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A dichotomy for three-dimensional vector fields

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Abstract. We prove that a generic C^1 vector field on a closed 3-manifold either has infinitely many sinks or sources or else is singular Axiom A without cycles. Singular Axiom A means that the non-wandering set of the vector field has a decomposition into compact invariant sets, each being either a hyperbolic basic set or a singular hyperbolic attractor (like the Lorenz-like ones) or a singular hyperbolic repeller. An attractor is a transitive set which attracts all nearby future orbits, and a repeller is an attractor for the time-reversed flow. Our result implies that generic C^1 vector fields on closed 3-manifolds do exhibit either attractors or repellers.

1. *Introduction*

Let M be a closed 3-manifold and $\mathcal{X}^1(M)$ be the set of C^1 vector fields on M endowed with the C^1 topology. Given an open subset $\mathcal{A}$ of $\mathcal{X}^1(M)$ a subset $\mathcal{R} \subset \mathcal{A}$ is residual if it coincides with a countable intersection of open-dense subsets of $\mathcal{A}$. We say that *a generic vector field in $\mathcal{A}$ satisfies a property (P)* if there is a residual subset $\mathcal{R}$ of $\mathcal{A}$ such that (P) holds for every $X \in \mathcal{R}$. A vector field $X \in \mathcal{X}^1(M)$ is non-singular if it has no singularities, i.e. $X(p) \neq 0$ for all $p \in M$. A compact invariant set Λ is *transitive* if $\Lambda = \omega_X(q)$ for some $q \in \Lambda$, and *attracting* if $\Lambda = \bigcap_{t \geq 0} X_t(U)$ for some neighborhood U of Λ satisfying $X_t(U) \subset U$, for all $t \geq 0$. An *attractor* of X is a transitive attracting set of X and a *repeller* is an attractor for $-X$. A *sink* of X is a trivial attractor of X, namely it reduces to a single orbit of X. A *source* of X is a trivial repeller of X. A regular orbit contained in the intersection of the stable and unstable manifolds associated with a hyperbolic singularity is called a *homoclinic loop*.

It is known that a generic non-singular vector field $X \in \mathcal{X}^1(M)$ either has infinitely many sinks or sources or else is Axiom A without cycles [**12, 13**] (this is not true in general [**7**]). Allowing singularities such a result can be improved as follows. Let $\mathcal{V}^1(M) \subset \mathcal{X}^1(M)$ be the set of vector fields *that can not be C^1 approximated by homoclinic loops*. The connecting lemma [**8**] implies that every $X \in \mathcal{V}^1(M)$ has its singularities separated from the non-wandering set. By using [**8, 22**] we conclude that a generic vector field

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in $\mathcal{V}^1(M)$ either has infinitely many sinks or sources or else it is Axiom A without cycles. Very recently, Arroyo [**1**] proved that every vector field in $\mathcal{V}^1(M)$ can be either approximated by one that is Axiom A without cycles or exhibits a homoclinic tangency associated with a periodic orbit.

In this paper we shall prove a singular version of [**13**] for flows on compact 3-manifolds. More precisely we consider a class of vector fields including both the Axiom A ones and the geometric Lorenz attractors. An element of this class is called singular Axiom A and is characterized by the fact that its non-wandering set can be written as a finite number of compact invariant sets, each being a hyperbolic basic set, a singular hyperbolic attractor or a singular hyperbolic repeller. This definition differs slightly from the one given in [**17**]. Our main result is as follows.

THEOREM A. *A generic vector field $X \in \mathcal{X}^1(M)$ satisfies (only) one of the following properties:*

(1) *X has infinitely many sinks or sources;*

(2) *X is singular Axiom A without cycles.*

Let us describe two immediate consequences of this theorem. The first is related to the abundance of three-dimensional vector fields exhibiting either attractors or repellers.

COROLLARY 1.1. *A generic vector field in $\mathcal{X}^1(M)$ does exhibit either attractors or repellers.*

As observed in [**13**], a generic C^1 diffeomorphism in the 2-sphere S^2 does exhibit either sinks or sources. It is then natural to ask whether such a result is valid for C^1 vector fields in the 3-sphere S^3 instead of C^1 diffeomorphisms in S^2. The answer is negative, as the following example shows. Write $S^3 = \mathbb{R}^3 \cup \{\infty\}$ and consider in $\mathbb{R}^3$ an unknotted 2-torus T^2. Then the closure in S^3 of each connected component of $S^3 \setminus T^2$ is a solid 2-torus. Now consider a Lorenz attractor in one of the solid 2-tori and a Lorenz repeller in the other. Since a fundamental domain for the Lorenz attractor (respectively repeller) is an unknotted solid 2-torus, we can glue two solid 2-tori through the unknotted one, obtaining a flow in S^3 which exhibits as the non-wandering set one Lorenz attractor and one Lorenz repeller. Such a flow is singular Axiom A but it can not be approximated by vector fields with either sinks or sources.

The second is related to a conjecture by Palis [**19**], asserting the denseness of vector fields exhibiting a finite number of attractors whose basin of attraction form a full Lebesgue measure set. Indeed, Theorem A gives an approach to this conjecture in the (open) set $\mathcal{N}^1(M)$ of C^1 vector fields on a closed 3-manifold M which can not be C^1 approximated by ones exhibiting infinitely many sinks or sources.

COROLLARY 1.2. *A generic vector field in $\mathcal{N}^1(M)$ exhibits a finite number of attractors whose basins of attraction form an open-dense set.*

We observe that it was proved recently that every C^1 diffeomorphism on a closed surface can be C^1 approximated by a diffeomorphism exhibiting a homoclinic tangency or by an Axiom A diffeomorphism without cycles [**21**]. This result and Theorem A lead us to formulate the following conjecture.

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CONJECTURE 1.3. *Every C^1 vector field on a closed 3-manifold can be C^1 approximated by a vector field exhibiting a homoclinic tangency or by a singular Axiom A one without cycles.*

2. *Preliminaries*

In this section we recall some definitions, and some results proved elsewhere, that will be used here. We shall also fix some notation.

If $X \in \mathcal{X}^1(M)$, we denote by X_t the flow generated by X. We say that $p \in M$ is *non-wandering* for X if for every $T > 0$ and every neighborhood U of p there is $t > T$ such that $X_t(U) \cap U \neq \emptyset$. The set of non-wandering points of X is denoted by $\Omega(X)$. If $q \in M$, we define $\omega_X(q)$ as the set of accumulation points of the positive orbit $\{X_t(q) : t \geq 0\}$ of q. We also define $\alpha_X(q) = \omega_{-X}(q)$, where $-X$ is the time-reversed vector field X. It is immediate that $\omega_X(q) \cup \alpha_X(q) \subset \Omega(X)$ for every $q \in M$.

An *orbit* of X is a set $O = O_X(q) = \{X_t(q) : t \in \mathbb{R}\}$ for some $q \in M$. A *singularity* of X is a point $\sigma \in M$ such that $X(\sigma) = 0$ (equivalently $O_X(\sigma) = \{\sigma\}$). A *periodic orbit* of X is an orbit $O = O_X(p)$ such that $X_T(p) = p$ for some minimal $T > 0$ (equivalently O is compact and $O \neq \{p\}$).

A subset Λ of M is *invariant* for X if $X_t(\Lambda) = \Lambda$, for all $t \in \mathbb{R}$. For every compact invariant set Λ of X, we define

$$W^s_X(\Lambda) = \{q \in M : \omega_X(q) \subset \Lambda\}$$

and

$$W^u_X(\Lambda) = \{q \in M : \alpha_X(q) \subset \Lambda\}.$$

Recall that a compact invariant set Λ is *transitive* if $\Lambda = \omega_X(q)$ for some $q \in \Lambda$, and *attracting* if $\Lambda = \bigcap_{t \geq 0} X_t(U)$ for some neighborhood U of Λ satisfying $X_t(U) \subset U$, for all $t \geq 0$. An *attractor* of X is a transitive attracting set of X and a *repeller* is an attractor for $-X$. A *sink* of X is a trivial attractor of X, namely it reduces to a single orbit of X. A *source* of X is a trivial repeller of X.

Denote by $m(T) = \inf_{\|v\|=1} \|T(v)\|$ the *minimum norm* of a linear operator T [**9**]. A compact invariant set H of X is *hyperbolic* if it exhibits a continuous invariant tangent bundle decomposition $T_H M = E^s_H \oplus E^X_H \oplus E^u_H$ and there are constants $\lambda, K > 0$ such that

- E^s_H *is* (K, λ)*-contracting*, i.e.

$$\|DX_t(x)/E^s_x\| \leq K^{-1}e^{-\lambda t}, \quad \text{for all } x \in H, \quad \text{for all } t \geq 0.$$

- E^u_H *is* (K, λ)*-expanding*, i.e.

$$m(DX_t/E^u_x) \geq Ke^{\lambda t}, \quad \text{for all } x \in H, \quad \text{for all } t \geq 0.$$

- E^X_x is the subspace in $T_x M$ generated by $X(x)$.

By invariant manifold theory [**9**] it follows that for every $p \in H$ the sets

$$U_X(p) = \{q \in M : \operatorname{dist}(X_t(q), X_t(p)) \to 0, t \to \infty\}$$

and

$$S_X(p) = \{q \in M : \operatorname{dist}(X_t(q), X_t(p)) \to 0, t \to -\infty\}$$

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are invariant C^r-manifolds tangent to E^s_p and E^u_p respectively. Here dist is the distance on M. If $O = O_X(p) \subset H$ is an orbit of X one has that

$$W^s_X(O) = \bigcup_{t\in\mathbb{R}} S_X(X_t(p))$$

and

$$W^u_X(O) = \bigcup_{t\in\mathbb{R}} U_X(X_t(p))$$

are invariant C^r-manifolds tangent to $E^s_p \oplus E^X_p$ and $E^X_p \oplus E^u_p$ respectively. We shall denote $W^s_X(p) = W^s_X(O_X(p))$ and $W^u_X(p) = W^u_X(O_X(p))$ for the sake of simplicity.

A singularity (or periodic orbit) of X is hyperbolic if its orbit is a hyperbolic set of X. Note that $S_X(\sigma) = W^s_X(\sigma)$ and $U_X(\sigma) = W^u_X(\sigma)$ for every hyperbolic singularity σ of X. A hyperbolic set H of X is called *basic* if it is transitive and *isolated*, namely $H = \bigcap_{t\in\mathbb{R}} X_t(U)$ for some neighborhood U of H. It follows from the shadowing lemma [**18**] that every hyperbolic basic set of X either reduces to a singularity or else has no singularities and is the closure of its periodic orbits.

We say that X is *Axiom A* if $\Omega(X)$ is both hyperbolic and the closure of its periodic orbits and singularities. The spectral decomposition theorem asserts that, if X is Axiom A, then there is a disjoint decomposition

$$\Omega(X) = H_1 \cup \cdots \cup H_k,$$

where each H_i is a hyperbolic basic set of X, $i = 1, \ldots, k$.

On the other hand, a compact invariant set Λ of X is *partial hyperbolic* if there are a continuous invariant tangent bundle decomposition $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ and constants $\lambda, K > 0$ such that

- E^s_Λ is (K, λ)-contracting.
- E^s_Λ (K, λ)-*dominates* E^c_Λ, i.e.

$$\|DX_t(x)/E^s_x\| \le K^{-1}e^{-\lambda t} m(DX_t(x)/E^c_x),$$

for all $x \in \Lambda$, $t \ge 0$.

We shall say that $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ is a (K, λ)-*splitting* for short. For $x \in \Lambda$ and $t \in \mathbb{R}$ we let $J^c_t(x)$ be the absolute value of the determinant of the linear map $DX_t(x)/E^c_x : E^c_x \to E^c_{X_t(x)}$. We say that the subbundle E^c_Λ of the partial hyperbolic set Λ is *volume expanding* if

$$J^c_t(x) \ge Ke^{\lambda t}$$

for every $x \in \Lambda$ and $t \ge 0$ (in this case we say that E^c_Λ is (K, λ)-*volume expanding* to indicate the dependence on K, λ). It follows from [**17**] that a non-singular partial hyperbolic set with volume expanding central direction is hyperbolic.

A partial hyperbolic set is *singular hyperbolic* if it has singularities (all of them hyperbolic) and it has volume expanding central direction [**16**]. A singular hyperbolic attractor is a singular hyperbolic set which is an attractor as well. A *singular hyperbolic repeller* of X is a singular hyperbolic attractor of $-X$. The most representative example of a singular hyperbolic attractor is the Lorenz attractor.

