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A rigidity theorem for manifolds without conjugate points

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Abstract. In this paper we show that some nonsimply connected manifolds without conjugate points exhibit rigidity phenomena similar to that studied in [**BGS**, section I.5]. This is a companion paper to [**CrKl1**] that deals with the simply connected case. In particular, we show that one cannot make a nontrivial, compactly supported, change to a complete flat metric without introducing conjugate points.

1. *Introduction*

In a companion paper [**CrKl1**] we showed that if (X, g_X) is a complete simply connected manifold without conjugate points then the product manifold $(M, g_0) = (X \times R, g_X \times dt^2)$ has the property that any metric g_1 on M without conjugate points that agrees with g_0 on the complement of a compact set K must in fact be isometric to g_0. We refer to this situation by the phrase 'there are no nontrivial NCP perturbations of (M, g_0) supported on K'. The purpose of this paper is to extend this result to certain nonsimply connected manifolds.

Our first theorem is that all noncompact flat manifolds have a similar rigidity property.

THEOREM 1.1. *Let (M, g_0) be a complete, noncompact flat manifold. Then any metric without conjugate points on M which agrees with g_0 outside a compact set $K \subset M$ is isometric to g_0.*

The case where (M, g_0) is compact is contained in Burago and Ivanov's recent proof of the Hopf conjecture that an n-torus without conjugate points must be flat [**BuIv**] (proved after the first draft of this paper was written), but the general noncompact case does not follow from their method‖. The case where M is isometric to E^n was studied for $n = 2$ in [**GrGu**] and for all n in [**Cr1**] (also see [**Cr2**]).

Many of our results involve a growth condition on the fundamental group. So we need the following definition.

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‖ Their approach makes essential use of the finiteness of volume.

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Definition 1.2. Let Γ be a finitely generated group. Then we will say that Γ *grows more slowly than* r^n if for any choice of finite generating set $\Sigma \subseteq \Gamma$, we have

$$\lim_{r\to\infty} \frac{\mathrm{Card}(\{g \in \Gamma \mid |g|_\Sigma \leq r\})}{r^n} \to 0$$

where $|\cdot|_\Sigma : \Gamma \longrightarrow R$ denotes the word norm on Γ defined by the generating set Σ (see [**Grm**]).

We call a Riemannian manifold (M, g_0) an NCP manifold if it is complete and has no conjugate points. In particular, the universal cover $\tilde{M}$ is diffeomorphic to R^n. For $K \subset M$, an NCP perturbation of M supported on K, or an NCP perturbation of (M, K, g_0), we mean a Riemannian metric g such that (M, g) is an NCP manifold and $g = g_0$ on $M - K$. Such a perturbation is called trivial if g is isometric to g_0.

Our second theorem states a special case of what we can show in a more general setting.

THEOREM 1.3. *Assume that (X, g_X) is a noncompact manifold without conjugate points where $\pi_1(X)$ has polynomial growth. If $K \subset X \times R$ is a compact totally convex subset (i.e. any geodesic segment with endpoints in K lies in K), then there are no nontrivial NCP perturbations of $(X \times R, g_X \times dt^2)$ supported on K.*

The condition of total convexity on K above is quite strong and there are many other cases where we can prove the nonexistence of NCP perturbations. However, every case that we can handle does involve a growth condition on the fundamental group.

To consider compact subsets K of a Riemannian manifold (M, g) that are not totally convex, we will need to consider the union, $C_1(K, g)$, of the geodesic segments in M with endpoints in K. We can now state a general theorem.

THEOREM 1.4. *Let (X, g_X) be a noncompact (n-1)-dimensional NCP manifold such that $\pi_1(X)$ grows slower than r^n and g an NCP perturbation of $(X \times R, K, g_0)$ where K is compact. Then if $C_1(K)$ is bounded and if at least one of the following hold:*

(1) *(X, g_X) has no focal points;*
(2) *$\pi_1(X)$ grows slower than r^{n-1};*
(3) *$(X, g_X) = (Y \times R, g_Y + dt^2)$;*
(4) *K is totally convex;*

then g is isometric to g_0.

One can see some of the added complications when $\pi_1(X)$ is not trivial by considering compact X and the metric $g_1 = f^2(t)g_X \times dt^2$ where $f(t) = 1$ for $t \notin [-1, 1]$ and $f(t) > 1$ for $t \in (-1, 1)$. Although g_1 will have conjugate points, it is not hard to see that geodesics that do not stay in the perturbed region have no conjugate points (in fact, they are lines when lifted to the universal cover). The existence of geodesics that stay in the perturbed region (or more generally return to it for arbitrarily large parameter values) will be a new complication. In particular, it will become important to show that their unit tangent vectors form a set of measure zero in the unit tangent bundle.

Hence a crucial ingredient in the proof of Theorems 1.3 and 1.4 is Lemma 4.4. Later we state a version holding for compact smooth domains (i.e. compact sets with smooth boundaries that are the closure of an open set), which may be of independent interest.

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PROPOSITION 1.5. *Let (M^n, g) be a manifold without conjugate points, $K \subset M$ a connected compact smooth domain, and $\Gamma = i_*(\pi_1(K)) \subset \pi_1(M)$ where $i : K \to M$ is the inclusion. Assume that Γ grows more slowly than r^n. If $W \subset \pi^{-1}(K) \subset SM$ is a measurable subset of the unit sphere bundle of M which is invariant under the geodesic flow, then W has Liouville measure zero.*

2. *Notation and preliminaries*

We will always assume that geodesics are parametrized by arc length unless otherwise stated. We will often consider two NCP metrics g_0 and g on a given manifold M. $S(M, g_0)$ and $S(M, g)$ will represent the respective unit sphere bundles. If v is a g-unit vector of M (i.e. $v \in S(M, g)$) then γ_v will represent the g-geodesic with $\gamma'(0) = v$. We similarly define the g_0-geodesic, γ_{0v}, if $v \in S(M, g_0)$. We will let g^t (respectively, g_0^t) represent the geodesic flow for time t, i.e. $g^t(v) = \gamma_v'(t)$.

If K is a subset of M we let $\Omega_g^+(K) = \{v \in S(X, g) |$ there are t_i with $t_i \to +\infty$ and $\gamma_v(t_i) \in K\}$. We let $\Omega_g^-(K) = -\Omega_g^+(K)$, $\Omega_g^\pm(K) = \Omega_g^+(K) \cup \Omega_g^-(K)$, and $\Omega_g(K) = \Omega_g^+(K) \cap \Omega_g^-(K)$. We note that $\Omega_g^+(K)$, and hence $\Omega_g^-(K)$, $\Omega_g^\pm(K)$, and $\Omega_g(K)$ are invariant under the geodesic flow on $S(X, g)$. Furthermore, if $v \in \Omega_g(K)$ then the geodesic γ_v lies in the set $C_1(K)$ defined in the introduction. We can similarly define $\Omega_{g_0}^+(K)$, $\Omega_{g_0}^-(K)$, $\Omega_{g_0}^\pm(K)$, and $\Omega_{g_0}(K)$.

Let (M, g_0) be an NCP manifold, $K \subset M$ a closed subset, and g an NCP perturbation of (M, K, g_0). If γ_0 is a g_0-geodesic such that $\gamma_0'(t) \notin \Omega_{g_0}^\pm(K)$, i.e. there is number N such that $\gamma_0(t) \notin K$ for $|t| > N$, then a g-geodesic γ is said to be a compact perturbation of γ_0 if $\gamma(t) = \gamma_0(t)$ for all $|t| > N$. The point here is that γ_0 has a compact perturbation if the g-geodesic that coincides with $\gamma_0(t)$ when $t \in (-\infty, -N]$ also coincides (including parameter) when $t \in [N, \infty)$.

We will call the perturbation a *lens equivalent* perturbation if for $v \in S(M - K, g_0) \cap \Omega^+(K, g_0)$ then $v \in S(M - K, g) \cap \Omega^+(K, g)$ and for $v \in S(M - K, g_0) - \Omega^\pm(K, g_0)$ then γ_{0v} has a compact perturbation in g.

We will often be considering products $M = X \times R^l$ where X (which is not necessarily compact or simply connected) has an NCP metric g_X and M is given the product metric $g_0 = g_X \times g_E$, where g_E is the Euclidean metric. An important special case for us is when $l = 1$. $B(a)$ will represent the ball of radius a in R^l. We will use π_X and π_R to represent projection onto the factors (supressing the exponent l), and will call a vector v which is tangent to a Euclidean factor (i.e. $\pi_{X*}(v) = 0$) a *vertical* vector, and one tangent to an X factor (i.e. $\pi_{R*}(v) = 0$) a *horizontal* vector. We will let $S^{\text{nh}}(M) = \{v \in S(M) \mid v$ is not horizontal$\}$.

