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Comparing Assembly Maps in Algebraic K-Theory

by

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Abstract

Given a group G and a ring R, Loday [Lod, 1976] described an assembly map $\alpha_G : h_n(BG;\mathbf{L}(R)) \to K_n(RG)$ where $\mathbf{L}(R)$ is a spectrum with nth space $K_0(S^nR) \times BGL(S^nR)^+$ for $n \geq 0$ and $K_n(RG) = \pi_n(BGL(RG)^+ \times K_0(RG))$. Hambleton and Pederson, [HP, 2004], indicate a proof that this map is isomorphic to the map on homotopy groups from the assembly map $A_G : BG_+ \wedge \mathbb{K}^{-\infty}(R) \to \mathbb{K}^{-\infty}(RG)$. We will complete the proof of this.

Key Words: K-Theory spectra, assembly maps, pairings of spectra

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1. Introduction

Algebraic K-theory started off as the study of two functors, K_0 and K_1, from the category of rings to the category of Abelian groups. Later, an additional functor K_2 was created. To construct functors K_n for $n \geq 3$ there was a construction due to Quillen called the plus construction which is used to construct for a ring R, a space $BGL(R)^+$, whose homotopy groups π_1 and π_2 agreed with the corresponding K-groups, see Rosenberg,[Ros, 1994],section 5.2 for details. This construction eventually evolved into a spectrum, whose zeroth space was $BGL(R)^+ \times K_0(R)$. Other more general constructions have been made. In general, today algebraic K-theory considers functors from certain types of categories to spaces or spectra. For notation, we will use K to represent a functor to spaces and $\mathbf{K}$ for one to spectra. $K_0(R)$ will refer to the classical group.

Loday, [Lod, 1976], pg. 333, constructed a map $\hat{\gamma} : L(R) \wedge L(R') \to L(R \otimes R')$. In fact he was able to extend this to a weak pairing $\hat{\gamma} : \mathbf{L}(R) \wedge \mathbf{L}(R') \to \mathbf{L}(R \otimes R')$. He was then able, for a group G, to construct a map from $BG_+ \wedge \mathbf{L}(R) \to \mathbf{L}(RG)$ as follows. There is a natural map $i : G \to BGL(\mathbb{Z}G)$ that comes from sending an element g to the 1×1 matrix with g as the entry. This induces a map $i_* : BG \to BGL(\mathbb{Z}G)$ and thus $j_* : BG \to BGL(\mathbb{Z}G)^+$. So the map (for the zeroth space) is

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the composition

$$BG_+ \wedge BGL(R)^+ \to (BGL(\mathbb{Z}G)^+)_+ \wedge BGL(R)^+ \to BGL(\mathbb{Z}G \otimes R)^+$$

Loday then shows that this can be extended to a map $BG_+ \wedge \mathbf{L}(R) \to \mathbf{L}(RG)$. This map is called an assembly map.

Weiss and Williams, [WW, 1993], in Theorem 1.1, describe a general type of assembly map for a homotopy invariant functor from spaces to spectra. More specifically, given a functor $F(-)$ from spaces to spectra that is homotopy invariant, they construct a strongly excisive, homotopy invariant functor $F^{\%}(-)$ from spaces to spectra, and a natural transformation $\alpha_F : F^{\%} \to F$ so that $\alpha_F : F^{\%}(*) \to F(*)$ is a homotopy equivalence.

Many other assembly maps have been constructed. An argument that shows that Loday's assembly map and the one coming from the spectrum constructed in [PW2, 1983] are homotopic was outlined in [HP, 2004]. We will fill in the details to show this.

2. Definitions

Different authors have used different definitions of spectra. For our purposes we use the following definition.

Definition 2.1 A spectrum is a collection of pointed spaces $\{X_i\}$ for all non-negative integers i along with pointed maps $f_i : \Sigma X_i \to X_{i+1}$, or by adjointness, maps $f_i^* : X_i \to \Omega X_{i+1}$.

To define a category of spectra we define the morphisms as follows. A function between spectra $\mathbf{g} : \mathbf{X} \to \mathbf{Y}$ is a collection of pointed maps $g_i : X_i \to Y_i$ so that $g_{i+1} \circ f_i = f_i' \circ \Sigma g_i$ where f_i is the structure map for the spectrum $\mathbf{X}$ and f_i' is the structure map for the spectrum $\mathbf{Y}$.

This is referred to in some of the literature as the category of prespectra with spectra being prespectra that require the maps $f_i^* : X_i \to \Omega X_{i+1}$ to all be homeomorphisms. We now introduce the notion of (weak) pairings on spectra.

Definition 2.2 Let $\mathbf{X}, \mathbf{Y}$, and $\mathbf{Z}$ be spectra. A pairing is a set of maps: $X_n \wedge Y_m \to Z_{n+m}$ which commute with the structure maps. A weak pairing is a collection of such maps that commute up to homotopy with the structure maps.

In order to define Loday's K-theory spectrum, we need the notion of the *suspension* of a ring. For a ring R we define lR as $\mathbb{N} \times \mathbb{N}$ indexed matrices with entries in R that are locally finite, i.e. each column and each row only have finitely many non-zero entries. Then mR is the ideal of lR with only finitely many nonzero terms. We then can define $\mu R = lR/mr$. Then the suspension of a ring

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R is defined by $SR = \mu\mathbb{Z} \otimes R$. We can the inductively define $S^n R$ and $\mu^n R$ by $S^{n+1}R = S(S^n R)$ and $\mu^{n+1}R = \mu(\mu^n R)$. There is a natural map from $SR \to \mu R$ that comes from the map $\mathbb{Z} \otimes R \to R$. This then can inductively be used to define maps $S^n R \to \mu^n R$. Then we can define Loday's K-theory spectrum $\mathbf{L}(R)$ with nth space given by $L_n(R) = K_0(S^n R) \times BGL(S^n R)^+$.

To define Pedersen and Weibel's spectrum requires some definitions. For the following definitions, $\mathcal{A}$ will be an additive category and $\underline{\underline{\mathcal{A}}}$ will be the subcategory of $\mathcal{A}$ with the same class of objects, but restricting the morphisms to just the isomorphisms of $\mathcal{A}$. We say $\mathcal{A}$ is filtered if for any objects A, B the set $\mathrm{Hom}(A,B)$ has a filtration $F_0(A,B) \subseteq F_1(A,B) \subseteq \ldots F_n(A,B) \subseteq \ldots$ satisfying the following:

1. $F_n(A,B)$ is a subgroup of Hom(A,B).
2. Hom(A,B)=$\cup F_n(A,B)$.
3. 1_A and 0_A are in $F_0(A,A)$.
4. projections $A \oplus B \to A$ and inclusions $A \to A \oplus B$ and all coherence isomorphisms are in F_0.
5. If $f \in F_n(A,B)$ and $g \in F_m(B,C)$ then $gf \in F_{m+n}(A,C)$.

We say that the morphisms in $F_d(A,B)$ have filtration degree d.

Let (M,d) be a metric space and let $\mathcal{A}$ be a (filtered) additive category. We can define the filtered category $C_M(\mathcal{A})$ as in [PW1, 1986], definition 1.1 as follows. An object is a collection $\{A_x\}_{x \in M}$ of objects of $\mathcal{A}$ with the restriction that the set $\{x \in M | A_x \neq 0\}$ is locally finite. A morphism $f : A \to B$ is a collection of $\mathcal{A}$−morphisms $f_y^x : A_x \to B_y$ with the condition that for every morphism f, there is an $R > 0$ so that if $d(x,y) > R$ both f_y^x and f_x^y are zero. This gives a natural filtration where $F_m(A,B)$ are the morphisms that are bounded by m. If $\mathcal{A}$ is already filtered, we also demand that f_y^x has filtration degree m as well.

With these definitions, if M and N are metric spaces and we put the metric on $M \times N$ by $d((x_1,y_1),(x_2,y_2)) = \max\{d(x_1,x_2),d(y_1,y_2)\}$ then we have $C_{M\times N}(\mathcal{A}) = C_M(C_N(\mathcal{A}))$. For any additive category $\mathcal{A}$ we define the K-theory space of $\mathcal{A}$ by $K(\mathcal{A}) = B\underline{\underline{\mathcal{A}}}^{-1}\underline{\underline{\mathcal{A}}}$, where the category $\underline{\underline{\mathcal{A}}}^{-1}\underline{\underline{\mathcal{A}}}$ is as described by Grayson in [Gra, 1976], pg. 219. Here for a category $\mathcal{C}$, $B\mathcal{C}$ is the classifying space for this category. We now have enough to define the spectrum of Pedersen and Weibel from [PW2, 1983] using their definition of $C_i(\mathcal{A})$ in section 1. Let $\mathbb{Z}^i$ be endowed with the metric

$$d((x_1,x_2,\ldots,x_i),(y_1,y_2,\ldots,y_i)) = \max\{|x_j - y_j|\}.$$

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Let $\mathcal{R}$ be the category whose objects are finitely generated free R-modules and whose morphisms are R-homomorphisms. Let $C_i(\mathcal{R}) = C_{\mathbb{Z}^i}(\mathcal{R})$. We then define the spectrum $\mathbb{K}^{-\infty}(R)$ as follows. For $n \geq 1$, we have $K_n^{-\infty}(R) = K_{-n}(R) \times K(C_n(\mathcal{R}))$. For $n = 0$, we let $B_*(\mathcal{R})$ be the basepoint component of $K(\mathcal{R})$. Then $K_0^{-\infty}(R) = K_0(R) \times B_*(\mathcal{R})$. This is necessary to get $\pi_0(\mathbb{K}^{-\infty}(R)) = K_0(R)$. In fact since [PW2, 1983], Cor. 1.3.1, shows that for $n \geq 1$, $K(C_n(R))$ is path-connected we could always just define $K_n^{-\infty}(R) = K_{-n}(R) \times B_*(C_n(\mathcal{R}))$. We can also get an equivalent definition using the idempotent completion as in [PW2, 1983].

