernoulli shift (see [**4**, Corollary 1.6]); even a Bernoulli shift with infinite entropy. Moreover, if Γ contains a free group, then α can be taken to be any positive number [**3**]. In fact, even more is true: a recent result of Seward (see [**29**, Theorem 1.1]) implies that for every non-amenable sofic group Γ, there is an $\alpha > 0$ so that any probability measure-preserving action of Γ is a factor of an action having sofic entropy less than α. Thus, there is no simple proof in the non-amenable case based on factors, and we *must* use a direct proof. Even in the amenable case, this argument relies on the group being an infinite conjugacy class group and one needs general methods to handle the general case.

2. *Preliminaries on sofic entropy*

We use S_n for the symmetric group on n letters. If A is a set, we will use $\mathrm{Sym}(A)$ for the set of bijections of A.

Definition 2.1. Let Γ be a countable discrete group. A *sofic approximation of* Γ is a sequence $\Sigma = (\sigma_i : \Gamma \to S_{d_i})$ of functions (not assumed to be homomorphisms) so that

$$\lim_{i\to\infty} u_{d_i}(\{1 \leq k \leq d_i : \sigma_i(g)\sigma_i(h)(k) = \sigma_i(gh)(k)\}) = 1 \quad \text{for all } g, h \in \Gamma,$$
$$\lim_{i\to\infty} u_{d_i}(\{1 \leq k \leq d_i : \sigma_i(g)(k) \neq \sigma_i(h)(k)\}) = 1 \quad \text{for all } g \neq h \text{ in } \Gamma.$$

We will call Γ *sofic* if it has a sofic approximation.

All amenable groups and residually finite groups are sofic. Also, it is known that soficity is closed under free products with amalgamation over amenable subgroups (see [**10–12**, **25**, **28**]). Additionally, residually sofic groups and locally sofic groups are sofic. Thus, by Malcev's theorem, we know all linear groups are sofic. It is known that graph products of sofic groups are sofic, by [**7**]. If Λ is a subgroup of Γ, and Λ is sofic and $\Gamma \curvearrowright \Gamma/\Lambda$ is amenable (in the sense of having a Γ-invariant mean), then Γ is sofic. One can argue this by the same methods as in [**12**, Theorem 1] (for example consider the observations after [**6**, Definition 12.2.12]).

We recall the definition of entropy in the presence of a Polish model given in [**17**]. Let X be a Polish space and let Γ be a countable, discrete group with $\Gamma \curvearrowright X$ by homeomorphisms. We say that a continuous pseudometric Δ on X is *dynamically generating* if, for every $x \in X$ and in every neighborhood U of x in X, there exists a finite $F \subseteq \Gamma$ and a $\delta > 0$ so that if $y \in X$ and

$$\max_{g\in F} \Delta(gx, gy) < \delta,$$

then $y \in U$. Notice that our definition includes the hypothesis that Δ is continuous. We use $C_b(X)$ for the Banach space of bounded, continuous functions on X with norm

$$\|f\| = \sup_{x\in X} |f(x)|.$$

Given a pseudometric space (X, Δ), $A, B \subseteq X$ and $\varepsilon > 0$, we say that A is ε-contained in B and write $A \subseteq_\varepsilon B$ if, for any $a \in A$, there is a $b \in B$ so that $\Delta(a, b) \leq \varepsilon$. We say

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that $A \subseteq X$ is ε-dense if $X \subseteq_\varepsilon A$. We use $S_\varepsilon(X, \Delta)$ for the minimal cardinality of an ε-dense subset of X. We say that $A \subseteq X$ is ε-separated if, for any $a_1 \neq a_2$ in A, we have $\Delta(a_1, a_2) > \varepsilon$. We use $N_\varepsilon(A, \Delta)$ for the maximal cardinality of an ε-separated subset of X. We leave it as an exercise to show that

$$N_{2\varepsilon}(A, \Delta) \leq S_\varepsilon(A, \Delta) \leq N_\varepsilon(A, \Delta) \tag{1}$$

and that, if $\delta > 0$ and $A \subseteq_\delta B$, then

$$S_{2(\varepsilon+\delta)}(A, \Delta) \leq S_\varepsilon(B, \Delta).$$

We use Δ_2 for the metric on X^n defined by

$$\Delta_2(\phi, \psi)^2 = \frac{1}{n} \sum_{j=1}^{n} \Delta(\phi(j), \psi(j))^2.$$

If X is a Polish space we use $\mathrm{Prob}(X)$ for the space of all Borel probability measures on X. If Γ is a countable, discrete group with $\Gamma \curvearrowright X$ by homeomorphisms, we use $\mathrm{Prob}_\Gamma(X)$ for the space of all Γ-invariant elements of $\mathrm{Prob}(X)$. We are now ready to state our definition of sofic entropy from [**17**]. It is defined by counting the exponential growth of maps from $\{1, \ldots, d_i\} \to X$ which approximately preserve the measure-theoretic structure of X and are approximately equivariant. We call these maps *microstates* (this is a heuristic term and will not be defined rigorously).

Definition 2.2. Let Γ be a countable, discrete, sofic group with sofic approximation $\Sigma = (\sigma_i : \Gamma \to S_{d_i})$. Let X be Polish space with $\Gamma \curvearrowright X$ by homeomorphisms. Fix a bounded, continuous pseudometric on X. For a finite $F \subseteq \Gamma$ and $\delta > 0$ we let $\mathrm{Map}(\Delta, F, \delta, \sigma_i)$ be all $\phi \in X^{d_i}$ so that

$$\max_{g \in F} \Delta_2(g\phi, \phi \circ \sigma_i(g)) < \delta.$$

Notice that $\mathrm{Map}(\Delta, F, \delta, \sigma_i)$ only accounts for the group action and not the measure-theoretic structure of X. Recall that $C_b(X)$ denotes the space of bounded continuous functions on X. For $\mu \in \mathrm{Prob}(X)$, a finite $L \subseteq C_b(X)$ and a $\delta > 0$ we let

$$U_{L,\delta}(\mu) = \bigcap_{f \in L} \left\{ \nu \in \mathrm{Prob}(X) : \left| \int f \, d\mu - \int f \, d\nu \right| < \delta \right\}.$$

The sets $U_{L,\delta}(\mu)$ form a basis of open sets for a topology called the *weak topology*. We use this topology to account for the measure-theoretic structure in our microstates.

Definition 2.3. Let Γ be a countable, discrete, sofic group with sofic approximation $\Sigma = (\sigma_i : \Gamma \to S_{d_i})$. Let X be Polish space with $\Gamma \curvearrowright X$ by homeomorphisms, and fix $\mu \in \mathrm{Prob}_\Gamma(X)$. For finite $F \subseteq \Gamma$, $L \subseteq C_b(X)$ and $\delta > 0$ we set $\mathrm{Map}_\mu(\Delta, F, \delta, L, \sigma_i)$ to be the set of all $\phi \in \mathrm{Map}(\Delta, F, \delta, L, \sigma_i)$ so that $\phi_*(u_{d_i}) \in U_{L,\delta}(\mu)$.

Definition 2.4. Let Γ be a countable, discrete, sofic group with sofic approximation $\Sigma = (\sigma_i : \Gamma \to S_{d_i})$. Let X be Polish space with $\Gamma \curvearrowright X$ by homeomorphisms, and let

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$\mu \in \mathrm{Prob}_\Gamma(X)$. Fix a bounded, dynamically generating pseudometric on X. For finite $F \subseteq \Gamma$, $L \subseteq C_b(X)$ and $\delta, \varepsilon > 0$ we set

$$h_{\Sigma,\mu}(\Delta, F, \delta, L, \varepsilon) = \limsup_{i\to\infty} \frac{1}{d_i} \log N_\varepsilon(\mathrm{Map}_\mu(\Delta, F, \delta, L, \sigma_i), \Delta_2),$$
$$h_{\Sigma,\mu}(\Delta, \varepsilon) = \inf_{\substack{F\subseteq\Gamma \text{ finite},\\ L\subseteq C_b(X) \text{ finite}, \delta>0}} h_{\Sigma,\mu}(\Delta, F, \delta, L, \varepsilon),$$
$$h_{\Sigma,\mu}(X, \Gamma) = \sup_{\varepsilon>0} h_{\Sigma,\mu}(\Delta, \varepsilon).$$

We call $h_{\Sigma,\mu}(X, \Gamma)$ the entropy of $\Gamma \curvearrowright (X, \mu)$ with respect to Σ.

It is shown in [**17**] that this agrees with entropy as defined by [**1**, **19**].

3. *Generating sets and tightness*

Since $C_b(X)$ is not separable, we would like to reduce checking the approximate measure-preserving property of our microstates from all functions in $C_b(X)$ to a smaller class of functions. For example, $C_b(X)$ is separable in the topology of uniform convergence on compact sets, so, if we require that we have a family of functions dense in this topology, this will give us a sufficiently small family of functions to deal with. However, for this to work, we need to modify our microstates so that they have some uniform tightness. We proceed with the definitions.

Definition 3.1. Let X be a Polish space. A family $\mathcal{L} \subseteq C_b(X)$ is said to be *generating* if there is a $A > 0$ so that for every $g \in C_b(X)$, for every compact $K \subseteq X$ and for every $\varepsilon > 0$ there is a $f \in \mathrm{Span}(\mathcal{L})$ so that $\|f|_K - g|_K\| < \varepsilon$ and $\|f\| \leq A\|g\|$.

We now proceed with our modified version of sofic entropy in the case of a generating set of functions.

Definition 3.2. Let Γ be a countable discrete group and let $\sigma : \Gamma \to S_d$ be a function for some $d \in \mathbb{N}$. Let X be a Polish space with $\Gamma \curvearrowright X$ by homeomorphisms, preserving a Borel probability measure μ. Let Δ be a dynamically generating pseudometric on X. For an open subset U of X, for $\eta, \delta > 0$ and finite $L \subseteq C_b(X)$, $F \subseteq \Gamma$, we let $\mathrm{Map}_\mu^{U,\eta}(\Delta, F, \delta, L, \sigma)$ be the set of $\phi \in \mathrm{Map}_\mu(\Delta, F, \delta, L, \sigma)$ so that

$$\phi_*(u_d)(U) > 1 - \eta.$$

Suppose now that Γ is sofic. For a dynamically generating pseudometric Δ, $\mathcal{L} \subseteq C_b(X)$ and a sofic approximation $\Sigma = (\sigma_i : \Gamma \to S_{d_i})$ set

$$h^{U,\eta}_{\Sigma,\mu}(\Delta, \varepsilon, F, \delta, L) = \limsup_{i\to\infty} \frac{1}{d_i} \log N_\varepsilon(\mathrm{Map}^{U,\eta}_\mu(\Delta, F, \delta, L, \sigma_i)),$$
$$h^{K,\eta}_{\Sigma,\mu}(\Delta, \varepsilon, F, \delta, L) = \inf_{\text{open } U \supseteq K} h^{U,\eta}_{\Sigma,\mu}(\Delta, \varepsilon, F, \delta, L),$$
$$h^{\eta}_{\Sigma,\mu}(\Delta, \varepsilon, F, \delta, L) = \sup_{K\subseteq X \text{ compact}} h^{K,\eta}_{\Sigma}(\Delta, \varepsilon, F, \delta, L)$$
$$h_{\Sigma,\mu}(\Delta, \varepsilon, \mathcal{L}) = \inf_{\substack{\eta,\delta>0,\\ F\subseteq\Gamma \text{ finite},\\ L\subseteq\mathcal{L} \text{ finite}}} h^{\eta}_{\Sigma,\mu}(\Delta, \varepsilon, F, \delta, L)$$
$$h_{\Sigma,\mu}(\Delta, \mathcal{L}) = \sup_{\varepsilon>0} h_{\Sigma,\mu}(\Delta, \varepsilon, \mathcal{L}).$$

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In the above definition, the necessary trick is to add more quantifiers to the definition of sofic entropy. The reader may be concerned, already, by the number of quantifiers involved in the original definition of sofic entropy. When we compute sofic entropy of Gaussian actions in §4.1, it will be clear that this is the correct trade-off. The difficulty involved in the computation will not be in dealing with quantifiers but, instead, that we can only show the approximate measure-preserving property on a class of functions which is not norm dense. However, one can easily see that this class of functions is generating and so we will use the above definition.

THEOREM 3.3. *Let Γ be a countable discrete sofic group with sofic approximation Σ. Let X be a Polish space with $\Gamma \curvearrowright X$ by homeomorphisms, preserving a Borel probability measure μ. For any dynamically generating pseudometric Δ on X, $\mathcal{L} \subseteq C_b(X)$ generating and $\varepsilon > 0$,*

$$h_{\Sigma,\mu}(\Delta, \varepsilon) = h_{\Sigma,\mu}(\Delta, \varepsilon, \mathcal{L}).$$

In particular,

$$h_{\Sigma,\mu}(X, \Gamma) = h_{\Sigma,\mu}(\Delta, \mathcal{L}).$$

Proof. The 'in particular' part follows from [**17**, Theorem 3.12]. We first show that

$$h_{\Sigma,\mu}(\Delta, \varepsilon) \leq h_{\Sigma,\mu}(\Delta, \varepsilon, \mathcal{L}).$$

Let $\eta > 0$. Since X is Polish, we may find a compact $K \subseteq X$ with

$$\mu(X \backslash K) < \eta.$$

Fix an open subset U of X containing K. By Urysohn's lemma, we may find a $f \in C_b(X)$ with

$$\chi_K \leq f \leq \chi_U.$$

Note that if $\nu \in \mathrm{Prob}(X)$ and $\delta > 0$ with

$$\left| \int_X f \, d\mu - \int_X f \, d\nu \right| < \delta,$$

then

$$\nu(U) > 1 - \eta - \delta.$$

We now see that, for all finite $L \subseteq \mathcal{L}$, $F \subseteq \Gamma$, $0 < \delta < \eta$,

$$\mathrm{Map}^{U,\eta+\delta}(\Delta, F, \delta, L, \sigma_i) \supseteq \mathrm{Map}(\Delta, F, \delta, L \cup \{f\}, \sigma_i).$$

Thus for all $\delta < \eta$, and all finite $F' \supseteq F$, $L' \supseteq L \cup \{f\}$ and $\delta' < \delta$,

$$h_{\Sigma,\mu}(\Delta, \varepsilon, F', \delta', L') \leq h^{U,2\eta}_{\Sigma,\mu}(\Delta, \varepsilon, F, \delta, L).$$

Taking the infimum over F', δ', L', we find that

$$h_{\Sigma,\mu}(\Delta, \varepsilon) \leq h^{U,2\eta}_{\Sigma,\mu}(\Delta, \varepsilon, F, \delta, L).$$

