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Prevalence of rapid mixing in hyperbolic flows

DMITRY DOLGOPYAT
Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA
(e-mail: dolgop@math.berkeley.edu)

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Abstract. We provide necessary and sufficient conditions for a suspension flow, over a subshift of finite type, to mix faster than any power of time. Then we show that these conditions are satisfied if the flow has two periodic orbits such that the ratio of the periods cannot be well approximated by rationals.

1. *Introduction*

The Bowen–Ruelle conjecture states that every mixing Axiom A flow is also exponentially mixing. This is equivalent to the corresponding property of suspension flows over subshifts of finite type. However, it turns out that if the roof function is locally constant no such flow can be exponentially mixing. Moreover, given any function of time one can produce a flow whose mixing rate is not faster than this function. In this paper we prove that, nonetheless, most flows have a decay of correlations which is quite fast. This is done by showing that, given the lengths of two periodic orbits, the correlations cannot decay more slowly than in a locally constant roof function flow.

We begin by describing necessary and sufficient conditions for rapid mixing. For this we need some notation. Let (Σ, σ) be the one-sided subshift of finite type with transition matrix Q. (All theorems we prove are also valid for two-sided subshifts, but we consider the one-sided case here to simplify the formulae.) A word means an admissible sequence of finite length. We assume that σ is topologically mixing, that is, some power of Q is strictly positive. Consider the distance $d_\theta(\omega, \varpi) = \theta^k$, where $k = \max\{j : \omega_i = \varpi_i \text{ for } i \le j\}$, $\theta < 1$. Let $C_\theta(\Sigma)$ be the space of functions which are Lipschitzian with respect to d endowed with the norm $\|h\|_\theta = \max(\|h\|_0, L(h))$, where $L(h)$ is the Lipschitz constant of h. If $\tau \in C_\theta(\Sigma)$ is positive the suspension flow with roof function τ is defined on

$$\Sigma^\tau = \{(\omega, t) : 0 \le t \le \tau(\omega)\}/((\omega, \tau(\omega)) \sim (\sigma(\omega), 0))$$

by $S^t(\omega, s) = (\omega, t + s)$ subject to the identification above. We will denote points of Σ^τ by q. We write $\tau_1 \sim \tau_2$ if τ_1 is cohomologous to τ_2, that is $\tau_2(\omega) = \tau_1(\omega) + T(\omega) - T(\sigma\omega)$. We refer the reader to [**PP**] for background concerning the basic notion of the thermodynamic formalism, such as Gibbs measures, the Ruelle–Perron–Frobenius theorem, the Livshic theorem etc. Let us introduce the set of test functions.

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Definition. The space $D(\Sigma^\tau)$ consists of functions of the form $D(\Sigma^\tau) = \{A(\omega, t) = 1_{\mathcal{C}}(\omega)a(t)$, where $\mathcal{C}$ is a cylinder in Σ and $a(t)$ is a C^∞ function with a compact support$\}$. It is assumed in this definition that $\operatorname{supp} a(t)$ lies strictly inside the interval $[0, \min_{\mathcal{C}} \tau(\omega)]$.

On Σ^τ we have the distance $d_\theta((\omega, t), (\omega', t')) = d_\theta(\omega, \omega') + |t - t'|$. Let $F(q)$ be a d_θ-Lipschitz function and μ_F be its Gibbs measure. For $A, B \in C(\Sigma^\tau)$, denote $R_{A,B}(F, t) = \mu_F(A(q)B(S^t q)) - \mu_F(A)\mu_F(B)$.

Definition. $\{S^t\} \in RM(F)$ ($\{S^t\}$ is rapidly mixing with respect to F) if:

(1) for $A, B \in D(\Sigma^\tau)$, $R_{A,B}(F, t)$ belongs to Schwartz space;

(2) the map $(A, B) \to R_{A,B}$ is continuous in the sense that given natural n there exist $N(n)$, $C(n)$ and $K(n)$ such that

$$|R_{A,B}(F, t)t^n| \le C(n)\|a\|_{C^N}\|b\|_{C^N} K^{l(\mathcal{C}(A))+l(\mathcal{C}(B))}$$

where $A(\omega, t) = 1_{\mathcal{C}(A)}(\omega)a(t)$, $B(\omega, t) = 1_{\mathcal{C}(B)}(\omega)b(t)$ and $l(\mathcal{C})$ denotes the length of $\mathcal{C}$. $\{S^t\}$ is called rapidly mixing ($\{S^t\} \in RM$) if it is rapidly mixing with respect to any Lipschitz F.

Definition. Let $\{S^t\}$ be a semiflow on a metric space M. We write $\{S^t\} \in TPM(q_1, q_2, t_0, \alpha_0)$ if for any r and $t \ge \max((1/r)^{\alpha_0}, t_0)$

$$S^t\mathcal{B}(q_1, r) \cap \mathcal{B}(q_2, r) \ne \emptyset$$

where $\mathcal{B}(q, r)$ is the ball of radius r about q. $\{S^t\}$ is called topologically power mixing ($\{S^t\} \in TPM$) if there exist t_0, α_0 such that $\forall q_1, q_2$ $\{S^t\} \in TPM(q_1, q_2, t_0, \alpha_0)$.