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A vector field $X \in \mathcal{X}^1(M)$ is called *singular Axiom A* if there is a finite disjoint decomposition

$$\Omega(X) = \Lambda_1 \cup \cdots \cup \Lambda_n,$$

where each Λ_i is a hyperbolic basic set, a singular hyperbolic attractor or a singular hyperbolic repeller, $i = 1, \ldots, n$. This definition differs slightly from the one given in [**17**]. It follows from the spectral decomposition theorem mentioned above that every Axiom A vector field is singular Axiom A.

A *cycle* of a singular Axiom A vector field X is a subcollection $\{\Lambda_{i_0}, \ldots, \Lambda_{i_k}\}$ of $\{\Lambda_1, \ldots, \Lambda_n\}$ such that $i_0 = i_k$ and $W^u_X(\Lambda_{i_j}) \cap W^s_X(\Lambda_{i_{j+1}}) \neq \emptyset$, for all $0 \leq j \leq k-1$. It is clear that every Axiom A vector field without cycles is singular Axiom A without cycles.

Recall that a compact set $L \subset M$ is called *Lyapunov stable* for $X \in \mathcal{X}^1(M)$ if for every neighborhood U of L there is a neighborhood $V \subset U$ of L such that $X_t(V) \subset U$, for all $t \geq 0$. Every attractor is a transitive Lyapunov stable set but not conversely.

The following lemmas summarize some classical properties of Lyapunov stable sets (see Chapter V in [**2**]).

LEMMA 2.1. *Let Λ^+ be a Lyapunov stable set of X. Then:*

(1) *if $x_n \in M$ and $t_n \geq 0$ satisfy $x_n \to x \in \Lambda^+$ and $X_{t_n}(x_n) \to y$, then $y \in \Lambda^+$;*

(2) *$W^u_X(\Lambda^+) \subset \Lambda^+$;*

(3) *if Γ is a transitive set of X and $\Gamma \cap \Lambda^+ \neq \emptyset$, then $\Gamma \subset \Lambda^+$.*

LEMMA 2.2. *A Lyapunov stable set Λ of a vector field X is an attractor of X if and only if there is a neighborhood U of Λ such that $\omega_X(x) \subset \Lambda$, for all $x \in U$.*

3. *Proof of Theorem A*

Denote by $\mathcal{G}^r(M)$ the set of vector fields $X \in \mathcal{X}^r(M)$ such that every periodic orbit and singularity of every $Y \in \mathcal{X}^r(M)$ C^r-close to X is hyperbolic.

THEOREM B. *Generic vector fields in $\mathcal{G}^1(M)$ are singular Axiom A without cycles.*

As in [**13**], Theorem A follows from Theorem B. Indeed, let $\mathcal{S}^1(M)$ be the set of vector fields $X \in \mathcal{X}^1(M)$ such that every singularity of X is hyperbolic. It follows that $\mathcal{S}^1(M)$ is open-dense in $\mathcal{X}^1(M)$. If $X \in \mathcal{S}^1(M)$ let $\Lambda(X)$ be the set of periodic orbits and singularities of X that are sinks or sources, and let $\overline{\Lambda}(X)$ be its closure. Clearly the set-valued function $X \to \overline{\Lambda}(X)$ is lower semicontinuous. By well-known properties about set-valued maps, there is a residual subset $\mathcal{O} \subset \mathcal{S}^1(M)$ such that every $X \in \mathcal{O}$ is a point of upper-semicontinuity of $\overline{\Lambda}$. As in [**13**] we have that every $X \in \mathcal{O}$ not satisfying (1) of Theorem A is in $\mathcal{G}^1(M)$. Indeed, consider $X \in \mathcal{O} \setminus \mathcal{G}^1(M)$. If X does not satisfy (1) of Theorem A, then X has finitely many sinks and sources. It follows that $\overline{\Lambda}(X)$ reduces to a finite number of closed orbits. As $X \notin \mathcal{G}^1(M)$ there is a vector field X' C^1-close to X with a non-hyperbolic closed orbit O. Since $\overline{\Lambda}(X)$ reduces to a finite number of sinks and sources we have that O is far from $\overline{\Lambda}(X)$. On the other hand, since $\mathcal{S}^1(M)$ is open and $X \in \mathcal{S}^1(M)$ we can assume that $X' \in \mathcal{S}^1(M)$. It follows that O is periodic. Since M is three-dimensional we have that the Poincaré map associated with O has two eigenvalues, one of which has modulus 1. By pushing this eigenvalue in the right direction we can

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approximate X' by a vector field X'' such that O is either a sink or a source of X''. Since O is far from $\overline{\Lambda}(X)$ we obtain a contradiction by the upper-semicontinuity. This contradiction shows that every $X \in \mathcal{O}$ not satisfying (1) of Theorem A is in $\mathcal{G}^1(M)$. Now, by Theorem B there is $\mathcal{R} \subset \mathcal{G}^1(M)$ residual such that every vector field in $\mathcal{R}$ is singular Axiom A without cycles. Define

$$\mathcal{O}' = (\mathcal{O} \cap (\mathcal{S}^1(M) \setminus \mathrm{Cl}(\mathcal{G}^1(M)))) \cup (\mathcal{O} \cap \mathcal{R}),$$

where $\mathrm{Cl}(\mathcal{G}^1(M))$ is the closure of $\mathcal{G}^1(M)$ in $\mathcal{X}^1(M)$. It follows that $\mathcal{O}'$ is residual in $\mathcal{S}^1(M)$. If $X \in \mathcal{O}'$ does not satisfy the option (1) of Theorem A, then $X \in \mathcal{G}^1(M)$ since $X \in \mathcal{O}$. It follows that $X \in \mathcal{O} \cap \mathcal{R}$ since $X \in \mathcal{O}'$. Thus X is singular Axiom A without cycles by the definition of $\mathcal{R}$ and the proof follows.

We prove Theorem B in the next section using Theorems C and D below. Before we announce these theorems, let us recall that it was proved in [**15**] that for generic $X \in \mathcal{G}^1(M)$ every non-trivial attractor with singularities of X is singular hyperbolic. Theorem C asserts that the same holds replacing attractor by transitive Lyapunov stable set. The proof is the same as in [**15**] and we include it in Appendix A for completeness.

THEOREM C. *For generic vector fields $X \in \mathcal{G}^1(M)$ every non-trivial transitive Lyapunov stable set with singularities of X is singular hyperbolic for X.*

Now, the question of whether a Lyapunov stable singular hyperbolic transitive set is an attractor is interesting. A positive answer for this question is given as follows. Let Λ be a compact invariant set with singularities (all of them hyperbolic) of $X \in \mathcal{X}^1(M)$. We say that Λ *has dense singular unstable branches* if for every singularity σ of X in Λ one has

$$\Lambda = \omega_X(q), \quad \text{for all } q \in W^u_X(\sigma) \setminus \{\sigma\}.$$

Obviously a compact invariant set with dense singular unstable branches is a transitive set.

THEOREM D. *Every Lyapunov stable singular hyperbolic set with dense singular unstable branches of $X \in \mathcal{X}^1(M)$ is an attractor of X.*

We prove Theorem D in §6.

4. *Proof of Theorem B*

In this section we shall prove Theorem B using Theorems C and D. For this we shall use the following notation. If A is a subset of M we denote by $\mathrm{Cl}(A)$ the closure of A and by $\mathrm{int}(A)$ the interior of A. When A is a submanifold we denote by $\dim(A)$ the dimension of A.

Given $X \in \mathcal{X}^1(M)$ we denote by $\mathrm{Sing}(X)$ the set of singularities of X and by $\mathrm{Per}(X)$ the union of the periodic orbits of X. If A is invariant we denote $\mathrm{Per}_X(A) = \mathrm{Per}(X) \cap A$ and $\mathrm{Sing}_X(A) = \mathrm{Sing}(X) \cap A$.

Observe that if $\sigma \in \mathrm{Sing}(X) \cap \mathrm{Cl}(\mathrm{Per}(X))$ then σ is neither a sink nor a source of X. Note also that as $\dim M = 3$, if σ is hyperbolic then either $\dim(W^u_X(\sigma)) = 1$ or $\dim(W^s_X(\sigma)) = 1$.

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Let us recall some properties for generic vector fields $X \in \mathcal{X}^1(M)$ proved elsewhere:

(a) X is *Kupka–Smale*, i.e. every periodic orbit and singularity of X is hyperbolic and the corresponding invariant manifolds intersect transversally [**5**]. In particular, $\mathrm{Sing}(X)$ is a finite set;
(b) $\Omega(X) = \mathrm{Cl}(\mathrm{Per}(X) \cup \mathrm{Sing}(X))$ (see [**20**]);
(c) $\mathrm{Cl}(W^u_X(\sigma))$ is Lyapunov stable for X, for all $\sigma \in \mathrm{Sing}(X)$ (see [**4**]);
(d) $\mathrm{Cl}(W^s_X(\sigma))$ is Lyapunov stable for $-X$, for all $\sigma \in \mathrm{Sing}(X)$ (see [**4**]);
(e) if $\sigma \in \mathrm{Sing}(X)$ and $\dim(W^u_X(\sigma)) = 1$ then $\omega_X(q)$ is Lyapunov stable for X, for all $q \in W^u_X(\sigma) \setminus \{\sigma\}$ (see [**4**]);
(f) if $\sigma \in \mathrm{Sing}(X)$ and $\dim(W^s_X(\sigma)) = 1$ then $\alpha_X(q)$ is Lyapunov stable for $-X$, for all $q \in W^s_X(\sigma) \setminus \{\sigma\}$ (see [**4**]).

LEMMA 4.1. *For generic vector fields $X \in \mathcal{X}^1(M)$, a transitive Lyapunov stable set with singularities Λ of X satisfying*

$$\dim(W^u_X(\sigma)) = 1, \quad \forall \sigma \in \mathrm{Sing}_X(\Lambda)$$

has dense singular unstable branches.

Proof. As X is generic we can assume that X satisfies the Properties (a)–(f) above (in particular, X is Kupka–Smale). Let Λ be a transitive Lyapunov stable set of X, $\sigma \in \mathrm{Sing}_X(\Lambda)$ and $q \in W^u_X(\sigma) \setminus \{\sigma\}$. As Λ is Lyapunov stable we have that $W^u_X(\sigma) \subset \Lambda$. In particular, $\omega_X(q) \subset \Lambda$. On the other hand, we have that $\dim(W^u_X(\sigma)) = 1$ by assumption. Then, $\omega_X(q)$ is Lyapunov stable by Property (e). As $\Lambda \cap \omega_X(q) \neq \emptyset$ and Λ is transitive, we conclude, by Lemma 2.1(3), that $\Lambda \subset \omega_X(q)$. Then $\omega_X(q) = \Lambda$, proving that Λ has dense singular unstable branches as desired. □

Another preliminary lemma is given below.

LEMMA 4.2. *Let $X \in \mathcal{X}^1(M)$ and $\sigma \in \mathrm{Sing}(X) \cap \mathrm{Cl}(\mathrm{Per}(X))$ be such that*

$$\dim(W^u_X(\sigma)) = 1.$$

If $\omega_X(q)$ is Lyapunov stable for X, for all $q \in W^u_X(\sigma) \setminus \{\sigma\}$, then $\mathrm{Cl}(W^u_X(\sigma))$ is transitive.

Proof. As $\dim(W^u_X(\sigma)) = 1$ we have that

$$W^u_X(\sigma) \setminus \{\sigma\} = O_X(q_1) \cup O_X(q_2)$$

for every q_1, q_2 belonging to different connected components of $W^u_X(\sigma) \setminus \{\sigma\}$. As $\sigma \in \mathrm{Cl}(\mathrm{Per}(X))$ we can assume that $q_1 \in \mathrm{Cl}(\mathrm{Per}(X))$ without loss of generality. On the other hand, $\omega_X(q_1)$ is Lyapunov stable for X by assumption. This fact and $q_1 \in \mathrm{Cl}(\mathrm{Per}(X))$ imply that $\sigma \in \omega_X(q_1)$. Thus, $W^u_X(\sigma) \subset \omega_X(q_1)$ by the Lyapunov stability of $\omega_X(q_1)$ once more. In particular, $q_1, q_2 \in \omega_X(q_1)$ and so $W^u_X(\sigma) \subset \omega_X(q_1)$. Taking closure we conclude that $\mathrm{Cl}(W^u_X(\sigma)) \subset \omega_X(q_1)$ proving that $\mathrm{Cl}(W^u_X(\sigma)) = \omega_X(q_1)$ and so $\mathrm{Cl}(W^u_X(\sigma))$ is transitive. □

The following theorem improves the main result of [**14**].