Taking the infimum over all $U \supseteq K$, we find that

$$h_{\Sigma,\mu}(\Delta, \varepsilon) \leq h^{K,2\eta}_{\Sigma,\mu}(\Delta, \varepsilon, F, \delta, L).$$

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Now we take the supremum over all K and then the infimum over all $\eta, \delta, F, L \subseteq \mathcal{L}$ to find that

$$h_{\Sigma,\mu}(\Delta, \varepsilon) \leq h_{\Sigma,\mu}(\Delta, \varepsilon, \mathcal{L}).$$

We now show that

$$h_{\Sigma,\mu}(\Delta, \varepsilon, \mathcal{L}) \leq h_{\Sigma,\mu}(\Delta, \varepsilon).$$

Let $A > 0$ be as in the definition of generating. Let $F \subseteq \Gamma$ and $L \subseteq C_b(X)$ be given finite sets and $\delta > 0$. Let $\kappa > 0$ be sufficiently small depending upon F, L, δ in a manner to be determined later. Since X is Polish, we may find a compact $K_0 \subseteq X$ so that

$$\mu(K_0) \geq 1 - \kappa.$$

Let $K \subseteq X$ compact be given with $K \supseteq K_0$. Let $L = \{f_1, \ldots, f_l\}$. For $1 \leq j \leq l$, we can find a $g_j \in \mathrm{Span}(\mathcal{L})$ so that

$$\|g_j|_K - f_j|_K\| < \delta$$

and

$$\|g_j\| \leq A\|f_j\|.$$

We may find an open neighborhood U_0 of K so that

$$\|g_j|_{U_0} - f_j|_{U_0}\|_{C_b(U_0)} < \delta \quad \text{for } j = 1, \ldots, l.$$

Let $\delta' > 0$ and $F' \subseteq \Gamma$ and $L' \subseteq C_b(X)$ be finite sets which will depend upon F, δ, L in a manner to be determined shortly. Let U be an open neighborhood of K contained in U_0. For $\phi \in \mathrm{Map}_\mu^{U,\eta}(\Delta, F', \delta', L', \sigma_i)$,

$$\begin{aligned}
&\left|\int_X f_j \, d\mu - \int_X f_j \, d\phi_*(u_{d_i})\right| \\
&\quad \leq \left|\int_X f_j \, d\phi_*(u_{d_i}) - \int_X g_j \, d\phi_*(u_{d_i})\right| + \left|\int_X g_j \, d\mu - \int_X g_j \, d\phi_*(u_{d_i})\right| \\
&\qquad + \left|\int_X g_j \, d\mu - \int_X f_j \, d\mu\right| \\
&\quad \leq \eta(1+A)\|f_j\| + \kappa(1+A)\|f_j\| + \left|\int_U f_j \, d\phi_*(u_{d_i}) - \int_U g_j \, d\phi_*(u_{d_i})\right| \\
&\qquad + \left|\int_X g_j \, d\mu - \int_X g_j \, d\phi_*(u_{d_i})\right| + \left|\int_U g_j \, d\mu - \int_U f_j \, d\mu\right| \\
&\quad \leq \eta(1+A)\|f_j\| + \kappa(1+A)\|f_j\| + 2\delta + \left|\int_X g_j \, d\mu - \int_X g_j \, d\phi_*(u_{d_i})\right|.
\end{aligned}$$

If we choose $\delta' > 0$ sufficiently small and F', L' sufficiently large, we may force

$$\left|\int_X g_j \, d\mu - \int_X g_j \, d\phi_*(u_{d_i})\right| < \delta.$$

If we choose

$$\max_j 2\eta(1+A)\|f_j\| < \delta,$$
$$\max_j \kappa(1+A)\|f_j\| < \delta,$$

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then

$$\mathrm{Map}^{U,\eta}(\Delta, F', \delta', L', \sigma_i) \subseteq \mathrm{Map}(\Delta, F, 7\delta, L, \sigma_i).$$

Thus

$$h^{K,\eta}_{\Sigma,\mu}(\Delta, \varepsilon, F', \delta', L') \leq h^{U,\eta}_{\Sigma,\mu}(\Delta, \varepsilon, F', \delta', L') \leq h_{\Sigma,\mu}(\Delta, \varepsilon, F, 7\delta, L).$$

Since $K \supseteq K_0$ was arbitrary, we may take the supremum over all K to see that

$$h_{\Sigma,\mu}(\Delta, \varepsilon, \mathcal{L}) \leq h^{\eta}_{\Sigma,\mu}(\Delta, \varepsilon, F', \delta', L') \leq h_{\Sigma,\mu}(\Delta, \varepsilon, F, 7\delta, L).$$

Taking the infimum over all F', δ', L' proves that

$$h_{\Sigma,\mu}(\Delta, \varepsilon, \mathcal{L}) \leq h_{\Sigma,\mu}(\Delta, \varepsilon). \qquad \square$$

4. *Entropy of Gaussian actions*

Gaussian actions are a natural class of actions induced by orthogonal representations of a group. When the representation is the left-regular representation, the Gaussian action is simply the Bernoulli action on $(\mathbb{R}, \nu)^{\Gamma}$, where ν is the Gaussian measure. In [**17**], we prove some results which give structural results of the Koopman representation $\Gamma \curvearrowright L^2(X, \mu)$ of a probability measure-preserving action $\Gamma \curvearrowright (X, \mu)$ of a sofic group Γ, under the assumption that this action has positive entropy. For example, we could show that if $\Gamma \curvearrowright (X, \mu)$ has positive entropy, then $\Gamma \curvearrowright L^2(X, \mu)$ must contain a 'piece' of the left-regular representation (for more precise statements, see [**17**, Theorem 1.1 and Corollary 1.2]). We will exploit the connections to representation theory to apply our spectral consequences of positive entropy in [**17**] to Gaussian actions. We will also exploit the similarity with Bernoulli shifts to compute the entropy of Gaussian actions.

Let Γ be a countable discrete group. Recall that an orthogonal representation of Γ on a real Hilbert space $\mathcal{H}$ is a homomorphism $\rho : \Gamma \to \mathcal{O}(\mathcal{H})$, where $\mathcal{O}(\mathcal{H})$ is the group of orthogonal transformations of $\mathcal{H}$ (that is, the set of $O \in B(\mathcal{H})$ so that $\langle O\xi, O\eta\rangle = \langle \xi, \eta\rangle$ for $\xi, \eta \in \mathcal{H}$). We let $\mathcal{H}_{\mathbb{C}} = \mathcal{H} \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{H} + i\mathcal{H}$ be the complexification of $\mathcal{H}$ equipped with the unique sesquilinear inner product extending the one on $\mathcal{H}$. We let $\rho_{\mathbb{C}} : \Gamma \to \mathcal{U}(\mathcal{H} + i\mathcal{H})$ be the complexification of ρ: that is, $\rho_{\mathbb{C}}(g)$ for $g \in \Gamma$ is the unique unitary transformation so that $\rho_{\mathbb{C}}(g)(\xi) = \rho(g)\xi$ for $\xi \in \mathcal{H}$.

4.1. *Definition of Gaussian actions.*

The most natural way to define Gaussian actions is by von Neumann algebras.

Definition 4.1. Let $\mathcal{H}$ be a complex Hilbert space. A von Neumann algebra on $\mathcal{H}$ is a $*$-subalgebra of $B(\mathcal{H})$ containing the identity which is closed in the weak operator topology. We say that a vector $\xi \in \mathcal{H}$ is cyclic for M if $\overline{M\xi} = \mathcal{H}$.

Suppose that (X, μ) is a standard probability space. For $f \in L^{\infty}(X, \mu)$, let $M_f \in B(L^2(X, \mu))$ be given by

$$(M_f\xi)(x) = f(x)\xi(x).$$

The map $f \mapsto M_f$ allows us to view $L^{\infty}(X, \mu)$ as a von Neumann algebra. It turns out (see [**8**, Theorem 14.5]) that if $M \subseteq B(\mathcal{H})$ is a commutative von Neumann algebra with

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cyclic vector ξ with $\|\xi\|_2 = 1$ and $\mathcal{H}$ is separable, then there is a standard probability space (X, μ) and a unitary

$$U : \mathcal{H} \to L^2(X, \mu)$$

so that

$$U(\xi) = 1$$
$$UMU^* = \{M_f : f \in L^\infty(X, \mu)\}.$$

Additionally, if $\phi : M \to \mathbb{C}$ is a linear functional so that

$$\phi|_{\{T \in M : \|T\| \leq 1\}}$$

is weak operator topology continuous, then there is a complex measure $\nu \ll \mu$ so that if $UTU^* = M_f$, then

$$\phi(T) = \int_X f \, d\nu.$$

We leave it to reader the verify that if $UTU^* = M_f$, then

$$\langle T\xi, \xi \rangle = \int_X f \, d\mu.$$

Definition 4.2. Let $\mathcal{H}$ be a real Hilbert space. The Gaussian algebra associated to $\mathcal{H}$, denoted $\mathcal{A}(\mathcal{H})$, is a commutative von Neumann algebra with cyclic vector Ω with $\|\Omega\| = 1$, which is generated by unitaries $\{u(\xi) : \xi \in \mathcal{H}\}$ satisfying

$$u(\xi_1 + \xi_2) = u(\xi_1)u(\xi_2) \quad \text{for } \xi \in \mathcal{H},$$
$$\langle u(\xi)\Omega, \Omega \rangle = \exp(-\pi \|\xi\|^2).$$

For $a \in \mathcal{A}(\mathcal{H})$, we let

$$\phi(a) = \langle a\Omega, \Omega \rangle.$$

Suppose that Γ is a countable discrete group and $\rho : \Gamma \to \mathcal{O}(\mathcal{H})$ is a representation. Then there is a ϕ-preserving action α on $\mathcal{A}(\mathcal{H})$ determined uniquely by

$$\alpha(g)(u(\xi)) = u(\rho(g)\xi) \quad \text{for } \xi \in \mathcal{H}.$$

This action is called the Gaussian action.

By [**27**], the Gaussian algebra exists and is unique up to ϕ-preserving isomorphism. Also see [**27**], for the existence and uniqueness of the Gaussian action.

Let us sketch an alternative construction of the Gaussian algebra. First, consider the case when $\mathcal{H}$ is finite-dimensional. Say, $\mathcal{H} = \mathbb{R}^n$. In this case, we can simply take $\mathcal{A}(\mathcal{H}) = L^\infty(\mathbb{R}^n, \nu)$, where $d\nu = e^{-\pi \|x\|^2_{\ell^2(n)}} \, dx$, and

$$\phi(a) = \int_{\mathbb{R}^n} a(x) \, d\nu(x),$$
$$u(\xi)(x) = \exp(2\pi i \langle \xi, x \rangle).$$

In general, consider the universal $*$-algebra $\widetilde{\mathcal{A}}(\mathcal{H})$ generated by unitaries $\widetilde{u}(\xi)$, $\xi \in \mathcal{H}$ satisfying the relation $\widetilde{u}(\xi_1 + \xi_2) = \widetilde{u}(\xi_1)\widetilde{u}(\xi_2)$ for $\xi_1, \xi_2 \in \mathcal{H}$. Here one can make sense of

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a unitary in a $*$-algebra by saying that it is an element u so that $u^*u = uu^* = 1$. One then has to check that there is a well-defined linear function $\widetilde{\phi} : \widetilde{\mathcal{A}}(\mathcal{H}) \to \mathbb{C}$ defined by

$$\widetilde{\phi}(\widetilde{u}(\xi)) = \exp(-\pi \|\xi\|^2).$$

Checking that $\widetilde{\phi}$ is well defined reduces to showing that if we take $\xi_1, \ldots, \xi_n$, $\zeta_1, \ldots, \zeta_m \in \mathcal{H}$ and $\lambda_1, \ldots, \lambda_n, c_1, \ldots, c_m \in \mathbb{C}$ with

$$\sum_{j=1}^{n} \lambda_j \widetilde{u}(\xi_j) = \sum_{i=1}^{m} c_j \widetilde{u}(\zeta_j),$$

then

$$\sum_{j=1}^{n} \lambda_j \exp(-\pi \|\xi_j\|^2) = \sum_{j=1}^{m} c_j \exp(-\pi \|\zeta_j\|^2).$$

Replacing $\mathcal{H}$ with the span of $\{\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_m\}$ reduces the verification that $\widetilde{\phi}$ is well defined to the case where $\mathcal{H}$ is finite-dimensional, where we have already shown that $\widetilde{\phi}$ is well defined (by explicitly exhibiting it as integration with respect to the Gaussian measure, as explained above). By similar reasoning, one checks that $\widetilde{\phi}(a^*a) \geq 0$ for $a \in \widetilde{\mathcal{A}}(\mathcal{H})$. One now runs the Gelfand–Naimark–Segal construction. That is, we let $\mathcal{K}$ be the completion of $\widetilde{\mathcal{A}}(\mathcal{H})$ under the inner product

$$\langle a, b \rangle = \widetilde{\phi}(b^*a).$$

We then have an induced homomorphism $\rho : \widetilde{\mathcal{A}}(\mathcal{H}) \to B(\mathcal{K})$ given by

$$(\rho(a))(b) = ab \quad \text{for } b \in \widetilde{\mathcal{A}}(\mathcal{H}).$$

We then let $\mathcal{A}(\mathcal{H}) = \overline{\rho(\widetilde{\mathcal{A}}(\mathcal{H}))}^{\mathrm{SOT}}$, $\Omega = \widetilde{u}(0)$ and $u(\xi) = \rho(\widetilde{u}(\xi))$. It is then straightforward to verify that $\mathcal{A}(\mathcal{H})$ and Ω have the desired properties.