Definition 2.3 From [PW2, 1983], Definition 1.4 : For an additive category $\mathcal{A}$ we can define its *idempotent completion* $\hat{\mathcal{A}}$ as follows. The objects of $\hat{\mathcal{A}}$ are all morphisms $p : A \to A$ in $\mathcal{A}$ satisfying $p^2 = p$. A morphism in $\hat{\mathcal{A}}$ from $p_1 : A \to A$ to $p_2 : B \to B$ is a morphism $\phi : A \to B$ in $\mathcal{A}$ satisfying $\phi = p_2 \phi p_1$.

In [PW2, 1983], it is pointed out that $K_0(C_i(\hat{\mathcal{R}})) = K_{-i}(R)$ and that for $n \geq 1, K_n(C_i(\mathcal{A})) = K_n(C_i(\hat{\mathcal{A}})) = K_n(\widehat{C_i}(\mathcal{A}))$. In fact a homotopy equivlaent definition would be to use $K(\widehat{C_n}(\mathcal{R}))$. To get the structure maps of $\mathbb{K}^{-\infty}(R)$ we consider the following. Let $C^+(\mathcal{A})$ be the subcategory of $C_1(\mathcal{A})$ where an object A has $A_x = 0$ for $x < 0$ and $C^-(\mathcal{A})$ has objects that have $A_x = 0$ for $x > 0$. We can then define $C_i^+(\mathcal{A}) = C^+(C_{i-1}(\mathcal{A}))$ and $C_i^-(\mathcal{A}) = C^-(C_{i-1}(\mathcal{A}))$. An additive category $\mathcal{A}$ is said to be *flasque* if there is an endofunctor ∞ of $\mathcal{A}$ along with a natural transformation $1 \oplus \infty \to \infty$. It is a consequence of the additivity theorem that the K-theory of a flasque category is contractible. The following lemma is equivalent to Lemma 1.3 from [PW2, 1983].

Lemma 2.4 *For any category $\mathcal{A}$, $C^+(\mathcal{A})$ and $C^-(\mathcal{A})$ are flasque.*

Proof: For $C^+(\mathcal{A})$ we define first the shift functor T by $T(A)_x = A_{x-1}$. We then define $\infty(A) = \bigoplus_{n>0} T^n(A)$. Then T induces a natural equivalence $1 \oplus \infty(A) = \bigoplus_{n \geq 0} T^n(A) \to \infty(A)$. For $C^-(\mathcal{A})$ we similarly define a shift functor T' by $T'(A)_x = A_{x+1}$. Then a similar construction shows that $C^-(\mathcal{A})$ is also flasque. □

Thus we have that $K(C_i^+(\mathcal{A}))$ and $K(C_i^-(\mathcal{A}))$ are contractible. The inclusion $C_i(\mathcal{R}) \to C_{i+1}(\mathcal{R})$ factors both through $C_{i+1}^+(\mathcal{R})$ and $C_{i+1}^-(\mathcal{R}))$ and as both of these are contractible this gives a structure map for the Pedersen-Weibel spectrum. In fact Pedersen and Weibel [PW2, 1983],Theorem 3.2 shows that with this structure map you actually get an Ω-spectrum.

3. Pairings and Assembly maps

In order to get assembly maps on these K-theory spectra, we define a pairing on the Pedersen-Weibel spectrum and a weak pairing on the Loday spectrum.

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To get a weak pairing on the Loday spectrum, one first considers the map $\otimes : GL_n(R) \times GL_m(R') \to GL_{nm}(R \otimes R')$. This induces a map $\gamma_{n,m} : BGL_n(R)^+ \times BGL_m(R')^+ \to BGL(R \otimes R')^+$. Loday, [Lod, 1976], pg. 333, shows that the maps $\gamma_{n,n}$ commute up to homotopy with the maps induced by inclusion $GL_n(R) \to GL_{n+1}(R)$. He thus uses these maps to define a map $\gamma : BGL(R)^+ \times BGL(R')^+ \to BGL(R \otimes R')^+$ that makes the diagram

$$\begin{array}{ccc} BGL_n(R)^+ \times BGL_n(R')^+ & \xrightarrow{\gamma_{n,n}} & BGL(R \otimes R')^+ \\ \downarrow & & \downarrow \\ BGL(R)^+ \times BGL(R')^+ & \xrightarrow{\gamma} & BGL(R \otimes R')^+ \end{array}$$

commute up to homotopy. He then observes that γ is homotopically trivial on $BGL(R)^+ \vee BGL(R')^+$ and so defines a map $\hat{\gamma} : BGL(R)^+ \wedge BGL(R')^+ \to BGL(R \otimes R')^+$ with the property that the diagram

$$\begin{array}{ccc} BGL_n(R)^+ \times BGL_n(R')^+ & \xrightarrow{\gamma_{n,n}} & BGL(R \otimes R')^+ \\ \downarrow & & \downarrow \\ BGL(R)^+ \wedge BGL(R')^+ & \xrightarrow{\hat{\gamma}} & BGL(R \otimes R')^+ \end{array}$$

commutes up to homotopy. There is also a natural map $K_0(R) \times K_0(R') \to K_0(R \otimes R')$ induced by the map that sends a pair of modules (M, M') to the module $M \otimes M'$. This allows a construction of a map $(K_0(R) \times BGL(R)^+) \wedge (K_0(R') \times BGL(R')^+) \to K_0(R \otimes R') \times BGL(R \otimes R')^+$. To get a pairing of spectra, we observe that $S^n R \otimes S^m R' \cong S^{n+m}(R \otimes R')$. Thus for each n, m we have a map $L_n(R) \wedge L_m(R') \to L_{n+m}(R \otimes R')$. Loday, [Lod, 1976], pg. 346, showed that this map commutes up to homotopy with the structure map $\Sigma L_n(R) \to L_{n+1}(R)$. To define an assembly map $\alpha : BG_+ \wedge \mathbf{L}(R) \to \mathbf{L}(RG)$, one proceeds as follows. There is a map $G \to GL(\mathbb{Z}G)$ that sends an element g to the class of the 1×1 matrix $[g]$. This induces a map $BG \to BGL^+(\mathbb{Z}G)$. We can define α as the composition:

$$BG_+ \wedge L_n(R) \to (BGL(\mathbb{Z}G)^+)_+ \wedge L_n(R) \to L_n(\mathbb{Z}G \otimes R) = L_n(RG).$$

Definition 3.5 A symmetric monoidal category is a category $\mathcal{A}$ along with a functor

$$\square : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

satisfying conditions in Mac Lane[Mac, 1971],p.180.

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In order to define a pairing on the Pedersen-Weibel spectrum, we first note that any additive category $\mathcal{A}$ is also a symmetric monoidal category $(\mathcal{A},\square)$ with $A\square B = A \oplus B$. [May, 1980], Thm 2.1,showed that for symmetric monoidal categories $\mathcal{A},\mathcal{B},\mathcal{C}$, if there is a functor $\otimes : \mathcal{A}\times\mathcal{B}\to\mathcal{C}$ that satisfies:

1. For any object A in $\mathcal{A}$ and any object B in $\mathcal{B}$, $A\otimes 0 = 0$ and $0\otimes B = 0$.

2. For any objects A,A' of $\mathcal{A}$, and any objects B,B' of $\mathcal{B}$, there is a coherent natural bidistributivity isomorphism

$$(A\square A')\otimes(B\square B') \cong ((A\otimes B)\square(A\otimes B'))\square((A'\otimes B)\square(A'\otimes B'))$$

then there is an induced map $K(\mathcal{A})\wedge K(\mathcal{B})\to K(\mathcal{C})$. In particular the map $\otimes : \mathcal{R}\times\mathcal{R}'\to\mathcal{R}\otimes\mathcal{R}'$ induces such a map. We can use this pairing to produce a functor $\otimes : C_i(\mathcal{R})\times C_j(\mathcal{R}')\to C_{i+j}(\mathcal{R}\times\mathcal{R}')$ using the following lemma.