THEOREM 4.3. *For generic $X \in \mathcal{G}^1(M)$ every $\sigma \in \mathrm{Sing}(X) \cap \mathrm{Cl}(\mathrm{Per}(X))$ belongs to a singular hyperbolic attractor or a singular hyperbolic repeller of X.*

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Proof. Let $\sigma_0 \in \mathrm{Sing}(X) \cap \mathrm{Cl}(\mathrm{Per}(X))$ of a generic $X \in \mathcal{G}^1(M)$. By Property (a) σ_0 is hyperbolic, and so either $\dim(W^u_X(\sigma_0)) = 1$ or $\dim(W^s_X(\sigma_0)) = 1$. Assuming that $\dim(W^u_X(\sigma_0)) = 1$ we shall prove that $\Lambda = \mathrm{Cl}(W^u_X(\sigma_0))$ is a singular hyperbolic attractor of X. Similar arguments (reversing the flow direction of X) show that $\mathrm{Cl}(W^s_X(\sigma_0))$ is a singular hyperbolic repeller when $\dim(W^s_X(\sigma_0)) = 1$. Thus, suppose that $\dim(W^u_X(\sigma_0)) = 1$ and define $\Lambda = \mathrm{Cl}(W^u_X(\sigma_0))$. By Property (c) we have that Λ is Lyapunov stable for X. As $\sigma_0 \in \mathrm{Sing}_X(\Lambda) \cap \mathrm{Cl}(\mathrm{Per}(X))$ and $\dim(W^u_X(\sigma_0)) = 1$, Lemma 4.2 and Property (e) imply that Λ is transitive. Clearly Λ is non-trivial since σ_0 is neither a sink nor a source. In conclusion, Λ is a non-trivial transitive Lyapunov stable set of X. As X is generic, Theorem C implies that Λ is singular hyperbolic. By [**17**] (see also Theorem 5.4) we have $\dim(W^u_X(\sigma)) = 1$ for every $\sigma \in \mathrm{Sing}_X(\Lambda)$. We conclude that Λ has dense singular unstable branches by Lemma 4.1 since X is generic. Then, Λ is an attractor by Theorem D and the proof follows. □

Proof of Theorem B. For $X \in \mathcal{G}^1(M)$ we denote

$$\mathrm{Sing}^*(X) = \mathrm{Sing}(X) \cap \mathrm{Cl}(\mathrm{Per}(X)) = \{\sigma_1, \ldots, \sigma_k\}.$$

Assume that X is generic. Theorem 4.3 implies that for every $i = 1, \ldots, k$ there is a compact invariant set Λ_i of X such that $\sigma_i \in \Lambda_i$ and Λ_i is either a singular hyperbolic attractor or a singular hyperbolic repeller of X. Denote

$$H^* = \Omega(X) \setminus \bigcup_{i=1}^{k} \Lambda_i.$$

We claim that H^* splits into a finite disjoint union of hyperbolic basic sets. Indeed, we have that $H^* \setminus \mathrm{Sing}(X)$ is closed in M for, otherwise, there would exist $\sigma \in \mathrm{Sing}(X)$ and a sequence $x_n \in H^* \setminus \mathrm{Sing}(X)$ converging to σ. By Property (b) we would have $\sigma \in \mathrm{Cl}(\mathrm{Per}(X))$ since $\mathrm{Sing}(X)$ is a finite set. In particular, $\sigma \in \mathrm{Sing}^*(X)$ and so $\sigma \in \Lambda_i$ for some $i = 1, \ldots, k$. As Λ_i is either an attractor or a repeller we would have that $x_n \in \Lambda_i$ for n large, a contradiction since $x_n \in H^* = \Omega(X) \setminus \bigcup_{i=1}^{k} \Lambda_i$ for all n. This proves that $H^* \setminus \mathrm{Sing}(X)$ is closed in M. Applying [**22**] we obtain that H^* is a hyperbolic set of X. On the other hand, Property (b) implies that $H^* = \mathrm{Cl}(\mathrm{Per}_X(H^*)) \cup \mathrm{Sing}_X(H^*)$. Then, the claim follows from the spectral decomposition theorem (see [**18**]). This proves the claim.

By the claim $\Omega(X)$ splits into a disjoint union of compact invariant sets each being either a hyperbolic basic set or Λ_i for some i. As each Λ_i is either a singular hyperbolic attractor or a singular hyperbolic repeller for all i we conclude that X is singular Axiom A. As X is Kupka–Smale (Property (a)) it follows that X has no cycles. This proves the result.

5. *Proof of Theorem D*

Theorem D follows from the following propositions.

PROPOSITION 5.1. *Let Λ be a Lyapunov stable singular hyperbolic set with dense singular unstable branches of $X \in \mathcal{X}^r(M)$. If there is $q \in \Lambda$ such that $\omega_X(q)$ has no singularities, then Λ is an attractor of X.*

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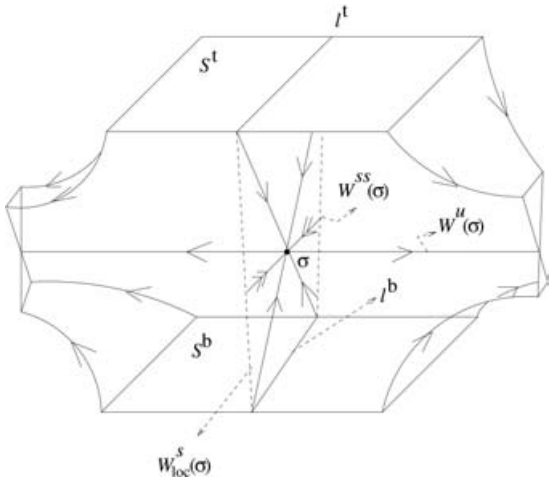


FIGURE 1. Lorenz-like singularity.

PROPOSITION 5.2. *Let Λ be a Lyapunov stable singular hyperbolic set with dense singular unstable branches of $X \in \mathcal{X}^r(M)$. Then, either Λ is an attractor of X or else there is $q \in \Lambda$ such that $\omega_X(q)$ has no singularities.*

5.1. *Proof of Proposition 5.1.* In this section we shall prove Proposition 5.1. Before we start with the proof, let us recall some results and notation given elsewhere.

The next definition is in [**16**].

Definition 5.3. A singularity σ of X is *Lorenz-like* if σ has three real eigenvalues $\lambda_1, \lambda_2, \lambda_3$ satisfying

$$\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1.$$

By invariant manifold theory it follows that the eigenspace associated with λ_2 is tangent to a one-dimensional invariant manifold of X denoted by $W^{ss}_X(\sigma)$. This manifold is called the *strong stable manifold* of σ. As usual we denote by $W^s_{\mathrm{loc}}(\sigma)$ a small ball in $W^s_X(\sigma)$ centered at σ. Similarly, we define $W^u_{\mathrm{loc}}(\sigma)$.

In the remainder of this section we let Λ be a transitive singular hyperbolic set of X.

THEOREM 5.4. [**16**] *Every singularity σ of X in Λ is Lorenz-like and satisfies*

$$\Lambda \cap W^{ss}_X(\sigma) = \{\sigma\}.$$

This result and the Hartman–Grobman Theorem [**5**] allow us to describe the flow of X around a singularity $\sigma \in \Lambda$ (see Figure 1). Indeed, σ has a two-dimensional stable manifold $W^s_X(\sigma)$, a strong stable manifold $W^{ss}_X(\sigma)$ and an unstable manifold $W^u_X(\sigma)$ (both one-dimensional).

It follows that $W^{ss}_X(\sigma)$ separates $W^s_{\mathrm{loc}}(\sigma)$ into two connected components, denoted the top and the bottom respectively. In one of these components, say the top one, we consider a

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cross section S^{t} together with a curve l^{t} as in Figure 1. Similarly we consider a cross section S^{b} and a curve l^{b} located in the bottom component of $W^s_{\mathrm{loc}}(\sigma)$, see Figure 1. Both S^{t} and S^{b} are homeomorphic to $[0,1]\times[0,1]$. S^{t} and S^{b} are chosen in such a way that l^{t} and l^{b} are contained in $W^s_X(\sigma)\setminus W^{ss}_X(\sigma)$. The positive flow lines of X starting at $S^{\mathrm{t}}\cup S^{\mathrm{b}}\setminus(l^{\mathrm{t}}\cup l^{\mathrm{b}})$ exit a small neighborhood of σ passing through the cusp region as indicated in Figure 1. The positive orbits starting at $l^{\mathrm{t}}\cup l^{\mathrm{b}}$ go directly to σ. We note that the boundary of S^{t} is formed by four curves, two of them transverse to l^{t} and two of them parallel to l^{t}. The union of the curves in the boundary of S^{t} which are transverse (parallel) to l^{t} is denoted by $\partial^{\mathrm{h}}S^{\mathrm{t}}$ ($\partial^{\mathrm{h}}S^{\mathrm{b}}$).

Definition 5.5. Let σ be a Lorenz-like singularity of X. The cross sections S^{t}, S^{b} introduced above are called *singular cross sections* associated with σ. The curves l^{t}, l^{b} are called *singular curves* of S^{t}, S^{b} respectively.

Remark 5.6. A first consequence of Theorem 5.4 is the following. Let U a neighborhood of Λ and $\sigma\in\Lambda$ be a singularity of X. Then there are singular cross sections S^{t}, S^{b} associated with σ such that $S^{\mathrm{t}}\cup S^{\mathrm{b}}\subset U$ and

$$\Lambda\cap(\partial^{\mathrm{h}}S^{\mathrm{t}}\cup\partial^{\mathrm{h}}S^{\mathrm{b}})=\emptyset.$$

Another property to be used later is given below. A hyperbolic set is *saddle-type* if $\dim E^s=\dim E^u=1$.

LEMMA 5.7. [**16**] *Every compact invariant non-singular set of X in Λ is a saddle-type hyperbolic set.*

Next we shall assume that Λ is Lyapunov stable. Denote by $T_\Lambda M=E^s_\Lambda\oplus E^c_\Lambda$ the splitting of Λ. It follows that E^s_Λ extends to an invariant contracting splitting $E^s_{U(\Lambda)}$ on a neighborhood $U(\Lambda)$ of Λ and $E^s_{U(\Lambda)}$ is integrable in $U(\Lambda)$ [**9**]. We extend E^c_Λ continuously to a subbundle $E^c_{U(\Lambda)}$ of $T_{U(\Lambda)}M$. *In what follows we fix such a neighborhood $U(\Lambda)$ of Λ.*

We denote by $\mathcal{F}^s$ the foliation obtained by integrating $E^s_{U(\Lambda)}$ and by $\mathcal{F}^s(x)$ the leaf of $\mathcal{F}^s$ passing through x. Denote by D^2 the two-disc in $\mathbb{R}^2$. If S is contained in an embedding $D^2\to M$ transverse to X, we denote by $\mathcal{F}^s_S$ the foliation of S obtained by projecting $\mathcal{F}^s(x)$ into S along the flow of X.

The next definition follows closely [**3**].

Definition 5.8. An *s-rectangle* of Λ is a closed set $S\subset M$, contained in the image of an embedding $D^2\to M$ transverse to X, such that there is a homeomorphism $h:[0,1]\times[0,1]\to S$ mapping $\{s\}\times[0,1]$, for all $s\in[0,1]$, into leaves of $\mathcal{F}^s_S$.

If $S=h([0,1]\times[0,1])$ is a an s-rectangle, we define the *interior* and the *boundary* of S by $\mathrm{int}(S)=h((0,1)\times(0,1))$ and $\partial S=h([0,1]\times\{0,1\}\cup\{0,1\}\times[0,1])$ respectively. It follows that ∂S is formed by four segments

$$\partial^{\mathrm{h}}S=[0,1]\times\{0,1\},\quad \partial^{\mathrm{v}}S=\{0,1\}\times[0,1].$$

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An s-rectangle is always assumed to be contained in $U(\Lambda)$. We define $\operatorname{diam}(S)$, the diameter of S, by

$$\operatorname{diam}(S) = \sup_{x \in \partial^{\mathrm{v}} S} \inf\{\operatorname{dist}(x, y) : x, y \text{ are not in the same component of } \partial^{\mathrm{v}} S\},$$

where dist is the distance in M.