Let K be a subset of M such that $\pi_R(K)$ is compact. We note that $\Omega^\pm(K, g_0)$ consists of only horizontal vectors. An NCP perturbation g of (M, K, g_0) will be called a *weak lens equivalence* if for each nonhorizontal $v \in S(M, g_0)$, γ_{0v} has a compact perturbation in g. It will be called a *local lens equivalence* if there is an $\epsilon > 0$ such that for every $v \in S(M)$ such that $|\pi_{X*}(v)| < \epsilon$, γ_{0v} has a compact perturbation in g. A lens equivalent perturbation is of course a weak lens equivalence, which is in turn a local lens equivalence.

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LEMMA 2.1. *Let (M, g_0) be an NCP manifold whose universal cover $(\tilde{M}, g_0)$ is $(\tilde{X} \times R^l, g_0)$, where $(\tilde{X}, g_X)$ is a simply connected NCP manifold and $g_0 = g_X \times g_E$. Let K be a compact subset of M whose lift $\tilde{K} \subset \tilde{M}$ is a subset of $\tilde{X} \times B(a)$. Let g be an NCP perturbation of (M, K, g_0) such that the induced metric (also called g) on $\tilde{M}$ is a weak lens equivalent perturbation of $(\tilde{M}, \tilde{K}, g_0)$. If $\mathrm{Vol}(C_1(K, g_0))$ is finite and $\mathrm{Vol}(\Omega_g(K)) = 0$ (i.e. $\Omega_g(K)$ has Liouville measure zero) then $\mathrm{Vol}(K, g_0) = \mathrm{Vol}(K, g)$.*

Proof. Since $\mathrm{Vol}(C_1(K, g_0))$ is finite and $K \subset C_1(K)$, it is sufficient to show that $\mathrm{Vol}(C_1(K, g_0)) = \mathrm{Vol}(C_1(K, g))$. We define a smooth map $F : S^{\mathrm{nh}}(\tilde{M}, g_0) \to S(\tilde{M}, g) - \Omega_g(\tilde{K})$ defined by $F(v) = \gamma'(0)$, where γ is the g-geodesic which is the compact perturbation of the g_0-geodesic γ_{0v}. It is easy to see that F is the identity on $S^{\mathrm{nh}}(\tilde{M} - C_1(\tilde{K}), g_0)$ and maps $S^{\mathrm{nh}}(C_1(\tilde{K}), g_0)$ injectively to $S(C_1(\tilde{K}), g)$. It is also clear that F is a conjugacy of the geodesic flows, i.e. $F \circ g_0^t = g^t \circ F$. From this and the invariance of the Liouville measure under the geodesic flow one sees that F preserves the Liouville measure. Later we will show that $W = S(C_1(\tilde{K}, g)) - F(S^{\mathrm{nh}}(C_1(\tilde{K}, g_0)))$ has Liouville measure zero. This, combined with the invariance of F under the deck group, and the fact that the horizontal vectors form a set of Liouville measure zero, tells us that $\mathrm{Vol}(C_1(K, g_0)) = \mathrm{Vol}(C_1(K, g))$ and hence yields the result.

To see that W has Liouville measure zero, we note that W can be described as $\{v \in S(C_1(\tilde{K}), g) \mid \pi_R(\gamma_v(t)) \in B(a) \text{ for all times } t\} = \Omega_g(K) \cup \bigcup_{i=0}^{\infty} W_i^+ \cup \bigcup_{i=0}^{\infty} W_i^-$ where $W_i^+ = \{v \in W \mid \gamma_v(t) \notin K \text{ for all } t \geq i\}$, and $W_i^- = -W_i^+$. Thus we need only show that for every integer $i > 0$ the set W_i^+ (and hence W_i^-) has g-Liouville measure zero. For $v \in W_i^+$ and $t > i$, $\gamma_v(t)$ is a g_0-geodesic such that $\pi_R(\gamma_v(t)) \in B(a)$. Hence $\pi_R(\gamma_v(t))$ for $t > i$ is in fact a single point. Thus for $t > i$, $\pi_{R*}(\gamma_v'(t)) = 0$. Now by the invariance of the Liouville measure under the geodesic flow g^t we see $\mathrm{Vol}(W_i^+) = \mathrm{Vol}(g^i(W_i^+))$. Now $g^i(W_i^+) \subset \{v \in S(\tilde{M} - \tilde{K}, g_0) \mid \pi_{R*}(v) = 0\}$ which has g_0 and hence g-Liouville measure zero. The lemma follows. □

PROPOSITION 2.2. *Let (M, g_0) be an NCP manifold whose universal cover $(\tilde{M}, g_0) = (\tilde{X} \times R^l, g_0)$, where $(\tilde{X}, g_X)$ is a simply connected NCP manifold and $g_0 = g_X \times g_E$. Let K be a compact subset of M whose lift $\tilde{K} \subset \tilde{M}$ is a subset of $\tilde{X} \times B(a)$. Let g be an NCP perturbation of (M, K, g_0) such that the induced metric (also called g) on $\tilde{M}$ is a local lens equivalent perturbation of $(\tilde{M}, \tilde{K}, g_0)$. Then $\mathrm{Vol}_g(K) \geq \mathrm{Vol}_{g_0}(K)$ with equality if and only if g is isometric to g_0.*

A proof of Proposition 2.2 may be found in §7.

Combining Proposition 2.2 with Lemma 2.1 we get the following.

COROLLARY 2.3. *Let (M, g_0) be an NCP manifold whose universal cover $(\tilde{M}, g_0)$ is $(\tilde{X} \times R^l, g_0)$, where $(\tilde{X}, g_X)$ is a simply connected NCP manifold and $g_0 = g_X \times g_E$. Let K be a compact subset of M such that $\mathrm{Vol}(C_1(K, g_0))$ is finite and whose lift $\tilde{K} \subset \tilde{M}$ is a subset of $\tilde{X} \times B(a)$. Let g be an NCP perturbation of (M, K, g_0) such that the induced metric (also called g) on $\tilde{M}$ is a weak lens equivalent perturbation of $(\tilde{M}, \tilde{K}, g_0)$. Then g is isometric to g_0 if and only if $\mathrm{Vol}_g(\Omega_g(K)) = 0$.*

Remark. The proof of Proposition 2.2 and Corollary 2.3 hold more generally than for compact subsets K. In particular, they hold if K has finite volume and $\pi_R(\tilde{K})$ is compact.

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However, Proposition 2.2 does not hold for arbitrary finite volume sets since there are closed (and open) finite volume subsets, K, of $\mathbb{R}^2$ such that every line intersects K in a noncompact set and hence any NCP perturbation is a lens equivalence. Furthermore, there are $K \subset \mathbb{R}^2$ of finite volume that admit NCP perturbations. On the other hand, we have no counterexample to Corollary 2.3 when it is only assumed that $C_1(K, g_0)$ has finite volume.

3. *Busemann functions and lens equivalence*

We will say $v \in S(M, g)$ is a ray if the geodesic γ_v is a ray in the sense that $d(\gamma(0), \gamma(t)) = t$ for all $t > 0$. We can then define the Busemann function $b_v : M \to R$ by $b_v(x) = \lim_{t\to\infty} d(x, \gamma_v(t)) - t$. We note that $b_v(\gamma_v(t)) = -t$ for $t > 0$ (or for all t if γ_v is a line, i.e. minimizes between any two points on it). In the case when M is a simply connected NCP manifold then b_v is defined for all $v \in S(M, g)$. For $v \in S(M, g_0)$ we use the notation b_{0v} for the Busemann function. If $v \in S(M, g)$ is a ray and $x \in M$ we can define the horoball $B_v(x) = \{y \in M \mid b_v(y) \le b_v(x)\}$ and the horosphere $\partial B_v(x) = \{y \in M \mid b_v(y) = b_v(x)\}$. Similarly, for $v \in S(M, g_0)$ a ray we can define $B_{0v}(x)$ and $\partial B_{0v}(x)$.

The reader is referred to [**E**] for results on Busemann functions on simply connected NCP manifolds. In particular, Busemann functions are $C^{1,1}$, and the vector field $V(x) = \nabla b_v(x)$ is a unit vector field whose integral curves are precisely those geodesics γ such that $db_v(\gamma(t))/dt = 1$.

If M is a simply connected NCP manifold, $C \subset M$ is closed, and $v \in S(M - C, g)$ is a ray, then the shadow $S(C, v)$ of C is the set $\{x \in M \mid \gamma_{\nabla b_v(x)}(t) \in C \text{ for some } t \le 0\}$. Now if g is a perturbation of (M, C, g_0) then we also have that $v \in S(M - C, g_0)$ and it is easy to check that for $y \in M - S(C, v)$ we have $b_v(y) = b_{0v}(y)$.

On simply connected NCP manifolds, if ν is the outward normal at x to $B_v(x)$ (i.e. $\nu = \nabla b_v(x)$), then, by an easy triangle inequality argument, $B_v(x) \cap B_\nu(x) \subset \partial B_v(x) \cap \partial B_\nu(x)$. We will use this fact extensively. We will also use the fact that on product manifolds $(X_1 \times X_2, g_1 \times g_2)$ we have $b_v(x_1, x_2) = a_1 b_{v_1}(x_1) + a_2 b_{v_2}(x_2)$, where $\pi_{i*}(v) = a_i v_i$ and $v_i \in S(X_i, g_i)$.