We write $h_n(\omega) = \sum_{j=0}^{n-1} h(\sigma^j\omega)$. Given $f \in C_\theta(\Sigma)$, denote by $\mathcal{L}_f$ the transfer operator on $C_\theta(\Sigma)$

$$\mathcal{L}_f h(\omega) = \sum_{\sigma\varpi=\omega} e^{f(\varpi)}h(\varpi)$$

and by $\mathcal{V}_b$ the operator

$$(\mathcal{V}_b h)(\omega) = e^{ib\tau(\omega)}h(\sigma\omega).$$

Below we give several conditions equivalent to rapid mixing. Our result is a quantitative version of the similar statement for mixing. Let us recall this theorem to the reader.

PROPOSITION 1. *The following conditions are equivalent:*

(i) $\exists F$ *such that* $(\Sigma^\tau, \{S^t\}, \mu_F)$ *is mixing;*

(ii) $\forall F$ $(\Sigma^\tau, \{S^t\}, \mu_F)$ *is mixing;*

(iii) $\exists F$ *such that* $(\Sigma^\tau, \{S^t\}, \mu_F)$ *is weak mixing;*

(iv) $\forall F$ $(\Sigma^\tau, \{S^t\}, \mu_F)$ *is weak mixing;*

(v) $\{S^t\}$ *has no Lipschitz continuous eigenfunction;*

(vi) $\{S^t\}$ *is topologically mixing;*

(vii) $\exists f$ *such that* $\mathcal{L}_f 1 = 1$ *and for all real* $b \ne 0$ $1 - \mathcal{L}_{f+ib\tau}$ *is invertible;*

(viii) $\forall f$ *such that* $\mathcal{L}_f 1 = 1$ *for all real* $b \ne 0$ $1 - \mathcal{L}_{f+ib\tau}$ *is invertible;*

(ix) $\exists f$ *such that* $\mathcal{L}_f 1 = 1$ *and for all real* $b \ne 0$ $\mathcal{L}_{f+ib\tau}$ *is not conjugated to* $\mathcal{L}_f$*;*

(x) $\forall f$ *such that* $\mathcal{L}_f 1 = 1$ *for all real* $b \ne 0$ $\mathcal{L}_{f+ib\tau}$ *is not conjugated to* $\mathcal{L}_f$*;*

(xi) *for all real* $b \ne 0$ *the operator* $\mathcal{V}_b$ *has no Lipschitz eigenfunction with eigenvalue 1.*

To introduce our conditions we need two definitions.

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Definition. Let $\mathcal{L} : C_\theta(\Sigma) \to C_\theta(\Sigma)$ be a linear operator. h is called a (λ, N, ϵ)-approximate eigenfunction if $1 \le \|h\|_\theta \le N$ and

$$\forall\omega \quad |(\mathcal{L}h - \lambda h)(\omega)| \le \epsilon.$$

Definition. $H \in C_\theta(\Sigma^\tau)$ is called a (λ, N, ϵ)-approximate eigenfunction for $\{S^t\}$ if $1 \le \|h\|_\theta \le N$, $H(t, q) = H(S^t q)$ is a differentiable function of t and $\forall q, t$

$$|(\partial_t - \lambda)H(t, q)| \le \epsilon.$$

THEOREM 1. *The following conditions are equivalent:*

(i) $\{S^t\} \in RM$;

(ii) $\exists F$ *such that* $\{S^t\} \in RM(F)$;

(iii) $\{S^t\} \in TPM$;

(iv) $\exists q_1, q_2$ *such that* $\{S^t\} \in TPM(q_1, q_2, t_0, \alpha_0)$;

(v) *given real* $f \in C_\theta(\Sigma)$ *such that* $\mathcal{L}_f 1 = 1$, *there are constants* α_1, C_1 *such that for real* b, $|b| > 1$, $\|(1 - \mathcal{L}_{f+ib\tau})^{-1}\| \le C_1|b|^{\alpha_1}$;

(vi) (v) *holds for some* f;

(vii) *given real* $f \in C_\theta(\Sigma)$ *such that* $\mathcal{L}_f 1 = 1$, *there is a constant* α_2 *such that for* ξ *satisfying* $|\Im\xi| > 1$, $|\Re\xi| \le |\Im\xi|^{-\alpha_2}$ *the operator* $1 - \mathcal{L}_{f+\xi\tau}$ *is invertible;*

(viii) (vii) *holds for some* f.

The theorem above may be considered as a quantitative version of the complex Ruelle–Perron–Frobenius theorem [**P1**]. The equivalence of (i) and (ii) to (vii) and (viii) can be regarded as the Paley–Wiener theorem for suspension flows.

Set $n(\beta, b) = [\beta \ln |b|]$.