LEMMA 5.9. *Let Λ be a Lyapunov stable singular hyperbolic set with dense singular unstable branches of X. If Λ exhibits an s-rectangle S satisfying*

$$\operatorname{int}(S) \cap \mathcal{F}^s_S(x) \cap \Lambda \neq \emptyset, \quad \text{for all } x \in \operatorname{int}(S), \tag{1}$$

then Λ is an attractor of X.

Proof. By Lemma 2.2 it suffices to prove that if $x_n \in M$ is a sequence converging to $p \in \Lambda$, then $\omega_X(x_n) \subset \Lambda$ for every n large. By contradiction, assume that $\omega_X(x_n)$ is not contained in Λ. We have the following two cases.

Case 1: $\omega_X(p)$ has no singularities. Then $\omega_X(p)$ is a saddle-type hyperbolic set by Lemma 5.7. Choose $y \in \omega_X(p)$. Then, $W^u_X(y)$ is well defined and has dimension two. Let S_y be a cross section of X containing y. As Λ is Lyapunov stable we have that $W^u_X(y) \subset \Lambda$. Choosing S_y small enough, there is $\epsilon > 0$ such that for all $x \in V_y = X_{(0,\epsilon)} S_y$ there is $z \in W^u_X(y) \subset \Lambda$ so that $x \in W^s_X(z)$. As $y \in \omega_X(p)$ we have that the positive orbit of p intersects V_y. This implies that the positive orbit of x_n also intersects V_y for n large because $x_n \to p$. In particular, for every n large there is $z_n \in \Lambda$ such that $x_n \in W^s_X(z_n)$. Thus, $\omega_X(x_n) = \omega_X(z_n) \subset \Lambda$, contradicting the assumption that $\omega_X(x_n)$ is not contained in Λ.

Case 2: $\omega_X(p)$ contains a singularity σ of X. As $p \in \Lambda$ we have that $\sigma \in \Lambda$. As $x_n \to p$ and $\omega_X(x_n)$ is not contained in Λ, there is $q \in W^u_X(\sigma) \setminus \{\sigma\}$ such that the positive orbit of x_n passes close to q for n large. As Λ has dense singular unstable branches we have $\Lambda = \omega_X(q)$, and so the positive orbit of q intersects $\operatorname{int}(S)$. Since S is an s-rectangle of Λ satisfying (1), we conclude as before that for every n large there is $z_n \in \Lambda$ such that $x_n \in W^s_X(z_n)$. This contradicts once more the assumption that $\omega_X(x_n)$ is not contained in Λ. This completes the proof. □

COROLLARY 5.10. *Let Λ be a Lyapunov stable singular hyperbolic set with dense singular unstable branches. Suppose that Λ is not an attractor. If $S^{\mathrm{t}}_0, S^{\mathrm{b}}_0$ are singular cross sections associated with a singularity $\sigma \in \Lambda$ and*

$$\Lambda \cap (\partial^{\mathrm{h}} S^{\mathrm{t}}_0 \cup \partial^{\mathrm{h}} S^{\mathrm{b}}_0) = \emptyset,$$

then there are singular cross sections $S^{\mathrm{t}}, S^{\mathrm{b}}$ of σ contained in $S^{\mathrm{t}}_0, S^{\mathrm{b}}_0$ respectively, such that

$$\Lambda \cap (\partial S^{\mathrm{t}} \cup \partial S^{\mathrm{b}}) = \emptyset.$$

Proof. Let Λ be a singular hyperbolic set of X and $S^{\mathrm{t}}_0, S^{\mathrm{b}}_0$ be singular cross sections associated with $\sigma \in \Lambda$ as in the statement. Let F be a connected component of $\partial^{\mathrm{v}} S^{\mathrm{t}}_0$. If there is no $F^{\mathrm{t}} \in \mathcal{F}^s_S$ close to F satisfying $\Lambda \cap F^{\mathrm{t}} = \emptyset$, then one could construct an s-rectangle $S \subset S^{\mathrm{t}}_0$ close to F satisfying (1). As Λ has dense singular unstable branches

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it would follow from Lemma 5.9 that Λ is an attractor. This is a contradiction because we are assuming that Λ is not an attractor. This proves the existence of a leaf F^{t} of $\mathcal{F}^s_S$ arbitrarily close to F such that $\Lambda \cap F^{\mathrm{t}} = \emptyset$. Similarly, if G denotes another connected component of $\partial^{\mathrm{v}} S^{\mathrm{t}}_0$, there is a leaf $G^{\mathrm{t}} \in \mathcal{F}^s_S$ close to G such that $\Lambda \cap G^{\mathrm{t}} = \emptyset$. Let S^{t} be the s-rectangle of Λ such that $S^{\mathrm{t}} \subset S^{\mathrm{t}}_0$, $\partial^{\mathrm{h}} S^{\mathrm{t}} \subset \partial^{\mathrm{h}} S^{\mathrm{t}}_0$ and $\partial^{\mathrm{v}} S^{\mathrm{t}} = F^{\mathrm{t}} \cup G^{\mathrm{t}}$. Then S^{t} satisfies the required properties. We construct S^{b} in a similar way. This finishes the proof. □

Proof of Proposition 5.1. Let Λ be a Lyapunov stable singular hyperbolic set with dense singular unstable branches of X. Suppose that there is $q \in \Lambda$ such that $\omega_X(q) \cap \mathrm{Sing}_X(\Lambda) = \emptyset$. Then, by Lemma 5.7 $H = \omega_X(q)$ is a saddle-type hyperbolic set of X. Choose $y \in H$. As H is saddle-type we have that y has a non-trivial two-dimensional unstable manifold $W^u_X(y)$. As Λ is Lyapunov stable we have $W^u_X(y) \subset \Lambda$. Using $W^u_X(y)$ as in the proof of Lemma 5.9 (Case 1), we construct an s-rectangle $S = S_y$ of Λ satisfying (1). So, Λ is an attractor by Lemma 5.9 and the result follows. □

6. *Proof of Proposition 5.2*

Let us introduce some definitions and preliminary facts before the proof.

Notation 6.1. Let X be a C^r vector field and Λ a transitive Lyapunov stable singular hyperbolic set of X. Let $\mathcal{S} = \{R_1, \ldots, R_r\}$ be a finite disjoint collection of s-rectangles of Λ. We shall use the following notation:

- $\mathcal{S} = \bigcup_{j=1}^r R_j$;
- $\sigma_{\mathcal{S},\Lambda} = \{q \in \Lambda : X_{\mathbb{R}}(q) \cap \mathcal{S} = \emptyset\}$;
- $l_{\mathcal{S},\Lambda} = \{q \in \Lambda \cap \mathcal{S} : X_t(q) \notin \mathcal{S}, \text{for all } t > 0\}$;
- $\mathrm{int}(\mathcal{S}) = \bigcup_{j=1}^r \mathrm{int}(R_j)$;
- $\partial\mathcal{S} = \bigcup_{j=1}^r \partial R_j$, $\partial^{\mathrm{v}}\mathcal{S} = \bigcup_{j=1}^r \partial^{\mathrm{v}} R_j$ and $\partial^{\mathrm{h}}\mathcal{S} = \bigcup_{j=1}^r \partial^{\mathrm{h}} R_j$.

We shall denote

$$\mathrm{Sing}_X(\Lambda) = \{\sigma_1, \ldots, \sigma_k\},$$

the set of singularities of X in Λ.

Definition 6.2. Given $X \in \mathcal{X}^r(M)$ and a transitive Lyapunov stable singular hyperbolic set Λ of X, a *singular partition* of Λ is a disjoint collection

$$\mathcal{S} = \{S^{\mathrm{t}}_1, S^{\mathrm{b}}_1, \ldots, S^{\mathrm{t}}_k, S^{\mathrm{b}}_k\}$$

of s-rectangles of Λ satisfying (see Figure 2):

(1) for every $i = 1, \ldots, k$ the s-rectangles S^{t}_i, S^{b}_i are singular cross sections associated with σ_i (recall Definition 5.5);
(2) $\Lambda \cap \partial\mathcal{S} = \emptyset$;
(3) $\sigma_{\mathcal{S},\Lambda} = \mathrm{Sing}_X(\Lambda)$.

Remark 6.3. Note that Properties 1, 2 and 3 above imply:

(4) for every $i = 1, \ldots, k$ if $l^{\mathrm{t}}_i, l^{\mathrm{b}}_i$ denote the singular curves of S^{t}_i, S^{b}_i respectively, then $l^{\mathrm{t}}_i, l^{\mathrm{b}}_i$ are leaves of $\mathcal{F}^s_{\mathcal{S}}$ and

$$l_{\mathcal{S},\Lambda} = \Lambda \cap l,$$

where $l = l_{\mathcal{S}} = \bigcup_{i=1}^k (l^{\mathrm{t}}_i \cup l^{\mathrm{b}}_i)$.

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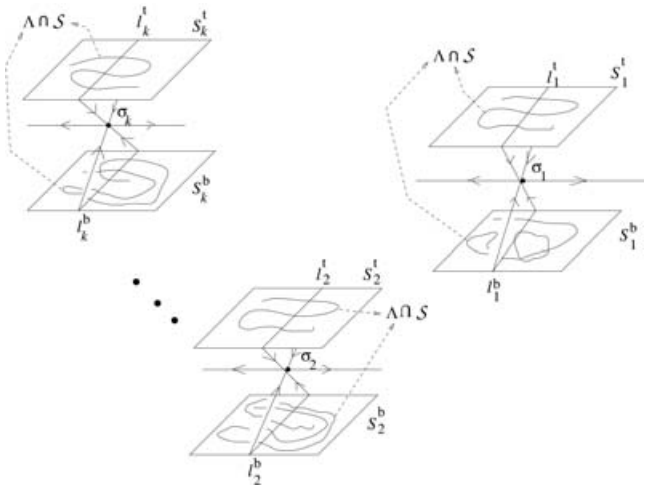


FIGURE 2. Singular partition.

Given a singular partition $\mathcal{S}$ we set

$$\operatorname{diam}(\mathcal{S}) = \sup\{\operatorname{diam}(S) : S \in \mathcal{S}\}.$$

A singular partition $\mathcal{S}$ of Λ defines a return map

$$\Pi : \operatorname{Dom}(\Pi) \subset \mathcal{S} \to \mathcal{S}$$

by

$$\Pi(x) = X_{t(x)}(x),$$

where $\operatorname{Dom}(\Pi)$ denotes the domain of Π and $t(x)$ denotes the return time of x. Clearly Π preserves $\mathcal{F}^s_{\mathcal{S}}$ and $\operatorname{Dom}(\Pi) \cap l = \emptyset$.

LEMMA 6.4. *Let Λ be a Lyapunov stable singular hyperbolic set of X and $\mathcal{S}$ be a singular partition of Λ with return map Π. Then there is a neighborhood O of $\Lambda \cap \mathcal{S}$ in $\mathcal{S}$ such that if $B \subset O$ and $n \in \mathbb{N}$ satisfy*

$$B, \Pi(B), \Pi^2(B), \ldots, \Pi^{n-1}(B) \subset \operatorname{Dom}(\Pi),$$

then $\Pi^n(B) \setminus \operatorname{Dom}(\Pi) \subset l$.

Proof. By Property 4 in Remark 6.3 we have $(\Lambda \cap \mathcal{S}) \setminus l \subset \operatorname{Dom}(\Pi)$ and $\Pi((\Lambda \cap \mathcal{S}) \setminus l) \subset \operatorname{int}(\mathcal{S})$. Definition 6.2(2) and (3) imply that we can choose for every $i \in \{1, \ldots, k\}$ and $* = t, b$ an s-rectangle $V_i^* \subset \operatorname{int}(S_i^*)$ satisfying

- $l \cap V_i^*$ is a leaf of $\mathcal{F}^s_{V_i^*}$;
- $l \cap \partial^{\mathrm{v}} V_i^* = \emptyset$;
- $V_i^* \setminus l \subset \operatorname{Dom}(\Pi)$.