If $(\tilde{M}, g_0) = (\tilde{X} \times R^l, g_X \times g_E)$, where $(\tilde{X}, g_X)$ is a simply connected NCP manifold, we will let $S_a^+(\tilde{M}) = \{v \in S(\tilde{M}) \mid \pi_R(\gamma_{0v}(t)) \notin B(a) \text{ for any } t \ge 0\}$. The point is that if g is any NCP perturbation of $(\tilde{M}, \tilde{X} \times B(a), g_0)$ then for $v \in S_a^+(M)$ we have that $\gamma_v(t) = \gamma_{0v}(t)$ for all $t > 0$.

LEMMA 3.1. *Let (M, g_0) be an NCP manifold and $K \subset M$ compact. Assume that $(\tilde{M}, g_0) = (\tilde{X} \times R^l, g_X \times g_E)$, where $(\tilde{X}, g_X)$ is a simply connected NCP manifold, and further assume that $\tilde{K} \subset \tilde{X} \times B(a)$. Let g be an NCP perturbation of (M, K, g_0). Then there is a constant $C(g_0, g)$ such that for every $x, y \in \tilde{M}$ we have $|d(x, y) - d_0(x, y)| < C(g, g_0)$.*

In particular, for each nonhorizontal vector $v \in S(\tilde{M} - \tilde{X} \times B(a))$ and for every $x \in \tilde{M}$ we have $|b_v(x) - b_{0v}(x)| < C(g, g_0)$. Furthermore, for every $v \in S_a^+(M)$ and every $x \in \tilde{M} - \tilde{X} \times B(a)$, we have $\pi_{R*}(\nabla b_v(x)) = \pi_{R*}(\nabla b_{0v}(x)) = \pi_{R*}(-v)$. In the special case that $(\tilde{X}, g_X)$ is also Euclidean we have that $\nabla b_v(x)$ is a parallel field on $\tilde{M} - \tilde{X} \times B(a)$.

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Proof. Since $\pi_1(M)$ acts co-compactly on $\tilde{K}$, we have $c_1^{-2}\tilde{g}_0 \leq \tilde{g} \leq c_1^2\tilde{g}_0$ for a suitable constant c_1. Let x_1, x_2 be such that $\pi_R(x_i) \in \partial B(a)$. Let $(\gamma_0(t), \tau_0(t)) \in \tilde{X} \times R^l$ be the geodesic from x_1 to x_2. The length of τ_0 is less than $2a$. Let $\bar{\gamma}(t) = (\gamma(t), \pi_R(x_1))$ and $\bar{\tau}_0(t) = (\pi_X(x_2), \tau_0(t))$ be curves in $\tilde{M}$. $\bar{\gamma}$ is both a g and a g_0 geodesic (and hence mimimizing) of length L, while the g_0 length of $\bar{\tau}$ is $\leq 2a$, and the g length of $\bar{\tau}$ is $\leq 2ac_1$. The path $\bar{\gamma} \cup \bar{\tau}$ goes from x_1 to x_2 so we see by the triangle inequality that $L - 2a \leq d_0(x_1, x_2) \leq L + 2a$ and $L - 2ac_1 \leq d(x_1, x_2) \leq L + 2ac_1$, and hence $|d(x_1, x_2) - d_0(x_1, x_2)| \leq 2a + 2ac_1 = c_2$.

Let x and y be in M such that $\pi_R(x)$ and $\pi_R(y)$ lie outside $B(a) - \partial B(a)$. Let $\gamma : [0, d_0(x, y)] \to \tilde{M}$ be the g_0-geodesic from x to y. By the convexity of $\tilde{X} \times B(a)$ either γ misses $\tilde{X} \times B(a)$ or there is an interval $[t_1, t_2]$ such that $\gamma \cap X \times B(a) = \gamma([t_1, t_2])$. Now

$$\begin{aligned} d_0(x, y) &= d_0(x, \gamma(t_1)) + d_0(\gamma(t_1), \gamma(t_2)) + d_0(\gamma(t_2), y) \\ &\geq d(x, \gamma(t_1)) + d(\gamma(t_1), \gamma(t_2)) - c_2 + d(\gamma(t_2), y) \geq d(x, y) - c_2. \end{aligned}$$

Similarly starting with the g-geodesic from x to y we conclude that $d(x, y) \geq d_0(x, y) - c_2$.

If $\pi_R(x) \in B(a) - \partial B(a)$, there is a point $\bar{x}$ such that $\pi_R(\bar{x}) \in \partial B(a)$, $d_0(x, \bar{x}) \leq a$, and $d(x, \bar{x}) \leq ac_1$. Similarly for y. Hence the lemma will hold for all x and y if we take $C(g, g_0) = c_2 + 2a + 2ac_1$.

The Busemann functions b_v and b_{0v} are limits as $t \to \infty$ of functions $b_t(x) = d(\gamma_v(t), x) - t$ and $b_{0t} = d(\gamma_v(t), x) - t$, respectively. Since for $t > 0$, $\gamma_v(t) = \gamma_{0v}(t)$, we see that $|b_t(x) - b_{0t}(x)| < C(g, g_0)$ and hence $|b_v(x) - b_{0v}(x)| < C(g, g_0)$.

We now compare the gradients of b_v and b_{0v} at a point x for $v \in S_a^+(\tilde{M})$. Let $w = \nabla b_v((x))$ and assume that $w \in S_a^+(\tilde{M})$ (otherwise $-w \in S_a^+(\tilde{M})$ and the proof goes through). Since $b_v(\gamma_w(s)) = s + b_v(x)$, we see that $|b_{0v}(\gamma_w(s)) - s| < 2C(g, g_0)$ and hence $\gamma_w'(s)$ is asymptotic to ∇b_{0v}; thus $\pi_{R*}(w) = \pi_{R*}(-v)$. In the case when $(\tilde{X}, g_X)$ is Euclidean we see that w is parallel to $-v$. □

LEMMA 3.2. *Let (X, g_X) be an NCP manifold, $K \subset X \times B(a) \subset X \times R^l$ be a (not necessarily compact) subset, and g an NCP perturbation of (M, K, g_0). Assume further that for each $v \in S_a^+(\tilde{M})$ and for every $x \in \tilde{M} - \tilde{X} \times B(a)$ we have $b_v(x) = b_{0v}(x)$. Then g is a weak lens equivalence.*

Proof. It is sufficient to prove weak lens equivalence for the induced metrics on $\tilde{M}$. Let γ_0 be a non-horizontal g_0-geodesic. $\pi_R(\gamma_0(t)) = L(t)$ for some line $L(t) \in R^l$. We may assume that $L(t) \notin B(a)$ whenever $t \notin [-d, d]$. In particular, $v = \gamma_0'(d) \in S_a^+(\tilde{M})$. Let $x = \gamma_0(-d)$, $y = \gamma_0(d)$, γ be the g-geodesic from x to y with $\gamma(d) = y$, $\gamma(-d') = x$, and $w = \gamma'(d)$.

We first show that $w \in S_a^+(\tilde{M})$. If $\pi_R(\gamma([-d', d])) \cap B(a) = \emptyset$ then $\gamma(t) = \gamma_0(t)$ (and $d = d'$) for all t since then γ is a g_0-geodesic from x to y. If $\pi_R(\gamma([-d', d])) \cap B(a) \neq \emptyset$ then, since $B(a)$ is convex, $\pi_R(\gamma(t)) \notin B(a)$ for $t \notin [-d', d]$ so $w \in S_a^+(\tilde{M})$.

We see that $2d = d_0(x, y) = b_{0v}(x) = b_v(x) \leq d(x, y)$, with the equality holding if and only if $x = \gamma_v(-2d)$. On the other hand, $d(x, y) = b_w(x) = b_{0w}(x) \leq d_0(x, y)$.

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Thus the equality holds and we see that $\gamma(t) = \gamma_v(t)$ is a compact perturbation of $\gamma_0(t)$ and thus that g is a weak lens equivalence. □

4. *Liouville measure of* $\Omega(K)$

When considering NCP perturbations, g, of $(\tilde{X} \times R^l, \tilde{K}, g_X \times g_E)$ in the case where M is not simply connected, one of the major problems is to show that $\mathrm{Vol}(\Omega_g(K)) = 0$. When $(\tilde{X}, g_X)$ has negative curvature we can show that $\Omega_g(K)$ contains no open sets, but this gives us no information about the Liouville measure. In general, this seems to be a very hard problem and it is the principal obstacle to extending Theorem 1.3 to a much broader class of products $X \times R$. In this section we show that $\mathrm{Vol}(\Omega_g(K)) = 0$ under auxiliary assumptions concerning volume or π_1 growth.

Let $C \subset M$, and $\tilde{C} \subset \tilde{M}$ be the pre-image of C in the universal cover $\tilde{M}$ of M. We will say that C has volume growth less than r^n if there is a function $f(r) > 0$ with $\lim_{r\to\infty} f(r) = 0$ such that, for every $\tilde{x} \in \tilde{C}$, $\mathrm{Vol}(\tilde{C} \cap B(\tilde{x}, r)) < f(r)r^n$.