THEOREM 2. *The following conditions are equivalent:*

(i) $\{S^t\} \notin TPM$;

(ii) *there is a function* f *satisfying* $\mathcal{L}_f 1 = 1$ *and a constant* α_3 *such that given* α *there exists a constant* β *and sequences* $b_k \to \infty$, $k \to \infty$ *and* $G_k(\omega)$ *such that* $|G_k| = 1$, $\|G_k\|_\theta \le |b_k|^{\alpha_3}$ *and if* $\mathcal{M}_k$ *denotes*

$$\mathcal{M}_k h = \frac{1}{G_k}\mathcal{L}^{n(\beta,b_k)}_{f+ib_k\tau}(G_k h), \tag{1}$$

then $\|\mathcal{M}_k - \mathcal{L}^{n(\beta,b_k)}_f\|_0 \le |b_k|^{-\alpha}$;

(iii) (ii) *holds for all* f *satisfying* $\mathcal{L}_f 1 = 1$;

(iv) *there exists* α_4, *such that, for any* α, $\{S^t\}$ *has a* $(ib_k, |b_k|^{\alpha_4}, |b_k|^{-\alpha})$*-approximate eigenfunction of absolute value 1 for some sequence* $b_k \to \infty$, $k \to \infty$;

(v) *There exists* α_5 *such that given* α *there exist a constant* β *and a sequence* $b_k \to \infty$, $k \to \infty$ *such that* $\mathcal{V}^{n(\beta,b_k)}_{b_k}$ *has a* $(1, |b_k|^{\alpha_5}, |b_k|^{-\alpha})$*-approximate eigenfunction of absolute value* 1.

Usually it is hard to verify any of the conditions above. However, there is a quite simple sufficient condition.

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Example. [**R2**, **P3**] Let (Σ, σ) be the full shift on two symbols

$$\tau(x) = \begin{cases} l_1 & \text{if } x_0 = 0 \\ l_2 & \text{if } x_0 = 1 \end{cases}$$

and $f(x) \equiv \ln 2$. One readily sees that $\mathcal{L}_{f+ib\tau}1 = \frac{1}{2}(e^{ibl_1} + e^{ibl_2})1$. Moreover, it is shown in the above-mentioned papers that

$$\|(1 - \mathcal{L}_{f+ib\tau})^{-1}\| \le \max\left(1 - \varepsilon, \frac{2}{2 - (e^{ibl_1} + e^{ibl_2})}\right).$$

The conditions of Theorem 1 are satisfied if l_1/l_2 is a Diophantine number, that is there exist C_2 and α_6 such that

$$\left|\frac{l_1}{l_2} - \frac{m_1}{m_2}\right| \ge C_2 m_2^{-\alpha_6}$$

for all integers m_1, m_2. This condition also works in a more general set-up.

THEOREM 3. *Assume that $\{S^t\}$ has two periodic points $\omega_1 = \sigma\omega_1$ and $\omega_2 = \sigma\omega_2$ with periods $l_1 = \tau(\omega_1)$, $l_2 = \tau(\omega_2)$, such that l_1/l_2 is a Diophantine number, then $\{S^t\}$ is rapidly mixing.*

This statement is proven in §13.

COROLLARY 1. *Let $\{\tau_s\}$ be continuous one-parameter family of roof functions and $\{S_s^t\}$ be the corresponding suspension flows. Assume that $\{\{S_s^t\}\} \cap RM = \emptyset$. Then there exists a function $\Gamma(s)$ such that all flows $\{S_s^{t/\Gamma(s)}\}$ are Holder conjugated to each other.*

Proof. Let $(\omega_1, 0)$ be a periodic point $\omega_1 = \sigma^n\omega_1$. Set $\Gamma(s) = (\tau_s)_n(\omega_1)$. Then all flows $\{S_s^{t/\Gamma(s)}\}$ have the periodic point $(\omega_1, 0)$ of period one. Therefore, by Theorem 3 all flows $S_s^{t/\Gamma(s)}$ have only periodic points of non-Diophantine length. However, any continuous function of the parameter which assumes only non-Diophantine values is a constant. Hence, all flows $\{S_s^{t/\Gamma(s)}\}$ have the same length spectrum so the statement follows by Livshic's theorem. □

COROLLARY 2. *In a generic one-parameter family* $\{\tau_s\}$ $\operatorname{mes}\{s : \{S_s^t\} \notin RM\} = 0$.

The counterparts of these results are also valid for smooth flows. The proofs are very similar. Let $\{S^t\}$ be a flow with basic hyperbolic set Λ. We say that S^t is rapidly mixing on Λ with respect to the Holder continuous potential F ($\{S^t\} \in RM(\Lambda, F)$) if the map $(A, B) \to R_{A,B}(F, t)$ is continuous from $C^\infty(M) \times C^\infty(M)$ to the Schwartz space.

THEOREM 4. *The following conditions are equivalent:*
(i) $\{S^t\} \in RM(\Lambda)$;
(ii) $\exists F$ *such that* $\{S^t\} \in RM(\Lambda, F)$;
(iii) $\{S^t\} \in TPM(\Lambda)$.

THEOREM 5. *Assume that $\{S^t\}$ has two periodic points γ_1 and γ_2 in Λ such that the ratio of their periods is a Diophantine number, then $\{S^t\}$ is rapidly mixing on Λ.*

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COROLLARY 3. *Let $\{\{S_s^t\}\}$ be a continuous one-parameter family of the flows such that $\{S_s^t\}$ has a basic hyperbolic set Λ_s near Λ_0. Then, if no $\{S_s^t\}$ is rapidly mixing on Λ_s, there is a function $\Gamma(s)$ such that $S_s^{t/\Gamma(s)}|_{\Lambda_s}$ is Holder conjugated to $S_0^t|_{\Lambda_0}$.*

COROLLARY 4. *For any $k \geq 1$ in a generic one-parameter C^k family $\{\{S_s^t\}\}$,* mes$\{s : \{S_s^t\} \notin RM(\Lambda_s)\} = 0$.