Lemma 3.6 *Let $\otimes : \mathcal{A}\times\mathcal{B}\to\mathcal{C}$ be a tensor product pairing. Then for all $i,j\geq 0$ there is an induced tensor product pairing $\otimes : C_i(\mathcal{A})\times C_j(\mathcal{B})\to C_{i+j}(\mathcal{C})$.*

Proof: Let A be an object of $C_i(\mathcal{A})$ and let B be an object of $C_j(\mathcal{B})$. Let x be a point in $\mathbb{Z}^{i+j}$. Write x as (x_i,x_j) where $x_i\in\mathbb{Z}^i$, $x_j\in\mathbb{Z}^j$. Then A_{x_i} is an object of $\mathcal{A}$ and B_{x_j} is an object of $\mathcal{B}$. Define $(A\otimes B)_x = A_{x_i}\otimes B_{x_j}$. To see that this is a pairing of the appropriate type we see first that $(A\otimes 0)_x = A_{x_i}\otimes 0 = 0$ and $(0\otimes B)_x = 0\otimes B_{x_j} = 0$. Also

$$((A\oplus A')\otimes(B\otimes B'))_x = (A\oplus A')_{x_i}\otimes(B\oplus B')_{x_j}$$

$$= (A_{x_i}\oplus A'_{x_i})\otimes(B_{x_j}\oplus B'_{x_j})$$

$$\cong ((A_{x_i}\otimes B_{x_j})\oplus(A_{x_i}\otimes B'_{x_j}))\oplus((A'_{x_i}\otimes B_{x_j})\oplus(A'_{x_i}\otimes B'_{x_j}))$$

$$= ((A\otimes B)_x\oplus(A\otimes B')_x)\oplus((A'\otimes B)_x\oplus(A'\otimes B')_x).$$

Thus we have the required equalities and congruences. □

In our particular case with the categories $\mathcal{R}$ and $\mathcal{R}'$, we can then pass to pairing on the idempotent completions $\hat{\otimes} : \widehat{C_n}(\mathcal{R})\times\widehat{C_m}(\mathcal{R}')$ to get maps $K_n^{-\infty}(R)\wedge K_m^{-\infty}(R')\to K_{n+m}^{-\infty}(R\otimes R')$. To see that we have a pairing we show the following.

Lemma 3.7 *Let $\otimes : \mathcal{A}\times\mathcal{B}\to\mathcal{C}$ be a tensor product pairing. Then the pairing of Lemma 3.6 gives a pairing $\hat{\otimes} : \mathbb{K}^{-\infty}(\mathcal{A})\wedge\mathbb{K}^{-\infty}(\mathcal{B})\to\mathbb{K}^{-\infty}(\mathcal{C})$.*

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Proof: The tensor product pair gives a pairing

$$K(\widehat{C}_i(\mathcal{A})) \wedge K(\widehat{C_j}(\mathcal{B})) \to K(\widehat{C_{i+j}}(\mathcal{C}))$$

so to see that we have a pairing of spectra we must check compatibility with the suspension maps $\Sigma K(C_n(-)) \to K(C_{n+1}(-))$. To see this we observe the following. If A is an object in $C_i^+(\mathcal{A})$ then we can see that $A_{x_i} = 0$ if the first coordinate of x_i is negative. Thus $A_{x_i} \otimes B_{x_j} = 0$ if the first coordinate of x_i is negative. But then this means that that $A \otimes B$ is an object of $C_{i+j}^+(\mathcal{C})$. Similarly, if A is an object of $C_i^-(\mathcal{A})$ we get that $A \otimes B$ is in $C_{i+j}^-(\mathcal{C})$. Also, the inclusion map $C_{i-1}(\mathcal{A}) \to C_i(\mathcal{A})$ is compatible with the tensor product. This shows compatibility in the first coordinate. For the second coordinate, we consider that to make the diagram

$$\begin{array}{ccc} C_i(\mathcal{A}) \times C_{j-1}(\mathcal{B}) & \longrightarrow & C_i(\mathcal{A}) \times C_j(\mathcal{B}) \\ \downarrow & & \downarrow \\ C_{i+j-1}(\mathcal{C}) & \longrightarrow & C_{i+j}(\mathcal{C}) \end{array}$$

commute we use the change of coordinates in Euclidean space that allows the diagram

$$\begin{array}{ccc} \mathbb{R}^i \times \mathbb{R}^{j-1} & \longrightarrow & \mathbb{R}^{i+j-1} \\ \downarrow & & \downarrow \\ \mathbb{R}^i \times \mathbb{R}^j & \longrightarrow & \mathbb{R}^{i+j} \end{array}$$

to commute. With this change of coordinates the same argument applies. □

To get an assembly map, we apply a similar technique to that for the Loday assembly map. First we need some categorical definitions from [HP, 2004]. In 5.2, given a category $\mathcal{C}$ and a ring, R, they define the category $R\mathcal{C}$ whose objects are those of $\mathcal{C}$, but for a pair of objects x, y, $\mathrm{Mor}_{R\mathcal{C}}(x,y)$ is the free R-module generated by $\mathrm{Mor}_{\mathcal{C}}(x,y)$. Then, for any groupoid, $\mathcal{G}$, they define $R(\mathcal{G})_\oplus$, the free additive category generated by $R\mathcal{G}$.

Lemma 3.8 *If $\mathcal{G}$ has one object and $Mor(\mathcal{G})$ is the group G, then $R(\mathcal{G})_\oplus$ is equivalent to the category whose objects are finitely generated free RG-modules and whose morphisms are RG-homomorphisms.*

Proof: If $\mathcal{G}$ has only one object, then $R\mathcal{G}$ also has one object and by definition $\mathrm{Mor}(R\mathcal{G}) = RG$. Thus $R(\mathcal{G})_\oplus$ has objects indexed by $0,1,2,\ldots$ with $\mathrm{Mor}(m,n)$ the collection of m by n matrices with entries in RG. Thus we have a natural equivalence with the category of finitely generated free RG-modules by sending n to $(RG)^n$. □

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From [HP, 2004] ,Thm 4.6, we have the following:

Theorem 3.9 *For any groupoid $\mathcal{G}$ there is a natural transformation*

$$A_{\mathcal{G}} : B\mathcal{G}_+ \wedge \mathbb{K}^{-\infty}(\mathcal{R}) \to \mathbb{K}^{-\infty}(R(\mathcal{G})_{\oplus})$$

such that $A_\bullet$ is a homotopy equivalence.

The proof uses a tensor product pairing $\otimes : \mathbb{Z}(\mathcal{G})_{\oplus} \times \mathcal{R} \to R(\mathcal{G})_{\oplus}$, where $\mathcal{R}$ is the category of finitely generated R-modules. This is just an extension of the isomorphism $\mathbb{Z} \otimes R \cong R$. In the case where $\mathcal{G}$ is a group G, we get a map $BG_+ \wedge \mathbb{K}^{-\infty}(\mathcal{R}) \to \mathbb{K}^{-\infty}(\mathcal{R}\mathcal{G})$ from Lemma 3.8. To get an assembly map, we let $c : X \to B\pi(X)$ be the classifying homotopy natural transformation of the universal covering where $\pi(X)$ is the fundamental groupoid of X. Then we have the following homotopy natural transformation:

$$\hat{\alpha}_X : X_+ \wedge \mathbb{K}^{-\infty}(\mathcal{R}) \xrightarrow{c\wedge 1} B\pi(X)_+ \wedge \mathbb{K}^{-\infty}(\mathcal{R}) \xrightarrow{A_{\pi(X)}} \mathbb{K}^{-\infty}(R(\pi(X))_{\oplus}).$$

If X is a one-point space then $B\pi(X)_+ \cong S^0$ and $\pi(X)$ is just the trivial category so we get a homotopy equivalence, thus by Observation 1.3 of [WW, 1993] this is homotopic to a generalized assembly map.