By our remarks before this definition, if ρ and Γ are as in the definition, then there is a standard probability space (X_ρ, μ_ρ) and a measure-preserving action $\Gamma \curvearrowright (X_\rho, \mu_\rho)$ so that

$$\Gamma \curvearrowright L^\infty(X_\rho, \mu_\rho) \cong \Gamma \curvearrowright (\mathcal{A}(\mathcal{H}), \phi).$$

Furthermore, this action is uniquely determined up to isomorphism. Note that, for two orthogonal representations

$$\rho_j : \Gamma \to \mathcal{O}(\mathcal{H}_j), \quad j = 1, 2,$$

we have

$$\Gamma \curvearrowright (X_{\rho_1 \oplus \rho_2}, \mu_{\rho_1 \oplus \rho_2}) \cong (X_{\rho_1} \times X_{\rho_2}, \mu_{\rho_1} \otimes \mu_{\rho_2}).$$

The definition via von Neumann algebras may be abstract, so let us mention a simple version of the definition in the case of a cyclic representation. Recall that, if $t \in c_c(\Gamma, \mathbb{R})$ and $x \in \mathbb{R}^\Gamma$, we use

$$t \cdot x = \sum_{g \in \Gamma} t(g) x(g).$$

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PROPOSITION 4.3. *Let Γ be a countable discrete group and let $\rho : \Gamma \to \mathcal{O}(\mathcal{H})$ be an orthogonal representation. Suppose that there is a vector $\xi \in \mathcal{H}$ so that $\mathcal{H} = \overline{\mathrm{Span}\{\rho(g)\xi : g \in \Gamma\}}$. Then the Gaussian action is isomorphic to the shift action on $\mathbb{R}^\Gamma$ with measure determined by*

$$\int_{\mathbb{R}^\Gamma} \exp(2\pi i t \cdot x)\, d\mu_\xi(x) = \exp\left(-\pi \left\| \sum_{g\in\Gamma} t(g)\rho(g)\xi \right\|^2\right) \quad \textit{for all } t \in c_c(\Gamma, \mathbb{R}).$$

Proof. Choose a realization $\Gamma \curvearrowright (\mathcal{A}(\mathcal{H}), \phi) \cong \Gamma \curvearrowright L^\infty(X_\rho, \mu_\rho)$. By [**30**, Proposition 5.3], a sequence $a_n \in \mathcal{A}(\mathcal{H})$ converges in the strong operator topology to $a \in \mathcal{A}(\mathcal{H})$ if and only if

$$\phi((a - a_n)^*(a - a_n)) \to 0.$$

Thus, $t \mapsto u(t\xi)$ is a strongly continuous one-parameter group in $\mathcal{A}(\mathcal{H})$. Fix $\zeta \in \mathcal{H}$. By Stone's Theorem, there is a closed, densely defined self-adjoint operator $\omega(\zeta)$ on $L^2(X_\rho, \mu_\rho)$ so that, for all $s \in \mathbb{R}$,

$$\exp(2\pi i s\omega(\zeta)) = u(s\zeta),$$

in the sense of functional calculus. Recall that $L^\infty(X_\rho, \mu_\rho)$ can be viewed as a von Neumann algebra (by multiplication operators). It is not hard to see that the elements $u \in L^\infty(X_\rho, \mu_\rho)$, whose multiplication operators are unitaries, are almost everywhere equal to measurable functions

$$u : X \to \{z \in \mathbb{C} : |z| = 1\}.$$

Thus we can identify $u(s\zeta)$ as a measurable function $X_\rho \to \{z \in \mathbb{C} : |z| = 1\}$. For similar reasons, we can identify $\omega(\zeta)$ with a measurable function $X_\rho \to \mathbb{R}$. Thus, for every $s \in \mathbb{R}$, it is true that, for almost every $x \in X_\rho$,

$$\exp(2\pi i s\omega(\zeta)(x)) = u(s\zeta)(x).$$

Define

$$\Phi : X_\rho \to \mathbb{R}^\Gamma$$

by

$$\Phi(x)(g) = \omega(\xi)(g^{-1}x).$$

Then it is not hard to see that, for every $s \in \mathbb{R}$ and for all $g \in \Gamma$,

$$\exp(2\pi i s\Phi(x)(g)) = u(s\rho(g)\xi)(x)$$

for almost every $x \in X_\rho$. Define $\mu_\xi = \Phi_*\mu_\rho$. Since the unitaries of the form

$$u(s\rho(g)\xi), \quad s \in \mathbb{R}, g \in \Gamma$$

generate all of $\mathcal{A}(\mathcal{H})$, we see that Φ gives an Γ-equivariant isomorphism

$$(\mathbb{R}^\Gamma, \mu_\xi) \cong (X_\rho, \mu_\rho).$$

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Additionally, for all $t \in c_c(\Gamma, \mathbb{R})$,

$$\begin{aligned}\int_{\mathbb{R}^\Gamma} \exp(2\pi i t \cdot x)\, d\mu_\xi(x) &= \int_X \exp\left(2\pi i \sum_{g\in\Gamma} t(g)\Phi(x)(g)\right) d\mu_\rho(g)\\ &= \phi\left(\omega\left(\sum_{g\in\Gamma} t(g)\rho(g)\xi\right)\right)\\ &= \exp\left(-\pi\left\|\sum_{g\in\Gamma} t(g)\rho(g)\right\|^2\right).\end{aligned}$$

□

4.2. *Preliminaries on the group von Neumann algebra and embedding sequences.* For our purposes, we will need to 'linearize' a sofic approximation to approximations of algebras associated to Γ. Let $\mathbb{C}(\Gamma)$ be the ring of finite formal linear combinations of elements of Γ, with addition defined naturally, and multiplication defined by

$$\left(\sum_{g\in\Gamma} a_g g\right)\left(\sum_{h\in\Gamma} b_h h\right) = \sum_{g\in\Gamma}\left(\sum_{h\in\Gamma} a_h b_{h^{-1}g}\right) g.$$

We will also define a conjugate-linear involution on $\mathbb{C}(\Gamma)$ by

$$\left(\sum_{g\in\Gamma} a_g g\right)^* = \sum_{g\in\Gamma} \overline{a_{g^{-1}}}\, g.$$

Given a sofic approximation $\Sigma = (\sigma_i : \Gamma \to S_{d_i})$ and $\alpha = \sum_{g\in\Gamma} \alpha_g g \in \mathbb{C}(\Gamma)$, we define $\sigma_i(\alpha) \in M_{d_i}(\mathbb{C})$ by

$$\sigma_i(\alpha) = \sum_{g\in\Gamma} \alpha_g \sigma_i(g).$$

In order to talk about the asymptotic properties of this extended sofic approximation, we will need a more analytic object associated to Γ.

Let $\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ be the left-regular representation defined by $(\lambda(g)\xi)(h) = \xi(g^{-1}h)$. We will continue to use λ for the linear extension to $\mathbb{C}(\Gamma) \to B(\ell^2(\Gamma))$. The *group von Neumann algebra* of Γ is defined by

$$\overline{\lambda(\mathbb{C}(\Gamma))}^{\mathrm{WOT}},$$

where WOT denotes the weak operator topology. We will use $L(\Gamma)$ to denote the group von Neumann algebra. Define $\tau : L(\Gamma) \to \mathbb{C}$ by

$$\tau(x) = \langle x\delta_e, \delta_e\rangle.$$

We leave it as an exercise to the reader to verify that τ has the following properties:

(1) $\tau(1) = 1$;
(2) $\tau(x^*x) \geq 0$, with equality if and only if $x = 0$;
(3) $\tau(xy) = \tau(yx)$, for all $x, y \in M$; and
(4) τ is weak operator topology continuous.

We call the third property the tracial property. We will, typically, view $\mathbb{C}(\Gamma)$ as a subset of $L(\Gamma)$. In particular, we will use τ as well for the functional on $\mathbb{C}(\Gamma)$ which is just the restriction of τ on $L(\Gamma)$.

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In order to state our linearization of a sofic approximation properly, we give a general definition. By definition, $*$-algebra is a complex algebra equipped with an involution $*$ which is conjugate linear and antimultiplicative.

Definition 4.4. A tracial $*$-algebra is a pair (A, τ) where A is a $*$-algebra equipped with a linear functional $\tau : A \to \mathbb{C}$ so that:
(1) $\tau(1) = 1$;
(2) $\tau(x^*x) \geq 0$, with equality if and only if $x = 0$;
(3) $\tau(xy) = \tau(yx)$, for all $x, y \in M$; and
(4) for all $a \in A$, there is a $C_a > 0$ so that, for all $x \in A$, $|\tau(x^*a^*ax)| \leq C_a\tau(x^*x)$.
For $a, b \in A$ we let $\langle a, b\rangle = \tau(b^*a)$ and we let $\|a\|_2 = \tau(a^*a)^{1/2}$. Define $L^2(A, \tau)$ to be the Hilbert space completion of A in this inner product. By condition (4) of the definition, we have a representation $\lambda : A \to B(L^2(A, \tau))$ defined densely by $\lambda(a)x = ax$ for $x \in A$. We let $\|a\|_\infty = \|\lambda(a)\|$.

We make $M_n(\mathbb{C})$ into a tracial $*$-algebra using $\mathrm{tr} = (1/n)\,\mathrm{Tr}$, where Tr is the usual trace. In particular, we use $\|A\|_2 = \mathrm{tr}(A^*A)^{1/2}$ and $\|A\|_\infty$ will denote the operator norm.

We let $\mathbb{C}[X_1, \ldots, X_n]$ be the free $*$-algebra on n-generators $X_1, \ldots, X_n$. We will call elements of $\mathbb{C}[X_1, \ldots, X_n]$ $*$-polynomials in n indeterminates. For a $*$-algebra A, for elements $a_1, \ldots, a_n \in A$, there is a unique $*$-homomorphism $\mathbb{C}[X_1, \ldots, X_n] \to A$ sending X_j to a_j. If $P \in \mathbb{C}[X_1, \ldots, X_n]$, we use $P(a_1, \ldots, a_n)$ for the image of P under this $*$-homomorphism.

Definition 4.5. Let (A, τ) be a tracial $*$-algebra. An *embedding sequence* is a sequence $\Sigma = (\sigma_i : A \to M_{d_i}(\mathbb{C}))$ such that

$$\begin{aligned}
&\sup_i \|\sigma_i(a)\|_\infty < \infty \quad \text{for all } a \in A,\\
&\|P(\sigma_i(a_1), \ldots, \sigma_i(a_n)) - \sigma_i(P(a_1, \ldots, a_n))\|_2 \to 0\\
&\qquad \text{for all } a_1, \ldots, a_n \in A \text{ and all } P \in \mathbb{C}[X_1, \ldots, X_n],\\
&\mathrm{tr}(\sigma_i(a)) \to \tau(a) \quad \text{for all } a \in A.
\end{aligned}$$

We will frequently use the following fact: if $x_1, \ldots, x_n \in A$ and $P \in \mathbb{C}[X_1, \ldots, X_n]$, then

$$\|P(\sigma_i(x_1), \ldots, \sigma_i(x_n))\|_2 \to \|P(x_1, \ldots, x_n)\|_2. \tag{2}$$

To see this, we notice that, since

$$\|P(\sigma_i(x_1), \ldots, \sigma_i(x_n)) - \sigma_i(P(x_1, \ldots, x_n))\|_2 \to 0,$$

it suffices to handle the case $n = 1$ and $P(X) = X$. Then

$$\|\sigma_i(x)\|_2^2 = \mathrm{tr}(\sigma_i(x)^*\sigma_i(x))$$

and, since $\|\sigma_i(x)^*\sigma_i(x) - \sigma_i(x^*x)\|_2 \to 0$,

$$|\mathrm{tr}(\sigma_i(x)^*\sigma_i(x)) - \mathrm{tr}(\sigma_i(x^*x))| \to 0.$$

As

$$\mathrm{tr}(\sigma_i(x^*x)) \to \|x\|_2^2,$$

we have proved (2).

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The proof of the next two propositions will be left to the reader.

PROPOSITION 4.6. *Let Γ be a countable discrete sofic group with sofic approximation $\Sigma = (\sigma_i : \Gamma \to S_{d_i})$. Extend Σ to maps $\sigma_i : \mathbb{C}(\Gamma) \to M_{d_i}(\mathbb{C})$ linearly. Then Σ is an embedding sequence of $(\mathbb{C}(\Gamma), \tau)$.*

PROPOSITION 4.7. *Let (A, τ) be a tracial $*$-algebra and $\Sigma = (\sigma_i : A \to M_{d_i}(\mathbb{C}))$ be an embedding sequence. If $\Sigma' = (\sigma_i' : A \to M_{d_i}(\mathbb{C}))$ is another sequence of functions so that*

$$\sup_i \|\sigma_i'(a)\|_\infty < \infty \quad \text{for all } a \in A,$$
$$\|\sigma_i(a) - \sigma_i'(a)\|_2 \to 0 \quad \text{for all } a \in A,$$

then Σ' is an embedding sequence.

We will, in fact, need to extend our sofic approximation to the group von Neumann algebra. For this, we use the following.

LEMMA 4.8. [**16**, Lemma 5.5] *Let Γ be a countable discrete group. Then any embedding sequence for $\mathbb{C}(\Gamma)$ extends to one for $L(\Gamma)$.*

We will use the preceding lemma when Γ is sofic, in combination with Proposition 4.6.

4.3. *Preliminaries on real subspaces of the left-regular representation.* We define the *Fourier algebra* of Γ as all functions $\phi : \Gamma \to \mathbb{C}$ so that there is a linear functional $\Phi : L(\Gamma) \to \mathbb{C}$ with $\Phi|_{\{x \in L(\Gamma) : \|x\|_\infty \leq 1\}}$ being weak operator topology continuous so that

$$\Phi(\lambda(g)) = \phi(g).$$

We will call Φ the continuous extension of ϕ (note that by continuity and Kaplansky's density theorem, Φ, as above, must be unique). We let $A(\Gamma)_+$ consist of all such ϕ where the continuous extension of ϕ is a positive linear functional. We let $\|\phi\|_{A(\Gamma)} = \|\Phi\|$. By [**30**] Theorem II.2.6, $\|\cdot\|_{A(\Gamma)}$ is a norm on $A(\Gamma)$ which makes $A(\Gamma)$ a Banach space. By [**30**] Theorem V.3.15, $A(\Gamma)$ consists of all functions of the form $g \mapsto \langle \lambda(g)\xi, \eta \rangle$, where $\xi, \eta \in \ell^2(\Gamma)$. Moreover,

$$\|\phi\| = \inf \|\xi\| \|\eta\|,$$

where the infimum is over all ξ, η so that $\phi(g) = \langle \lambda(g)\xi, \eta \rangle$.

For intuition, we leave it to the reader to verify that, when Γ is abelian, $A(\Gamma)$ consists of all $\widehat{f}$, where $f \in L^1(\widehat{\Gamma})$, and that $A(\Gamma)_+$ consists of all $\widehat{f}$, where $f \in L^1(\widehat{\Gamma})$ and $f \geq 0$. We state a few basic (and well-known) properties of $A(\Gamma)$ in the following proposition. Lastly, if we are given $x \in L(\Gamma)$ we set

$$\widehat{x} = x\delta_e \in \ell^2(\Gamma).$$

PROPOSITION 4.9. *Let Γ be a countable discrete group.*

(1) *If $\rho : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation with $\rho \ll \lambda_\Gamma$ and $\xi \in \mathcal{H}$, then $\phi(g) = \langle \rho(g)\xi, \xi \rangle$ is in $A(\Gamma)_+$. In particular, there is a $\zeta \in \ell^2(\Gamma)$ with $\langle \rho(g)\xi, \xi \rangle = \langle \lambda(g)\zeta, \zeta \rangle$.*
(2) *If $\phi \in A(\Gamma)$, then $\overline{\phi} \in A(\Gamma)$ and $\|\overline{\phi}\| = \|\phi\|$. If $\phi \in A(\Gamma)_+$, then so is $\overline{\phi}$.*

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(3) *For $x \in L(\Gamma)$, we define $\phi_x : \Gamma \to \mathbb{C}$ by*

$$\phi_x(g) = \tau(x\lambda(g)).$$

Then $\phi_x \in A(\Gamma)$ and $\phi_x \in A(\Gamma)_+$ if and only if $x \in L(\Gamma)_+$. Additionally,

$$\|\phi_x\| = \tau(|x|),$$

and $\{\phi_x : x \in L(\Gamma)_+\}$ is dense in $A(\Gamma)_+$.