2. *Scheme of the proof of Theorems 1 and 2*

From now on we write simply (i)–(viii) for conditions (i)–(viii) of Theorem 1 and (2(i))–(2(v)) for conditions (i)–(v) of Theorem 2. (ix)–(xii) will mean the converses of (2(ii))–(2(v)), respectively.

The plan of the proof of Theorems 1 and 2 is as follows. It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv), (v) $\Rightarrow$ (vi), (vii) $\Rightarrow$ (viii) and (ix) $\Rightarrow$ (x). Also, (v) $\Rightarrow$ (vii) and (vi) $\Rightarrow$ (viii) since

$$(1-\mathcal{L}_{f+\xi\tau})^{-1} = (1-(1-\mathcal{L}_{f+i\Im\xi\tau})^{-1}(\mathcal{L}_{f+\xi\tau}-\mathcal{L}_{f+i\Im\xi\tau}))^{-1}(1-\mathcal{L}_{f+i\Im\xi\tau})^{-1}$$

and $\|(\mathcal{L}_{f+\xi\tau}-\mathcal{L}_{f+i\Im\xi\tau})\| \leq C_3|\Re\xi|$. In §3 we prove that (xi) $\Leftrightarrow$ (xii), in §4 we show that (x) $\Rightarrow$ (xii) and in §5 we demonstrate that (iv) $\Rightarrow$ (xi). §6 contains an auxiliary estimate. The implication (xii) $\Rightarrow$ (v) is proven in §7–9. The implication (vi) $\Rightarrow$ (ii) where F and f are related by

$$\bar{F}(x) = f(x) + G(x) - G(\sigma x) + Pr(F)\tau \tag{2}$$

(where $\bar{F}(x) = \int_0^{\tau(x)} F(x,s)\,ds$ and $Pr(F)$ is a constant) is due to Pollicott [**P2**, **P3**]. For convenience to the reader we provide the proof in §10. This gives (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (xi) $\Leftrightarrow$ (xii). Hence, (i) is also equivalent to (ii)–(vi) because every F has decomposition (2) (see, for example, [**PP**]). Thus, we get

$$\text{(ix)} \Rightarrow \text{(x)} \Rightarrow (\text{(i)} - \text{(vi)}, \text{(xi)}, \text{(xii)}) \Rightarrow \text{(vii)} \Rightarrow \text{(viii)}.$$

In §11 we prove that (vii) $\Rightarrow$ (ix) with the same function f. Therefore, also (viii) $\Rightarrow$ (x), which completes the proof of Theorems 1 and 2.

3. *Eigenfunctions of $\{S^t\}$ and $\mathcal{V}$*

In this section we show that (2(iv)) $\Leftrightarrow$ (2(v)) or in other words that (xi) $\Leftrightarrow$ (xii).

(2(iv)) $\Rightarrow$ (2(v)). Let $J_k(q)$ be the approximate eigenfunction for $\{S^t\}$. Set $H_k(\omega) = 1/J_k((\omega,0))$. Then

$$\begin{aligned}
|((\mathcal{V}_{b_k}^{n(\beta,b_k)}H_k) - H_k)(\omega)| &= |e^{ib_k\tau_{n(\beta,b_k)}(\omega)}H_k(\sigma^{n(\beta,b_k)}\omega) - H(\omega)| \\
&= |e^{ib_k\tau_{n(\beta,b_k)}(\omega)}J_k((\omega,0)) - J_k((\sigma^{n(\beta,b_k)}\omega,0))| \\
&= |e^{ib_k\tau_{n(\beta,b_k)}(\omega)}J_k((\omega,0)) - J_k(S^{\tau_{n(\beta,b_k)}(\omega)}(\omega,0))| \\
&\leq |b_k|^{-\alpha}\tau_{n(\beta,b_k)}(\omega) \leq C_4|b_k|^{-\alpha}n(\beta,b_k).
\end{aligned}$$

(2(v)) $\Rightarrow$ (2(iv)). Let H_k be the approximate eigenfunction of $\mathcal{V}_{b_k}^{n(\beta,b_k)}$, then if b_k is large enough we can find a branch $\Delta(\omega)$ of

$$\text{Arg}\left(\frac{H_k(\omega)}{H_k(\sigma^{n(\beta,b_k)}\omega)}e^{ib_k\tau_{n(\beta,b_k)}(\omega)}\right)$$

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which takes values in $(-2/|b_k|^\alpha, 2/|b_k|^\alpha)$. Consider the space

$$\tilde\Sigma^\tau = \{(\omega, t) : 0 \le t \le \tau_n(\omega)\}/((\omega, \tau_n(\omega)) \sim (\sigma^n\omega, 0)).$$

$\mathbb{Z}_n$ acts on $\tilde\Sigma^\tau$ by shifts $T^k(\omega, t) = (\sigma^k\omega, t + \tau_k(\omega))$ and $\tilde\Sigma^\tau/\mathbb{Z}_n \cong \Sigma^\tau$. Denote by $\{\tilde S^t\}$ the flow $\tilde S^t(\omega, s) = (\omega, s + t)$ and by P the natural projection $P : \tilde\Sigma^\tau \to \Sigma^\tau$. We may assume, without loss of generality, that $\tau \ge 1$. Let $\varphi(t)$ be a cut-off C^∞ function such that $\operatorname{supp}\varphi \subset [\frac12, 1]$ and $\int_{\frac12}^1 \varphi(s)\,ds = 1$. If $\tilde q = (\omega, t)$, $0 \le t < \tau_{n(b_k)}(\omega)$, set

$$\tilde J_k(\tilde q) = \frac{1}{H_k(\omega)} \exp\left(i\left[\int_0^t (b_k + \Delta(x)\varphi(s))\,ds\right]\right).$$