4. Comparing K-theories

Now we wish to construct a homotopy equivalence from the Pedersen-Weibel spectrum to Loday's spectrum. This means that we want to construct functors $\theta_n : K_n^{-\infty}(R) \to L_n(R)$ that are homotopy equivalences and commute at least up to homotopy with the structure maps. If $F : \mathcal{A} \to \mathcal{B}$ is a functor between additive categories, we will say that F induces a homotopy equivalence in K-theory if the induced map of spectra $K(F) : K(\mathcal{A}) \to K(\mathcal{B})$ is a homotopy equivalence of spectra. To construct the functors we will use a couple of intermediate spectra. The first will be the spectrum with nth space $K'_n(R) = K_{-n}(R) \times B_*(\mu^n\mathcal{R})$. Here $\mu^n\mathcal{R}$ is the category of free $\mu^n R$ modules and $\mu^n R$-homomorphisms. We will in fact show that this is an Ω-spectrum, that is that $K'_n \cong \Omega K'_{n+1}$. First we define a new category. For a ring R we let $\mathcal{R}^*$ be the category of countably generated free R-modules and locally finite homomorphisms. We can observe here that $\mathrm{End}_{\mathcal{R}^*}(R^\infty)$ is lR. Karoubi, [Kar, 1976], Definition 1.5, defines the notion of a category $\mathcal{U}$ being filtered by a category $\mathcal{A}$. If $\mathcal{U}$ is $\mathcal{A}$-filtered, then on page 126 he defines the category $\mathcal{U}/\mathcal{A}$ whose objects are the same as those in $\mathcal{U}$ but whose morphisms are equivalence classes under the equivalence relation that $f \sim g$ if $f - g$ factors through an object in $\mathcal{A}$. In particular $\mathcal{R}^*$ is $\mathcal{R}$-filtered. The category $\mathcal{R}^*/\mathcal{R}$ has one non-zero object,

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namely R^∞ and $\mathrm{End}_{\mathcal{R}^*/\mathcal{R}}(R^\infty) = \mu R$. In [PW1, 1986], Prop 6.1 shows that there is a natural equivalence of $\mathcal{R}^*/\mathcal{R}$ and $\mu\mathcal{R}$. The functor $\alpha : \mathcal{R}^*/\mathcal{R} \to \mu\mathcal{R}$ that sends any finitely generated module to the zero module and sends R^∞ to μR and acts as the identity on morphisms induces this natural equivalence. Thus for $n > 1$ we can observe that $K'_n(R) \cong K_{-n}(R) \times B_*(\mu^{n-1}\mathcal{R}^*/\mu_{n-1}\mathcal{R})$. Since for any ring R, $\mathcal{R}^*$ includes countably generated modules, it can be seen to be flasque using the functor $\infty^* : \mathcal{R}^* \to \mathcal{R}^*$, $\infty^*(M) = \oplus_{i=1}^\infty M$, with a natural equivalence from ∞^* to $1 \oplus \infty^*$ coming from the natural isomorphism of $\infty^*(M)$ with $M \oplus \infty^*(M)$. Thus this fits with the spectrum defined in [PW1, 1986], definition 5.9. We also will use in our process the intermediate spectrum $\hat{K}_n(R) = K_{-n}(R) \times BGL^+(\mu^n R)$ described by Wagoner in [Wag, 1972], page 359. We note here that K' here is essentially based on Wagoner's spectrum replacing the space $BGL^+(\mu^n R)$ with the basepoint component of the classifying space for the category $\mu^n\mathcal{R}$. We will then construct functors $D_n : C_n(\mathcal{R}) \to \mu^n\mathcal{R}$ that induce homotopy equivalences on K-theory that commute with the structure maps. We define $D_n : C_n(\mathcal{R}) \to \mu^n\mathcal{R}$ inductively as follows. D_0 is just the identity. We can observe that there is a functor $\beta : C^+(\mathcal{R}) \to \mathcal{R}^*$ that forgets the control structure and just sends an object A to $\oplus A_i$ and acts as the identity on morphisms. β acts as the identity on $\mathcal{R}$ and induces a functor $\hat{\beta} : C^+(\mathcal{R})/\mathcal{R} \to \mathcal{R}^*/\mathcal{R}$. [PW1, 1986], Thm. 6.2 shows that the functor $\tau : C_1(\mathcal{R}) \to C^+(\mathcal{R})/\mathcal{R}$ that sends an object $\{\ldots, A_{-2}, A_{-1}, A_0, A_1, A_2, \ldots\}$ to $\{A_0, A_1, A_2, \ldots\}$ and sends a morphism ϕ to the equivalence class of the matrix induced by the truncation of objects induces a homotopy equivalence on K-theory. For $n \geq 0$ we inductively define $D_{n+1} = \alpha \circ \hat{\beta} \circ \tau \circ C_1(D_n)$. Here we use the fact that $C_{n+1}(\mathcal{R}) = C_1(C_n(\mathcal{R}))$. All the functors except $C_1(D_n)$ are known to induce homotopy equivalences in K-theory by previous results. For the first arrow $C_1(D_n)$ we proceed by induction. For $n = 0$ this is just the identity functor. To use induction we need the following lemma.

Lemma 4.10 *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor that induces a homotopy equivalence $K(\hat{\mathcal{A}}) \to K(\hat{\mathcal{B}})$. Then $C_1(F) : C_1(\mathcal{A}) \to C_1(\mathcal{B})$ induces a homotopy equivalence $K(C_1(\mathcal{A})) \to K(C_1(\mathcal{B}))$.*

Proof: F induces a map of commutative squares:

$$\begin{array}{ccc} \hat{\mathcal{A}} & \longrightarrow & C^+(\hat{\mathcal{A}}) \\ \downarrow & & \downarrow \\ C^-(\hat{\mathcal{A}}) & \longrightarrow & C_1(\hat{\mathcal{A}}) \end{array} \longrightarrow \begin{array}{ccc} \hat{\mathcal{B}} & \longrightarrow & C^+(\hat{\mathcal{B}}) \\ \downarrow & & \downarrow \\ C^-(\hat{\mathcal{B}}) & \longrightarrow & C_1(\hat{\mathcal{B}}). \end{array}$$

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The map on the upper left corners induces a homotopy equivalence by hypothesis. The maps on the upper right and lower left corners induce homotopy equivalences on K-theory since those categories are all flasque. Thus the map from $K(C_1(\hat{\mathcal{A}})) \to K(C_1(\hat{\mathcal{B}}))$ must also be a homotopy equivalence. Since $K(C_1(\mathcal{A})$ is homotopy equivalent to the basepoint component of $K(C_1(\hat{\mathcal{A}})$ the conclusion follows. □

Thus we now have maps $D_n : C_n(\mathcal{R}) \to \mu^n\mathcal{R}$ that induce homotopy equivalences on K-theory. In order to have a homotopy equivalence of spectra, we must verify the fact that homotopy equivalences commute with the structure maps. To do this we have to verify that the following diagram of functors commutes:

$$\begin{array}{ccc} C_i(\mathcal{R}) & \longrightarrow & C_{i+1}^+(\mathcal{R}) \\ \downarrow & & \downarrow \\ C_{i+1}^-(\mathcal{R}) & \longrightarrow & C_{i+1}(\mathcal{R}) \end{array} \longrightarrow \begin{array}{ccc} \mu^i\mathcal{R} & \longrightarrow & \mu^i\mathcal{R}^* \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mu^{i+1}\mathcal{R}. \end{array}$$

Here, 0 is the zero category or trivial category with just one object and the identity morphism. Here the functor for the upper left and lower right corners is D_n and D_{n+1} respectively. On the lower left corner it is just the zero functor. On the upper right we have a functor D_n^+ defined by $D_n^+ = \beta \circ C^+(D_{n-1})$. We now prove the following.

Proposition 4.11 *The above map between commutative squares commutes with the squares.*

Proof: The map of left vertical arrows commutes since both directions end in the zero category. For the bottom horizontal arrows, the commutativity follows from the fact that τ applied to an object in $C_i^-(\mathcal{R})$ gives the zero object. We can then consider the square

$$\begin{array}{ccc} C_i(\mathcal{R}) & \longrightarrow & C_{i+1}^+(\mathcal{R}) \\ \downarrow & & \downarrow \\ \mu^i\mathcal{R} & \longrightarrow & \mu^i\mathcal{R}^*. \end{array}$$

To see this commutes, going across is the inclusion map, and that the restriction of D_{i+1}^+ to C_i is the same as D_i followed by the inclusion map into $\mu^i\mathcal{R}^*$. Finally we

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need to check that the square

$$\begin{array}{ccc} C^+_{i+1}(\mathcal{R}) & \longrightarrow & \mu^i\mathcal{R}^* \\ \downarrow & & \downarrow \\ C_{i+1}(\mathcal{R}) & \longrightarrow & \mu^{i+1}\mathcal{R} \end{array}$$

commutes. To see this, we note that going across and down is the composition

$$C^+_{i+1}(\mathcal{R}) \overset{C^+(D_i)}{\longrightarrow} C^+(\mu^i\mathcal{R}) \overset{\beta}{\longrightarrow} \mu^i\mathcal{R}^* \longrightarrow \mu^i\mathcal{R}^*/\mu^i\mathcal{R} \overset{\alpha}{\longrightarrow} \mu^{i+1}\mathcal{R}.$$

Going the other way we have

$$C^+_{i+1}(\mathcal{R}) \overset{C_1(D_i)}{\longrightarrow} C_1(\mu^i\mathcal{R}) \overset{\tau}{\longrightarrow} C^+(\mu^i\mathcal{R})/\mu^i\mathcal{R} \overset{\hat{\beta}}{\longrightarrow} \mu^i\mathcal{R}^*/\mu^i\mathcal{R} \overset{\alpha}{\longrightarrow} \mu^{i+1}\mathcal{R}$$

since by definition the inclusion map followed by $C_1(D_i)$ is the same as $C^+(D_i)$. Since the diagram

$$\begin{array}{ccc} C^+(\mu\mathcal{R}) & \overset{\beta}{\longrightarrow} & \mu\mathcal{R}^* \\ \downarrow & & \downarrow \\ C^+(\mu\mathcal{R})/\mu\mathcal{R} & \overset{\hat{\beta}}{\longrightarrow} & \mu\mathcal{R}^*/\mathcal{R} \end{array}$$

commutes, we can replace the first line with

$$C^+_{i+1}(\mathcal{R}) \overset{C^+(D_i)}{\longrightarrow} C^+(\mu^i\mathcal{R}) \longrightarrow C^+(\mu^i\mathcal{R})/\mu^i\mathcal{R} \overset{\hat{\beta}}{\longrightarrow} \mu^i\mathcal{R}^*/\mu^i\mathcal{R} \overset{\alpha}{\longrightarrow} \mu^{i+1}\mathcal{R}.$$

Also τ restricted to $C^+(\mathcal{R})$ is just the quotient map, thus this square commutes as well. □

From this we get the following.