For (M, g) an NCP manifold, $W \subset SM$ a subset of the unit tangent bundle, $x \in M$, and $r > 0$ we will let $B^W(x, r) = \{\exp_x(tV) \mid V \in W \cap T_xM \text{ and } 0 \le t \le r\}$ be the W-ball at x of radius r. We similarly define the W-sphere, $S^W(x, r)$. Let $\pi(W)$ represent the projection of W to M, and $\tilde{B}^W(x, r)$ represent a W-ball in the universal cover centered at a point that projects to x (hence only defined up to the deck group).

The proof of Theorem 1 of [**Cr3**] involves integration over the unit sphere bundle $S(M)$ of a compact NCP manifold M. Restricting these arguments to a subset W of $S(M)$ yields the following.

THEOREM 4.1. *Let (M, g) be an NCP manifold and $W \subset SM$ be a measurable subset such that:*

(1) $W = -W$;

(2) *W is geodesic flow invariant;*

(3) *$\pi(W)$ has finite volume and bounded curvature.*

Then for each fixed $r > 0$

$$\mathrm{Ave}(\mathrm{Vol}(\tilde{B}^W(x, r))) \ge \frac{\mathrm{Vol}(W)}{\mathrm{Vol}(\pi(W))} \frac{r^n}{n}$$

where the average is taken over all $x \in \pi(W)$ with respect to the Riemannian measure and $\mathrm{Vol}(W)$ *represents the Liouville measure of W.*

The proof is given in §7 and involves the volumes of W-spheres.

Remark. Both Theorems 1 and 3 of [**Cr3**], including the equality discussions, have analogues in this setting; however, they will not be needed in this paper.

We will apply this theorem when $W = \Omega(K)$. We let $C_2(K)$ denote the projection of $\Omega(K)$ to M and note that $C_2(K) \subset C_1(K)$.

COROLLARY 4.2. *Let K be a compact subset of an n-dimensional complete NCP manifold, M. If $C_1(K)$ (or $C_2(K)$) has finite volume, bounded curvature, and volume growth less than r^n, then $\Omega(K)$ has Liouville measure zero.*

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Proof. Since $C_2(K) \subset C_1(K)$, we need only prove the $C_2(K)$ version. Let $W = \Omega(K)$, $C = C_2(K)$, $\tilde{M}$ be the universal cover of M with covering projection p, $\tilde{C} = p^{-1}(C)$, and $\tilde{W} = (p_*)^{-1}(W)$. Now for $x \in \pi(W)$, and $\tilde{x} \in \tilde{M}$ such that $p(\tilde{x}) = x$ we see that $B^{\tilde{W}}(\tilde{x}, r) \subset \tilde{C} \cap B(\tilde{x}, r)$ and hence $\mathrm{Vol}(B^{\tilde{W}}(\tilde{x}, r) \leq \mathrm{Vol}(\tilde{C} \cap B(\tilde{x}, r)) < f(r)r^n$, where $f(r)$ comes from the definition of volume growth less than r^n. Hence Theorem 4.1 along with the assumption that $C_2(K)$ has volume growth less than r^n implies that $\mathrm{Vol}(W) = 0$. □

If g is an NCP perturbation of (M, K, g_0), then $C_1(K, g) = C_1(K, g_0)$; however, it is not in general the case that $C_2(K, g) = C_2(K, g_0)$ even though both are subsets of the set $C_1(K, g) = C_1(K, g_0)$.

COROLLARY 4.3. *Let g be an NCP perturbation of (M, K, g_0), where K is a compact subset of an n-dimensional complete NCP manifold M. If $C_1(K, g_0)$ has finite volume, bounded curvature, and volume growth less than r^n with respect to g_0, then $\Omega_g(K)$ has Liouville measure zero.*

Proof. We first note that $C_1(K, g)$ has finite volume, and bounded curvature since $C_1(K, g_0)$ and $C_1(K, g)$ are the same subset C and disagree metrically only on the compact set K. Thus by Corollary 4.2 we need only show that C has volume growth less than r^n. Since g and g_0 only disagree on a compact set there is a constant c_0 such that for every vector v, $(1/c_0)g_0(v, v) \leq g(v, v) \leq c_0 g_0(v, v)$. Hence for any $\tilde{x} \in \tilde{M}$, $B_g(x, r) \subset B_{g_0}(\tilde{x}, c_0 r)$, and for any subset $S \subset \tilde{M}$, $\mathrm{Vol}_g(S) \leq c_0^n \mathrm{Vol}_{g_0}(S)$. Hence $\mathrm{Vol}_g(\tilde{C} \cap B_g(\tilde{x}, r)) \leq \mathrm{Vol}_g(\tilde{C} \cap B_{g_0}(\tilde{x}, c_0 r)) \leq c_0^n \mathrm{Vol}_{g_0}(\tilde{C} \cap B_{g_0}(\tilde{x}, c_0 r)) \leq c_0^n f_0(c_0 r)(c_0 r)^n$, where f_0 comes from the definition of C having growth rate less than r^n in g_0. So we can take $f(r) = c_0^{2n} f_0(c_0 r)$ and the corollary follows. □

These results are related to π_1 growth via the following lemma.

LEMMA 4.4. *Let M be an n-dimensional NCP manifold such that $\pi_1(M)$ grows more slowly than r^n, and B a bounded subset. Then the set $W = \{v \in SM \mid \gamma_v(t) \in B \text{ for all } t\}$ has Liouville measure zero.*

Proof. Fix $x_0 \in B$ and let D be such that $B \subset B(x_0, D)$. Let Σ be any generating set of $\pi_1(M)$ which includes all elements of π_1 which can be represented by loops at x_0 of length less than $3D$. We will let $p : \tilde{M} \to M$ be the projection from the universal cover, and $\tilde{x}_0 \in p^{-1}(x_0)$. For $r > 0$ let $N(r) = \{\tilde{x}_i \in p^{-1}(x_0) \mid \tilde{x}_i \text{ can be connected to } \tilde{x}_0 \text{ via a path in } p^{-1}(B(x_0, D)) \text{ of length less than } r + 2D\}$. Let $x \in B$ and $\tilde{x} \in p^{-1}(x) \cap B(\tilde{x}_0, D)$. Then each $\tilde{y} \in B^W(\tilde{x}, r)$ can be connected to $\tilde{x}$ by a path of length less than r in $p^{-1}(B)$, and hence in $p^{-1}B(x_0, D)$. Since there is some $\tilde{x}_i \in p^{-1}(x_0)$ with $\tilde{y}$ connected to $\tilde{x}_i$ in $B(\tilde{x}_i, D)$ we see that $\tilde{x}_i \in N(r)$. Thus

$$B^W(\tilde{x}, r) \subset \bigcup_{\tilde{x}_i \in N(r)} B(\tilde{x}_i, D).$$

Hence $\mathrm{Vol}(B^W(\tilde{x}, r)) \leq \#N(r)\, \mathrm{Vol}(B(\tilde{x}_0, D))$.

By Theorem 4.1, we need only show $\lim_{r\to\infty} \#N(r)/r^n = 0$ to complete the proof. To do this we will show that each $\tilde{x}_i \in N(r)$ is of the form $\alpha(\tilde{x}_0)$ where $\alpha \in \pi_1(M)$ and $|\alpha|_\Sigma \leq r/D + 2$, and then the lemma follows from the growth rate of $\pi_1(M)$.

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Fix $\tilde{x}_i \in N(r)$ and let $\tau(t)$, $0 \leq t \leq L \leq r + 2D$, be a path with arc length parameter from $\tilde{x}_0$ to $\tilde{x}_i$ in $p^{-1}(B(x_0, D))$. For each integer $1 \leq j \leq [L/D]$ let σ_j be a path of length less than D from $\tau(Dj)$ to an $\tilde{x}_{i_j} \in p^{-1}(x_0)$. Let $\tilde{x}_{i_0} = \tilde{x}_0$ and $\tilde{x}_{i_{[L/D]+1}} = \tilde{x}_i$. Thus it is easy to see that for each $0 \leq j \leq [L/D]$ there is a path of length less than $3D$ from x_{i_j} to $x_{i_{j+1}}$ and hence there is an $\alpha_j \in \Sigma$ such that $x_{i_{j+1}} = \alpha_j(x_{i_j})$. So we see that for $\alpha = \alpha_{[L/D]} \ldots \alpha_1 \alpha_0$, $\tilde{x}_i = \alpha(\tilde{x}_0)$, and $|\alpha|_\Sigma \leq r/D + 2$. The lemma follows. □

Applying the above lemma with $B = C_2(K)$ we have the following.

COROLLARY 4.5. *Let M be an n-dimensional NCP manifold such that $\pi_1(M)$ grows more slowly than r^n, and K a compact subset. If $C_2(K)$ is bounded then $\Omega(K)$ has Liouville measure zero.*

Of course the same holds if $C_1(K)$ is bounded, since then $C_2(K)$ is also bounded.