It follows that

$$\Lambda_0 = (\Lambda \cap \mathcal{S}) \Big\backslash \bigcup_{i \in \{1,\ldots,k\}, * = t,b} \operatorname{int}(V_i^*)$$

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is a compact subset of $\mathcal{S}$ contained in $(\Lambda \cap \mathcal{S}) \setminus l$. Thus,

$$\Pi(\Lambda_0) \subset \Pi(\Lambda \setminus l) \subset \operatorname{int}(\mathcal{S}).$$

As $\Lambda_0 \cap \partial\mathcal{S} = \emptyset$, the tubular flow box theorem [**5**] implies that for all $x \in \Lambda_0$ there is a neighborhood $V_x \subset \operatorname{int}(\mathcal{S})$ of x such that $V_x \subset \operatorname{Dom}(\Pi)$ and $\Pi(V_x) \subset \operatorname{int}(\mathcal{S})$. The family $\{V_x : x \in \Lambda_0\}$ is an open covering of Λ_0 and so it has a finite subcovering $\{V_{x_1}, \ldots, V_{x_r}\}$ by compactness.

Define

$$O_0 = \Big(\bigcup_{j=1}^{r} V_{x_j}\Big) \cup \Big(\bigcup \{\operatorname{int}(V_i^*) : i \in \{1, \ldots, k\}, * = t, b\}\Big).$$

Then, O_0 is a neighborhood of $\Lambda \cap \mathcal{S}$ in $\mathcal{S}$ satisfying $O_0 \setminus l \subset \operatorname{Dom}(\Pi)$ and

$$\begin{aligned} O_0 \setminus \operatorname{Dom}(\Pi) &= \{(O_0 \setminus l) \cup l\} \setminus \operatorname{Dom}(\Pi) = \{(O_0 \setminus l) \setminus \operatorname{Dom}(\Pi)\} \cup \{l \setminus \operatorname{Dom}(\Pi)\} \\ &\subset \{\operatorname{Dom}(\Pi) \setminus \operatorname{Dom}(\Pi)\} \cup \{l \setminus \operatorname{Dom}(\Pi)\} \subset \emptyset \cup l = l. \end{aligned}$$

So,

$$O_0 \setminus \operatorname{Dom}(\Pi) \subset l.$$

On the other hand, as Λ is Lyapunov stable, there is a neighborhood $O \subset O_0$ of $\Lambda \cap \mathcal{S}$ such that if $x \in O$ and $t > 0$ satisfy $X_t(x) \in \mathcal{S}$, then $X_t(x) \in O_0$. We claim that O satisfies the required properties.

Indeed, let $B \subset O$ and $n \in \mathbb{N}$ satisfying

$$B, \Pi(B), \Pi^2(B), \ldots, \Pi^{n-1}(B) \subset \operatorname{Dom}(\Pi)$$

and $y \in \Pi^n(B) \setminus \operatorname{Dom}(\Pi)$. It follows that $y = \Pi^n(x)$ for some $x \in B$. Denoting by $t(\cdot)$ the return time we have

$$\Pi^n(x) = X_{t(x)+t(\Pi(x))+\cdots+t(\Pi^{n-1}(x))}(x).$$

We conclude that $y \in O_0$ since $t = t(x) + t(\Pi(x)) + \cdots + t(\Pi^{n-1}(x)) > 0$. As $y \notin \operatorname{Dom}(\Pi)$ we conclude that $y \in O_0 \setminus \operatorname{Dom}(\Pi)$. Thus, $\Pi^n(B) \setminus \operatorname{Dom}(\Pi) \subset O_0 \setminus \operatorname{Dom}(\Pi) \subset l$, as desired. □

We shall denote by TM the tangent space of M. It follows that TM is endowed with a metric $\|\cdot\|$ induced by an inner product $\langle , \rangle$.

If $x \in M$ we denote by $\angle(v_x, w_x)$ the tangent of the angle between $v_x, w_x \in T_xM$. If L_x is a linear subspace of T_xM, we define

$$\angle(v_x, L_x) = \inf_{w_x \in L_x} \angle(v_x, w_x).$$

Given $\alpha > 0$ we define

$$C_\alpha(L_x) = \{v_x \in T\mathcal{S}_x : \angle(v_x, L_x) \leq \alpha\}.$$

We shall use the linear Poincaré flow [**6**]. If x is a regular point of X (i.e. $X(x) \neq 0$), we denote by $N_x = \{v_x \in T_xM : \langle v_x, X(x)\rangle = 0\}$ the orthogonal complement of $X(x)$

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in T_xM. Denote by $O_x : T_xM \to N_x$ the orthogonal projection onto N_x. For every $t \in \mathbb{R}$ we define $P_x^t : N_x \to N_{X_t(x)}$ by

$$P_x^t = O_{X_t(x)} \circ DX_t(x).$$

It follows that $P = \{P_x^t : t \in \mathbb{R}, X(x) \neq 0\}$ satisfies the cocycle relation

$$P_x^{s+t} = P_{X_t(x)}^s \circ P_x^s,$$

for every $t, s \in \mathbb{R}$. The parameterized family P is called the *linear Poincaré flow* of X.

We denote by $\mathrm{vol}(v_x, w_x)$ the area of the parallelogram in T_xM generated by $v_x, w_x \in T_xM$. As M is a compact manifold, there is a constant $V \geq 1$ such that $V^{-1} \leq \mathrm{vol}(v_x, w_x) \leq V$, for all $x \in M$, $v_x, w_x \in T_xM$ satisfying $\|v_x\| = \|w_x\| = 1$ and $\langle v_x, w_x \rangle = 0$. For simplicity we shall assume that $V = 1$. In other words,

$$\mathrm{vol}(v_x, w_x) = \|v_x\| \|w_x\|,$$

for all $x \in M$, $v_x, w_x \in T_xM$ with $\langle v_x, w_x \rangle = 0$.

In addition,

$$\mathrm{vol}(v_x, X(x)) = \|O_x v_x\| \|X(x)\|,$$

for all $x \in M$ regular, $v_x \in T_xM$. In particular,

$$\mathrm{vol}(DX_t(x)v_x, X(X_t(x))) = \|P_x^t(v_x)\| \|X(X_t(x))\|, \tag{2}$$

for all $x \in M$ regular, $t \in \mathbb{R}$, $v_x \in N_x$.

Recall that if Λ is a singular hyperbolic set of X with (K, λ)-splitting $T_\Lambda M = E_\Lambda^s \oplus E_\Lambda^c$, then

$$J_t^c(x) \geq K e^{\lambda t},$$

for all $x \in \Lambda$, $t \geq 0$, where $J_t^c(x)$ is the absolute value of the determinant of the linear map $DX_t(x)/E_x^c$. So, if Λ is a singular hyperbolic set as above one has

$$\mathrm{vol}(DX_t(x)v_x^c, DX_t(x)w_x^c) \geq K e^{\lambda t} \mathrm{vol}(v_x^c, w_x^c),$$

for all $x \in \Lambda$, $t \geq 0$, $v_x^c, w_x^c \in E_x^c$.

Recall that $U(\Lambda)$ denotes a neighborhood of Λ where the splitting $E_\Lambda^s \oplus E_\Lambda^c$ extends to $E_{U(\Lambda)}^s \oplus E_{U(\Lambda)}^c$.

LEMMA 6.5. *Let Λ be a Lyapunov stable singular hyperbolic set of X. Then for every $\alpha \in (0, 1]$ there are a neighborhood $U_\alpha \subset U(\Lambda)$ of Λ and constants $T_\alpha, K_\alpha, \lambda_\alpha > 0$ such that:*

(1) *if $x \in U_\alpha$ and $t \geq T_\alpha$, then*

$$DX_t(x)(C_\alpha(E_x^c)) \subset C_{\alpha/2}(E_{X_t(x)}^c);$$

(2) *if $x \in U_\alpha$ is regular, $X(x) \in C_\alpha(E_x^c)$, $t \geq T_\alpha$ and $v_x \in C_\alpha(E_x) \cap N_x$, then*

$$\|P_x^t(v_x)\| \|X(X_t(x))\| \geq K_\alpha e^{\lambda_\alpha t} \|v_x\| \|X(x)\|.$$

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Proof. Let Λ and $\alpha \in (0, 1]$ be as in the statement. As mentioned above $T_{U(\Lambda)}M = E^s_{U(\Lambda)} \oplus E^c_{U(\Lambda)}$ denotes the extension of the (K, λ)-splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ of Λ to a neighborhood $U(\Lambda)$ of Λ. Let π^s the projection of T_Λ on E^s_Λ and π^c be the projection of T_Λ on E^c_Λ. Denote $v_x = v^s_x + v^c_x \in E^s_x \oplus E^c_x = T_xM$, for all $x \in U(\Lambda)$, for all $v_x \in T_xM$. In other words, $v^s_x = \pi^s(v_x)$ and $v^c_x = \pi^c(v_x)$.

As E^s_Λ (K, λ)-dominates E^c_Λ we have that

$$\|DX_t(x)/E^s_x\| \le K^{-1}e^{-\lambda t}m(DX_t(x)/E^c_x), \tag{3}$$

for all $x \in \Lambda$, $t \ge 0$.

Fix $R > 4$ such that

$$\frac{K}{R} < 1. \tag{4}$$

Choosing $T^1 = T^1_\alpha > 0$ large enough one has

$$\|DX_{T^1}(x)/E^s_x\| \le \frac{K\alpha}{2R}m(DX_t(x)/E^c_x), \tag{5}$$

for all $x \in \Lambda$, $t \ge 0$.

Since $E^s_\Lambda \oplus E^c_\Lambda$ is invariant we have $\pi^{s(c)}_{X_t(x)} \circ DX_t(x) = DX_t(x) \circ \pi^{s(c)}_x$, and so

$$\angle(DX_t(x)v_x, E^c_{X_t(x)}) = \frac{\|DX_t(x)v^s_x\|}{\|DX_t(x)v^c_x\|}, \quad \text{for all } x \in \Lambda, t \ge 0. \tag{6}$$

Recall that $\angle$ denotes the tangent of the angle. The inequality (5) and the last equality imply that

$$\angle(DX_{T^1}(x)v_x, E^c_{X_{T^1}(x)}) \le \frac{K\alpha}{2R}, \quad \text{for all } x \in \Lambda, v_x \in C_\alpha(E^c_x).$$

So,

$$DX_{T^1}(x)(C_\alpha(E^c_x)) \subset C_{K\alpha/2R}(E^c_{X_{T^1}}(x)), \quad \text{for all } x \in \Lambda.$$

Choose a neighborhood $U^1 = U^1_\alpha \subset U(\Lambda)$ of Λ sufficiently close to Λ such that

$$DX_{T^1}(x)(C_\alpha(E^c_x)) \subset C_{K\alpha/R}(E^c_{X_{T^1}}(x)), \quad \text{for all } x \in U^1. \tag{7}$$

On the other hand, using (3) we get

$$\frac{\|DX_t(x)v^s_x\|}{\|v^s_x\|} \le K^{-1}e^{-\lambda t}\frac{\|DX_t(x)v^c_x\|}{\|v^c_x\|},$$

and so

$$\frac{\|DX_t(x)v^s_x\|}{\|DX_t(x)v^c_x\|} \le K^{-1}e^{-\lambda t}\frac{\|v^s_x\|}{\|v^c_x\|} = K^{-1}e^{-\lambda t}\angle(v_x, E^c_x).$$

So, by (6), we get

$$\angle(DX_r(x)v_x, E^c_{X_r(x)}) \le K^{-1}e^{-\lambda r}\angle(v_x, E^c_x) \le K^{-1}\angle(v_x, E^c_x),$$

for all $x \in \Lambda$, $r \in [0, T^1]$, $v_x \in T_xM$. This implies that

$$DX_r(x)(C_{K\alpha/R}(E^c_x)) \subset C_{\alpha/R}(E^c_{X_r(x)}), \quad \text{for all } x \in \Lambda, r \in [0, T^1].$$

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Choose a neighborhood $V^2 = V^2_\alpha \subset U^1$ of Λ sufficiently close to Λ such that

$$DX_r(x)(C_{K\alpha/R}(E^c_x)) \subset C_{2\alpha/R}(E^c_{X_r(x)}), \quad \text{for all } x \in V^2, r \in [0, T^1]. \tag{8}$$