$\tilde J(\tilde q)$ can be lifted to a $(ib_k, C_5|b_k|^{\alpha_7}, 1/|b_k|^{-\alpha})$-approximate eigenfunction for $\{\tilde S^t\}$ ($\alpha_7 = \max(1, \alpha_5)$). Let

$$J(q) = \prod_{P\tilde q = q} \tilde J(\tilde q),$$

then J is a $(in_k(\beta, b_k)b_k, C_5|b_k|^{\alpha_7} n(\beta, b_k), n(\beta, b_k)/|b_k|^{-\alpha})$-approximate eigenfunction for $\{S^t\}$.

4. *Almost conjugacy*

In this section we show that (2(v)) $\Rightarrow$ (2(iii)) or in other words (x) $\Rightarrow$ (xii). This follows immediately from the following statement.

LEMMA 1. *Assume that* $|G| \equiv 1$ *and* $\|\mathcal{V}_b^n G - e^{i\Theta}G\|_0 \le \epsilon$. *Define* $\mathcal{M}h = G\mathcal{L}^n_{f+ib\tau}(h/G)$, *then* $\|(\mathcal{M} - e^{i\Theta}\mathcal{L}_f^n)h\|_0 \le \epsilon\|h\|_0$.

Proof.

$$\begin{aligned} |\mathcal{M} - e^{i\Theta}\mathcal{L}_f^n h(\omega)| &= \left|\sum_{\sigma^n\varpi=\omega} e^{f_n(\varpi)}\left[\frac{G_k(\sigma^n\varpi)}{G_k(\varpi)} e^{ib_k\tau_n(\varpi)} - e^{i\Theta}\right]h(\varpi)\right| \\ &\le \sum_{\sigma^n\varpi=\omega} e^{f_n(\varpi)}\epsilon\|h\|_0 = \epsilon\|h\|_0(\mathcal{L}_f^n 1)(\omega) = \epsilon\|h\|_0. \end{aligned}$$

□

5. *Topological power mixing and approximate eigenfunctions*

In this section we show that (iv) $\Rightarrow$ (xi). Assume that there exist α_0, t_0 so that for $t > \min((1/r)^{\alpha_0}, t_0)$, $S^t\mathcal{B}(q_1, r)\bigcap\mathcal{B}(q_2, r) \ne \emptyset$. We prove that, for any α_4, $\{S^t\}$ has no approximate eigenvalue of unit absolute value if α and $|b|$ are large enough. Indeed, assume that the converse is true. Let $J(q)$ be such an eigenfunction, $J(q_1) = e^{i\Theta_1}$, $J(q_2) = e^{i\Theta_2}$. Let $r = 1/|b|^\gamma$, then for $q \in \mathcal{B}(q_1, r)$, $|J(q) - e^{i\Theta_1}| \le |b|^{\alpha_4}/|b|^\gamma$ and for $q \in \mathcal{B}(q_2, r)$, $|J(q) - e^{i\Theta_2}| \le |b|^{\alpha_4}/|b|^\gamma$. Also for $q \in S^t\mathcal{B}(q_1, r)$,

$$|J(q) - e^{i(\Theta_1 - bt)}| \le \frac{t}{|b|^\alpha} + \frac{|b|^{\alpha_4}}{|b|^\gamma}.$$

So if $t < 2/r^{\alpha_0} = 2|b|^{\gamma\alpha_0}$ then

$$|J(q) - e^{i(\Theta_1 - bt)}| \le \frac{2|b|^{\gamma\alpha_0}}{|b|^\alpha} + \frac{|b|^{\alpha_4}}{|b|^\gamma}.$$

Thus if, for any t, $1/r^{\alpha_0} \le t \le 2/r^{\alpha_0}$ $\exists q \in S^t\mathcal{B}(q_1, r)\bigcap\mathcal{B}(q_2, r)$ then for any such t

$$|e^{i(\Theta_1 + tb)} - e^{i\Theta_2}| \le 2|b|^{\alpha_4 - \gamma} + 2|b|^{\gamma\alpha_0 - \alpha}$$

which cannot be true if $\gamma > \alpha_4$ and $\alpha > \gamma\alpha_0$.

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6. *A priori bounds*

This section contains useful bounds for iterations of complex transfer operators.

PROPOSITION 2. (See [**PP**]) *Let* $\mathcal{L}_f 1 = 1$, *then:*

(a) $\|\mathcal{L}^n_{f+ib\tau}\| \le 1$;

(b) *there exists a constant* C_6 *such that*

$$L(\mathcal{L}^n_{f+ib\tau}h) \le C_6(|b|\|h\|_0 + \theta^n L(h)).$$

Proof. By direct calculation. □

Consider a new norm on $C_\theta(\Sigma)$

$$\|h\|_{(N)} = \max\left(\|h\|_0, \frac{L(h)}{2C_6N}\right),$$

then $\|\mathcal{L}^n_{f+ib\tau}\|_{(|b|)} \le 1$.