Proof of Proposition 1.5. The proof is the same as that of Lemma 4.4 except that we take D to be the intrinsic diameter of K (i.e. for every two points in K there is a path between them in K of length $\leq D$), take Σ to be a generating set of Γ including all elements representable by loops lying in K of length less than $3D$, and make sure that all the paths we choose lie in $\tilde{K}$ instead of $p^{-1}(B(x_0, D))$. □

5. *Flat case*

In this section we will prove Theorem 1.1.

We begin by recalling some facts about flat manifolds (see [**Cha**]). If (M, g_0) is a complete flat manifold, then we may identify its universal cover $\tilde{M}$ with Euclidean space E^n, and there is a $\pi_1(M)$-invariant k-dimensional affine subspace $A^k \subseteq E^n$ with $A^k/\pi_1(M)$ compact. We will write $E^n = A^k \times R^l$. Consequently, for every $a > 0$ the a-tubular neighborhood of $A^k/\pi_1(M) \subset M$ is compact, and so we may assume without loss of generality that (M, g) is an NCP perturbation of (M, C, g_0), where C is such an a-tubular neighborhood. Passing to universal covers, we find that $(\tilde{M}, \tilde{g})$ is an NCP perturbation of $(E^n, \tilde{C}, \tilde{g}_0)$ where $\tilde{C} = \{x \in E^n \mid d(x, A^k) \leq a\} = A^k \times B(a)$. In particular, since $\tilde{C}$ is convex we see that $C_1(K) \subset C$ and hence Corollary 4.5 tells us that $\Omega_g(K)$ has Liouville measure zero.

By Lemma 3.1 we see that if $v \in S(\tilde{M} - \tilde{C})$, then ∇b_v is a constant vector field on $E^n - \tilde{C}$.

In the case *codimension*$(A^k) \geq 2$, $E^n - \tilde{C}$ is connected, so $\nabla(b_v - b_v^0) = 0$ forces $b_v = b_v^0$. By Lemma 3.2 $(\tilde{M}, \tilde{g})$ is thus a weak lens equivalent perturbation of $(\tilde{M}, \tilde{C}, \tilde{g}_0)$. Corollary 2.3 now gives the result.

When A^k has codimension one, $E^n - \tilde{C}$ has two components, each isometric to a half-space. We know that a g-geodesic starting in one half-space which is not parallel to the hyperplanes bounding $\tilde{C}$ will intersect $\tilde{C}$ in a finite segment, and re-emerge in the other half-space with the same direction (measured with respect to E^n).

After passing to a twofold cover if necessary, we may assume that M has two ends, and that (M, g_0) is isometric to a Riemannian product $R \times N$ where N is a compact flat manifold. Our set C is isometric to $I \times N$ for some interval $I = [a, b]$. If the perturbation of g_0 is supported away from ∂C, then we may form a smooth closed

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Riemannian manifold X by gluing together the two (isometric) boundary components of (C, g) by the obvious map.

We now claim that X has no conjugate points. To see this, note that if $v \in S(\tilde{M}, \tilde{g})$ with $\pi(v) \in \partial\tilde{C}$, then γ_v carries a global solution of the Riccati equation which is zero on the portion of γ_v lying outside $\tilde{C}$. Hence any geodesic in $\tilde{X}$ which hits a copy of $\partial\tilde{C}$ will carry a global solution of the Riccati equation which is zero near each copy of $\partial\tilde{C}$ in $\tilde{X}$; in particular, such a geodesic has no conjugate points. On the other hand, if a geodesic in $\tilde{X}$ never hits a copy of $\partial\tilde{C}$, then it lies in a copy of $(\tilde{C}, \tilde{g})$, so it has no conjugate points.

Note that for each choice of $v \in S(C, g)|_{\partial C}$, we obtain a vector field on X which looks like the gradient of a Busemann function. The union of the images of these (smoothly varying) vector fields contains everything but $\Omega(C) \cup \mathrm{Vert}$, which has measure zero in SX. Hence the results of [**CrKl2**] imply that X—and consequently $(\tilde{M}, \tilde{g})$—is flat.

Alternatively, one could apply the recent result of Burago and Ivanov [**BuIv**] that a torus without conjugate points must be flat. □

6. $X \times R$

In this section we consider perturbations g of $(X \times R, K, g_0)$ where K is compact and $g_0 = g_X \times dt^2$ for more general (X, g_X). When X is compact and not flat there is very little more we can say. So we will be most interested in the case where (X, g_X) is not simply connected, not flat, and not compact. The proof of such rigidity theorems breaks up into showing that $\mathrm{Vol}(\Omega_g(K)) = 0$, and that g is a weak lens equivalence.

PROPOSITION 6.1. *Let (X, g_X) be a noncompact (n-1)-dimensional NCP manifold and g an NCP perturbation of $(X \times R, K, g_0)$ where K is compact. Then g will be a weak lens equivalence if at least one of the following hold:*

(1) *(X, g_X) has no focal points;*
(2) *$\pi_1(X)$ grows slower than r^{n-1};*
(3) *$(X, g_X) = (Y \times R, g_Y \times dt^2)$;*
(4) *K is totally convex and $\pi_1(X)$ has polynomial growth.*

Before proving the proposition we need a preliminary lemma.

LEMMA 6.2. *Let (X, g_X) be a simply connected manifold with no focal points, and $v \in S(X)$. If $f : X \to R$ is a $C^{1,1}$ function on X such that $|\nabla f(x)| = 1$ and $|b_v(x) - f(x)| < C$ for all $x \in X$ and some constant C, then $f(x) = b(x) + c$ for some other constant c.*

Proof. We first note that the integral curves $\gamma(t)$ of ∇f are geodesics. To see this, let $\tau(s)$, $0 \le s \le d(\gamma(t_1), \gamma(t_2))$ be the minimizing geodesic from $\gamma(t_1)$ to $\gamma(t_2)$. Then integrating $g_X(\nabla f(\tau(s)), \tau'(s))$ we see that $t_2 - t_1 = f(\gamma(t_2)) - f(\gamma(t_1)) \le d(\gamma(t_1), \gamma(t_2))$, and hence γ is a geodesic.

Fix $x \in X$ and let $w = -\nabla f(x)$ and $u = -\nabla b_v(x)$. Let $g(t) = b_v(\gamma_w(t)) - f(\gamma_w(t)) = b_v(\gamma_w(t)) + t - f(x)$. Now if $w \neq u$ we see that $g'(0) > 0$. Furthermore, since there are no focal points, b_v is convex, and hence $g''(t) \ge 0$, and thus g is

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unbounded. This contradicts the assumption of the lemma and hence we conclude that $\nabla f = \nabla b_v$ and the lemma follows. □

Proof of Proposition 6.1. We will let $M = X \times R$, $K \subset X \times [-a, a]$, and $\tilde{M} = \tilde{X} \times R$ represent the universal covering space. Let $v \in S_a^+(\tilde{M})$ and fix $t_0 \notin [-a, a]$. By Lemma 3.1, the gradients of the restrictions of b_v and b_{0v} to $\tilde{X} \times \{t_0\}$ have the same constant norm α. They thus, after multiplying by $1/\alpha$, induce functions $b_v^{t_0}$ and $b_{0v}^{t_0}$ on $\tilde{X}$ with gradients of norm one and of a bounded distance from each other. Furthermore, since g_0 is the product metric $b_{0v}^{t_0}$ is a Busemann function of $\tilde{X}$.

Proof of (1). By Lemma 6.2 we see that $b_v^{t_0}(x) = b_{0v}^{t_0}(x) + c$ for some constant c. On the other hand, since X is noncompact and K is compact, there is an $x_0 \in \tilde{X}$ such that (x_0, t_0) is not in the shadow $S(\tilde{K}, v)$; hence $b_v(x_0) = b_{0v}(x_0)$ and hence $c = 0$. Thus Lemma 3.2 gives the result.

Proof of (2). For $D > 0$ and $C \subset \tilde{X}$ let $T_D(C)$ represent the D tubular neighborhood of C. Let $W^D = \{w \in S(X) \mid \gamma_w(t) \in T_D(\pi_X(K)) \text{ for all } t\}$, where π_X is the projection from M to X. Assumption (2) of Proposition 6.1 along with Lemma 4.4 tells us that W^D (and hence its lift $\tilde{W}^D$ to $S(\tilde{X})$) has Liouville measure zero for all $D > 0$.