(4) *If $x, y \in L(\Gamma)$ and $\phi(g) = \langle \lambda(g)\widehat{x}, \widehat{x}\rangle$, $\psi(g) = \langle \lambda(g)\widehat{y}, \widehat{y}\rangle$, then*

$$\|\widehat{x} - \widehat{y}\|_2^2 \leq \|\phi - \psi\|_{A(\Gamma)}.$$

Proof. (1) The assumption $\rho \ll \lambda_\Gamma$ implies that we may extend ρ to a $*$-homomorphism $\rho : L(\Gamma) \to B(\mathcal{H})$ so that

$$\rho|_{\{x \in L(\Gamma) : \|x\|_\infty \leq 1\}}$$

is weak operator topology continuous. Thus the continuous extension Φ of ϕ is given by

$$\Phi(x) = \langle \rho(x)\xi, \xi\rangle.$$

The 'in particular' part follows from the discussion preceding the proposition.

(2) Write

$$\phi(x) = \langle \lambda(x)\xi, \eta\rangle,$$

with $\xi, \eta \in \ell^2(\Gamma)$. Then

$$\overline{\phi(x)} = \langle \lambda(x)\overline{\xi}, \overline{\eta}\rangle,$$

and the conclusion follows easily from this equality.

(3) If $x \in L(\Gamma)_+$, then the continuous extension Φ of ϕ_x is given by

$$\Phi(y) = \tau(xy).$$

For $y \geq 0$,

$$\Phi(y) = \tau(xy) = \tau(y^{1/2}xy^{1/2}) \geq 0$$

as $x \geq 0$.

Conversely, if $\phi_x \in A(\Gamma)_+$, let Φ be the continuous extension of ϕ_x to $L(\Gamma)$. Then, for all $y \in L(\Gamma)$,

$$\langle xy\delta_e, y\delta_e\rangle = \tau(y^*xy) = \tau(xyy^*) = \Phi(yy^*) \geq 0.$$

Since $\overline{L(\Gamma)\delta_e} = \ell^2(\Gamma)$, we see that $x \in L(\Gamma)_+$. The norm equality and density statement are contained in [**31**, Lemma IX.2.12].

(4) This is the Powers–Størmer inequality (see [**6**, Proposition 6.2.4] for a proof which generalizes to our situation). □

In order to compute the entropy of Gaussian actions, we will need to discuss real subspaces of $\ell^2(\Gamma)$, and for this we need a real version of $L(\Gamma)$. We let $L_{\mathbb{R}}(\Gamma)$ be all $x \in L(\Gamma)$ so that $\widehat{x} \in \ell^2(\Gamma, \mathbb{R})$. Recall that the convolution between $f \in c_c(\Gamma)$, $g \in \mathbb{C}^\Gamma$ is defined by

$$f * g(x) = \sum_{h \in \Gamma} f(h)g(h^{-1}x).$$

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We, similarly, define $f * g$ if $f \in \mathbb{C}^\Gamma$ and $g \in c_c(\Gamma)$. If $f \in c_c(\Gamma)$, we use

$$\check{f} = \sum_{g \in \Gamma} f(g)g \in \mathbb{C}(\Gamma).$$

Note that, for all $f \in c_c(\Gamma)$, $x \in L(\Gamma)$, $xf = \widehat{x} * f$. Thus

$$\|x\|_\infty = \sup_{f \in c_c(\Gamma), \|f\|_2 = 1} \|\widehat{x} * f\|_2.$$

Define $J : \ell^2(\Gamma) \to \ell^2(\Gamma)$ by $(Jf)(x) = \overline{f(x^{-1})}$, and observe that $J\widehat{x} = \widehat{x^*}$. Thus, for all $f \in c_c(\Gamma)$,

$$\|f * \widehat{x}\|_2 = \|J\widehat{x} * Jf\|_2 = \|\widehat{x^*} * f\|_2 \leq \|x^*\|_\infty \|f\|_2 = \|x\|_\infty \|f\|_2.$$

Replacing x with x^* we see that

$$\|x\|_\infty = \sup_{f \in c_c(\Gamma), \|f\|_2 = 1} \|f * \widehat{x}\|_2.$$

Thus there is a unique bounded operator on $\ell^2(\Gamma)$ extending $f \mapsto f * \widehat{x}$ of norm equal to that of x. We write the image of $\xi \in \ell^2(\Gamma)$ under this operator as ξx. We use

$$\ell^2(\Gamma)x = \{\xi x : \xi \in \ell^2(\Gamma)\}.$$

By [**8**, Theorem 43.11 and Proposition 43.10],

$$\{\widehat{x} : x \in L(\Gamma)\} = \left\{\xi \in \ell^2(\Gamma) : \sup_{f \in c_c(\Gamma), \|f\| \leq 1} \|\xi * f\|_2 < \infty\right\}.$$

For $\xi \in \ell^2(\Gamma)$, $f \in c_c(\Gamma)$,

$$\|\xi * f\|_2 = \|\overline{\xi} * \overline{f}\|_2,$$

where the bar denotes complex conjugation. It follows that there is a weak operator topology continuous, norm-one operator

$$C : L(\Gamma) \to L(\Gamma)$$

so that

$$\widehat{Cx} = \overline{\widehat{x}}.$$

LEMMA 4.10. *Let Γ be a countable discrete group and $\rho : \Gamma \to \mathcal{O}(\mathcal{H})$ be an orthogonal representation on a real separable Hilbert space $\mathcal{H}$ with $\rho_\mathbb{C} \ll \lambda_\Gamma$. Suppose that ρ has a cyclic vector. Then there is an orthogonal projection $p \in L_\mathbb{R}(\Gamma)$ so that*

$$\Gamma \curvearrowright \mathcal{H} \cong \Gamma \curvearrowright \ell^2(\Gamma, \mathbb{R})p.$$

Proof. Let ξ be a cyclic vector for $\mathcal{H}$, and define $\phi : \Gamma \to \mathbb{R}$ by

$$\phi(g) = \langle \rho(g)\xi, \xi \rangle.$$

Then $\phi(g) \in A(\Gamma)_+$. By (3) of the preceding proposition, we may find $x_n \in L(\Gamma)_+$ so that

$$\|\phi - \phi_{x_n}\| \to 0.$$

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Letting $y_n = (x_n + Cx_n)/2$ and using the fact that $\overline{\phi} = \phi$, we find that

$$\|\phi - \phi_{y_n}\| \to 0.$$

Let $\zeta_n = \widehat{y_n^{1/2}}$. Approximating the square root function by polynomials, we see that $\zeta_n \in \ell^2(\Gamma, \mathbb{R})$. Note that

$$\phi_{y_n}(g) = \tau(\lambda(g)y_n^{1/2}y_n^{1/2}) = \tau(y_n^{1/2}\lambda(g)y_n^{1/2}) = \langle \lambda(g)\zeta_n, \zeta_n\rangle.$$

By the Powers–Størmer inequality (that is, (4) of the preceding proposition),

$$\|\zeta_n - \zeta_m\| \leq \|\phi_{y_n} - \phi_{y_m}\|^{1/2}$$

so that ζ_n is a Cauchy sequence. Hence ζ_n converges to a $\zeta \in \ell^2(\Gamma, \mathbb{R})$. Also,

$$\phi(g) = \lim_{n\to\infty} \phi_{y_n}(g) = \lim_{n\to\infty} \langle \lambda(g)\zeta_n, \zeta_n\rangle = \lim_{n\to\infty} \langle \lambda(g)\zeta, \zeta\rangle.$$

Thus

$$\Gamma \curvearrowright \mathcal{H} \cong \Gamma \curvearrowright \overline{\mathrm{Span}\{\lambda(g)\zeta : g \in \Gamma\}}.$$

Let P be the projection from $\ell^2(\Gamma, \mathbb{R})$ onto $\overline{\mathrm{Span}\{\lambda(g)\zeta : g \in \Gamma\}}$ and let $P_{\mathbb{C}}$ denote its complexification as an operator on $\ell^2(\Gamma)$. As $P_{\mathbb{C}}$ commutes with $\lambda(g)$, [**8**, Theorem 43.11] shows that there is a unique orthogonal projection $p \in L(\Gamma)$ so that, for all $\xi \in \ell^2(\Gamma)$, $P_{\mathbb{C}}\xi = \xi p$. Moreover,

$$\widehat{p}(g) = \langle \widehat{p}, \delta_g\rangle = \langle P(\delta_e), \delta_g\rangle \in \mathbb{R},$$

so $p \in L_{\mathbb{R}}(\Gamma)$. Thus

$$\overline{\left\{\sum_{g\in\Gamma} f(g)\lambda(g)\zeta : f \in c_c(\Gamma, \mathbb{R})\right\}} = P(\ell^2(\Gamma, \mathbb{R})) = \ell^2(\Gamma, \mathbb{R})p. \qquad \square$$

We will need to extend a sofic approximation to an embedding sequence of $L(\Gamma)$, as in [**16**, Lemma 5.5]. However, we will also want $\sigma_i(L_{\mathbb{R}}(\Gamma)) \subseteq M_{d_i}(\mathbb{R})$.

PROPOSITION 4.11. *Let Γ be a countable discrete group and $\Sigma = (\sigma_i : \Gamma \to S_{d_i})$ a sofic approximation.*

(i) *There exists an embedding sequence $\Sigma = (\sigma_i : L(\Gamma) \to M_{d_i}(\mathbb{C}))$ such that*

$$\sigma_i(\alpha) = \sum_{g\in\Gamma} \widehat{\alpha}(g)\sigma_i(g)$$

and $\sigma_i(L_{\mathbb{R}}(\Gamma)) \subseteq M_{d_i}(\mathbb{R})$.

(ii) *If $\Sigma = (\sigma_i : L(\Gamma) \to M_{d_i}(\mathbb{C}))$ is as in (i), and $p \in L_{\mathbb{R}}(\Gamma)$ is an orthogonal projection, then there are orthogonal projections $p_i \in M_{d_i}(\mathbb{R})$ so that*

$$\|\sigma_i(p) - p_i\|_2 \to 0.$$

Proof. (i) By Lemma 4.8, we may extend to some embedding sequence $\Sigma = (\sigma_i : L(\Gamma) \to M_{d_i}(\mathbb{C}))$ so that

$$\sigma_i(\alpha) = \sum_{g\in\Gamma} \widehat{\alpha}(g)\sigma_i(g).$$

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By Proposition 4.7, it suffices to show that, for all $x \in L_{\mathbb{R}}(\Gamma)\backslash\mathbb{R}(\Gamma)$, there are $x_i \in M_{d_i}(\mathbb{R})$ so that

$$\sup_i \|x_i\|_\infty < \infty$$

and

$$\|x_i - \sigma_i(x)\|_2 \to 0.$$

For $A \in M_n(\mathbb{C})$, define $\overline{A} \in M_n(\mathbb{C})$ by $\overline{A}_{ij} = \overline{A_{ij}}$. Then

$$\|\overline{A}\|_\infty = \|A\|_\infty,$$
$$\|\overline{A}\|_2 = \|A\|_2.$$

It suffices to show that

$$\|\sigma_i(Cx) - \overline{\sigma_i(x)}\|_2 \to 0$$

for all $x \in L(\Gamma)$. Indeed, assuming that we have the above convergence, we may then redefine $\sigma_i(x)$ for $x \in L_{\mathbb{R}}(\Gamma)$ by $\frac{1}{2}(\sigma_i(x) + \overline{\sigma_i(x)})$. So let $x \in L(\Gamma)$, let $\varepsilon > 0$ and let $\alpha \in \mathbb{C}(\Gamma)$ be such that

$$\|x\delta_e - \lambda(\alpha)\delta_e\|_2 < \varepsilon.$$

Since $\sigma_i(C\lambda(\alpha)) = \overline{\sigma_i(\alpha)}$,

$$\begin{aligned}\|\sigma_i(Cx) - \overline{\sigma_i(x)}\|_2 &\le \|\sigma_i(Cx) - \sigma_i(C\lambda(\alpha))\|_2 + \|\overline{\sigma_i(x)} - \overline{\sigma_i(\lambda(\alpha))}\|_2 \\ &= \|\sigma_i(Cx) - \sigma_i(C\lambda(\alpha))\|_2 + \|\sigma_i(x) - \sigma_i(\lambda(\alpha))\|_2.\end{aligned}$$

Letting $i \to \infty$ and using that σ_i is a sofic approximation, we find that

$$\limsup_{i\to\infty} \|\sigma_i(Cx) - \overline{\sigma_i(x)}\|_2 \le \|Cx\delta_e - C\lambda(\alpha)\delta_e\| + \|x\delta_e - \lambda(\alpha)\delta_e\|_2 < 2\varepsilon.$$

Letting $\varepsilon \to 0$ proves (i).

(ii) Since σ_i is an embedding sequence,

$$\||\sigma_i(p)|^2 - \sigma_i(p)\|_2 \to 0,$$
$$\||\sigma_i(p)|^2 - |\sigma_i(p)|^4\|_2 \to 0.$$

By functional calculus,

$$\begin{aligned}\|\chi_{[1/2,3/2]}(|\sigma_i(p)|^2) - (|\sigma_i(p)|^2)\|_2^2 &= \|\chi_{\{t:|t-1|>1/2\}}(|\sigma_i(p)|^2)\|_2^2 \\ &= \operatorname{tr}(\chi_{\{t:|t-1|>1/2\}}(|\sigma_i(p)|^2)) \\ &\le 16\operatorname{tr}(||\sigma_i(p)|^2 - |\sigma_i(p)|^4|^2) \\ &= 16\||\sigma_i(p)|^2 - |\sigma_i(p)|^4\|_2^2 \\ &\to 0.\end{aligned}$$

Thus setting $p_i = \chi_{[1/2,3/2]}(|\sigma_i(p)|^2)$ completes the proof. □

Lastly, we will need an analogous definition of singularity, as in the unitary case. If $\rho_j : \Gamma \to \mathcal{O}(\mathcal{H}_j)$, $j = 1, 2$ are two orthogonal representations, we use $\operatorname{Hom}_\Gamma(\rho_1, \rho_2)$ for the space of *real*, linear, bounded, Γ-equivariant maps from $\mathcal{H}_1 \to \mathcal{H}_2$. We say that ρ_1, ρ_2 are mutually singular, written $\rho_1 \perp \rho_2$, if, whenever ρ_j' is a non-zero subrepresentation of ρ_j for $j = 1, 2$, ρ_1', ρ_2' are not isomorphic. Similarly, we say that $\rho_1 \ll \rho_2$ if ρ_1 is embeddable into $\rho_2^{\oplus\infty}$.