For future use we provide an estimate of the resolvent of $\mathcal{L}_f$ in $\|\cdot\|_{(N)}$-norm. By the Ruelle–Perron–Frobenius theorem we can decompose $\mathcal{L}_f$ as $\mathcal{L}_f = \mathcal{P} + \mathcal{N}$ where $\mathcal{P}h = \nu_f(h)1$, $\mathcal{N}\mathcal{P} = \mathcal{P}\mathcal{N} = 0$ and $\|\mathcal{N}^k\|_\theta \le C_7\delta^k$ for some constants $\delta < 1, C_7$.

LEMMA 2. *If* $(1+\delta)/2 \le |\lambda| \le 1$ *then*

$$\|(\lambda - \mathcal{L}^n_f)^{-1}\|_{(N)} \le \frac{C_8}{|1-\lambda|}\left(\ln\frac{1}{|1-\lambda|} + \ln N + n\right).$$

Proof. Inequalities

$$\|\cdot\|_\theta \le \|\cdot\|_{(N)} \le 2C_6N\|\cdot\|_\theta$$

imply $\|\mathcal{N}^k\|_{(N)} \le C_9\delta^k N$. Now

$$(\lambda - (\mathcal{P})^k)^{-1} = (\lambda - \mathcal{P})^{-1} = \frac{\mathcal{P}}{\lambda - 1} + \frac{1-\mathcal{P}}{\lambda}.$$

Hence,

$$\left\|\left(1 - \frac{\mathcal{P}^k}{\lambda}\right)^{-1}\right\|_{(N)} \le \frac{C_{10}}{|\lambda - 1|}.$$

Take $k = [C_{11}(\ln(N) + \ln(1/|\lambda - 1|))]$. Then if C_{11} is large enough

$$\|\mathcal{N}^k\|_{(N)} \le \frac{C_{10}}{2|\lambda - 1|} \le \frac{1}{2\|(1 - (\mathcal{P}^k/\lambda))^{-1}\|_{(N)}}.$$

Therefore,

$$\|(\lambda - \mathcal{L}^{kn}_f)^{-1}\|_{(N)} \le \frac{C_{12}}{|\lambda - 1|}$$

and the proof is complete. □

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7. *Pointwise estimates*

In this section we begin with the proof of (xii) $\Rightarrow$ (v). We assume that (v) is false and obtain (2(v)). If (v) fails there is a function f, $\mathcal{L}_f 1 = 1$, such that given α there is an arbitrary large b such that $\|(1 - \mathcal{L}_{f+ib\tau})^{-1}\|_\theta > |b|^\alpha$. Choose such a function f. In the next three sections we will work only with this weight. We use $\mathcal{L}_b$ as a shorthand for $\mathcal{L}_{f+ib\tau}$. The proof of the above implication consists of two steps. First, we show that if (v) is false then, given α, there are $\beta(\alpha)$ and $b_k \to \infty$ such that $\mathcal{V}_{b_k}^{n(\beta,b_k)}$ has an approximate eigenfunction with approximate eigenvalue $e^{i\Theta}$. We then demonstrate that $e^{i\Theta}$ should be close to 1. The first step in turn, is divided into two parts. In this section we show that if (v) is violated there is a function h, $\|h\|_{(b)} = 1$, whose $\mathcal{L}_b$ iterations remain near the unit sphere for a long time; in the next section we show that some iteration of h gives the approximate eigenfunction we need.

LEMMA 3. *If there are α, β such that for any b, $|b| > 1$ and for any h, $\|h\|_{(|b|)} \le 1$, there exist $\omega_0 \in \Sigma$ and n, $0 \le n \le 3n(\beta, b)$, such that $|(\mathcal{L}_b{}^n(h))(\omega_0)| \le 1 - 1/|b|^\alpha$, then there exist constants C_{13}, α' such that $\|(1 - \mathcal{L}_b)^{-1}\|_\theta \le C_{13}|b|^{\alpha'}$.*

Proof.

$$|\mathcal{L}_b{}^N h(\omega)| = |(\mathcal{L}_b{}^{N-n}(\mathcal{L}_b{}^n h))(\omega)| \le (\mathcal{L}_f^{N-n}|\mathcal{L}_b{}^n h|)(\omega).$$

Let ν_f be the Gibbs measure for f (that is, $\mathcal{L}_f^* \nu_f = \nu_f$), then by the Ruelle–Perron–Frobenius theorem

$$(\mathcal{L}_f^{N-n}|\mathcal{L}_b{}^n h|)(\omega) \le \nu_f(|\mathcal{L}_b{}^n h|) + C_7 \delta^{N-n}\|\mathcal{L}_b{}^n h\|_\theta.$$

By Proposition 2, $\|\mathcal{L}_b{}^n h\|_\theta \le C_7|b|$. On the other hand, $|(\mathcal{L}_b{}^n h)(\omega)| \le 1 - (1/2|b|^\alpha)$ for $\omega \in \mathcal{B}(\omega_0, 1/2C_6|b|^{\alpha+1})$. Now there exist constants C_{14} and α_8 such that $\nu_f(\mathcal{B}(\omega_0, 1/2C_6|b|^{\alpha+1})) \ge C_{14}/|b|^{\alpha_8}$. Hence, $\nu_f(|\mathcal{L}_b{}^n h|) \le 1 - C_{15}/|b|^{\alpha_9}$ and

$$\|\mathcal{L}_b{}^N h\|_0 \le 1 - \frac{C_{15}}{|b|^{\alpha_9}} + C_7 \delta^{N-n}|b|.$$