Let $S^{t_0} = \pi_{\tilde{X}}(S(\tilde{K}, v) \cap \tilde{X} \times \{t_0\})$, i.e. the shadow at t_0. Now if $\gamma(s)$ is a gradient geodesic of $b_{0v}^{t_0}(x)$ on $\tilde{X}$ and for some s_0, $\gamma(s_0) \notin S^{t_0}$, then $b_{0v}^{t_0}(\gamma(s_0)) = b_v^{t_0}(\gamma(s_0))$ and $\nabla b_{0v}^{t_0}(\gamma(s_0)) = \nabla b_v^{t_0}(\gamma(s_0))$. Thus $\gamma(s)$ is also a gradient geodesic of $b_v^{t_0}$, and we see that $b_{0v}^{t_0}(\gamma(s)) = b_v^{t_0}(\gamma(s))$ for all s. On the other hand, if $\gamma(s_0) \in S^{t_0}$ then for some $t \in [-a, a]$ we have $(\gamma(s_0 + ((t_0 + t)/\sqrt{1-\alpha^2})), t) \in \tilde{K}$ so $\gamma(s_0 + (t_0/\sqrt{1-\alpha^2})) \in T_D\pi_{\tilde{X}}(\tilde{K})$ where $D = (a/\sqrt{1-\alpha^2})$. Thus if $\gamma(s) \in S^{t_0}$ for all s, then $\gamma(s) \in T_D\pi_{\tilde{X}}(\tilde{K})$ for all s, i.e. $\gamma'(s) \in \tilde{W}^D$.

Let γ be a g-geodesic with $v = \gamma'(0) \in S_a^+(\tilde{M})$ and $\pi_R(\gamma(t)) < -a$ for all $t < T$. If $\pi_{X*}(\gamma'(T))/|\pi_{X*}(\gamma'(T))| \notin \tilde{W}^D$, the above tells us that $b_v(\gamma(t)) = b_{0v}(\gamma(t))$ for all $t < T$. This also holds for γ_0 a g_0-geodesic with $v = \gamma_0'(0) \in S_a^+(\tilde{M})$ and $\pi_R(\gamma_0(t)) < -a$ for all $t < T$. That is, if $\pi_{X*}(\gamma_0'(T))/|\pi_{X*}(\gamma_0'(T))| \notin \tilde{W}^D$, then $b_v(\gamma_0(t)) = b_{0v}(\gamma_0(t))$ for all $t < T$.

We finish the proof by modifying the proof of Lemma 3.2. It is sufficient to prove weak lens equivalence for the induced metrics on $\tilde{M}$. Let γ_0 be a non-horizontal g_0-geodesic in $\tilde{M}$. We can assume (by changing the orientation and shifting the parameter of γ_0 if necessary) that $\pi_R(\gamma_0(t)) = bt$ for some $0 < b \leq 1$. Let $d > a/b$, $x = \gamma_0(-d)$, $y = \gamma_0(d)$, and $v = \gamma_0'(d)$. Since $\tilde{W}^D$ has measure zero we can choose $v_i \in S(\tilde{M})$ converging to v such that $\pi_{X*}(\gamma_{v_i}'(-2d))/|\pi_{X*}(\gamma_{v_i}'(-2d))| \notin \tilde{W}^D$. Let $x_i = \gamma_{0v_i}(-2d)$ and $y_i = \gamma_{0v_i}(0)$. We see that $2d = d_0(x_i, y_i) = b_{0v_i}(x_i) = b_{v_i}(x_i) \leq d(x_i, y_i)$, and hence taking limits $d_0(x, y) \leq d(x, y)$. On the other hand, if γ is the g-geodesic from x to y (with $\gamma(0) = y$), let $w = \gamma'(0)$ and as above choose w_i converging to w such that $\pi_{X*}(\gamma_{w_i}'(-2d))/|\pi_{X*}(\gamma_{w_i}'(-2d))| \notin \tilde{W}^D$. Let $x_i' = \gamma_{0w_i}(-2d)$ and $y_i' = \gamma_{0w_i}(0)$. Then $d(x_i', y_i') = b_{w_i}(x_i') = b_{0w_i}(x_i') \leq d_0(x_i', y_i')$, and taking limits $d(x, y) \leq d_0(x, y)$. Thus we get for every large d, $2d = d_0(\gamma_0(d), \gamma_0(-d)) = d(\gamma_0(d), \gamma_0(-d))$ and we see that γ_0 has a compact perturbation and thus that g is a weak lens equivalence.

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Proof of (3). As usual we will let $M = X \times R$, $K \subset Y \times B(a) \subset X \times [-a, a]$, and $\tilde{M} = \tilde{X} \times R = \tilde{Y} \times R^2$ represent the universal covering space. We will let $C = Y \times B(a)$ so $K \subset C$.

We will prove the result using Lemma 3.2. To that end let $v \in S_a^+(\tilde{M})$.

We first study the product metric g_0. For each $y \in Y$ let $P_y = \{y\} \times R^2$ be the totally geodesic two plane through $(y, 0, 0)$. Since $g_0 = g_Y \times dz^2$ where dz^2 is the flat metric on R^2, we see that, for every $y \in Y$, $b_{0v}|_{P_y}$ is linear and the level set, $LS_{0v}(r, s) = \partial B_{0v}(r, s) \cap P_y$, of $b_{0v}|_{P_y}$ through (r, s) is the straight line perpendicular to the projection of v to R^2. It is also easy to see that the shadow $S(C, v) \cap P_y$ is the a-tubular neighborhood of the ray τ through the origin parallel and opposite to the projection of v onto R^2.

For ease of discussion we will put coordinates (r, s) on the plane P_y such that τ is the positive r-axis. So $b_{0v}((r, s)) = L(r)$ for some linear function $L : R \to R$ with $L'(r) > 0$.

By applying Lemma 3.2, we need only show that b_{0v} and b_v agree on $P_y - B(a)$. Since b_{0v} and b_v agree on $M - S(C, v)$, we have for $(r, s) \in P_y - S(C, v)$, $b_v(r, s) = L(r)$. For $(r_0, s_0) \in P_y$ with $r_0 > a$ let $\psi = \nabla b_v(r_0, s_0) \in T_{(r_0,s_0)}M$ be the outward normal to the level set of b_v. Then $\partial B_\psi(r_0, s_0)$ intersects P_y in a line, $LS_\psi(r_0, s_0)$, through (r_0, s_0) (since it coincides with $\partial B_{0\psi}(r_0, s_0) \cap P_y$) which must lie 'to the right' of the level set $LS_v(r_0, s_0)$ of b_v (i.e. if $(r_1, s_1) \in LS_\psi(r_0, s_0)$ then $b_v(r_1, s_1) \geq b_v(r_0, s_0)$).

For $r_0 > a$ and $|s_0| > a$ we have $LS_v(r_0, s_0) \cap \{(r, s) || s| > a\} = \{(r_0, s) || s| > a\}$ and hence $LS_\psi(r_0, s_0) = \{(r_0, s)\}$. Hence it is easy to see that for any $(r_1, s_1) \in LS_v(r_0, s_0)$ we have $r_1 \leq r_0$. On the other hand, if r_0 is chosen large enough such that $LS_v(r_0, 2a) \subset \{(r, s) \mid r > a\}$ and $(r_1, s_1) \in LS_v(r_0, 2a)$, then the above argument forces the set $\{(r_0, s) \mid |s| > a\}$ to lie to the left of the line $LS_\psi(r_1, s_1)$ ($\psi = \nabla b_v(r_1, s_1)$) through (r_1, s_1) and hence $r_1 \geq r_0$. Thus for r_0 sufficiently large the lines $r = r_0$ are level sets of $b_v|_{P_y}$ and hence b_v and b_{0v} agree on all of $P_y - B(a)$. Case (3) follows.

Proof of (4). First note that $\pi_1(K) \equiv \pi_1(X \times R) \equiv \pi_1(X)$ (since X is connected) as loops in X based at some point $p \in K$ are homotopic to geodesic loops based at p, which lie entirely in K since K is totally convex. $\pi_1(K)$ has polynomial growth, so it is a virtually nilpotent group by [**Grm**], and hence it is virtually Abelian by [**CrS**]. The homological dimension of $\pi_1(K)$ is less than $n-1$ since $\pi_1(K) \equiv \pi_1(X)$ and X is noncompact; hence $\pi_1(K)$ is virtually Z^k with $k < n-1$, and we are reduced to case (2) of the proposition. □

Proof of Theorem 1.3. By the proof of part (4) of Proposition 6.1, we know that $\pi_1(K)$ has growth slower than $\dim(X)$. Hence we may apply Corollary 4.4, Proposition 6.1, and finally Corollary 2.3 to conclude that g is isometric to g_0. □

We can now combine the results of §4 and 6 with Corollary 2.3 to prove Theorem 1.4.

7. *Jacobi tensors and proofs*

We will be considering Lagrange tensors along a geodesic $\gamma(t)$. (For a detailed discussion see [**EsOS**] and [**E**].) These are tensors, A, which satisfy:

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(1) the Jacobi equation

$$A(t)'' + R(t)A(t) = 0$$

where R is the curvature tensor $R(., \gamma'(t))\gamma'(t)$;

(2) $A'(t)A^{-1}(t)$ is symmetric whenever A^{-1} exists;

(3) $A(t)$ is nondegenerate in the sense that $\ker A(t) \cap \ker A'(t) = 0$.