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LEMMA 4.12. *Let Γ be a countable discrete group and let $\rho_j : \Gamma \to \mathcal{O}(\mathcal{H}_j)$ be two orthogonal representations. The following are equivalent:*

(i) $\rho_1 \perp \rho_2$*;*

(ii) $\mathrm{Hom}_\Gamma(\rho_1, \rho_2) = \{0\}$*; and*

(iii) $\mathrm{Hom}_\Gamma(\rho_2, \rho_1) = \{0\}$.

Proof. The proof of (ii) being equivalent to (iii) and (ii) implies that (i) is the same as in [**17**, Proposition 4.2]. We can copy the proof of (i) implies (ii) in [**17**, Proposition 4.2] provided we prove an analog of the polar decomposition.

So let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded operator. Let

$$T_{\mathbb{C}} : \mathcal{H}_{1,\mathbb{C}} \to \mathcal{H}_{2,\mathbb{C}}$$

be the complexification of T. Using T^t for the real Banach space transpose of T,

$$T^*_{\mathbb{C}}|_{\mathcal{H}_2} = T^t.$$

Thus

$$T^*_{\mathbb{C}} T_{\mathbb{C}}(\mathcal{H}_1) \subseteq \mathcal{H}_1.$$

Approximating the square root function by polynomials, we see that

$$|T_{\mathbb{C}}|(\mathcal{H}_1) \subseteq \mathcal{H}_1.$$

Let

$$T_{\mathbb{C}} = U|T_{\mathbb{C}}|$$

be the polar decomposition. As

$$U = SOT - \lim_{\varepsilon \to 0} T_{\mathbb{C}}(|T_{\mathbb{C}}| + \varepsilon)^{-1},$$

we find that $U(\mathcal{H}_1) \subseteq \mathcal{H}_2$. The rest is as in [**17**, Proposition 4.2]. □

We now show that, in the case of the left-regular representation, the concepts of singularity and absolute continuity in the real case are related to the complex case.

LEMMA 4.13. *Let Γ be a countable discrete group. Let $\rho : \Gamma \to \mathcal{O}(\mathcal{H})$ be an orthogonal representation. Then $\rho_{\mathbb{C}} \ll \lambda_\Gamma$ (in the unitary sense) if and only if $\rho \ll \lambda_{\Gamma,\mathbb{R}}$. Similarly, $\rho_{\mathbb{C}} \perp \lambda_\Gamma$ if and only if $\rho \perp \lambda_{\Gamma,\mathbb{R}}$.*

Proof. Suppose $\rho_{\mathbb{C}} \ll \lambda_\Gamma$. Applying Zorn's lemma to write ρ as a direct sum of cyclic representations and applying Lemma 4.10, we see that $\rho \ll \lambda_{\Gamma,\mathbb{R}}$. The converse is even easier.

Suppose $\rho_{\mathbb{C}} \perp \lambda_\Gamma$. If $\mathcal{K} \subseteq \mathcal{H}$ is a closed, linear, Γ-invariant subspace so that $\rho|_{\mathcal{K}}$ embeds into $\lambda_{\Gamma,\mathbb{R}}$. Then, by complexification, $\rho_{\mathbb{C}}|_{\mathcal{K}_{\mathbb{C}}}$ embeds into λ_Γ. Conversely, suppose $T \in \mathrm{Hom}_\Gamma(\rho_{\mathbb{C}}, \lambda_\Gamma)$. Then $T|_{\mathcal{H}}$ is a Γ-equivariant, bounded, real-linear map

$$\mathcal{H} \to \ell^2(\Gamma).$$

As $\ell^2(\Gamma) \cong \ell^2(\Gamma, \mathbb{R})^{\oplus 2}$ as a real representation, we see that $T|_{\mathcal{H}} = 0$, by the previous lemma. Since $\mathcal{H}$ spans $\mathcal{H}_{\mathbb{C}}$ (as a complex vector space), we see that $T = 0$. □

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The following is proved in the same way as [**17**, Proposition 4.3].

PROPOSITION 4.14. *Let Γ be a countable discrete group and $\rho_j : \Gamma \to \mathcal{O}(\mathcal{H}_j)$, $j = 1, 2$ be two orthogonal representations. Then*

$$\rho_1 = \rho_{1,s} \oplus \rho_{1,c},$$

where $\rho_{1,s} \perp \rho_2$, and $\rho_{1,c} \ll \rho_2$.

4.4. *Sofic entropy of Gaussian actions.* In this section we compute the entropy of Gaussian actions. Let us first start with a very simple corollary of [**17**, Theorem 4.4].

COROLLARY 4.15. *Let Γ be a countable discrete sofic group with sofic approximation Σ. Let $\rho : \Gamma \to \mathcal{O}(\mathcal{H})$ be an orthogonal representation on a separable Hilbert space $\mathcal{H}$. Suppose that $\rho \perp \lambda_{\Gamma,\mathbb{R}}$. Then, if $\Gamma \curvearrowright (X_\rho, \mu_\rho)$ is the corresponding Gaussian action,*

$$h_{\Sigma,\mu_\rho}(X_\rho, \Gamma) \leq 0.$$

Proof. As in Proposition 4.3, for any $\xi \in \mathcal{H}$, we can find a unique

$$\omega(\xi) : X_\rho \to \mathbb{R}$$

so that, for all $t \in \mathbb{R}$,

$$\exp(2\pi i t \omega(\xi)(x)) = u(t\xi)(x)$$

for almost every $x \in X_\rho$. By uniqueness,

$$\omega(\xi + \eta) = \omega(\xi) + \omega(\eta)$$

almost everywhere. By [**27**], $\omega(\xi) \in L^2(X_\rho, \mu_\rho)$ for all $\xi \in \mathcal{H}$ and, in fact,

$$\Gamma \curvearrowright \overline{\mathrm{Span}}\{\omega(\xi) : \xi \in \mathcal{H}\} \cong \Gamma \curvearrowright \mathcal{H} \perp \lambda_\Gamma.$$

We have that $L^\infty(X_\rho, \mu_\rho)$ is generated as a von Neumann algebra by

$$u(t\rho(g)\xi) = \exp(2\pi i t \omega(\rho(g)\xi)) \quad \text{for } t \in \mathbb{R},\ \xi \in \mathcal{H},\ g \in \Gamma.$$

From this, it is not hard to argue that the sigma-algebra generated by

$$\{\omega(\xi)^{-1}(E) : E \subseteq \mathbb{C} \text{ is Borel}, \xi \in \mathcal{H}\}$$

is all measurable sets (up to measure zero). Thus, by [**17**, Theorem 4.4] and Lemma 4.13, we know that

$$h_{\Sigma,\mu_\rho}(X_\rho, \Gamma) \leq 0. \qquad \square$$

We turn to the computation of sofic entropy of Gaussian actions in the case where $\rho_{\mathbb{C}} \ll \lambda_\Gamma$. We will need the following general lemmas.

LEMMA 4.16. *Let Γ be a countable discrete sofic group with sofic approximation $\Sigma = (\sigma_i : \Gamma \to S_{d_i})$. Let X be a Polish space with a bounded, compatible (not necessarily complete) metric Δ', and fix $x_0 \in X$. Let $\Gamma \curvearrowright X^\Gamma$ be the Bernoulli action, and give X^Γ the dynamically generating pseudometric*

$$\Delta(x, y) = \Delta'(x(e), y(e)).$$

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For a finite $E \subseteq \Gamma$ and $x \in X^{d_i}$, define $\phi_x^{(E)} : \{1, \dots, d_i\} \to X^\Gamma$ by

$$\phi_x^{(E)}(j)(g) = \begin{cases} x(\sigma_i(g)^{-1}(j)) & \text{if } g \in E, \\ x_0 & \text{if } g \notin E. \end{cases}$$

For any finite $F \subseteq \Gamma$, there is a finite $E_0 \subseteq \Gamma$ so that, if $E \subseteq \Gamma$ is a finite set containing E_0, then, for all $\delta > 0$ and for all large i,

$$\phi_x^{(E)} \in \mathrm{Map}(\Delta, F, \delta, \sigma_i).$$

Proof. For $h \in F$,

$$\Delta_2(h\phi_x^{(E)}, \phi_x^{(E)} \circ \sigma_i(h))^2 = \frac{1}{d_i} \sum_{j=1}^{d_i} \Delta'(\phi_x^{(E)}(j)(h^{-1}), \phi_x^{(E)}(\sigma_i(h)(j))(e))^2.$$

If $E \supseteq F \cup F^{-1} \cup \{e\}$, then

$$\Delta_2(h\phi_x^{(E)}, \phi_x^{(E)} \circ \sigma_i(h))^2 = \frac{1}{d_i} \sum_{j : \sigma_i(h^{-1})^{-1}(j) \neq \sigma_i(h)(j)} \Delta'(x(\sigma_i(h^{-1})^{-1}(j)), x(\sigma_i(h)(j)))^2.$$

Let M be the diameter of (X, Δ'). Then

$$\Delta_2(h\phi_x^{(E)} \circ \sigma_i(h), h\phi_x^{(E)})^2 \leq M u_{d_i}(\{1 \leq j \leq d_i : \sigma_i(h^{-1})^{-1}(j) \neq \sigma_i(h)(j)\}) \to 0$$

by soficity. Setting $E_0 = F \cup F^{-1} \cup \{e\}$ completes the proof. □

LEMMA 4.17. *Let Γ be a countable discrete sofic group with sofic approximation $\Sigma = (\sigma_i : \Gamma \to S_{d_i})$. By Lemma 4.8, extend Σ to an approximation sequence $\Sigma = (\sigma_i : L(\Gamma) \to M_{d_i}(\mathbb{C}))$ so that, for $\alpha \in \mathbb{C}(\Gamma)$,*

$$\sigma_i(\alpha) = \sum_{g \in \Gamma} \widehat{\alpha}(g)\sigma_i(g).$$

Fix a finite $F \subseteq \Gamma$ and a $x \in L(\Gamma)$, and let $x_i \in M_{d_i}(\mathbb{C})$ be such that

$$\sup_i \|x_i\|_\infty < \infty,$$
$$\|\sigma_i(x) - x_i\|_2 \to 0.$$

Then the following statements hold.

(i) *There is a sequence $C_i \subseteq \{1, \dots, d_i\}$ so that*

$$u_{d_i}(C_i) \to 1$$

with

$$\lim_{i \to \infty} \sup_{\substack{t \in \mathbb{R}^F, s \in \mathbb{R}^F, \\ j \in C_i}} \frac{|\|\sigma_i(\check{t})x_i\sigma_i(\check{s})e_j\|_{\ell^2(d_i)}^2 - \|\lambda(\check{t})x\lambda(\check{s})\delta_e\|_2^2|}{\|t\|_{\ell^2(F)}^2 \|s\|_{\ell^2(F)}^2} = 0.$$

(ii) *There is a sequence $A_i \subseteq \{1, \dots, d_i\}^2$ so that*

$$u_{d_i} \otimes u_{d_i}(A_i) \to 1$$

and

$$\lim_{i \to \infty} \sup_{\substack{t \in \mathbb{R}^F, s \in \mathbb{R}^F, \\ (j,k) \in A_i}} \frac{|\|\sigma_i(\check{t})^* x_i e_j - \sigma_i(\check{s})^* x_i e_k\|_{\ell^2(d_i)}^2 - \|\lambda(\check{t})^* x \delta_e\|_2^2 - \|\lambda(\check{s})x\delta_e\|_2^2|}{(\|t\|_{\ell^2(F)} + \|s\|_{\ell^2(F)})^2} = 0.$$

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Proof. (i) We first handle the case where $x \in \mathbb{C}(\Gamma)$ and $x_i = \sigma_i(x)$. We may choose a C_i so that

$$u_{d_i}(C_i) \to 1,$$
$$\sigma_i(\check{t}x\check{s})e_j = \sigma_i(\check{t})\sigma_i(x)\sigma_i(\check{s})e_j,$$

for $j \in C_i$. By soficity, we may also force that

$$\sigma_i(g)e_j \neq \sigma_i(h)e_j \quad \text{for } g \neq h \text{ in } \operatorname{supp}(\check{t}x\check{s}), \text{ and } j \in C_i,$$

while still having

$$u_{d_i}(C_i) \to 1.$$

In this case, it is easy to see that

$$\|\sigma_i(\check{t})\sigma_i(x)\sigma_i(\check{s})e_j\|^2_{\ell^2(d_i)} = \|\sigma_i(\check{t}x\check{s})e_j\|^2_{\ell^2(d_i)} = \|\lambda(\check{t})x\lambda(\check{s})\delta_e\|^2_{\ell^2(d_i)}$$

for all $j \in C_i$.