As Λ is Lyapunov stable there is a neighborhood $U^2 \subset V^2$ of Λ such that

$$X_t(U^2) \subset V^2, \quad \text{for all } t \geq 0.$$

Now, let $x \in U^2$ and $t \geq T^1$ be given. Then $t = nT^1 + r$ for some integer $n \geq 1$ and some $r \in [0, T^1]$. Thus,

$$DX_t(x)(C_\alpha(E^c_x)) = DX_{nT^1+r}(x)(C_\alpha(E^c_x)) = DX_r(X_{nT^1}(x))(DX_{nT^1}(x)(C_\alpha(E^c_x))). \tag{9}$$

Using (7) and (9) recursively, that $K\alpha/R < \alpha$ and that $n \geq 1$, we obtain

$$\begin{aligned}
&DX_{nT^1}(x)(C_\alpha(E^c_x)) \\
&\quad = DX_{(n-1)T^1}(X_{T^1}(x))(DX_{T^1}(x)C_\alpha(E^c_x)) \\
&\quad \subset DX_{(n-1)T^1}(X_{T^1}(x))(C_{K\alpha/R}(E^c_{X_{T^1}(x)})) \subset DX_{(n-1)T^1}(X_{T^1}(x))(C_\alpha(E^c_{X_{T^1}(x)})) \\
&\quad \subset \cdots \subset DX_{T^1}(X_{(n-1)T^1}(x))(C_\alpha(E^c_{X_{(n-1)T^1}(x)})) \subset C_{K\alpha/R}(E^c_{X_{nT^1}(x)}).
\end{aligned}$$

Hence

$$DX_{nT^1}(x)(C_\alpha(E^c_x)) \subset C_{K\alpha/R}(E^c_{X_{nT^1}(x)}).$$

Applying $DX_r(X_{nT^1}(x))$ to both sides of the last expression, substituting in (9) and using (8) we obtain

$$DX_t(x)(C_\alpha(E^c_x)) \subset DX_r(X_{nT^1}(x))(C_{K\alpha/R}(E^c_{X_{nT^1}(x)})) \subset C_{2\alpha/R}(E^c_{X_t(x)}).$$

As $R > 4$ we have $2\alpha/R < \alpha/2$ and so

$$DX_t(x)(C_\alpha(E^c_x)) \subset C_{\alpha/2}(E^c_{X_t(x)}),$$

for all $x \in U^2$, $t \geq T^1$, proving (1) of Lemma 6.5.

Throughout we fix the neighborhood U^2 of Λ and the constant $T^1 > 0$ obtained above.

As E^c_Λ is (K, λ)-volume expanding we have

$$\operatorname{vol}(DX_t(x)v^c_x, DX_t(x)w^c_x) \geq Ke^{\lambda t} \operatorname{vol}(v^c_x, w^c_x),$$

for all $x \in \Lambda$, $t \geq 0$, $v^c_x, w^c_x \in E^c_x$.

Clearly there is $L > 1$ such that

$$L^{-1} \operatorname{vol}(v^c_x, w^c_x) \leq \operatorname{vol}(v_x, w_x) \leq L \operatorname{vol}(v^c_x, w^c_x),$$

for all $x \in \Lambda$, $v_x, w_x \in C_\alpha(E^c_x)$, $\alpha \in (0, 1]$. Applying the last two relations and the invariance of $E^s_\Lambda \oplus E^c_\Lambda$ we obtain

$$\begin{aligned}
\operatorname{vol}(DX_t(x)v_x, DX_t(x)w_x) &\geq L^{-1} \operatorname{vol}(DX_t(x)v^c_x, DX_t(x)w^c_x) \\
&\geq L^{-1}Ke^{\lambda t} \operatorname{vol}(v^c_x, w^c_x) \geq L^{-2}Ke^{\lambda t} \operatorname{vol}(v_x, w_x),
\end{aligned}$$

for all $x \in \Lambda$, $t \geq T^1$, $v_x, w_x \in C_\alpha(E^c_x)$ (note that $DX_t(x)v_x, DX_t(x)w_x \in C_\alpha(E^c_{X_t(x)})$ since $t \geq T^1$).

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Choose $S > 0$ large so that

$$\frac{S}{L^{-2}K} > 1. \tag{10}$$

It follows that there is $T^2 = T^2_\alpha > T^1$ such that

$$\operatorname{vol}(DX_{T^2}(x)v_x, DX_{T^2}(x)w_x) \geq \frac{2S}{L^{-2}K} \operatorname{vol}(v_x, w_x),$$

for all $x \in \Lambda$, $v_x, w_x \in C_\alpha(E^c_x)$. In particular,

$$\inf\{\operatorname{vol}(DX_{T^2}(x)v_x, DX_{T^2}(x)w_x) : x \in \Lambda, v_x, w_x \in C_\alpha(E^c_x), \\ \|v_x\| = \|w_x\| = 1, \langle v_x, w_x\rangle = 0\} \geq \frac{2S}{L^{-2}K}.$$

Since Λ is compact there is a neighborhood $V^3 = V^3_\alpha \subset U^2$ of Λ so that

$$\inf\{\operatorname{vol}(DX_{T^2}(x)v_x, DX_{T^2}(x)w_x) : x \in V^3, v_x, w_x \in C_\alpha(E^c_x), \\ \|v_x\| = \|w_x\| = 1, \langle v_x, w_x\rangle = 0\} \geq \frac{S}{L^{-2}K}.$$

Then,

$$\operatorname{vol}(DX_{T^2}(x)v_x, DX_{T^2}(x)w_x) \geq \frac{S}{L^{-2}K} \operatorname{vol}(v_x, w_x), \tag{11}$$

for all $x \in V^3$, $v_x, w_x \in C_\alpha(E^c_x)$ with $\langle v_x, w_x\rangle = 0$. As Λ is Lyapunov stable there is a neighborhood $U^3 \subset V^3$ of Λ such that

$$X_t(U^3) \subset V^3, \qquad \text{for all } t \geq 0.$$

We have

$$\|P^{nT^2}_x v_x\| \|X(X_{nT^2}(x))\| = \|P^{T^2}_{X_{(n-1)T^2}(x)}(P^{(n-1)T^2}_x v_x)\| \|X(X_{T^2}(X_{(n-1)T^2}(x)))\|.$$

Call $z = X_{(n-1)T^2}(x)$ and $v_z = P^{(n-1)T^2}_x v_x$. From the last equality, using $X(X_{nT^2}(x)) = DX_{nT^2}(x)(X(x))$, v_z is orthogonal to z, and combining (2) with (11) we get

$$\begin{aligned}
&\|P^{nT^2}_x v_x\| \|X(X_{nT^2}(x))\| \\
&\quad = \|P^{T^2}_z v_z\| \|X(X_{T^2}(z))\| \\
&\quad = \operatorname{vol}(DX_{T^2}(z)v_z, X(X_{T^2}(z))) = \operatorname{vol}(DX_{T^2}(z)v_z, DX_{T^2}(z)(X(z)) \\
&\quad \geq \frac{S}{L^{-2}K} \operatorname{vol}(v_z, X(z)) = \frac{S}{L^{-2}K} \operatorname{vol}(P^{(n-1)T^2}_x v_x, X(X_{(n-1)T^2}(x))).
\end{aligned}$$

Thus,

$$\|P^{nT^2}_x v_x\| \|X(X_{nT^2}(x))\| \geq \left\{\frac{S}{L^{-2}K}\right\}^n \|v_x\| \|X(x)\|, \tag{12}$$

for all $x \in U^3$ regular with $X(x) \in C_\alpha(E^c_x)$, $n \in \mathbb{N}$, $v_x \in C_\alpha(E^c_x) \cap N_x$ (recall that N_x denotes the orthogonal complement of $X(x)$ in $T_x M$).

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On the other hand,

$$\operatorname{vol}(DX_r(x)v_x, DX_r(x)w_x) \geq L^{-2}K \operatorname{vol}(v_x, w_x),$$

for all $x \in \Lambda$, $r \in [0, T^2]$, $v_x, w_x \in C_\alpha(E^c_x)$.

As before there is a neighborhood $V^4 = V^4_\alpha \subset U^3$ of Λ such that

$$\operatorname{vol}(DX_r(x)v_x, DX_r(x)w_x) \geq \frac{L^{-2}K}{2} \operatorname{vol}(v_x, w_x), \tag{13}$$

for all $x \in V^4$, $v_x, w_x \in C_\alpha(E^c_x)$ with $\langle v_x, w_x \rangle = 0$, $r \in [0, T^2]$. As Λ is Lyapunov stable there is a neighborhood $U^4 \subset V^4$ of Λ such that

$$X_t(U_4) \subset V^4, \quad \text{for all } t \geq 0.$$

Now, let $x \in U^4$ be regular with $X(x) \in C_\alpha(E^c_x)$, $t \geq T^2$ and $v_x \in C_\alpha(E^c_x) \cap N_x$. Then $t = nT^2 + r$ for some integer $n \geq 1$ and some $r \in [0, T^2]$.

Applying (12) and (13), and using (2) and (10) we obtain

$$\begin{aligned}
\|P^{\mathrm{t}}_x v_x\| \|X(X_t(x))\| &= \|P^r_{X_{nT^2}(x)} P^{nT^2}_x v_x\| \|DX_r(X_{nT^2}(x))X(X_{nT^2}(x))\| \\
&= \operatorname{vol}(DX_r(X_{nT^2}(x))P^{nT^2}_x v_x, DX_r(X_{nT^2}(x))X(X_{nT^2}(x))) \\
&\geq \frac{L^{-2}K}{2} \operatorname{vol}(P^{nT^2}_x v_x, X(X_{nT^2}(x))) \\
&= \frac{L^{-2}K}{2} \|P^{nT^2}_x v_x\| \|X(X_{nT^2}(x))\| \\
&\geq \left(\frac{L^{-2}K}{2}\right) \left\{\frac{SN}{L^{-2}K}\right\}^n \|v_x\| \|X(x)\| \\
&= \left(\frac{L^{-2}K}{2}\right) \left(\frac{S}{L^{-2}K}\right)^{-r/T^2} \left\{\left(\frac{S}{L^{-2}K}\right)^{1/T^2}\right\}^{\mathrm{t}} \|v_x\| \|X(x)\| \\
&\geq \left(\frac{L^{-4}K^2}{2S}\right) \left\{\left(\frac{S}{L^{-2}K}\right)^{1/T^2}\right\}^{\mathrm{t}} \|v_x\| \|X(x)\|.
\end{aligned}$$

Thus, choosing $U_\alpha = U^4$, $T_\alpha = T^2$, $K_\alpha = L^{-4}K^2/2S$ and $\lambda_\alpha = \ln(S/L^{-2}K)^{1/T^2} > 0$ we obtain (2) of Lemma 6.5. □

LEMMA 6.6. *Let Λ a Lyapunov stable singular hyperbolic set of a vector field X, $\alpha > 0$ and $U_\alpha \subset U(\Lambda)$ be a neighborhood of Λ. Let $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ be the splitting of Λ and $E^c_{U_\alpha}$ be a continuous extension of E^c_Λ to U_α. Then, for every singularity σ of X in Λ there are singular cross sections S^{t}, S^{b} associated with σ such that $S^{\mathrm{t}} \cup S^{\mathrm{b}} \subset U_\alpha$,*

$$\Lambda \cap (\partial^{\mathrm{h}} S^{\mathrm{t}} \cup \partial^{\mathrm{h}} S^{\mathrm{b}}) = \emptyset \quad \text{and} \quad X(x) \in C_\alpha(E^c_x),$$

for all $x \in S^{\mathrm{t}} \cup S^{\mathrm{b}}$.

Proof. Let Λ, α be as in the statement. By Remark 5.6 applied to $U = U_\alpha$ there are singular cross sections $S^{\mathrm{t}}_0, S^{\mathrm{b}}_0$ associated with σ such that $S^{\mathrm{t}}_0, S^{\mathrm{b}}_0 \subset U(\Lambda)$ and

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$\Lambda \cap (\partial^{\mathrm{h}} S_0^{\mathrm{t}} \cup \partial^{\mathrm{h}} S_0^{\mathrm{b}}) = \emptyset$. Recall that $U(\Lambda)$ is the neighborhood of Λ where the (K, λ)-splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ has an extension to $T_{U(\Lambda)} M = E^s_{U(\Lambda)} \oplus E^c_{U(\Lambda)}$, with $E^s_{U(\Lambda)}$ invariant and contracting. We denote by $l_0^{\mathrm{t}}, l_0^{\mathrm{b}}$ the singular curves of $S_0^{\mathrm{t}}, S_0^{\mathrm{b}}$ respectively (Definition 5.5).