For $v \in SM$ there are two such Lagrange tensors along γ_v (the geodesic with $\gamma'(0) = v$) of interest to us. J_v will be the Lagrange tensor with $J_v(0) = 0$ and $J_v'(0) = \mathrm{Id}$. A_v will be the Lagrange tensor with $A_v(0) = \mathrm{Id}$ and $A_v'(0) = 0$. J_v and A_v are Lagangian since Jacobi tensors (i.e. tensors satisfying (1) above) which satisfy (2) (respectively, (3)) for some t must satisfy (2) (respectively, (3)) for all t. If we fix an orthonormal basis for $v^\perp$, then with respect to parallel bases along γ_v we can think of $J_v(t)$ and $A_v(t)$ as matrices where the columns represent Jacobi fields with respect to these bases.

It is not hard to check, since A_v is Lagrange (see [**EsOS**]), that if $A_v(t)$ is nonsingular for all $t \in [0, s]$ we have

$$J_v(s) = A_v(s) \int_0^s A_v^{-1}(t) A_v^{-1*}(t)\, dt. \tag{1}$$

When γ_v is a geodesic in an NCP manifold we define a tensor $U(v) = \lim_{t\to\infty} J'_{g^t v}(-t) J^{-1}_{g^t v}(-t)$. Note that $J'_{g^t v}(-t) J^{-1}_{g^t v}(-t)$ is a tensor at $\gamma_v(0)$ for all t that represents the second fundamental form of the sphere of radius t centered at $\gamma_v(-t)$. See [**EsOS**] for a proof that $U(v)$ exists. We will let $u(v) = tr(U(v))$. Then it is easy to see, by looking at $u(v)$ as the limit of mean curvatures of spheres, that $u(v) + u(-v) \geq 0$.

If M is an NCP manifold and $W \subset SM$ is a measurable set, invariant under the geodesic flow, such that $\pi(W)$ has finite volume and bounded sectional curvatures, then (as was shown in [**EsOS**] and [**G**]) $u(v)$ is bounded and varies measurably in v for $v \in W$. Hence we can define $h_\mu^W = \int_W u(v)\, \overline{dv}$ where $\overline{dv}$ is the normalized Liouville measure, i.e. $\overline{dv} = (1/\mathrm{Vol}(W))\, dv$. We note, in particular, that when $-W = W$ we have $h_\mu^W \geq 0$. We remark in passing that when $W = SM$ then h_μ^W is the measure entropy of the geodesic flow (see [**FrMa**]).

The following lemma is Lemma 1 of [**Cr3**].

LEMMA 7.1. *Let J_v and u be as above. Then*

$$\det(J_v(l)) \geq \left[\int_0^l \exp\left\{ \frac{1}{n-1} \left[\int_0^l u(g^z v)\, dz - 2 \int_0^s u(g^z v)\, dz \right] \right\} ds \right]^{n-1}.$$

Proof of Theorem 4.1.

$$\begin{aligned} &\mathrm{Ave}(\mathrm{Vol}(\tilde{S}^W(x, l))) \\ &= \frac{1}{\mathrm{Vol}(\pi(W))} \int_W \det(J_v(l))\, dv \\ &\geq \frac{\mathrm{Vol}(W)}{\mathrm{Vol}(\pi(W))} \left[\int_W \int_0^l \exp\left\{ \frac{1}{n-1} \left[\int_0^l u(g^z v)\, dz - 2 \int_0^s u(g^z v)\, dz \right] \right\} ds\, \overline{dv} \right]^{n-1} \end{aligned}$$

by Lemma 7.1, the definition of $\overline{dv}$, and a Höelder inequality. Hence by changing the order of integration, and then applying Jensen's inequality to the convex function exp

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we see

$$\left[\frac{\text{Ave}(\text{Vol}(\tilde{S}^W(x,l)))\,\text{Vol}(\pi(W))}{\text{Vol}(W)}\right]^{1/(n-1)}$$
$$\geq \int_0^l \int_W \exp\left\{\frac{1}{n-1}\left[\int_0^l u(g^z v)\,dz - 2\int_0^s u(g^z v)\,dz\right]\right\}\overline{dv}\,ds$$
$$\geq \int_0^l \exp\left\{\int_W \frac{1}{n-1}\left[\int_0^l u(g^z v)\,dz - 2\int_0^s u(g^z v)\,dz\right]\overline{dv}\right\}ds.$$

Now the fact that $\overline{dv}$ and W are invariant under the geodesic flow g^z yields

$$\int_W \int_0^t u(g^z v)\,dz\,\overline{dv} = \int_0^t \int_W u(g^z v)\,\overline{dv}\,dz = \int_0^t \int_W u(v)\,\overline{dv}\,dz = t h_\mu^W.$$

Applying this to the above inequality yields:

$$\left[\frac{\text{Ave}(\text{Vol}(\tilde{S}(x,l)))\,\text{Vol}(\pi(W))}{\text{Vol}(W)}\right]^{1/(n-1)} \geq \int_0^l \exp\left\{\frac{l-2s}{n-1}h_\mu^W\right\}ds \geq l$$

where the last inequality follows (after performing the integral) since $h_\mu^W \geq 0$. Thus we see that $\text{Ave}(\text{Vol}(\tilde{S}(x,l))) \geq (\text{Vol}(W)/\text{Vol}(\pi(W)))l^{n-1}$. Integrating both sides as l runs from 0 to r yields the theorem. □

The next lemma is the primary ingredient in the proof of Proposition 2.2. We will let $B^{n-1}(\epsilon)$ represent the ϵ-ball in E^{n-1}, dx represent the standard volume form, and $\pi : B^{n-1}(\epsilon) \times R \to B^{n-1}(\epsilon)$ represent the projection.

LEMMA 7.2. *Let (M, g_0) be an NCP manifold and $C \subset M$ compact. Let $F_0 : B^{n-1}(\epsilon) \times R \to M$ be a smooth injective map such that:*

(1) *$F_0^*(g_0)$ is a product metric;*
(2) *for each fixed x, $\gamma_{0x}(t) = F_0(x,t)$ is a geodesic parametrized by arc length;*
(3) *there is a T such that $F_0^{-1}(C) \subset B^{n-1}(\epsilon) \times [-T, T]$.*

Let g be an NCP perturbation of g_0 supported on C, which is a local lens equivalence in the sense that all 'nearly vertical' g_0-geodesics (i.e. those with initial tangents sufficiently close to $\gamma'_{0x}(t)$ for some x and t), have g-geodesics that are compact perturbations. Let $F : B^{n-1}(\epsilon) \times R \to M$ be the corresponding map to (M, g) using the compact perturbations $\gamma_x(t)$. Let μ_0 (respectively, μ) be the pullback under F_0 (respectively, F) of the g_0-Riemannian measure (respectively, g-Riemannian measure). Define $f_0 : B^{n-1}(\epsilon) \to R^+$ (respectively, f) by $f_0(x)\,dx = \pi_*(\mu_0 \mid_{F_0^{-1}(C)})$ (respectively, f).

Then $f_0(x) \geq f(x)$ for all x with equality holding if and only if $F^*(g)$ is also a product metric (hence isometric to $F_0^*(g_0)$).

Proof. Since g and g_0 agree outside C we can assume that $F^{-1}(C) = F_0^{-1}(C) = B^{n-1}(\epsilon) \times [-T, T]$. For $x \in B^{n-1}(\epsilon)$ we will use the notation $A(x,t)$ (respectively, $J(x,t)$) for the matrix representation of $A_v(t+T)$ (respectively, $J_v(t+T)$), where $v = \gamma'_{0x}(-T)) = \gamma'_x(-T)$, along the g-geodesic $\gamma_x(t)$ with respect to a parallel orthonormal basis. We similarly define $A_0(x,t)$ and $J_0(x,t)$ along the g_0-geodesic γ_{0v};

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we choose the orthonormal basis to agree at $(x, -T)$. Since g_0 is the product metric $A_0(x,t) = \mathrm{Id}$ and $J_0(x,t) = (t+T)\,\mathrm{Id}$.

If for some t_0, $A(x,t_0)$ is singular (say $A(x,t_0)\vec{r} = 0$ for some $\vec{r} \in R^{n-1}$) then there would be a Jacobi field, j (i.e. $j(t) = A(x,t)\vec{r}$), along γ_x such that $j(t_0) = 0$ and $j(t)$ is parallel and hence bounded for $t \le -T$. However, since the curvatures are bounded along γ_x, Theorem 1 of [**EsOS**] says that there are no such Jacobi fields. Hence $A(x,t)$ is always nonsingular.