Now we handle the general case. Let $\varepsilon > 0$, and choose $\alpha \in \mathbb{C}(\Gamma)$ so that

$$\|(\lambda(\alpha) - x)\delta_e\|_2 < \varepsilon.$$

Since σ_i is an approximation sequence, by (2), for all $g, h \in \Gamma$,

$$\lim_{i\to\infty} \|\sigma_i(g)\sigma_i(\alpha)\sigma_i(h) - \sigma_i(g)x_i\sigma_i(h)\|_2 = \|\lambda(g)(\lambda(\alpha) - x)\lambda(h)\delta_e\|_2 < \varepsilon.$$

Thus, for all large i,

$$\begin{aligned}
&\frac{1}{d_i}\sum_{g,h\in F}\sum_{j=1}^{d_i} \|\sigma_i(g)\sigma_i(\alpha)\sigma_i(h)e_j - \sigma_i(g)x_i\sigma_i(h)e_j\|^2_{\ell^2(d_i)} \\
&\quad = \sum_{g,h\in F} \|\sigma_i(g)\sigma_i(\alpha)\sigma_i(h) - \sigma_i(g)\sigma_i(\alpha)\sigma_i(h)\|^2_{L^2(M_{d_i}(\mathbb{C}),\mathrm{tr})} \\
&\quad < \varepsilon|F|^2.
\end{aligned}$$

For such i, we may find a $C_i \subseteq \{1, \ldots, d_i\}$ with

$$u_{d_i}(C_i) \geq (1 - \sqrt{\varepsilon}|F|^2)$$

so that

$$\|\sigma_i(g)\sigma_i(\alpha)\sigma_i(h)e_j - \sigma_i(g)x_i\sigma_i(h)e_j\|^2_{\ell^2(d_i)} < \sqrt{\varepsilon}$$

for all $g, h \in F$, all large i, and $j \in C_i$. Thus, for all $j \in C_i$ and all $s, t \in \mathbb{R}^F$,

$$\begin{aligned}
&\|\sigma_i(\check{t})x_i\sigma_i(\check{s})e_j - \sigma_i(\check{t})\sigma_i(\alpha)\sigma_i(\check{s})e_j\|_{\ell^2(d_i)} \\
&\quad \leq \sum_{g,h\in\Gamma} |t(g)s(h)|\|\sigma_i(g)x_i\sigma_i(h)e_j - \sigma_i(g)\sigma_i(\alpha)\sigma_i(h)e_j\|_{\ell^2(d_i)} \\
&\quad \leq \varepsilon^{1/4}|F|^2\|t\|_{\ell^2(F)}\|s\|_{\ell^2(F)}.
\end{aligned}$$

Therefore, for $j \in C_i$ and all large i, and all $t, s \in \mathbb{R}^F$,

$$|\|\sigma_i(\check{t})x_i\sigma_i(\check{s})e_j\|_{\ell^2(d_i)} - \|\sigma_i(\check{t})\sigma_i(\alpha)\sigma_i(\check{s})e_j\|_2| \leq \varepsilon^{1/4}|F|^2\|t\|_{\ell^2(F)}\|s\|_{\ell^2(F)}.$$

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By the first part, we can find a $C_i' \subseteq \{1, \dots, d_i\}$ with

$$u_{d_i}(C_i') \to 1$$

and

$$\|\sigma_i(\check{t})\sigma_i(\alpha)\sigma_i(\check{s})e_j\|^2_{\ell^2(d_i)} = \|\lambda(\check{t})\lambda(\alpha)\lambda(\check{s})\delta_e\|_2^2$$

for all $j \in C_i'$ and $t, s \in \mathbb{R}^F$. Thus, for all large i and all $t, s \in \mathbb{R}^F$, $j \in C_i \cap C_i'$,

$$\begin{aligned}
&|\|\sigma_i(\check{t})x_i\sigma_i(\check{s})e_j\|_{\ell^2(d_i)} - \|\lambda(\check{t})x\lambda(\check{s})\delta_e\|_{\ell^2(\Gamma)}| \\
&\quad \le \varepsilon^{1/4}|F|^2\|t\|_{\ell^2(F)}\|s\|_{\ell^2(F)} + \varepsilon\|\lambda(\check{t})\|_\infty\|\lambda(\check{s})\|_\infty \\
&\quad \le \varepsilon^{1/4}|F|^2\|t\|_{\ell^2(F)}\|s\|_{\ell^2(F)} + \varepsilon\|t\|_{\ell^1(F)}\|s\|_{\ell^1(F)} \\
&\quad \le \varepsilon^{1/4}|F|\|t\|_{\ell^2(F)}\|s\|_{\ell^2(F)} + |F|\varepsilon\|t\|_{\ell^2(F)}\|s\|_{\ell^2(F)}.
\end{aligned}$$

Since

$$u_{d_i}(C_i \cap C_i') \ge (1 - 2\sqrt{\varepsilon})|F|$$

for all large i, we can use a diagonal argument to complete the proof of (i).

(ii) Let C_i be as (i). Note that

$$\begin{aligned}
\|\sigma_i(\check{t})^*x_ie_j - \sigma_i(\check{s})^*x_ie_k\|^2_{\ell^2(d_i)} &= \|\sigma_i(\check{t})^*x_ie_j\|^2_{\ell^2(d_i)} + \|\sigma_i(\check{s})^*x_ie_k\|^2_{\ell^2(d_i)} \\
&\quad - 2\,\mathrm{Re}(\langle x_i^*\sigma_i(\check{s})\sigma_i(\check{t})^*x_ie_j, e_k\rangle_{\ell^2(d_i)}).
\end{aligned}$$

We look for A_i with $A_i \subseteq C_i \times C_i$. By (i), it is enough to find an $A_i \subseteq C_i \times C_i$ with

$$\lim_{i\to\infty} \sup_{s,t\in\mathbb{R}^F,(j,k)\in A_i} \frac{|\langle x_i^*\sigma_i(\check{s})\sigma_i(\check{t})^*x_ie_j, e_k\rangle_{\ell^2(d_i)}|}{(\|t\|_{\ell^2(F)} + \|s\|_{\ell^2(F)})^2} = 0,$$
$$u_{d_i} \otimes u_{d_i}(A_i) \to 1.$$

We prove that

$$\lim_{i\to\infty} \frac{1}{d_i^2(\|t\|_{\ell^2(F)} + \|s\|_{\ell^2(F)})^2} \sum_{1\le j,k\le d_i} |\langle x_i^*\sigma_i(\check{s})\sigma_i(\check{t})^*x_ie_j, e_k\rangle_{\ell^2(d_i)}| = 0, \tag{3}$$

which will clearly prove the existence of such an A_i. Again, as in (i), we first do this for $x_i = \sigma_i(\alpha)$ for $\alpha \in \mathbb{C}(\Gamma)$. Let D_i be the set of all j so that

$$\sigma_i(\alpha)^*\sigma_i(\check{s})\sigma_i(\check{t})^*\sigma_i(\alpha)e_j = \sigma_i(\alpha^*st^*\alpha)e_j.$$

We use $o(1)$ for any expression which tends to zero as $i \to \infty$. Then

$$\lim_{i\to\infty} u_{d_i}(D_i) = 1$$

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and

$$\begin{aligned}
&\frac{1}{d_i^2}\sum_{1\le j,k\le d_i}|\langle\sigma_i(\alpha)^*\sigma_i(\check{s})\sigma_i(\check{t})^*\sigma_i(\alpha)e_j,\,e_k\rangle_{\ell^2(d_i)}|^2 \le o(1)\\
&\quad+\frac{1}{d_i^2}\sum_{1\le j,k\le d_i,\,j\in D_i}\\
&\quad\times\sum_{g,h\in\Gamma}(\alpha^*\check{s}\check{t}^*\alpha)(g)\overline{(\alpha^*\check{s}\check{t}^*\alpha)(h)}\langle\sigma_i(g)e_j,\,e_k\rangle_{\ell^2(d_i)}\langle e_k,\,\sigma_i(h)e_j\rangle_{\ell^2(d_i)}\\
&=o(1)+\frac{1}{d_i^2}\sum_{j\in D_i}\sum_{g\in\Gamma}|(\alpha^*\check{s}\check{t}^*\alpha)(g)|^2\\
&=o(1)+\frac{1}{d_i}u_{d_i}(D_i)\|\alpha^*\check{s}\check{t}^*\alpha\|_2^2\\
&\le o(1)+\frac{1}{d_i}u_{d_i}(D_i)\|\alpha\|_\infty\|s\|_{\ell^1(F)}^2\|t\|_{\ell^1(F)}^2\|\alpha\|_\infty\\
&\le o(1)+\frac{1}{d_i}|F|^2\|t\|_{\ell^2(F)}^2\|s\|_{\ell^2(F)}^2\|\alpha\|_\infty^2(1+o(1))\\
&\to 0.
\end{aligned}$$

This proves (3), by the Cauchy–Schwartz inequality. The general case follows by approximation, as in (i). □

For notation, if $E\subseteq\Gamma$ is finite and $f:\mathbb{R}^E\to\mathbb{C}$ is measurable, we let $f\otimes 1_{\mathbb{R}^{\Gamma\setminus E}}:\mathbb{R}^\Gamma\to\mathbb{C}$ be defined by

$$f\otimes 1_{\mathbb{R}^{\Gamma\setminus E}}(x)=f(x|_E).$$

We say that $f\in S(\mathbb{R}^\Gamma)$ if there is a finite $E\subseteq\Gamma$ and a Schwartz function $f_0:\mathbb{R}^E\to\mathbb{C}$ so that

$$f=f_0\otimes 1_{\mathbb{R}^{\Gamma\setminus E}}.$$

By standard Fourier analysis, there is a $\theta\in S(\mathbb{R}^E)$ so that

$$f(x)=\int_{\mathbb{R}^E}\exp(2\pi i t\cdot x)\,\theta(t)\,dt.$$

We prove that if we choose $x\in\mathbb{R}^{d_i}$ with respect to the Gaussian measure on $p_i\mathbb{R}^{d_i}$, then, with high probability, the microstate $\phi_x^{(E)}$ will approximately preserve the measure $\mu_{p\delta_e}$ when integrated against Schwartz functions.

We need the following notation: if $\phi\in X^{d_i}$, $\psi\in Y^{d_i}$, we define $\phi\otimes\psi\in(X\times Y)^{d_i}$ by

$$(\phi\otimes\psi)(j)=(\phi(j),\,\psi(j)).$$

LEMMA 4.18. *Let Γ be a countable discrete sofic group with sofic approximation $\Sigma=(\sigma_i:\Gamma\to S_{d_i})$. Let $p\in L_{\mathbb{R}}(\Gamma)$ be an orthogonal projection. Fix a sequence of orthogonal projections $p_i\in M_{d_i}(\mathbb{R})$ such that*

$$\|p_i-\sigma_i(p)\|_2\to 0.$$

Define the Gaussian measure ν_i on $p_i\ell^2_{\mathbb{R}}(d_i)$ by

$$d\,\nu_i(x)=e^{-\pi\|x\|^2_{\ell^2(d_i)}}\,dx.$$

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Here dx is the Lebesgue measure on $p_i\ell^2_{\mathbb{R}}(d_i)$. Let $\phi_x^{(E)}$ be defined, as in Lemma 4.16, for $X=\mathbb{R}$, $x_0=0$. Let $\mu_{p\delta_e}$ be defined as in Proposition 4.3. Let $F\subseteq E$ be finite subsets of Γ and $\delta>0$. Then, for any compact Hausdorff space Y, sequence $\psi_i\in Y^{d_i}$ and $f\in S(\mathbb{R}^F)$ and $g\in C(Y)$,

$$\nu_i\left(\left\{x\in p_i\mathbb{R}^{d_i}:\left|\int f\otimes 1_{\mathbb{R}^{\Gamma\setminus F}}\otimes g\,d(\phi_x^{(E)}\otimes\psi_i)_*(u_{d_i})\right.\right.\\ \left.\left.-\int_{\mathbb{R}^\Gamma} f\otimes 1_{\mathbb{R}^{\Gamma\setminus F}}\,d\mu_{p\delta_e}\int g\,d(\psi_i)_*(u_{d_i})\right|>\delta\right\}\right)\to 0.$$

Proof. Define $G: p_i\mathbb{R}^{d_i}\to\mathbb{R}$ by

$$G(x)=\int f\otimes 1_{\mathbb{R}^{\Gamma\setminus E}}\otimes g\,d(\phi_x^{(E)}\otimes\psi_i)_*(u_{d_i}).$$

We will show that

$$\left|\int_{p_i\mathbb{R}^{d_i}} G\,d\nu_i-\left(\int_X f\otimes 1_{\mathbb{R}^{\Gamma\setminus E}}\,d\mu_{p\delta_e}\right)\left(\int_Y g\,d(\psi_i)_*(u_{d_i})\right)\right|\to 0 \tag{4}$$

and

$$\left|\int_{p_i\mathbb{R}^{d_i}}|G|^2\,d\nu_i-\left|\int_X f\otimes 1_{\mathbb{R}^{\Gamma\setminus E}}\,d\mu_{p\delta_e}\right|^2\left|\int_Y g\,d(\psi_i)_*(u_{d_i})\right|^2\right|\to 0. \tag{5}$$

As

$$\left\|G-\int G\,d\nu_i\right\|^2_{L^2(d\nu_i)}=\int_{p_i\mathbb{R}^{d_i}}|G|^2\,d\nu_i-\left|\int_{p_i\mathbb{R}^{d_i}} G\,d\nu_i\right|^2,$$

the lemma will then follow, from Chebyshev's inequality.

Write

$$f(x)=\int_{\mathbb{R}^F}\exp(2\pi it\cdot x)\theta(t)\,dt$$

with $\theta\in S(\mathbb{R}^F)$. Note that, by Proposition 4.3 and by the fact that $\theta\in L^1(\mathbb{R}^F)$,

$$\begin{aligned}\int_{\mathbb{R}^\Gamma} f\otimes 1_{\mathbb{R}^{\Gamma\setminus E}}\,d\mu_{p\delta_e}&=\int_{\mathbb{R}^F}\int_{\mathbb{R}^\Gamma}\exp(2\pi it\cdot x)\theta(t)\,d\mu_{p\delta_e}dt\\&=\int_{\mathbb{R}^F}\theta(t)\exp(-\pi\|\lambda(\check{t})p\|_2^2)\,dt.\end{aligned} \tag{6}$$

We have

$$\begin{aligned}\int_{p_i\mathbb{R}^{d_i}} G(x)\,d\nu_i(x)&=\frac{1}{d_i}\sum_{j=1}^{d_i}\int_{p_i\mathbb{R}^{d_i}}\int_{\mathbb{R}^F}\theta(t)\exp(2\pi it\cdot\phi_x^{(E)}(j))g(\psi_i(j))\,dt\,d\nu_i(x)\\&=\frac{1}{d_i}\sum_{j=1}^{d_i}\int_{\mathbb{R}^F}\int_{p_i\mathbb{R}^{d_i}}\theta(t)\exp(2\pi it\cdot\phi_x^{(E)}(j))g(\psi_i(j))\,d\nu_i(x)\,dt,\end{aligned}$$

the interchanges of integrals being valid as g is bounded and $\theta\in L^1(\mathbb{R}^F)$. If $E\supseteq F$, then

$$t\cdot\phi_x^{(E)}(j)=\langle\sigma_i(\check{t})x,e_j\rangle_{\ell^2(d_i)}.$$

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Thus

$$\begin{aligned}
&\int_{p_i\mathbb{R}^{d_i}} G(x)\,dv_i(x)\\
&\quad=\frac{1}{d_i}\sum_{j=1}^{d_i}\int_{\mathbb{R}^F}\int_{p_i\mathbb{R}^{d_i}}\theta(t)\exp(2\pi i\langle\sigma_i(\check{t})x,\,e_j\rangle_{\ell^2(d_i,\mathbb{R})})g(\psi_i(j))\,dv_i(x)\,dt\\
&\quad=\frac{1}{d_i}\sum_{j=1}^{d_i}\int_{\mathbb{R}^F}\int_{p_i\mathbb{R}^{d_i}}\theta(t)\exp(2\pi i\langle x,\,p_i\sigma_i(\check{t})^*e_j\rangle_{\ell^2(d_i,\mathbb{R})})g(\psi_i(j))\,dv_i(x)\,dt\\
&\quad=\frac{1}{d_i}\sum_{j=1}^{d_i}\int_{\mathbb{R}^F}\theta(t)\exp(-\pi\|p_i\sigma_i(\check{t})^*e_j\|^2_{\ell^2(d_i)})g(\psi(j))\,dt.
\end{aligned}$$