Corollary 4.12 *The collection of maps $\{D_n\}$ induce a homotopy equivalence of spectra.*

We now construct a homotopy equivalence of spectra

$$F_n : BGL(S^nR)^+ \times K_0(S^nR) \to K'_n(R).$$

To do this we use a mapping telescope model of $BGL(R)$ for a ring, that is $BGL(R)$ is the infinite mapping telescope of $BGL_1(R) \to BGL_2(R) \to \ldots \to BGL_n(R) \to \ldots$. There are maps $\eta_n : BGL_n(R) \to B_*(\mathcal{R})$ induced by the map that sends an

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element $M \in GL_n(R)$ to the equivalence class of $(M,1) \in Aut_{\underline{\underline{\mathcal{R}}}^{-1}\underline{\underline{\mathcal{R}}}}(R^n, R^n)$. The diagram

$$\begin{array}{ccc} GL_n R & \xrightarrow{\eta_n} & Aut_{\underline{\underline{\mathcal{R}}}^{-1}\underline{\underline{\mathcal{R}}}}(R^n, R^n) \\ \downarrow & & \downarrow \\ GL_{n+1}(R) & \xrightarrow{\eta_{n+1}} & Aut_{\underline{\underline{\mathcal{R}}}^{-1}\underline{\underline{\mathcal{R}}}}(R^{n+1}, R^{n+1}) \end{array}$$

commutes, which allows us to construct a map $\eta : BGL(R) \to B_*(\mathcal{R})$. This map can be used to induce a map $BGL(R)^+ \to B_*(\mathcal{R})$ which is a homotopy equivalence as shown by Grayson in [Gra, 1976],pg. 224. Wagoner, in [Wag, 1972], Prop. 5.3, shows that the natural map $i_n : S^n R \to \mu^n R$ induces a homotopy equivalence $i_n : BGL(S^n R)^+ \to BGL(\mu^n R)^+$. Thus if we let $F_n = \eta \circ i_n$ we have $F_n : BGL(S^n R)^+ \to B_*(\mu^n R)$ a homotopy equivalence. If we let J_n be a homotopy inverse of F_n and let $\theta_n = J_n \circ D_n$ we have the following theorem.

Theorem 4.13 *The collection θ_n is a homotopy equivalence of spectra from $\mathbb{K}^{-\infty}(R)$ to Loday's spectrum $\mathbf{L}(R)$.*

5. Comparing Assembly Maps

We now aim to compare the assembly maps for the Pedersen-Weibel spectrum and the Loday spectrum. Since both of these assembly maps can be thought of as coming from the (weak) pairings, we first wish to show that there are diagrams

$$\begin{array}{ccc} K_n^{-\infty}(\mathcal{R}) \wedge K_m^{-\infty}(\mathcal{R}') & \longrightarrow & K_{n+m}^{-\infty}(\mathcal{R} \otimes \mathcal{R}') \\ \downarrow & & \downarrow \\ L_n(R) \wedge L_m(R') & \longrightarrow & L_{n+m}(R \otimes R') \end{array}$$

that commute up to homotopy. We then show that these maps also commute with the structure maps. This shows that the equivalent diagram in the category of spectra commutes up to homotopy. This will show that we have a homotopy commuting diagram for the assembly maps. Just as in the previous section, we will use the intermediate spectrum $K'(R)$. First we consider the diagram

$$\begin{array}{ccc} C_n(\mathcal{R}) \times C_m(\mathcal{R}') & \xrightarrow{\otimes} & C_{n+m}(\mathcal{R} \otimes \mathcal{R}') \\ \downarrow{\scriptstyle D_n \times D'_m} & & \downarrow{\scriptstyle D''_{n+m}} \\ \mu^n \mathcal{R} \times \mu^m \mathcal{R}' & \xrightarrow{\otimes} & \mu^{n+m}(\mathcal{R} \otimes \mathcal{R}'). \end{array}$$

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Here, the lower $\otimes$ is the map induced from the map:$\mu^n R \otimes \mu^m R' \to \mu^{n+m}(R \otimes R')$ First we prove the following.

Theorem 5.14 *The previous diagram of functors commutes.*

Proof: The proof is by induction first on one variable, then the other. We start by assuming that $n = 0$ and do an induction on m. For the following, we use $\big(A\big)$ to denote a matrix in lR for some ring R and $\big[A\big]$ to denote an equivalence class of matrices in μR for some ring R. When $n = m = 0$ the vertical arrows are all the identity and there is no problem. Suppose that it commutes for m, we wish to show it commutes for $m+1$. Recall that $D_{m+1} = \alpha \circ \hat{\beta} \circ \tau \circ C_1(D_m)$. Let A be an object in $C_{m+1}(\mathcal{R}')$ with $A = \{\ldots, A_{-1}, A_0, A_1, \ldots\}$. Let M represent an object in $\mathcal{R}$. Then

$$D_{m+1}(M \otimes A) = \alpha \circ \hat{\beta} \circ \tau \circ C_1(D_m(M \otimes A))$$

$$= \alpha \circ \hat{\beta} \circ \tau(\ldots, M \otimes D_m(A_{-1}), M \otimes D_m(A_0), M \otimes D_m(A_1), \ldots)$$

$$= \alpha \circ \hat{\beta}(M \otimes D_m(A_0), M \otimes D_m(A_1), \ldots) = \alpha(\bigoplus_{i \geq 0} M \otimes (D_m(A_i)))$$

If we consider $M \otimes D_{m+1}(A)$ we get

$$M \otimes D_{m+1}(A) = M \otimes (\alpha \circ \hat{\beta} \circ \tau \circ C_1(D_m(A))) = M \otimes (\alpha(\bigoplus_{i \geq 0} D_m(A_i))).$$

If M is the zero module, clearly these are equal. If M is a non-zero module, then whether these represent the zero object in $\mu^{n+1}\mathcal{R} \otimes \mathcal{R}'$ depends on whether or not $\alpha(\bigoplus_{i \geq 0}(D_m(A_i)))$ represents the zero object. In either case, we get the same thing. For morphisms we note that α is just the identity on morphisms. Also on a morphism $\hat{\beta}$ also essentially acts as the identity if we represent a morphism in $C^+(\mu^m\mathcal{R}/\mu^m\mathcal{R})$ as an equivalence class of a $\mathbb{N} \times \mathbb{N}$ indexed matrix and likewise a morphism in $(\mu^m\mathcal{R})^*/\mu^m\mathcal{R}$ as the same type of matrix. We think of $\phi = \big(\phi_{ij}\big)$ where ϕ_{ij} is a morphism in $C_m(\mathcal{R}')$. But $\tau(\big(D_m(\phi_{ij})\big))$ is the equivalence class of the matrix whose entries are $D_m(\phi_{ij})$ with $i, j \geq 0$. We can think of each $D_m(\phi_{ij})$ being an element of $\mu^n R'$. Then $\hat{\beta}(\tau(\big(D_m(\phi_{ij})\big)))$ is the equivalence class of the matrix consisting of those elements. Thus we have

$$\psi \otimes \hat{\beta}(\tau(\big(D_m(\phi_{ij})\big)))$$

$$= \psi \otimes \begin{bmatrix} D_m(\phi_{00}) & D_m(\phi_{01}) & \ldots \\ D_m(\phi_{10}) & D_m(\phi_{11}) & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

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$$= \begin{bmatrix} \psi \otimes D_m(\phi_{00}) & \psi \otimes D_m(\phi_{01}) & \cdots \\ \psi \otimes D_m(\phi_{10}) & \psi \otimes D_m(\phi_{11}) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

On the other hand we have

$$\hat{\beta}(\tau(D_m(\psi \otimes (\phi_{ij})))) = \hat{\beta}(\tau(D_m((\psi \otimes \phi_{ij}))))$$

$$= \hat{\beta}(\tau((D_m(\psi \otimes \phi_{ij})))) = \hat{\beta}(\tau((\psi \otimes D_m(\phi_{ij}))))$$

$$= \begin{bmatrix} \psi \otimes D_m(\phi_{00}) & \psi \otimes D_m(\phi_{01}) & \cdots \\ \psi \otimes D_m(\phi_{10}) & \psi \otimes D_m(\phi_{11}) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

Here the second to last equality uses the induction hypothesis along with the fact that D_0 is just the identity.