With respect to our chosen orthonormal bases, $A(x,t)$ represents $dF(x,t)$ restricted to vectors perpendicular to $\gamma_x'(t)$. So we see that F is an immersion and

$$f_0(x) = \int_{-T}^{T} \mathrm{Det}(A_0(x,t))\,dt\,dx = 2T \quad \text{and} \quad f(x) = \int_{-T}^{T} \mathrm{Det}(A(x,t))\,dt\,dx.$$

By the local lens equivalence, nearly vertical g_0-geodesics have g-geodesics as compact perturbations. Hence, by taking variations of such geodesics, we see that Jacobi fields $j_0(t)$ and $j(t)$ along γ_x and γ_{0x}, respectively, with $j_0(-T) = j(-T)$ and $j_0'(-T) = j'(T)$ have $j_0(t) = j(t)$ for all $t \notin [-T, T]$ (note $\gamma_x(t) = \gamma_{0x}(t)$ for $t \notin [-T, T]$). Hence we see that Jacobi tensors along γ_x and γ_{0x} with the same initial conditions at $-T$ must agree for $t \notin [-T, T]$. (Note that this does not necessarily mean that they have the same matrix representation since parallel bases along γ_x and γ_{0x} that are the same for $t \le -T$ may differ by a fixed orthogonal transformation for $t \ge 2r$. However, the determinants will be the same.) Thus we see from equation (1)

$$\begin{aligned} \mathrm{Det}\left[\frac{1}{2T}\int_{-T}^{T} A^{-1}(x,t)A^{-1*}(x,t)\,dt\right] &= \mathrm{Det}\left[\frac{1}{2T}A^{-1}(x,T)J(x,T)\right] \\ &= \mathrm{Det}\left[\frac{1}{2T}A_0^{-1}(x,T)J_0(x,T)\right] = 1. \end{aligned}$$

Using the above equation and the convexity of $\mathrm{Det}^{-1/2}$ on the space of self-adjoint positive definite matrices (see [**B**, 11.8.9.5]) we see that

$$\begin{aligned} f(x) &= 2T\frac{1}{2T}\int_{-T}^{T} \mathrm{Det}(A(x,t))\,dt \\ &\ge 2T\,\mathrm{Det}^{-1/2}\left[\frac{1}{2T}\int_{-T}^{T} A^{-1}(x,t)A^{-1*}(x,t)\,dt\right] = 2T = f_0(x). \end{aligned}$$

The equality holds if and only if $A^{-1}(x,t)A^{-1*}(x,t)$ is constant in t, and hence is the identity. But this is the case if and only if $A(x,t)$ is orthogonal, hence $A'(x,t)A^{-1}(x,t)$ is skew symmetric. On the other hand, we know that $A'(x,t)A^{-1}(x,t)$ is symmetric, thus $A'(x,t) = 0$ and $A(x,t) = \mathrm{Id}$. Hence equality holds if and only if $A(x,t) = \mathrm{Id}$. If equality holds for all x we see that $F^*(g)$ is isometric to $F^*(g_0)$. □

Proof of Proposition 2.2. If M is itself a product with a line then Lemma 7.2 gives the result rather easily. In our more general setting we need to be somewhat more careful. We will assume that l is the maximum such that $\tilde{M} = \tilde{X} \times R^l$ and $\tilde{K} \subset \tilde{X} \times B(a)$ for some choice of $\tilde{X}$ (if $\tilde{M}$ is Euclidean, for example, there are a lot of choices for splittings). This maximality implies that the deck group, $\pi_1(M)$, acts so as to preserve the splitting.

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We let $\tilde{\mathcal{H}}_0$ be the collection of vertical $v \in S(\tilde{M}, g_0)$. $\tilde{\mathcal{H}}_0$ carries the natural n-dimensional foliation $\tilde{\mathcal{F}}_0$ with leaves consisting of parallel vectors. We define $\tilde{\mathcal{H}}$ to be the collection of $v \in S(\tilde{M}, g)$ such that there is a $t \in R$ with $\gamma_v(t) \in \tilde{M} - \tilde{X} \times B(a)$ and $\gamma_v'(t) \in \tilde{\mathcal{H}}_0$. $\tilde{\mathcal{H}} \subset S(\tilde{M}, g)$ is invariant under the geodesic flow of g, and since the amount of time that γ_v spends in $\tilde{X} \times B(a)$ is uniformly bounded for $v \in \tilde{\mathcal{H}}$ (see Lemma 3.1), it is clear that $\tilde{\mathcal{H}}$ is a closed submanifold of $S(\tilde{M}, g)$. We may define a foliation $\tilde{\mathcal{F}}$ of $\tilde{\mathcal{H}}$ by declaring the leaves to be the sets of $v \in \tilde{\mathcal{H}}$ such that $\gamma_v'(t) \in S(\tilde{M} - \tilde{X} \times B(a))$ is parallel to a fixed $w \in S(\tilde{M} - \tilde{X} \times B(a), g_0)$. $\tilde{\mathcal{H}}$ is invariant under the natural action of $\pi_1(M)$ on $S(\tilde{M}, g)$, and we set $\mathcal{H} = \tilde{\mathcal{H}}/\pi_1(M) \subset S(M, g)$.

It is obvious that π_0 maps each leaf of $\tilde{\mathcal{F}}_0$ diffeomorphically onto $\tilde{M}$, where $\pi_0 : S(\tilde{M}, g_o) \to \tilde{M}$ is the projection. It is also true that π is a diffeomorphism on each leaf of $\tilde{\mathcal{F}}$. This follows since for each leaf of $\tilde{\mathcal{F}}$ there is a natural diffeomorphism, d, to the corresponding leaf of $\tilde{\mathcal{F}}_0$, and the composition $\pi \circ d^{-1} \circ \pi_0^{-1}$ is a proper (by Lemma 3.1) immersion (by the argument that $A(x, t)$ is nonsingular in the proof of Lemma 7.2) and hence is a diffeomorphism.

We now put a measure on each leaf of $\tilde{\mathcal{F}}_0$ (respectively, $\tilde{\mathcal{F}}$) by pulling back the Riemannian volume form from $(\tilde{M}, g_0)$ (respectively, $(\tilde{M}, g)$), and define a transverse measure on $\tilde{\mathcal{F}}_0$ (respectively, $\tilde{\mathcal{F}}$) by identifying the leaf space of $\tilde{\mathcal{F}}_0$ (respectively, $\tilde{\mathcal{F}}$) with the set of directions S^{l-1} in R^l, and using normalized Lebesgue measure on S^{l-1}. Combining these two measures, we get a measure $\tilde{\mu}_0$ on $\tilde{\mathcal{H}}_0$ (respectively, $\tilde{\mu}$ on $\tilde{\mathcal{H}}$) such that $\pi_{0*}\tilde{\mu}_0$ (respectively, $\pi_*\tilde{\mu}$) is the Riemannian volume on $(\tilde{M}, g_0)$ (respectively, $(\tilde{M}, \tilde{g})$). If we let $\tilde{\mathcal{G}}_0$ (respectively, $\tilde{\mathcal{G}}$) be the orbit space of $\tilde{\mathcal{H}}_0$ (respectively, $\tilde{\mathcal{H}}$) under the geodesic flow, then $\tilde{\mathcal{G}}_0$ and $\tilde{\mathcal{G}}$ are naturally diffeomorphic; moreover, using the local product structures of the measures and applying Lemma 7.2 locally on the leaves of $\tilde{\mathcal{F}}_0$ and $\tilde{\mathcal{F}}$ we see that the pushforward of $\mu|_{\pi^{-1}(\tilde{X} \times B(a))}$ to $\tilde{\mathcal{G}} \simeq \tilde{\mathcal{G}}_0$ under the quotient map $\tilde{\mathcal{H}} \longrightarrow \tilde{\mathcal{G}}$ is greater than or equal to the pushforward of $\mu_0|_{\pi^{-1}(\tilde{X} \times B(a))}$ to $\tilde{\mathcal{G}}_0$ under the projection $\tilde{\mathcal{H}}_0 \longrightarrow \tilde{\mathcal{G}}_0$, with equality only if the leaves of $\tilde{\mathcal{F}}$ correspond to parallel vector fields on $(\tilde{M}, g)$.

If we denote by μ_0 (respectively, μ) the corresponding measures in $\mathcal{H}_0$ (respectively, $\mathcal{H}$), then $\pi_*\mu_0$ (respectively, $\pi_*\mu$) will be the Riemannian measure for (M, g_0) (respectively, (M, g)). In particular, we have $\int_{\mathcal{H} \cap \pi^{-1}(K)} \mu = \mathrm{Vol}_g(K)$ and $\int_{\mathcal{H}_0 \cap \pi_0^{-1}(K)} \mu = \mathrm{Vol}_{g_0}(K)$. If we now let $\mathcal{G}_0$ (respectively, $\mathcal{G}$) be the orbit space of $\mathcal{H}_0$ (respectively, $\mathcal{H}$) under the geodesic flow, then again $\mathcal{G}_0$ and $\mathcal{G}$ are naturally diffeomorphic; and the pushforward of $\mu|_{\pi^{-1}(K)}$ to $\mathcal{G} \simeq \mathcal{G}_0$ under the quotient map $\mathcal{H} \longrightarrow \mathcal{G}$ is greater than or equal to the pushforward of $\mu_0|_{\pi^{-1}(K)}$ to $\mathcal{G}_0$ under the projection $\mathcal{H}_0 \longrightarrow \mathcal{G}_0$, with equality only if the leaves of $\tilde{\mathcal{F}}$ correspond to parallel vector fields on $(\tilde{M}, \tilde{g})$.

Thus we see that $\mathrm{Vol}_g(K) \geq \mathrm{Vol}_{g_0}(K)$, with equality holding only if g is isometric to g_0. □

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