Here we are using that

$$\int_{p_i\mathbb{R}^{d_i}}\exp(2\pi i\langle x,\,p_i\sigma_i(t)^*e_j\rangle_{\ell^2(d_i,\mathbb{R})})\,dv_i(x)=\exp(-\pi\|p_i\sigma_i(\check{t})^*e_j\|^2_{\ell^2(d_i)}),$$

(this is an obvious generalization of [**13**, Proposition 8.24]). By the preceding lemma, there is a $C_i\subseteq\{1,\dots,d_i\}$ with

$$\frac{|C_i|}{d_i}\to 1 \tag{7}$$

and

$$\limsup_{i\to\infty}\sup_{j\in C_i,t\in\mathbb{R}^F}\frac{|\|p_i\sigma_i(\check{t})^*e_j\|^2_{\ell^2(d_i)}-\|\lambda(\check{t})p\|_2^2|}{\|t\|_2^2}=0.$$

As

$$\|\lambda(\check{t})p\delta_e\|_2^2=\tau(p\lambda(\check{t})^*\lambda(\check{t})p)=\tau(\lambda(\check{t})p\lambda(\check{t})^*)=\|p\lambda(\check{t})^*\delta_e\|_2^2,$$

we have

$$\begin{aligned}
&\sup_{j\in C_i}\left|\int_{\mathbb{R}^F}\theta(t)\exp(-\pi\|p_i\sigma_i(\check{t})^*e_j\|^2_{\ell^2(d_i)})\,dt-\int_{\mathbb{R}^F}\theta(t)\exp(-\pi\|\lambda(\check{t})p\delta_e\|^2)\,dt\right|\\
&\quad\le o(1)\int_{\mathbb{R}^F}|\theta(t)|\|t\|^2_{\ell^2(F)}\,dt,
\end{aligned}\tag{8}$$

where we use $o(1)$ for any expression that goes to zero as $i\to\infty$. Additionally,

$$\sup_{j\in\{1,\dots,d_i\}\setminus C_i}\left|\int_{\mathbb{R}^F}\theta(t)\exp(-\pi\|p_i\sigma_i(\check{t})^*e_j\|^2_{\ell^2(d_i)})\right|\le\|\theta\|_1. \tag{9}$$

Since

$$\int_{\mathbb{R}^F}\theta(t)\|t\|^2_{\ell^2(F)}\,dt<\infty,$$

equations (8), (9), (7) and the fact that g is bounded imply that

$$\left|\int_{p_i\mathbb{R}^{d_i}}G(x)\,dv_i(x)-\int_{\mathbb{R}^F}\theta(t)\exp(-\pi\|\lambda(\check{t})p\|_2^2)\,dt\int g\,d\psi_*(u_{d_i})\right|\to 0.$$

By (6), we have proved (4).

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We now turn to the proof of (5). By the same computations as above,

$$\begin{aligned}
&\int_{p_i\mathbb{R}^{d_i}} |G|^2 \, d\nu_i \\
&\quad = \frac{1}{d_i^2} \sum_{1\le j,k\le d_i} \int_{p_i\mathbb{R}^{d_i}} \int_{\mathbb{R}^F} \int_{\mathbb{R}^F} \theta(t)\overline{\theta(s)} e^{2\pi i(\langle \sigma_i(\check{t})x, e_j\rangle_{\ell^2(d_i)} - \langle \sigma_i(\check{s})x, e_k\rangle_{\ell^2(d_i)})} \\
&\qquad \times g(\psi_i(j))\overline{g(\psi_i(k))}\, dt\, ds\, d\nu_i(x) \\
&\quad = \frac{1}{d_i^2} \sum_{1\le j,k\le d_i} \int_{\mathbb{R}^F} \int_{\mathbb{R}^F} \theta(t)\overline{\theta(s)} e^{-\pi(\|p_i\sigma_i(\check{t})^* e_j - p_i\sigma_i(\check{s})^* e_k\|^2_{\ell^2(d_i)})} \\
&\qquad \times g(\psi_i(j))\overline{g(\psi_i(k))}\, dt\, ds.
\end{aligned}$$

Again the interchanges of integrals are valid as $\theta \in L^1(\mathbb{R}^F)$. By the preceding lemma, there are $A_i \subseteq \{1, \dots, d_i\}^2$ so that

$$u_{d_i} \otimes u_{d_i}(A_i) \to 1 \tag{10}$$

and

$$\lim_{i\to\infty} \sup_{\substack{(j,k)\in A_i,\\ t,s\in\mathbb{R}^F}} \frac{|\|p_i\sigma_i(\check{t})^* e_j - p_i\sigma_i(\check{s})^* e_j\|_2^2 - \|\lambda(\check{t})p\delta_e\|_2^2 - \|\lambda(\check{s})p\delta_e\|_2^2|}{(\|t\|_{\ell^2(F)} + \|s\|_{\ell^2(F)})^2}\, dt = 0.$$

Thus

$$\begin{aligned}
&\sup_{(j,k)\in A_i} \left| \int_{\mathbb{R}^F} \int_{\mathbb{R}^F} \theta(t)\overline{\theta(s)} e^{-\pi(\|p_i\sigma_i(\check{t})^* e_j - p_i\sigma_i(\check{s})^* e_k\|^2_{\ell^2(d_i)})}\, dt\, ds \right. \\
&\qquad \left. - \left| \int_{\mathbb{R}^F} \theta(t) e^{(-\pi\|p_i\sigma_i(\check{t})^* e_j\|^2_{\ell^2(d_i)})}\, dt \right|^2 \right| \to 0,
\end{aligned} \tag{11}$$

as

$$\int_{\mathbb{R}^F} \int_{\mathbb{R}^F} |\theta(t)||\theta(s)|(\|t\|_{\ell^2(F)} + \|s\|_{\ell^2(F)})^2\, dt\, ds < \infty.$$

Additionally,

$$\begin{aligned}
&\sup_{(j,k)\in\{1,\dots,d_i\}^2\setminus(A_i\cap C_i\times C_i)} \left| \int_{\mathbb{R}^F} \int_{\mathbb{R}^F} \right. \\
&\qquad \left. \times\, \theta(t)\overline{\theta(s)} e^{-\pi(\|p_i\sigma_i(t)^* e_j\|^2_{\ell^2(d_i)} - 2\langle \sigma_i(s)p_i\sigma_i(t)^* e_j, e_k\rangle_{\ell^2(d_i)} + \|p_i\sigma_i(t)e_k\|^2_{\ell^2(d_i)})} \right| \\
&\qquad\qquad \le \|\theta\|_1^2.
\end{aligned} \tag{12}$$

Equations (8), (12), (10), (7) and the fact that g is bounded imply that

$$\begin{aligned}
&\left| \int_{p_i\mathbb{R}^{d_i}} |G|^2\, d\nu_i - \left| \int_{\mathbb{R}^F} \theta(t) \exp(-\pi\|\lambda(\check{t})p\delta_e\|_2^2)\, dt \right|^2 \right. \\
&\qquad \left. \cdot \frac{1}{d_i^2} \sum_{1\le j,k\le d_i} g(\psi_i(j))\overline{g(\psi_i(k))} \right| \to 0.
\end{aligned}$$

Since

$$\frac{1}{d_i^2} \sum_{1\le j,k\le d_i} g(\psi_i(j))\overline{g(\psi_i(k))} = \left| \int g\, d\psi_*(u_{d_i}) \right|^2,$$

we have proved (5). □

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LEMMA 4.19. *Let A be an infinite set. Then the space $S(\mathbb{R}^A)$ generates $C_b(\mathbb{R}^A)$, in the sense of Definition 3.1.*

Proof. Let $\varepsilon > 0$, $f \in C_b(\mathbb{R}^A)$ and $K \subseteq \mathbb{R}^A$ be compact. We say that a function $\psi : K \to \mathbb{C}$ depends upon a finite number of coordinates if there is a finite $E \subseteq A$ so that, if $x, y \in K$ and

$$x(a) = y(a) \quad \text{for all } a \in E,$$

then

$$f(x) = f(y).$$

By the Stone-Weierstrass theorem there is a function $\psi : K \to \mathbb{C}$ depending upon a finite number of coordinates so that

$$\|\psi\|_{C(K)} \leq \|f|_K\|_{C(K)} \leq \|f\|_{C_b(X)}$$

and

$$\|\psi - f|_K\|_{C(K)} < \varepsilon.$$

Let $E \subseteq A$ be such that if $x, y \in K$ and

$$x(a) = y(a) \quad \text{for all } a \in E,$$

then $\psi(x) = \psi(y)$. Let

$$K_E = \{x \in \mathbb{R}^E : \text{ there is a } y \in K \text{ with } x(a) = y(a) \text{ for all } a \in E\}.$$

Then K_E is a compact, being the continuous image of K under the projection map $\mathbb{R}^A \to \mathbb{R}^E$. There is a well-defined function $\widetilde{\psi} : K_E \to \mathbb{C}$ such that

$$\widetilde{\psi}(x) = \psi(x')$$

whenever $x' \in \mathbb{R}^\Gamma$ has $x'(g) = x(g)$ for all $g \in E$. It is well known that there is a $\widetilde{\phi} \in C_c^\infty(\mathbb{R}^E)$ so that

$$\|\widetilde{\phi}|_{K_E} - \widetilde{\psi}\|_{C(K_E)} < \varepsilon,$$
$$\|\widetilde{\phi}\|_{C_b(\mathbb{R}^E)} \leq \|f\|_{C_b(\mathbb{R}^A)}.$$

Let

$$\phi = \widetilde{\phi} \otimes 1_{\mathbb{R}^{A \setminus E}}.$$

Since $\widetilde{\phi}$ is a Schwartz function, $\phi \in S(\mathbb{R}^A)$. Finally,

$$\|\phi|_K - f|_K\|_{C(K)} \leq \|\phi|_K - \psi\|_{C(K)} + \|\psi - f|_K\|_{C(K)} < 2\varepsilon.$$

This completes the proof. □

THEOREM 4.20. *Let Γ be a countable discrete sofic group with sofic approximation $\Sigma = (\sigma_i : \Gamma \to S_{d_i})$. Let $\rho : \Gamma \to \mathcal{O}(\mathcal{H})$ be an orthogonal representation on a real, separable Hilbert space $\mathcal{H}$ with $\rho \ll \lambda_{\Gamma,\mathbb{R}}$. Let $\Gamma \curvearrowright (X_\rho, \mu_\rho)$ be the corresponding Gaussian action. Let (Y, ν) be a standard probability space and let $\Gamma \curvearrowright (Y, \nu)$ be a measure-preserving action with $h_{\Sigma,\nu}(Y, \Gamma) \geq 0$. Then $h_{\Sigma,\mu_\rho \otimes \nu}(X_\rho \times Y, \mu_\rho \otimes \nu) = \infty$.*

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Proof. We shall first reduce to the case when ρ is cyclic: that is, there is a vector $\xi \in \mathcal{H}$ so that

$$\mathcal{H} = \overline{\text{Span}\{\rho(g)\xi : g \in \Gamma\}}.$$

Suppose we can prove the Theorem in the cyclic case. Let

$$\rho = \rho_1 \oplus \rho_2,$$

where ρ_1 is cyclic and $\rho_2 \ll \lambda_{\Gamma,\mathbb{R}}$. Since we are assuming the theorem in the cyclic case and

$$\Gamma \curvearrowright (X_\rho \times Y, \mu_\rho \otimes \nu) \cong \Gamma \curvearrowright (X_{\rho_1} \times X_{\rho_2} \times Y, \mu_{\rho_1} \otimes \mu_{\rho_2} \otimes \nu),$$

it suffices to show that

$$h_{\Sigma,\mu_{\rho_2}\otimes\nu}(X_{\rho_2} \times Y, \Gamma) \geq 0, \tag{13}$$

as ρ_1 is cyclic. Since $\mathcal{H}$ is separable, we may write

$$\rho_2 = \bigoplus_{n=1}^{\infty} \rho_{2,n},$$

where $\rho_{2,n}$ is cyclic. For $n \in \mathbb{N}$, let

$$\rho_{2,\leq N} = \bigoplus_{n=1}^{N} \rho_{2,n}.$$

By the cyclic case and induction, we have

$$h_{\Sigma,\mu_{\rho_{2,\leq N}}\otimes\nu}(X_{\rho_{2,\leq N}} \times Y, \Gamma) = \infty$$

for any $N \in \mathbb{N}$. Using that

$$\Gamma \curvearrowright (X_{\rho_2} \times Y, \mu_{\rho_2} \otimes \nu) \cong \varprojlim \Gamma \curvearrowright (X_{\rho_{2,\leq N}} \times Y, \mu_{\rho_{2,\leq N}} \otimes \nu),$$

it is not hard to show (13).

We use the preceding Lemma and Theorem 3.3. By Proposition 4.3 we may regard $\Gamma \curvearrowright (X_\rho, \mu_\rho)$ as $\mathbb{R}^\Gamma$ with the measure $\mu_{p\delta_e}$ defined by

$$\int_{\mathbb{R}^\Gamma} \exp(2\pi i t \cdot x)\, d\mu_{p\delta_e}(x) = \exp(-\pi \|\lambda(t) p\delta_e\|_2^2)$$

for $t \in c_c(\Gamma, \mathbb{R})$ and some orthogonal projection $p \in L_{\mathbb{R}}(\Gamma)$. Extend Σ to an approximation sequence of $L(\Gamma)$, still denoted by $\sigma_i : L(\Gamma) \to M_{d_i}(\mathbb{C})$. By Proposition 4.11, there exists a sequence p_i of orthogonal projections in $M_{d_i}(\mathbb{R})$ so that

$$\|p_i - \sigma_i(p)\|_2 \to 0.$$

Let ν_i be the Gaussian measure on $p_i\mathbb{R}^{d_i}$ defined by

$$d\nu_i(x) = e^{-\pi \|x\|^2_{\ell^2(d_i)}}\, dx.$$

Choosing a compact model, we may assume that Y is a compact metrizable space and that $\Gamma \curvearrowright Y$ by homeomorphisms. Let Δ_Y be a compatible metric on Y. Let $\Delta_{\mathbb{R}^\Gamma}$ be the dynamic pseudometric on $\mathbb{R}^\Gamma$ defined by

$$\Delta_{\mathbb{R}^\Gamma}(x, y) = \min(|x(e) - y(e)|, 1).$$

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We shall use the generating set $\mathcal{L} = \{f \otimes g : f \in S(\mathbb{R}^\Gamma), g \in C(Y)\}$. We use the dynamically generating pseudometric on $\mathbb{R}^\Gamma \times Y$ defined by

$$\Delta((x_1, y_1), (x_2, y_2)) = \Delta_{\mathbb{R}^\Gamma}(x_1, x_2) + \Delta_Y(y_1, y_2).$$

Let $\varepsilon, \delta, \eta > 0$ and finite $F \subseteq \Gamma$,$L \subseteq \mathcal{L}$ be given.