Take $N = n(\bar{\beta}, b)$ where $\bar{\beta} \gg \beta$, then

$$\|\mathcal{L}_b{}^N h\|_0 \le 1 - \frac{C_{15}}{2|b|^{\alpha_9}}.$$

Take $\tilde{N} > N$, then

$$\|\mathcal{L}_b{}^{\tilde{N}} h\|_0 \le \|\mathcal{L}_b{}^N h\|_0 \le 1 - \frac{C_{15}}{2|b|^{\alpha_9}},$$

whereas

$$L(\mathcal{L}_b{}^{\tilde{N}} h) \le \left(1 - \frac{C_{15}}{2|b|^\alpha}\right) C_6|b| + C_6\theta^{\tilde{N}-N}|b|.$$

So if $\tilde{N} = n(\tilde{\beta}, b)$ where $\tilde{\beta} \gg \bar{\beta}$ then

$$\|\mathcal{L}^{\tilde{N}} h\|_{(b)} \le 1 - \frac{C_{15}}{4|b|^{\alpha_9}}.$$

The estimate of the lemma now follows easily. □

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8. *Construction of approximate eigenfunctions*

In this section we use Lemma 3 to construct approximate eigenfunctions for $\mathcal{L}_b$.

LEMMA 4. *Assume that* (v) *fails, then given* α *there is* $\beta(\alpha)$ *such that one can find an arbitrary large* b *such that*

$$\|(1-\mathcal{L}_b)^{-1}\|_\theta > |b|^\alpha$$

and $\mathcal{V}_b^{n(\beta,b)}$ *has a* $(e^{i\Theta}, 2C_6|b|, 1/|b|^\alpha)$*-approximate eigenfunction of absolute value 1.*

The proof proceeds by a series of lemmas. Take some α', β, then by Lemma 3 one can find an arbitrary large b such that

$$\|(1-\mathcal{L}_b)^{-1}\|_\theta > |b|^\alpha$$

and there is a function $h(\omega)$ such that

$$\forall n : 0 \le n \le 3n(\beta, b) \quad \forall \omega \ |(\mathcal{L}_b{}^n h)(\omega)| \ge 1 - \frac{1}{|b|^{\alpha'}}. \tag{3}$$

Write $\bar{h} = \mathcal{L}_b{}^{n(\beta,b)}h$, $\bar{\bar{h}} = \mathcal{L}_b{}^{2n(\beta,b)}h$, $h(\omega) = r(\omega)e^{i\gamma(\omega)}$, $\bar{h}(\omega) = \bar{r}(\omega)e^{i\bar{\gamma}(\omega)}$ and $\bar{\bar{h}}(\omega) = \bar{\bar{r}}(\omega)e^{i\bar{\bar{\gamma}}(\omega)}$.

LEMMA 5. $\forall \bar{\alpha}, \beta \ \exists \alpha'$ *such that if (3) holds then* $\forall(\varpi, \omega) : \omega = \sigma^{n(\beta,b)}\varpi$

$$|\exp(i[b\tau_{n(\beta,b)}(\varpi) + \gamma(\varpi) - \bar{\gamma}(\omega)]) - 1| \le \frac{1}{|b|^{\bar{\alpha}}}, \tag{4}$$

$$|\exp(i[b\tau_{n(\beta,b)}(\varpi) + \bar{\gamma}(\varpi) - \bar{\bar{\gamma}}(\omega)]) - 1| \le \frac{1}{|b|^{\bar{\alpha}}}. \tag{5}$$

Proof. We show only (4), the proof of (5) being the same. By definition,

$$\bar{r}(\omega)e^{i\bar{\gamma}(\omega)} = \sum_{\sigma^{n(\beta,b)}\varpi=\omega} e^{(f+ib\tau)_{n(\beta,b)}(\varpi)} r(\varpi)e^{i\gamma(\varpi)}.$$

Thus

$$1 - \bar{r}(\omega) = \sum_{\sigma^{n(\beta,b)}\varpi=\omega} e^{f_{n(\beta,b)}(\omega)}(1 - r(\varpi)\exp(i[b\tau_{n(\beta,b)}(\varpi) + \gamma(\varpi) - \bar{\gamma}(\omega)]))$$

so that

$$\sum_{\sigma^{n(\beta,b)}\varpi=\omega} e^{f_{n(\beta,b)}(\omega)}(1 - \exp(i[b\tau_{n(\beta,b)}(\varpi) + \gamma(\varpi) - \bar{\gamma}(\omega)])) = O\left(\frac{1}{|b|^{\alpha'}}\right).$$

Since the real part of each term is positive $\forall(\varpi, \omega)$

$$e^{f_{n(\beta,b)}(\omega)}\Re(1 - \exp(i[b\tau_{n(\beta,b)}(\varpi) + \gamma(\varpi) - \bar{\gamma}(\omega)])) = O\left(\frac{1}{|b|^{\alpha'}}\right).$$