Now we do a similar induction on n given the result above. For $n = 0$ this is just the result above. Now, let A be an object of $C_{n+1}(\mathcal{R})$ and B an object of $C_m(\mathcal{R}')$. As above we treat $C_{n+1}(\mathcal{R})$ as $C_1(C_n(\mathcal{R}))$. $A \otimes B = \{\ldots, A_{-1}, A_0, A_1, \ldots\} \otimes B = \{\ldots, A_{-1} \otimes B, A_0 \otimes B, A_1 \otimes B, \ldots\}$ So we have $\alpha(\hat{\beta}(\tau((C_1(D_{m+n}(A \otimes B))))))) =$

$$\alpha(\hat{\beta}(\tau(D_{m+n}(\{\ldots, A_{-1} \otimes B, A_0 \otimes B, A_1 \otimes B, \ldots\}))))$$

$$= \alpha(\hat{\beta}(\{D_{m+n}(A_0 \otimes B), D_{m+n}(A_1 \otimes B), \ldots\}))$$

$$= \alpha(\bigoplus_{i \geq 0} D_{n+m}(A_i \otimes B)) = \alpha(\bigoplus_{i \geq 0} D_n(A_i) \otimes D_n(B))$$

$$= \alpha((\bigoplus_{i \geq 0} D_n(A_i)) \otimes D_m(B)).$$

If we take the other direction, we get $\alpha(\beta(\tau(C_1(D_n(A))))) \otimes D_m(B) =$

$$\alpha(\hat{\beta}(\tau(\{\ldots, D_n(A_{-1}), D_n(A_0), D_n(A_1), \ldots\})))$$

$$= \alpha(\hat{\beta}(\{D_n(A_0), D_n(A_1), \ldots\})) \otimes D_m(B) = \alpha(\bigoplus_{i \geq 0} D_n(A_i)) \otimes D_m(B).$$

As before if $D_m(B)$ is the zero object, then both the above are the zero object, otherwise they both depend solely on whether $\alpha(\bigoplus_{i \geq 0} D_n(A_i))$ is the zero object. For morphisms again α is just the identity, and by the natural association of bases of objects in $C^+(\mathcal{R})$ and $\mathcal{R}^*$, so is $\hat{\beta}$. Now we let ψ be a morphism in $C_{n+1}(\mathcal{R}) =$

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$C_1(C_n(\mathcal{R}))$ and let ϕ be a morphism in $C_m(\mathcal{R}')$. We will denote ψ by $\left(\psi_{ij}\right)$ where each ψ_{ij} is a morphism in $C_n(\mathcal{R})$. Then we consider first

$$\hat{\beta}(\tau(C_1(D_{m+n}(\psi \otimes \phi)))) = \hat{\beta}(\tau(C_1((D_{m+n}(\left(\psi_{ij}\right) \otimes \phi)))))$$

$$= \hat{\beta}(\tau(C_1(D_{m+n}(\left(\psi_{ij} \otimes \phi\right))))) = \beta(\tau(\left(D_{m+n}(\psi_{ij} \otimes \phi)\right)))$$

$$= \begin{bmatrix} D_{m+n}(\psi_{00} \otimes \phi) & D_{m+n}(\psi_{01} \otimes \phi) & \cdots \\ D_{m+n}(\psi_{01} \otimes \phi) & D_{m+n}(\psi_{11} \otimes \phi) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

Going the other way we get

$$\hat{\beta}(\tau(C_1(D_n(\psi)))) \otimes D_m(\phi) = \hat{\beta}(\tau(C_1(D_n(\left(\psi_{ij}\right))))) \otimes D_m(\phi)$$

$$= \begin{bmatrix} D_n(\psi_{00}) & D_n(\psi_{01}) & \cdots \\ D_n(\psi_{10}) & D_n(\psi_{11}) & \cdots \\ \vdots & \vdots & \cdots \end{bmatrix} \otimes D_m(\phi)$$

$$= \begin{bmatrix} D_n(\psi_{00}) \otimes D_m(\phi) & D_n(\psi_{01}) \otimes D_m(\phi) & \cdots \\ D_n(\psi_{10}) \otimes D_m(\phi) & D_n(\psi_{01}) \otimes D_m(\phi) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

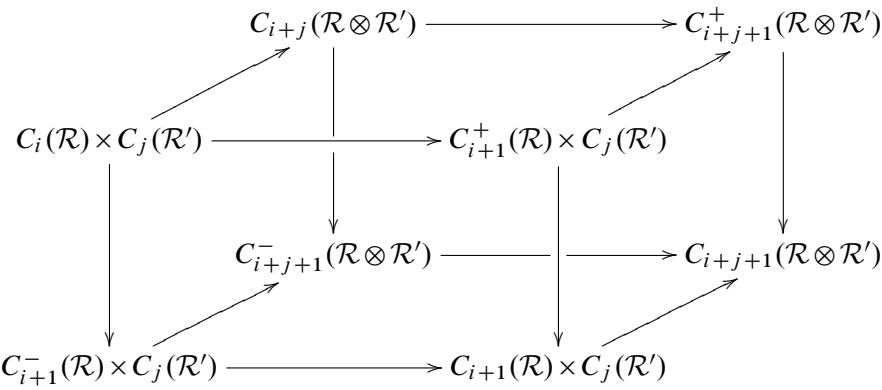
$$= \begin{bmatrix} D_{n+m}(\psi_{00} \otimes \phi) & D_{n+m}(\psi_{01} \otimes \phi) & \cdots \\ D_{n+m}(\psi_{10} \otimes \phi) & D_{n+m}(\psi_{11} \otimes \phi) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

Here the last equality comes from the inductive hypothesis. Thus the square of functors given commutes. □

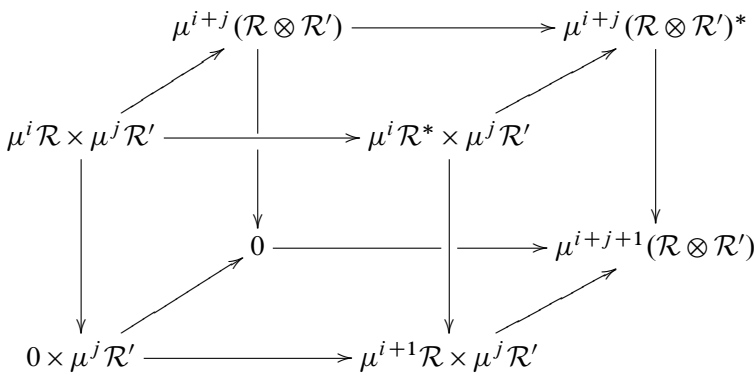
Now, we want to see that in fact the pairings induced by the two tensor product pairings also commute with the structure maps. As before, we will do this by looking at the diagrams at the categorical level and show that the tensor product pairings commute with the relevant diagrams.

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That is to say we have to look at the diagram:



and the corresponding diagram:



and the transformation between them.

We prove the following theorem.

Theorem 5.15 *The above diagrams are commutative and the functors previously defined induce a map between the diagrams.*

Proof: First we note that the commutativity of the top diagram was given in the proof of Lemma 3.7, since this is what allows us to show that we actually get a pairing of spectra. For the bottom diagram, the commutativity of the left face is obvious since both maps end in the zero category. The front and back faces are just the pullback diagrams that give the fibration sequence. In the top face, this is just the fact that if we take the tensor product of a pair (A,B) in $\mu^i\mathcal{R}\times\mu^j\mathcal{R}'$ that this will be finitely generated. For the bottom face, going one way factors through the zero category, so that is the zero functor. Going the other way we get

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$(0,A) \to (0,A) \to 0 \otimes A = 0$ so we get the zero functor as well. The only face left is the right face:

$$\begin{array}{ccc} \mu^i\mathcal{R}^* \times \mu^j\mathcal{R} & \longrightarrow & \mu^{i+j}(\mathcal{R}\otimes\mathcal{R}')^* \\ \downarrow & & \downarrow \\ \mu^{i+1}\mathcal{R} \times \mu^j\mathcal{R} & \longrightarrow & \mu^{i+j+1}(\mathcal{R}\otimes\mathcal{R}'). \end{array}$$