Choose two sequences of singular cross sections $S_n^{\mathrm{t}} \subset S_0^{\mathrm{t}}$, $S_n^{\mathrm{b}} \subset S_0^{\mathrm{b}}$ associated with σ satisfying:

(a) $\Lambda \cap (\partial^{\mathrm{h}} S_n^{\mathrm{t}} \cup \partial^{\mathrm{h}} S_n^{\mathrm{b}}) = \emptyset$;

(b) $\operatorname{diam}(S_n^{\mathrm{t}}), \operatorname{diam}(S_n^{\mathrm{b}}) \to 0$ as $n \to \infty$;

(c) the singular curves $l_n^{\mathrm{t}}, l_n^{\mathrm{b}}$ of $S_n^{\mathrm{t}}, S_0^{\mathrm{b}}$ satisfy $l_n^{\mathrm{t}} = l_0^{\mathrm{t}}, l_n^{\mathrm{b}} = l_0^{\mathrm{b}}$, for all n.

The properties (b) and (c) imply that for all n large there is $T = T_n > 0$ such that

$$\angle(X(X_T(x)), E^c_\sigma) < \frac{\alpha}{2},$$

for all $x \in S_n^{\mathrm{t}} \cup S_n^{\mathrm{b}}$. As $E^c_{U_\alpha}$ is a continuous extension of E^c_Λ, we have that $E^c_{X(X_T(x))}$ is close to E^c_σ, for all $x \in S_n^{\mathrm{t}} \cup S_n^{\mathrm{b}}$. Then,

$$X_T(x) \in U_\alpha \quad \text{and} \quad X(X_T(x)) \in C_\alpha(E^c_{X(X_T(x))}),$$

for all n large, for all $x \in S_n^{\mathrm{t}} \cup S_n^{\mathrm{b}}$.

By Property (a) and the above relation we have that for n large enough $S^{\mathrm{t}} = X_T(S_n^{\mathrm{t}})$ and $S^{\mathrm{b}} = X_T(S_n^{\mathrm{b}})$ are singular cross sections associated with σ satisfying the required properties. □

Definition 6.7. Let $\epsilon > 0$ and Λ be a Lyapunov stable transitive singular hyperbolic set. *An ϵ-singular partition* of Λ is a singular partition $\mathcal{S}$ of Λ for which there is $\alpha > 0$ such that the family $\mathcal{C} = \{C_\alpha(E^c_x) \cap T\mathcal{S}_x : x \in \mathcal{S}\}$ satisfies:

(1) if $x \in \operatorname{Dom}(\Pi)$, then

$$D\Pi(x)(C_\alpha(E^c_x) \cap T\mathcal{S}_x) \subset \operatorname{int}(C_{\alpha/2}(E^c_{\Pi(x)}) \cap T\mathcal{S}_{\Pi(x)});$$

(2) if $x \in \operatorname{Dom}(\Pi)$ and $v_x \in C_\alpha(E^c_x) \cap T\mathcal{S}_x$, then

$$\|D\Pi(x)(v_x)\| \geq \epsilon^{-1} \|v_x\|;$$

(3) $\inf\{\angle(v_x, E^s_x) : x \in \mathcal{S}, v_x \in C_\alpha(E^c_x) \cap T\mathcal{S}_x\} > 0$.

We have the following lemma.

LEMMA 6.8. *Let Λ be a Lyapunov stable singular hyperbolic set with dense singular unstable branches of X. Suppose that Λ is not an attractor of X and*

$$\omega_X(q) \cap \operatorname{Sing}_X(\Lambda) \neq \emptyset,$$

for all $q \in \Lambda$. Then Λ exhibits an ϵ-singular partition for every $\epsilon > 0$.

Proof. Let Λ be a Lyapunov stable singular hyperbolic set of X as in the statement and $\epsilon > 0$ be fixed. As already mentioned the splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ of Λ extends to a splitting $T_{U(\Lambda)} M = E^s_{U(\Lambda)} \oplus E^c_{U(\Lambda)}$ on a neighborhood $U(\Lambda)$ of Λ in such a way that $E^s_{U(\Lambda)}$ is invariant and contracting. Recall that $k = \#\operatorname{Sing}_X(\Lambda)$.

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Choose $\alpha \in (0, 1]$ such that

$$\inf\{\angle(v_x, E^s_x) : x \in U(\Lambda), v_x \in C_\alpha(E^c_x) \cap T\mathcal{S}_x\} > 0.$$

This α exists because $T_{U(\Lambda)}M = E^s_{U(\Lambda)} \oplus E^c_{U(\Lambda)}$. Let $U_\alpha \subset U(\Lambda)$ and $T_\alpha, K_\alpha, \lambda_\alpha > 0$ be given by Lemma 6.5 for that α.

By Lemma 6.6 we can choose for every $\sigma_i \in \mathrm{Sing}_X(\Lambda)$ two singular cross sections $S^{\mathrm{t},1}_i, S^{\mathrm{b},1}_i \subset U(\Lambda)$ associated with σ_i satisfying

$$\Lambda \cap (\partial^{\mathrm{h}} S^{\mathrm{t},1}_i \cup \partial^{\mathrm{h}} S^{\mathrm{b},1}_i) = \emptyset \quad \text{and} \quad X(x) \in C_\alpha(E^c_x),$$

for all $x \in S^{\mathrm{t},1}_i \cup S^{\mathrm{b},1}_i$, for all $i = 1, \ldots, k$. Define

$$\mathcal{S}^1 = \{S^{\mathrm{t},1}_i, S^{\mathrm{b},1}_i\}_{i\in\{1,\ldots,k\}}.$$

We denote by $l^{\mathrm{t},1}_i, l^{\mathrm{b},1}_i$ the singular curves of $S^{\mathrm{t},1}_i, S^{\mathrm{b},1}_i$ (recall Definition 5.5). For simplicity we shall assume that

$$T\mathcal{S}^1_x = N_x, \quad \text{for all } x \in \mathcal{S}^1.$$

Let $D > 0$ (depending on $\mathcal{S}^1$ only) be such that

$$\frac{\|X(x)\|}{\|X(y)\|} \geq D, \quad \text{for all } x, y \in \mathcal{S}^1. \tag{14}$$

Choose $T_\epsilon > T_\alpha$ large enough so that

$$K_\alpha e^{\lambda_\alpha t} D > \epsilon^{-1}, \quad \text{for all } t \geq T_\epsilon. \tag{15}$$

Next we consider, for every $\delta > 0$ small, $i \in \{1, \ldots, k\}$ and $* = t, b$, a s-rectangle $S^{*,\delta}_i \subset S^{*,1}_i$ containing $l^{*,1}_i$ such that

$$l^{*,1}_i \cap \partial^{\mathrm{v}} S^{*,\delta}_i = \emptyset$$

and

$$\mathrm{diam}(S^{*,\delta}_i) \leq 2\delta.$$

By Corollary 5.10, as Λ is not an attractor, we can assume that

$$\Lambda \cap (\partial S^{\mathrm{t},\delta}_i \cup \partial S^{\mathrm{b},\delta}_i) = \emptyset.$$

In particular, the collection

$$\mathcal{S} = \mathcal{S}^\delta = \{S^{\mathrm{t},\delta}_i, S^{\mathrm{b},\delta}_i\}_{i\in\{1,\ldots,k\}}$$

satisfies the conditions (1) and (2) of Definition 6.2.

By hypothesis

$$\omega_X(q) \cap \mathrm{Sing}_X(\Lambda) \neq \emptyset, \quad \text{for all } q \in \Lambda.$$

This implies condition (3) of Definition 6.2 for $\mathcal{S} = \mathcal{S}^\delta$. Thus, $\mathcal{S} = \mathcal{S}^\delta$ is a singular partition of Λ, for all $\delta > 0$.

Next we show that for $\delta > 0$ small the family

$$\mathcal{C} = \{C_\alpha(E^c_x) \cap T\mathcal{S}_x : s \in \mathcal{S} = \mathcal{S}^\delta\}$$

satisfies the Properties (1), (2) and (3) of Definition 6.7.

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Indeed, as $T\mathcal{S}^1_x = N_x$ for every $x \in \mathcal{S}^1$ one has

$$D\Pi(x) = P_x^{t_\delta(x)}, \quad \text{for all } x \in \mathrm{Dom}(\Pi), \tag{16}$$

where Π is the return map of $\mathcal{S} = \mathcal{S}^\delta$, $\mathrm{Dom}(\Pi)$ is the domain of Π and $t_\delta(x)$ is the return time of $x \in \mathrm{Dom}(\Pi)$. Note that $t_\delta(x) \to \infty$ uniformly as $\delta \to 0^+$. So, there is $\delta > 0$ such that $t_\delta(x) \geq T_\epsilon$, for all $x \in \mathrm{Dom}(\Pi)$.

By (16) we have

$$D\Pi(x)(C_\alpha(E^c_x) \cap T\mathcal{S}_x) = P_x^{t_\delta(x)}(C_\alpha(E^c_x) \cap N_x).$$

Then, by Lemma 6.5(1) one has

$$D\Pi(x)(C_\alpha(E^c_x) \cap T\mathcal{S}_x) \subset C_{\alpha/2}(E^c_{\Pi(x)}) \cap T\mathcal{S}_{\Pi(x)},$$

for all $x \in \mathrm{Dom}(\Pi)$ because $t_\delta(x) \geq T_\epsilon > T_\alpha$. This proves Property (1) of Definition 6.7.

By Lemma 6.5(2), (14), (15) and the definition of T_ϵ one has

$$\begin{aligned}
\|D\Pi(x)(v_x)\| &= \|P_x^{t_\delta(x)}(v_x)\| \\
&= \|P_x^{t_\delta(x)}(v_x)\| \|X(X_{t_\delta(x)}(x))\| \|X(X_{t_\delta(x)}(x))\|^{-1} \\
&\geq K_\alpha e^{\lambda_\alpha t_\delta(x)} \frac{\|X(x)\|}{\|X(X_{t_\delta(x)}(x))\|} \|v_x\| \geq K_\alpha e^{\lambda_\alpha t_\delta(x)} D\|v_x\| \geq \epsilon^{-1}\|v_x\|,
\end{aligned}$$

for all $x \in \mathrm{Dom}(\Pi)$, for all $v_x \in C_\alpha(E^c_x) \cap T_x\mathcal{S}$ because $X(x) \in C_\alpha(E^c_x)$, for all $x \in \mathcal{S}^1$ and $t_\delta(x) > T_\alpha$, for all $x \in \mathrm{Dom}(\Pi)$. This proves Property (2) of Definition 6.7. The Property (3) follows from the choice of α. This finishes the proof. □

Proof of Proposition 5.2. Let Λ be a Lyapunov stable singular hyperbolic set with dense singular unstable branches of a C^r vector field X in M. By contradiction suppose that Λ is not an attractor and

$$\omega_X(q) \cap \mathrm{Sing}_X(\Lambda) \neq \emptyset,$$

for all $q \in \Lambda$. Then, by the previous lemma there is an ϵ-singular partition $\mathcal{S}$ of Λ with $\epsilon \in (0, 1/2)$. Recall that $\mathcal{S}$ consists of a finite family of singular cross sections equipped with singular curves $l = \bigcup_{i=1}^k (l_i^{\mathrm{t}} \bigcup l_i^{\mathrm{b}})$ and a family $\mathcal{C} = \{C_\alpha(E^c_x) \cap T\mathcal{S}_x : x \in \mathcal{S}\}$ (for some $\alpha > 0$) given by Definition 5.5. As before Π denotes the return map associated with $\mathcal{S}$.