Let $e = g_1, g_2, \ldots$ be an enumeration of the elements of Γ. Inductively, find positive real numbers $M_1, M_2, \ldots$ so that

$$\mu_{p\delta_e}(\{x \in \mathbb{R}^\Gamma : |x(g_j)| \leq M_j, j = 1, \ldots, l\}) \geq 1 - \sum_{j=1}^{l} 2^{-j}\eta.$$

Define $M : \Gamma \to (0, \infty)$ by $M(g_j) = M_j$. Set

$$K_M = \{x \in \mathbb{R}^\Gamma : |x(g)| \leq M(g) \forall g \in \Gamma\},$$

and note that K_M is compact. Let U be an open neighborhood of K_M in $\mathbb{R}^\Gamma$.

Let $L_1 \subseteq S(\mathbb{R}^\Gamma)$, $L_2 \subseteq C(Y)$ be finite sets so that

$$L \subseteq \{f \otimes g : f \in L_1, g \in L_2\}.$$

Since

$$h_{\Sigma,\nu}(Y, \Gamma) \geq 0,$$

for an infinite number of i, there is a $\psi_i \in \mathrm{Map}_\nu(\Delta_Y, F, \delta, L_2, \sigma_i)$. For these i, by Lemma 4.18, there is a finite set E of Γ containing the identity and an $B_i \subseteq p_i\mathbb{R}^{d_i}$ so that

$$\nu_i(B_i) \to 1,$$
$$\phi_x^{(E)} \otimes \psi_i \in \mathrm{Map}_{\mu\otimes\nu}(\Delta, F, \delta, L, \sigma_i)$$

for all $x \in B_i$. We will also assume that

$$U \supseteq \pi_E(K_M) \times \mathbb{R}^{\Gamma\setminus E}.$$

This may be done by enlarging E, as K_M is compact. Choose $f \in S(\mathbb{R}^E)$ so that

$$\int_{\mathbb{R}^E} f \, d(\pi_E)_*\mu_\rho \geq -\eta + (\pi_E)_*\mu(\pi_E(K_M))$$

and

$$0 \leq f \leq \chi_{\pi_E(K_M)}.$$

We may assume that

$$\phi_x^{(E)} \otimes \psi_i \in \mathrm{Map}_{\mu\otimes\nu}(\Delta, F, \delta, L \cup \{f \otimes 1_{\mathbb{R}^{\Gamma\setminus E}\otimes 1}\}, \sigma_i)$$

for all $x \in B_i$. Note that, in this case,

$$\begin{aligned}(\phi_x^{(E)} \otimes \psi_i)_*(u_{d_i})(U \times Y) &\geq (\phi_x^{(E)} \otimes \psi_i)_*(u_{d_i})(\pi_E(K_M) \times \mathbb{R}^{\Gamma\setminus E} \times Y)\\ &\geq -\delta + \int_{\mathbb{R}^\Gamma} f \otimes 1_{\mathbb{R}^{\Gamma\setminus E}} \, d\mu_p\\ &\geq -\delta - \eta + (\pi_E)_*\mu(\pi_E(K_M))\\ &\geq -\delta - 2\eta + 1.\end{aligned}$$

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So, if $\delta < \eta$,

$$\phi_x^{(E)} \otimes \psi_i \in \mathrm{Map}_{\mu\otimes\nu}^{U\times Y,3\eta}(\Delta,\, F,\, \delta,\, L \cup \{f \otimes 1_{\mathbb{R}^{\Gamma\setminus E}} \otimes 1\},\, \sigma_i).$$

Thus

$$h_{\Sigma,\mu_p\otimes\nu}^{U\times Y,\eta}(\Delta,\, \varepsilon,\, F,\, \delta,\, L \cup \{f \otimes 1_{\mathbb{R}^{\Gamma\setminus E}} \otimes 1\}) \geq \limsup_{i\to\infty} \frac{1}{d_i} \log S_\varepsilon(B_i,\, \Delta_2).$$

Suppose that $S \subseteq B_i$ is ε-dense. Then, for every $y \in B_i$, there exists $x \in S$ and a $C \subseteq \{1, \ldots, d_i\}$ so that

$$|x(j) - y(j)| < \sqrt{\varepsilon} \quad \text{for } j \in C,$$
$$|C| \geq (1 - \sqrt{\varepsilon})d_i.$$

Using m for Lebesgue measure,

$$\begin{aligned} \nu_i(B) &\leq \sum_{\substack{x\in S,\\ A\subseteq\{1,\ldots,d_i\},\\ |A|\geq(1-\sqrt{\varepsilon})d_i}} \nu_i((x_A + \sqrt{\varepsilon}\,\mathrm{Ball}(\ell^2_{\mathbb{R}}(A,\, u_A))) \times \mathbb{R}^{A^c}) \\ &\leq \sum_{\substack{x\in S,\\ A\subseteq\{1,\ldots,d_i\},\\ |A|\geq(1-\sqrt{\varepsilon})d_i}} m(\sqrt{\varepsilon}\,\mathrm{Ball}(\ell^2_{\mathbb{R}}(A,\, u_A))) \\ &= \sum_{\substack{x\in S,\\ A\subseteq\{1,\ldots,d_i\},\\ |A|\geq(1-\sqrt{\varepsilon})d_i}} \varepsilon^{|A|/2} m(\mathrm{Ball}(\ell^2_{\mathbb{R}}(A,\, u_A))). \end{aligned}$$

By [**13**] Corollary 2.55 and Stirling's formula, there is a $R > 0$ so that

$$m(\mathrm{Ball}(\ell^2_{\mathbb{R}}(A,\, u_A))) \leq R^{|A|/2}\left(\frac{d_i}{|A|}\right)^{|A|/2}.$$

Set

$$H(t) = \begin{cases} 0 & \text{if } t = 0,\, 1, \\ -t\log(t) - (1-t)\log(1-t) & \text{for } 0 < t < 1. \end{cases}$$

Then, for some $D > 0$,

$$\begin{aligned} \nu_i(B) &\leq \sum_{\substack{x\in S,\\ A\subseteq\{1,\ldots,d_i\},\\ |A|\geq(1-\sqrt{\varepsilon})d_i}} \varepsilon^{d_i/2} R^{|A|/2}\left(\frac{d_i}{|A|}\right)^{|A|/2} \\ &\leq |S| D\sqrt{\varepsilon} d_i R^{d_i/2} \exp(H(\sqrt{\varepsilon})d_i)\left(\frac{\varepsilon}{(1-\sqrt{\varepsilon})}\right)^{d_i/2}, \end{aligned}$$

the last line following by Stirling's formula. Thus

$$\limsup_{i\to\infty} \frac{1}{d_i} \log|S| \geq \frac{1}{2}\log\left(\frac{1-\sqrt{\varepsilon}}{\varepsilon}\right) - \frac{1}{2}\log R - H(\sqrt{\varepsilon}).$$

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So

$$h^{U\times Y,3\eta}_{\Sigma,\mu_p\otimes\nu}(\Delta,\varepsilon,F,\delta,L) \geq h^{U\times Y,3\eta}_{\Sigma,\mu_p\otimes\nu}(\Delta,\varepsilon,F,\delta,L\cup\{f\otimes\mathbb{R}^{\Gamma\setminus E}\otimes 1\})$$
$$\geq \frac{1}{2}\log\left(\frac{1-\sqrt{\varepsilon}}{\varepsilon}\right)-\frac{1}{2}\log R - H(\sqrt{\varepsilon}).$$

Taking the infimum over all $U \supseteq K_M$, then the supremum over all K_M, then the infimum over all F, δ, η and $L \subseteq \mathcal{L}$, we find that

$$h_{\Sigma,\mu_{p\delta_e}\otimes\nu}(\Delta,\varepsilon,\mathcal{L}) \geq \frac{1}{2}\log\left(\frac{1-\sqrt{\varepsilon}}{\varepsilon}\right)-\frac{1}{2}\log R - H(\sqrt{\varepsilon}).$$

If we now let $\varepsilon \to 0$, we see that

$$h_{\Sigma,\mu_{p\delta_e}\otimes\nu}(X\times Y,\Gamma)=\infty. \qquad \square$$

We now prove a general formula for the entropy of Gaussian actions.

COROLLARY 4.21. *Let Γ be a countable discrete sofic group with sofic approximation $\Sigma=(\sigma_i:\Gamma\to S_{d_i})$. Let $\rho:\Gamma\to\mathcal{O}(\mathcal{H})$ be an orthogonal representation. By Proposition 4.14, write $\rho=\rho_1\oplus\rho_2$, where $\rho_1\ll\lambda_{\Gamma,\mathbb{R}}$ and $\rho_2\perp\lambda_{\Gamma,\mathbb{R}}$. Let $\Gamma\curvearrowright(X_\rho,\mu_\rho)$, $\Gamma\curvearrowright(X_{\rho_j},\mu_{\rho_j})$, $j=1,2$ be the corresponding Gaussian actions. Then:*

(i) $h_{\Sigma,\mu_{\rho_2}}(X_{\rho_2},\mu_{\rho_2})\in\{0,-\infty\}$;

(ii)
$$h_{\Sigma,\mu_\rho}(X_{\rho_1},\mu_{\rho_1})=\begin{cases}-\infty & \text{if } h_{\Sigma,\mu_{\rho_2}}(X_{\rho_2},\mu_{\rho_1})=-\infty,\\ 0 & \text{if } \rho_1=0 \text{ and } h_{\Sigma,\mu_{\rho_2}}(X_{\rho_2},\mu_{\rho_2})=0,\\ \infty & \text{if } \rho_1\neq 0 \text{ and } h_{\Sigma,\mu_{\rho_2}}(X_{\rho_2},\mu_{\rho_2})=0.\end{cases}$$

Proof. Statement (i) is just a direct corollary of Corollary 4.15 and the fact that sofic entropy is always non-negative or $-\infty$. The first case of statement (i) follows from the general fact that if $\Gamma\curvearrowright(X,\mu)$, $\Gamma\curvearrowright(Y,\nu)$ are two measure-preserving actions on standard probability spaces, then

$$h_{\Sigma,\mu}(X,\Gamma)=-\infty$$

implies that

$$h_{\Sigma,\mu\otimes\nu}(X\times Y,\Gamma)=-\infty.$$

The second case of statement (ii) is also just a special case of statement (i). The last case of statement (ii) follows from Theorem 4.20. $\square$

We give some examples to show that $h_{\Sigma,X_\rho}(X_\rho,\mu_\rho)$ can be $-\infty$. Let Γ be a countable discrete sofic group with sofic approximation $\Sigma=(\sigma_i:\Gamma\to S_{d_i})$. We say that Σ is *ergodic* if, whenever $A_i\subseteq\{1,\ldots,d_i\}$ are such that

$$\lim_{i\to\infty} u_{d_i}(A_i\Delta\sigma_i(g)A_i)=0 \quad \text{for all } g\in\Gamma,$$

then

$$\lim_{i\to\infty} u_{d_i}(A_i)(1-u_{d_i}(A_i))=0.$$

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The following is a folklore result and we include a proof for completeness.

PROPOSITION 4.22. *Let Γ be a countable discrete sofic group with an ergodic sofic approximation Σ. Then, if $\Gamma \curvearrowright (X, \mu)$ is a non-ergodic probability measure-preserving action on a standard probability space,*

$$h_{\Sigma,\mu}(X, \Gamma) = -\infty.$$

Proof. Let $\Sigma = (\sigma_i : \Gamma \to S_{d_i})$. Let $E \subseteq X$ be a Γ-invariant set with $0 < \mu(E) < 1$. Let $\alpha : X \to \{0, 1\}$ be the finite observable given by

$$\alpha(x) = \chi_E(x).$$

Let $\varepsilon > 0$. By a diagonalization argument, it is easy to see that there is a $\delta > 0$ and a finite $F \subseteq \Gamma$ containing e so that, if $B_i \subseteq \{1, \ldots, d_i\}$ has

$$\limsup_{i\to\infty} \max_{g\in F} u_{d_i}(B_i \Delta \sigma_i(g) B_i) < \delta, \tag{14}$$

then

$$\limsup_{i\to\infty} u_{d_i}(B_i)(1 - u_{d_i}(B_i)) < \varepsilon. \tag{15}$$

Set $\widetilde{E} = \{a \in \{0, 1\}^F : a(e) = 1\}$. If δ is sufficiently small and $\phi \in \mathrm{AP}(\alpha, F, \delta', \sigma_i)$, then since, $\mu(gE\Delta E) = 0$,

$$\max_{g\in F} u_{d_i}(\sigma_i(g)\phi^{-1}(\widetilde{E})\Delta\phi(\widetilde{E})) < \delta$$

and we always have

$$\mu(E) + \delta' \geq u_{d_i}(\phi^{-1}(\widetilde{E})) \geq \mu(E) - \delta'.$$

If δ', ε are sufficiently small, then

$$u_{d_i}(\phi^{-1}(\widetilde{E}))(1 - u_{d_i}(\phi^{-1}(\widetilde{E}))) > \varepsilon.$$

Thus we see from (14) and (15) that $\mathrm{AP}(\alpha, F, \delta', \sigma_i) = \varnothing$. Thus $h_{\Sigma,\mu}(X, \Gamma) = -\infty$, using Kerr's definition of entropy via partitions. □

Combining with [**27**, Theorem 2.8] we have the following corollary.

COROLLARY 4.23. *Let Γ be a countable discrete sofic group with an ergodic sofic approximation Σ. Let $\rho : \Gamma \to \mathcal{O}(\mathcal{H})$ be an orthogonal representation which is not weakly mixing (e.g. ρ could be compact). Then, if $\Gamma \curvearrowright (X, \mu)$ is the associated Gaussian action,*

$$h_{\Sigma,\mu}(X, \Gamma) = -\infty.$$

We mention an example of an ergodic sofic approximation when $\Gamma = \mathbb{F}_2$ is the free group on two generators a, b. Here we can choose a sofic approximation randomly. Namely, let $\phi = (\phi_1, \phi_2) \in S_n^2$ be chosen uniformly at random, and let

$$\sigma_\phi : \mathbb{F}_2 \to S_n$$

be the unique homomorphism so that $\sigma_\phi(a) = \phi_1$, $\sigma_\phi(b) = \phi_2$. It is known that, with high probability, σ_ϕ is a sofic approximation (see [**23**], as well as [**18**, Lemma 3.1]). It is

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also known from the theory of expanders that, with high probability, σ_ϕ is a ergodic sofic approximation (see [**14**], and the remarks in [**26**, §5]). If we take an orthogonal representation $\rho : \mathbb{F}_2 \to \mathcal{O}(\mathcal{H})$ which is not weakly mixing (e.g. take $\mathcal{H}$ to be finite-dimensional), then

$$h_{\Sigma,\mu_\rho}(X_\rho, \Gamma) = -\infty.$$

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