To see the commutativity of this, on objects, we look at a pair (A,B). If A is finitely generated then going down and across $(A,B) \to (0,B) \to 0 \otimes B = 0$. If we go across and down (A,B) goes to $A \otimes B$ which will be finitely generated as well, so it will go to zero. If A is infinitely generated going across and then down (A,B) goes to $A \otimes B$ which is infinitely generated unless $B = 0$, thus it will go to $\mu^{i+j+1}(R \otimes R')$. Going down then across (A,B) goes to $(\mu^{i+1}R, B)$ which goes to $\mu^{i+1}R \otimes B$ which is isomorphic to $\mu^{i+j+1}(R \otimes R')$ unless $B = 0$. For morphisms this boils down to showing that if $M, M' \in lR, N \in mR'$ that $[M \otimes N] = [M' \otimes N]$ if $M - M' \in mR$. But if $M - M' \in mR$, then $(M - M') \otimes N \in m(R \otimes R')$ But $(M - M') \otimes N = M \otimes N - M' \otimes N$ and thus we have the necessary equality. This shows that we really did have a pairing of spectra for $K'(R)$. Now we show commutativity of the maps between the diagrams, one face at a time. First we note that for the front and back faces, this follows immediately from Proposition 4.11. Now, we consider the map from the left face of the top:

$$\begin{array}{ccc} C_i(\mathcal{R}) \times C_j(\mathcal{R}') & \longrightarrow & C_{i+j}(\mathcal{R}\otimes\mathcal{R}') \\ \downarrow & & \downarrow \\ C^-_{i+1}(\mathcal{R}) \times C_j(\mathcal{R}') & \longrightarrow & C^-_{i+j+1}(\mathcal{R}\otimes\mathcal{R}') \end{array}$$

to the left face of the bottom:

$$\begin{array}{ccc} \mu^i\mathcal{R} \times \mu^j\mathcal{R}' & \longrightarrow & \mu^{i+j}(\mathcal{R}\otimes\mathcal{R}') \\ \downarrow & & \downarrow \\ 0 \times \mu^j\mathcal{R} & \longrightarrow & 0. \end{array}$$

The proof that the map between top horizontal lines commute is the proof of Theorem 5.14. The commutativity of the bottom horizontal row and the right vertical row is immediate since they both end in the zero category. For the left vertical arrows, the commutativity follows from the proof of Proposition 4.11. Next,

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we consider the map from the right face of the top:

$$\begin{array}{ccc} C^+_{i+1}(\mathcal{R}) \times C_j(\mathcal{R}') & \longrightarrow & C^+_{i+j+1}(\mathcal{R} \otimes \mathcal{R}') \\ \downarrow & & \downarrow \\ C_{i+1}(\mathcal{R}) \times C_j(\mathcal{R}') & \longrightarrow & C_{i+j+1}(\mathcal{R} \otimes \mathcal{R}') \end{array}$$

to the right face on the bottom:

$$\begin{array}{ccc} \mu^i \mathcal{R}^* \times \mu^j \mathcal{R}' & \longrightarrow & \mu^{i+j}(\mathcal{R} \otimes \mathcal{R}')^* \\ \downarrow & & \downarrow \\ \mu^{i+1}\mathcal{R} \times \mu^j \mathcal{R}' & \longrightarrow & \mu^{i+j+1}(\mathcal{R} \otimes \mathcal{R}'). \end{array}$$

First we look at the map on top horizontal rows

$$\begin{array}{ccc} C^+_{i+1}(\mathcal{R}) \times C_j(\mathcal{R}') & \longrightarrow & C^+_{i+j+1}(\mathcal{R} \otimes \mathcal{R}') \\ \downarrow & & \downarrow \\ \mu^i \mathcal{R}^* \times \mu^j \mathcal{R}' & \longrightarrow & \mu^{i+j}(\mathcal{R} \otimes \mathcal{R})^*. \end{array}$$

Going down and across is the map $(\beta \circ C^+(D_i)) \otimes D'_j$ But we have

$$(\beta \circ C^+(D_i)) \otimes D'_j = \beta(C^+(D_i) \otimes D'_j) = \beta(C^+(D_{i+j})) \circ \otimes.$$

Going the other way we get $\beta \circ C^+(D_{i+j}) \circ \otimes$ so this square commutes. The fact that the bottom horizontal lines commute is also the result of Theorem 5.14. The fact that the right vertical arrows commute follows from the proof of Proposition 4.11. Also, the same proof shows that the left vertical arrows commute. Thus the map commutes on this face. Thus we have commutativity now on the front, back, left, and right faces. But all arrows are on at least one of those faces, thus we have a commutative map from the top cube to the bottom cube. □

As a result of this theorem we have the following corollary.

Corollary 5.16 *The maps D_n induce a map of pairings of spectra.*

Now we consider the map $F_n : BGL(S^n R)^+ \times K_{-n}(R) \to K'_n(\mu^n \mathcal{R})$. Again we use the infinite mapping telescope model for $BGL(S^n R)$ and $BGL(\mu^n R)$. First we consider the map $BGL(S^n R)^+ \to BGL(\mu^n R)^+$. We can see that the diagram

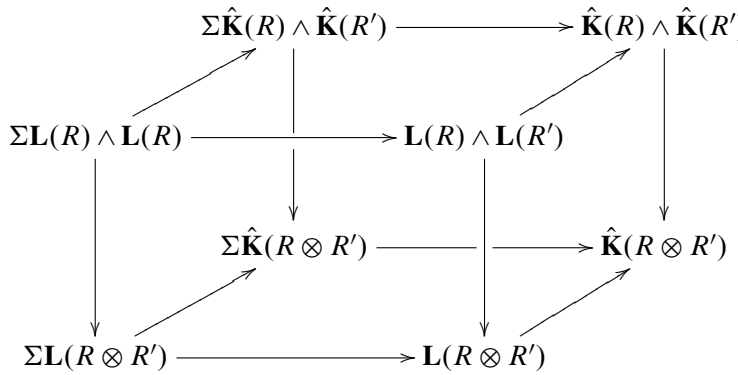
$$\begin{array}{ccc} BGL_k(S^n R) \times BGL_l(S^m R') & \xrightarrow{\otimes} & BGL_{k+l}(S^{n+m}(R \otimes R'))^+ \\ \downarrow & & \downarrow \\ BGL_k(\mu^n R) \times BGL_l(\mu^m R') & \xrightarrow{\otimes} & BGL_{k+l}(\mu^{n+m}(R \otimes R'))^+ \end{array}$$

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commutes since the vertical maps are simply the maps $S^n R \to \mu^n R$. Thus, up to homotopy the diagram

$$\begin{array}{ccc} BGL(S^n R)^+ \wedge BGL(S^m R') & \longrightarrow & BGL(S^{n+m}(R \otimes R'))^+ \\ \downarrow & & \downarrow \\ BGL(\mu^n R)^+ \wedge BGL(\mu^m R')^+ & \longrightarrow & BGL(\mu^{n+m}(R \otimes R'))^+ \end{array}$$

commutes where the top horizontal map is the one constructed by Loday more generally for $BGL(R)^+ \wedge BGL(R')^+ \to BGL(R \otimes R')^+$ and the bottom is that map followed by the map $\mu^n R \otimes \mu^m R' \to \mu^{n+m}(R \otimes R')$. To check that this map commutes with the structure map up to homotopy, we examine the diagram:



where $\mathbf{L}(R)$ is Loday's spectrum and $\hat{\mathbf{K}}(R)$ is the spectrum with $\hat{K}_n(R)$ the spectrum from [Wag, 1972], pg. 359.

Proposition 5.17 *The above diagram commutes up to homotopy.*

Proof: The fact that the front face commutes is the result of Loday,[Lod, 1976], pg. 346. His proof can also replace $S^n R$ with $\mu^n R$ and $S\mathbb{Z}$ with $\mu\mathbb{Z}$ to show that the back face commutes. To see that the top and bottom faces we first note that the homotopy equivalence $BGL(S^n R)^+ \to BGL(\mu^n R)^+$ fits in to the commutative diagram of fibrations:

$$\begin{array}{ccccc} BGL(S^n R)^+ \times K_{-n}(R) & \longrightarrow & BGL(CS^n R)^+ & \longrightarrow & BGL(S^{n+1} R)^+ \\ \downarrow & & \downarrow & & \downarrow \\ BGL(\mu^n R)^+ \times K_{-n}(R) & \longrightarrow & BGL(l\mu^n R)^+ & \longrightarrow & BGL(\mu^{n+1} R)^+ . \end{array}$$

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This gives a commutative diagram:

$$\begin{array}{ccc} L_n(R) & \longrightarrow & \Omega L_{n+1}(R) \\ \downarrow & & \downarrow \\ \hat{K}_n(R) & \longrightarrow & \Omega \hat{K}_{n+1}(R). \end{array}$$

The adjoint of this diagram gives us the result we need for the top and bottom faces. The previous argument shows that the right face commutes up to homotopy since that square comes from comparing the weak pairings of L_n and $\hat{K}_n$. For the left face, we consider the following. Loday showed that the diagram:

$$\begin{array}{ccc} \Sigma(L_n(R)) \wedge L_m(R') & \longrightarrow & L_{n+m+1}(R \otimes R') \\ \downarrow & \nearrow & \\ \Sigma(L_n(R) \wedge L_m(R')) & & \end{array}$$

commutes up to homotopy. Thus, the face can be replaced with

$$\begin{array}{ccc} \Sigma(L_n(R) \wedge L_m(R')) & \longrightarrow & \Sigma L_{n+m}(R \otimes R') \\ \downarrow & & \downarrow \\ \Sigma(\hat{K}_n(R) \wedge \hat{K}_m(R')) & \longrightarrow & \Sigma \hat{K}_{n+m}(R \otimes R') \end{array}$$

which commutes since this is just the suspension of the diagram given to show that the pairings commute. □

Now we consider the maps

$$\eta_n : BGL(\mu^n R)^+ \times K_{-n}(R) \to B_*(\mu^n \mathcal{R}) \times K_{-n}(R).$$

We want to show that these maps commute up to homotopy with the pairings induced by $\otimes$ and that in fact the maps of pairings commute up to homotopy with the structure maps. First we prove the following proposition.