A curve in $\mathcal{S}$ will be a C^1 map $c : I \to \mathcal{S}$ for some interval $I \subset \mathbb{R}$. We shall identify c with its image $c(I)$. In particular, c is a connected subset of $\mathcal{S}$. The length of c with respect to the metric of $\mathcal{S}$ is denoted by $L(c)$. A *$\mathcal{C}^u$-curve* is a curve c in $\mathcal{S}$ such that $T_x c$, the tangent vector of c at $x \in c$, satisfies $T_x c \in C_\alpha(E^c_x) \cap T\mathcal{S}_x$ for all $x \in c$. We have the following properties of $\mathcal{C}^u$-curves c of $\mathcal{S}$:

(i) if $c \subset \mathrm{Dom}(\Pi)$, then $\Pi(c)$ is a $\mathcal{C}^u$-curve in $\mathcal{S}$. This follows from Definition 6.7(1);

(ii) if $c \cap l \neq \emptyset$, then $c \cap l$ consists of a single point x_c. This follows from Definition 6.7(3). In this case we denote by c^+ the largest connected component of $c \setminus \{x_c\}$. Clearly

$$L(c^+) \geq \frac{L(c)}{2}.$$

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Let O be the neighborhood of $\Lambda \cap \mathcal{S}$ in $\mathcal{S}$ given by Lemma 6.4. Fix a $\mathcal{C}^u$-curve c_1 in $\mathcal{S}$ contained in O. As $\mathcal{S}$ is an ϵ-singular partition with $\epsilon > 0$ and c_1 is a $\mathcal{C}^u$-curve, we have that $L(\Pi^n(c_1))$ grows exponentially as $n \to \infty$. Then, as $\mathcal{S}$ has finite diameter, there is a first iterate $\Pi^{n_1}(c_1)$ of c_1 such that $\Pi^{n_1}(c_1) \setminus \mathrm{Dom}(\Pi) \neq \emptyset$. Therefore, by Lemma 6.4 applied to $B = c_1$ we have that $\Pi^{n_1}(c_1) \cap l \neq \emptyset$. Set $\beta_1 = \Pi^{n_1}(c_1)$. Then, $\beta_1 \cap l \neq \emptyset$ and so β_1^+ is well defined by (ii).

Observe that

$$L(\beta_1^+) \geq \frac{L(\beta_1)}{2} \geq \frac{L(c_1)}{2}.$$

Moreover, $\beta_1^+ \subset \mathrm{Dom}(\Pi)$ for otherwise β_1 would intersect l twice by Lemma 6.4, contradicting (ii). So, $\Pi(\beta_1^+)$ is well defined and

$$L(\Pi(\beta_1^+)) \geq \epsilon^{-1} L(\beta_1^+) \geq RL(c_1),$$

where $R = (2\epsilon)^{-1} > 1$. Note that $R > 1$ because $\epsilon \in (0, 1/2)$.

Define $c_2 = \Pi(\beta_1^+)$. Then c_2 is a $\mathcal{C}^u$-curve in $\mathcal{S}$. As before there is a first iterate $\Pi^{n_2}(c_2)$ of c_2 such that $\Pi^{n_2}(c_2) \cap l \neq \emptyset$. Set $\beta_2 = \Pi^{n_2}(c_2)$. Then, $\beta_2 \cap l \neq \emptyset$ and so β_2^+ is well defined by (ii). As before we have

$$L(\Pi(\beta_2^+)) \geq \epsilon^{-1} L(\beta_2^+) \geq R^2 L(c_1).$$

In this way we can construct a sequence of $\mathcal{C}^u$-curves $c_1, c_2, \ldots, c_j, \ldots$ in $\mathcal{S}$ such that $L(c_j) \geq R^j L(c_1)$. This is a contradiction since $R > 1$ and $\mathcal{S}$ has finite diameter. The proof is completed. □

A. *Appendix*

In this appendix we shall prove Theorem C. The proof of this theorem is essentially the same as the one of Theorem D in [**15**], where a similar result was proved for non-trivial attractors instead of non-trivial transitive Lyapunov stable sets.

We start by giving some notation, definitions and recall some results proved elsewhere, that will be used to obtain Theorem C. Consider a vector field $X \in \mathcal{X}^r(M)$ and a hyperbolic singularity σ of X, $r \geq 1$. Assume that the eigenvalues λ_1, λ_2, λ_3 of σ are real and satisfy $\lambda_2 < \lambda_3 < 0 < \lambda_1$. Recall that $W_X^{ss}(\sigma)$ denotes the unique invariant manifold which is tangent to the eigenspace associated with λ_2. By [**9**] there is a *center-unstable invariant manifold* $W_X^{cu}(\sigma)$ of class C^1 at least, non-unique and tangent to the eigenspace associated with $\{\lambda_1, \lambda_3\}$. A regular orbit $\Gamma \in W_X^s(\sigma) \cap W_X^u(\sigma)$ is called a *homoclinic loop* of σ. We have the following generic properties of a homoclinic loop Γ associated with σ:

(G1) for any centre-unstable $W_X^{cu}(\sigma)$ manifold of X at σ containing Γ we have that $W_X^{cu}(\sigma)$ is transverse to $W_X^s(\sigma)$ along Γ;

(G2) $\Gamma \notin W_X^{ss}(\sigma)$, the strong-unstable manifold associated to σ.

We say that Γ is an *inclination-flip* (respectively *orbit-flip*) if (G1) (respectively (G2)) above fails [**10**].

The following result is a consequence of Theorem 4.1 in [**11**]; see Theorem 7.1 in [**15**].

THEOREM A.1. *There is no $Z \in \mathcal{G}^1(M)$ exhibiting an inclination-flip homoclinic loop.*

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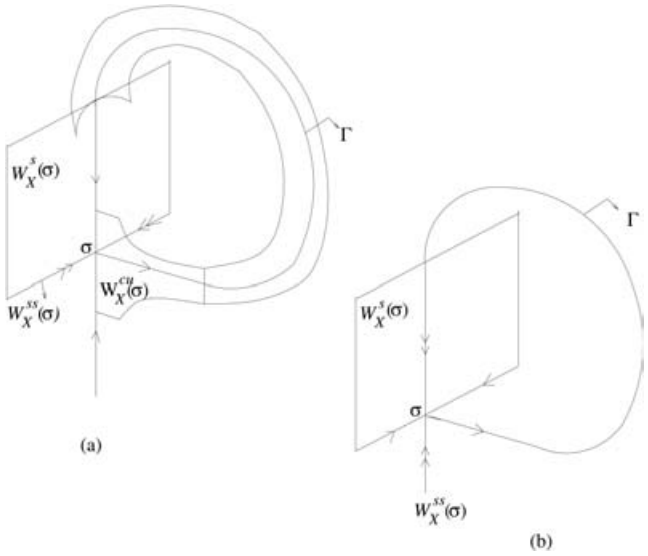


FIGURE 3. (a) Inclination-flip; (b) orbit-flip.

The following result and its corollary are given in [**15**, Theorem 7.2].

THEOREM A.2. *Let $X \in \mathcal{X}^1(M)$ and σ be a singularity of X exhibiting an orbit-flip homoclinic loop. Suppose that*

(H) *There are neighborhoods $\mathcal{U}_0 \subset \mathcal{X}^1(M)$ and $U_0 \subset M$ of X and σ (respectively) such that the Z-orbit of every sink of $Z \in \mathcal{U}_0$ does not intersect U_0.*

Then, for every neighborhood $\mathcal{U} \subset \mathcal{X}^1(M)$ of X there is $Y \in \mathcal{U}$ such that $\sigma \in \mathrm{Sing}(Y)$ has an inclination-flip homoclinic loop.

COROLLARY A.3. *Let $X \in \mathcal{G}^1(M)$ and $\sigma \in \mathrm{Sing}(X)$ with eigenvalues $\lambda_2 < \lambda_3 < 0 < \lambda_1$. If $W_X^{ss}(\sigma) \cap \Omega(X) \neq \{\sigma\}$, then (H) does not hold for X and σ.*

LEMMA A.4. *There is a residual subset $\mathcal{R} \subset \mathcal{G}^1(M)$ such that if $X \in \mathcal{R}$ and $\sigma \in \mathrm{Sing}(X)$ belongs to a non-trivial transitive Lyapunov stable set Λ of X, then σ is Lorenz-like for X and*

$$W_X^{ss}(\sigma) \cap \Lambda = \{\sigma\}.$$

Proof. For every $Z \in \mathcal{X}^1(M)$ let

$$\Phi_s(X) = \mathrm{Cl}(\{q \in M : \sigma \text{ is a sink of } Z\}).$$

Similarly we define $\Phi_u(Z)$ with source instead of sink.

Clearly

$$\Phi_s, \Phi_u : \mathcal{X}^1(M) \to 2_c^M$$

are lower semi-continuous functions. So, there is a residual subset $\mathcal{R}^*$ of $\mathcal{X}^1(M)$ such that Φ_s and Φ_u are semi-continuous at $X \in \mathcal{R}^*$.

Setting $\mathcal{R} = \mathcal{R}^* \cap \mathcal{G}^1(M)$ we have that $\mathcal{R}$ is residual in $\mathcal{G}^1(M)$.

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Let us prove that $\mathcal{R}$ satisfies the required property. Since $\mathcal{R} \subset \mathcal{G}^1(M)$ we have that every $X \in \mathcal{R}$ has finitely many sinks and sources [**11**].

Let $\sigma \in \mathrm{Sing}(X) \cap \Lambda$ for some non-trivial transitive Lyapunov stable set Λ of $X \in \mathcal{R}$. We claim that the eigenvalues of σ are real.

Indeed, suppose by contradiction that σ has a complex eigenvalue λ. Let λ' denote the real eigenvalue. We can suppose, without loss of generality, that λ' is positive.

We have that $-\lambda'/\mathrm{Re}(\lambda)$ is either >1 or ≤ 1. In the former case we can C^1-approximate X by a flow Z exhibiting a sink close to σ. On the other hand, $\sigma \notin \Phi^s(X) \cup \Phi^u(X)$, of Φ^s at X. This proves the claim in the former case. In the latter case, we can C^1-approximate X by a flow Z exhibiting a source close to σ contradicting now the upper semi-continuity of Φ^u at X, as before. This proves the claim. Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ be the eigenvalues of σ. We have either

(I) $\lambda_2 \leq \lambda_3 < 0 < \lambda_1$ or

(II) $\lambda_2 < 0 < \lambda_3 \leq \lambda_1$.

Let us prove that (II) can not occur. Indeed, suppose by contradiction that (II) holds. We can assume that $\lambda_3 < \lambda_1$ without loss of generality. Thus, $W^{uu}_X(\sigma)$ is well defined. As $W^u_X(\sigma) \subset \Lambda$, recall that Λ is Lyapunov stable, we have $W^{uu}_X(\sigma) \subset \Lambda$. Since Λ is non-trivial, applying the connecting lemma, we obtain an orbit-flip homoclinic loop associated with a singularity $\sigma(Z)$ for a vector field Z close to $-X$. Note that $\sigma(Z)$ is close to σ.

Applying Corollary A.3 to the vector field Z (which is obviously in $\mathcal{G}^1(M)$), we have that there is Z' C^1-close to Z exhibiting a sink arbitrarily close to $\sigma(Z)$, and so arbitrarily close to σ. As $-Z'$ is close to X, this contradicts the upper semi-continuity of Φ^u. So, (I) always holds. A similar argument implies that $\lambda_2 < \lambda_3$.

Thus, we have proved that

$$\lambda_2 < \lambda_3 < 0 < \lambda_1.$$

To prove that σ is Lorenz-like, we just need to prove the inequality $-\lambda_3 < \lambda_1$. If this were not true, we apply the connecting lemma to obtain X' C^1-close to X exhibiting a homoclinic loop associated with $\sigma(X')$. Hence, as before, we can approximate X' by Z exhibiting a sink close to $\sigma(X')$, and so to σ, contradicting the semi-continuity of Φ^s once more. Thus, σ is Lorenz-like.

It remains to prove that $W^{ss}_X(\sigma) \cap \Lambda = \{\sigma\}$. But this equality follows from Corollary A.3 since $\Lambda \subset \Omega(X)$. This proves the lemma. □

Proof of Theorem C. Let $\mathcal{R}$ be given by Lemma A.4. By construction, $\mathcal{R}$ has the following property.

(**) If $X \in \mathcal{R}$ and Λ is a non-trivial transitive Lyapunov stable set of X, then there are neighborhoods $\mathcal{U} \subset \mathcal{X}^1(M)$ and $U \subset M$ of X and Λ (respectively) such that $Z \in \mathcal{U}$ exhibits neither a sink nor a source whose orbit intersects U.

Now choose a vector field X and a non-trivial transitive Lyapunov stable set Λ of X with $\mathrm{Sing}_X(\Lambda) \neq \emptyset$. If $\sigma \in \mathrm{Sing}_X(\Lambda)$, Lemma A.4 implies that every $\sigma \in \Lambda \cap \mathrm{Sing}(X)$ is Lorenz-like and satisfies

$$W^{ss}_X(\sigma) \cap \Lambda = \{\sigma\}.$$

Then, replacing Lemma 1 in [**16**] by (**) and using the arguments there we conclude that Λ is singular hyperbolic for X. This finishes the proof. □

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