Proposition 5.18 *The following diagram commutes up to homotopy:*

$$\begin{array}{ccc} BGL(R)^+ \wedge BGL(\mu^m R')^+ & \xrightarrow{\otimes} & BGL(\mu^{n+m}(R \otimes R'))^+ \\ \downarrow & & \downarrow \\ B_*(\mu^n \mathcal{R}) \wedge B_*(\mu^m \mathcal{R}') & \xrightarrow{\otimes} & B_*(\mu^{n+m}(\mathcal{R} \otimes \mathcal{R}')). \end{array}$$

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Proof: For this we will use the infinite mapping telescope model for $BGL(\mu^n R)$ and thus also for $BGL(\mu^n R)^+$. The diagram

$$\begin{array}{ccc} BGL_k(\mu^n R) \times BGL_j(\mu^m R') & \xrightarrow{\otimes} & BGL_{kj}(\mu^{n+m}(R \otimes R')) \\ \downarrow & & \downarrow \\ B_*(\mu^n \mathcal{R}) \times B_*(\mu^m \mathcal{R}') & \xrightarrow{\otimes} & B_*(\mu^{n+m}(\mathcal{R} \otimes \mathcal{R}')) \end{array}$$

commutes by definition. We can then extend this to a homotopy commuting square

$$\begin{array}{ccc} BGL(\mu^n R)^+ \times BGL(\mu^m R')^+ & \longrightarrow & BGL(\mu^{n+m}(R \otimes R')) \\ \downarrow & & \downarrow \\ B_*(\mu^n \mathcal{R}) \times B_*(\mu^m \mathcal{R}') & \longrightarrow & B_*(\mu^{n+m}(\mathcal{R} \otimes \mathcal{R}')). \end{array}$$

We do this by using the fact that the following square homotopy commutes:

$$\begin{array}{ccc} BGL_k(\mu^n R)^+ \times BGL_j(\mu^m R') & \longrightarrow & BGL_{k+1}(\mu^n R)^+ \times BGL_j(\mu^m R') \\ \downarrow & & \downarrow \\ BGL(\mu^{n+m}(R \otimes R')) & \longrightarrow & BGL(\mu^{n+m}(R \otimes R')) \end{array}$$

as well as the similar diagram increasing j by 1. Since both of these are homotopy trivial along the wedge, we can in fact replace with the homotopy commutative square in the proposition. □

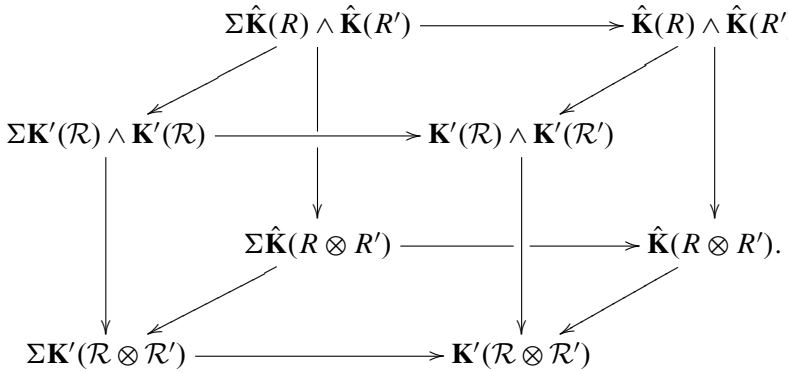
Thus we have the following corollary.

Corollary 5.19 *The pairings for $\hat{K}_n(R) \wedge \hat{K}_m(R')$ and $K'_n(\mathcal{R}) \wedge K'_m(\mathcal{R}')$ commute up to homotopy.*

We now wish to show that, up to homotopy, the map of pairings commutes with the structure map, that is to show the following.

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Proposition 5.20 *The following diagram commutes up to homotopy:*



Proof: To see that the top and bottom squares homotopy commute, we note that the homotopy equivalence $\hat{K}_n(R)\to K'_n(R)$ is compatible up to homotopy with the structure maps. The front and back commute up to homotopy since the pairings commute up to homotopy with the respective structure maps. The right square is the square from Corollary 5.19. To see that the left side commutes we note that as before we can replace this square with the diagram:

$$\begin{array}{ccc}\Sigma(\hat{K}_n(R)\wedge\hat{K}_m(R')) & \longrightarrow & \Sigma(K'_n(\mathcal{R})\wedge K'_m(\mathcal{R}'))\\ \downarrow & & \downarrow\\ \Sigma\hat{K}_{n+m}(R\otimes R') & \longrightarrow & \Sigma K'_{n+m}(\mathcal{R}\otimes\mathcal{R}')\end{array}$$

which is just the suspension of the diagram in Corollary 5.19. □

Combining the previous results we have a diagram of pairings:

$$\begin{array}{ccc}L_n(R)\wedge L_m(R') & \longrightarrow & L_{n+m}(R\otimes R')\\ \downarrow & & \downarrow\\ K'_n(\mathcal{R})\wedge K'_m(\mathcal{R}') & \longrightarrow & K'_{n+m}(\mathcal{R}\otimes\mathcal{R}')\end{array}$$

that commutes up to homotopy, which also commutes up to homotopy with the structure maps. Also the vertical maps are all homotopy equivalences so by taking homotopy inverses we get a square:

$$\begin{array}{ccc}K'_n(\mathcal{R})\wedge K'_m(\mathcal{R}') & \longrightarrow & K'_{n+m}(\mathcal{R}\otimes\mathcal{R}')\\ \downarrow & & \downarrow\\ L_n(R)\wedge L_m(R') & \longrightarrow & L_{n+m}(R\otimes R')\end{array}$$

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that homotopy commutes. Using the homotopy inverses, we can see that this diagram also commutes up to homotopy with the structure maps. Combining this with the result of Corollary 5.16 we have the following theorem.

Theorem 5.21 *There are homotopy equivalences $\theta_m : \mathbb{K}^{-\infty}(\mathcal{R}) \to \mathbf{L}(R)$ so that the following diagram commutes up to homotopy*

$$\begin{array}{ccc} K_n^{-\infty}(\mathcal{R}) \wedge K_m^{-\infty}\mathcal{R}' & \xrightarrow{\otimes} & K_{n+m}^{-\infty}(\mathcal{R} \otimes \mathcal{R}') \\ \downarrow{\scriptstyle \theta_n \wedge \theta_m} & & \downarrow{\scriptstyle \theta_{n+m}} \\ L_n(R) \wedge L_m(R') & \xrightarrow{\hat{\gamma}} & L_{n+m}(R \otimes R') \end{array}$$

and this diagram is compatible up to homotopy with the structure maps.

Here $\hat{\gamma}$ is Loday's pairing, and $\otimes$ is the Pedersen-Weibel pairing. We now show the following from [HP, 2004].

Theorem 5.22 *If the groupoid $\mathcal{G}$ is a group G, then the Loday assembly map α_G is isomorphic to the map on homotopy groups induced by the natural transformation A_G from Theorem 3.9.*

Proof: In this case we have $A_G : BG_+ \wedge \mathbb{K}^{-\infty}(\mathcal{R}) \to \mathbb{K}^{-\infty}(\mathcal{R}\mathcal{G})$ since we saw that in this case $R(\mathcal{G})_\oplus$ was just $\mathcal{R}\mathcal{G}$. Using the map of pairings from Theorem 5.21 gets us the desired isomorphism of maps on homotopy groups. □

Finally, we can get as a corollary to this, the desired result from [HP, 2004].

Corollary 5.23 *Let X be a connected space and $G = \pi_1(X, x_0)$. Then the Loday assembly map α_G is isomorphic to the map induced on homotopy groups by the Weiss-Williams assembly maps $\alpha_{B\pi(X)}$.*

The proof here uses the fact that the Pedersen-Weibel assembly map

$$\hat{\alpha}_X : X_+ \wedge \mathbb{K}^{-\infty}(\mathcal{R}) \xrightarrow{c \wedge 1} B\pi(X)_+ \wedge \mathbb{K}^{-\infty}(\mathcal{R}) \xrightarrow{A_{\pi(X)}} \mathbb{K}^{-\infty}(R(\pi(X))_\oplus)$$

where $c : X \to B\pi(X)$ is the classifying homotopy natural transformation of the universal covering is homotopy equivalent to the generalized assembly map α_X. When $X = BG$, c induces a homotopy equivalence, so we get that A_G is homotopy equivalent to $\alpha_{B\pi(X)}$. This shows that for BG we have an isomorphism between Loday's assembly map and the generalized Weiss-Williams assembly map.

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