Using the estimate

$$\exp[f_{n(\beta,b)}(\omega)] \ge \exp[-n(\beta, b)\|f\|_0] \ge \frac{C_{16}}{|b|^{\beta\|f\|_0}}$$

we get

$$\Re(1 - \exp(i[b\tau_{n(\beta,b)}(\varpi) + \gamma(\varpi) - \bar{\gamma}(\omega)])) = O\left(\frac{1}{|b|^{\alpha'-\beta\|f\|_0}}\right). \qquad \square$$

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LEMMA 6. *If β in the previous lemma is large enough, then there exists Θ such that $\forall\omega$*

$$|\exp(i[\bar{\bar{\gamma}}(\omega) - \bar{\gamma}(\omega) - \Theta]) - 1| \le \frac{4}{|b|^{\bar{\alpha}}}.$$

Proof. Let $\mathcal{B}$ be a ball of radius $1/2C_6 b^{\bar{\alpha}+1}$ in Σ. Then $\exists\Theta_1, \Theta_2$ such that $\forall\varpi \in \mathcal{B}$,

$$|e^{i\gamma(\varpi)} - e^{i\Theta_1}| \le \frac{1}{|b|^{\bar{\alpha}}}, \quad |e^{i\bar{\gamma}(\varpi)} - e^{i\Theta_2}| \le \frac{1}{|b|^{\bar{\alpha}}}.$$

If β is large enough $\forall\omega\,\exists\varpi \in \mathcal{B}$ such that $\sigma^{n(\beta,b)}\varpi = \omega$ and the statement follows by Lemma 5. □

Let $\bar{\alpha} > \alpha$. We claim that $H(\omega) = e^{-i\gamma(\omega)}$ is the required eigenfunction. Indeed

$$\begin{aligned}\frac{\mathcal{V}_b^{n(\beta,b)} H}{H}(\omega) &= \exp(i[b\tau_{n(\beta,b)}(\omega) - \bar{\gamma}(\sigma^{n(b,\beta)}\omega) + \bar{\gamma}(\omega)]) \\ &= \exp(i[\Theta + b\tau_{n(\beta,b)}(\omega) - \bar{\bar{\gamma}}(\sigma^{n(b,\beta)}\omega) + \bar{\gamma}(\omega)])\left(1 + O\left(\frac{1}{|b|^{\bar{\alpha}}}\right)\right) \\ &= \exp(i\Theta)\left(1 + O\left(\frac{1}{|b|^{\bar{\alpha}}}\right)\right)\end{aligned}$$

by Lemma 5.

9. *Perturbation estimates*

Combining the result of the previous section with Lemma 1 we obtain the following.

LEMMA 7. *If* (v) *fails, then given α there exists $\beta(\alpha)$ such that one can find an arbitrary large b, a constant Θ and a function H satisfying $|H| \equiv 1$, $\|H\|_{(b)} = 1$ such that if $\mathcal{K}$ denotes the operator $\mathcal{K}h = He^{i\Theta}\mathcal{L}_f^{n(\beta,b)}(h/H)$ then*

$$\|(1 - \mathcal{L}_b)^{-1}\|_\theta \ge |b|^\alpha$$
$$\|(\mathcal{L}_b{}^{n(\beta,b)} - \mathcal{K})h\|_0 \le \|h\|_0 \frac{1}{|b|^\alpha}.$$

We now show that the last two inequalities impose some restrictions on the value of Θ.

PROPOSITION 3.

$$L((\mathcal{L}_b{}^{n(\beta,b)} - \mathcal{K})h) \le \frac{1}{|b|^\alpha} L(h) + C_{17}|b|\|h\|_0.$$

Proof. By direct calculation. □

Recall that $\|h\|_{(N)} = \max(\|h\|_0, L(h)/C_6 N)$. By Proposition 3

$$\|(\mathcal{L}_b{}^{n(\beta,b)} - \mathcal{K})h\|_{(N)} \le \frac{1}{|b|^\alpha} + \frac{C_{17}}{N}|b|.$$

We show that if b is large enough, then $|e^{i\Theta} - 1| \le 1/|b|^{\alpha^*}$, $\alpha^* = (\alpha - 3)/2$. Indeed, assume that this inequality is false. Then by Lemma 2

$$\|(1 - e^{i\Theta}\mathcal{L}_f^{n(\beta,b)})^{-1}\|_{(N)} \le C_{18}|b|^{\alpha^*}(\ln|b| + \ln N).$$

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Now if $N > 2C_6|b|$, multiplication by H has norm less than 2 and so

$$\|(1-\mathcal{K})^{-1}\|_{(N)} \le C_{19}|b|^{\alpha^*}.$$

On the other hand, if $N = |b|^{\alpha^*+2}$ then $\|\mathcal{L}_b - \mathcal{K}\|_{(N)} \le C_{20}|b|^{\alpha^*+1}$. Therefore, $\|(1-\mathcal{L}_b)^{-1}\|_{(N)} \le C_{21}|b|^{\alpha}$ and

$$\|(1-\mathcal{L}_b)^{-1}\|_{\theta} \le 2C_6 N\|(1-\mathcal{L}_b)^{-1}\|_{(N)} = C_{22}|b|^{2\alpha^*+2} = C_{22}|b|^{\alpha-1},$$

a contradiction. Hence in fact $|e^{i\Theta}-1| \le 1/|b|^{\alpha^*(\alpha)}$. Since $\alpha^* \to \infty$ as $\alpha \to \infty$, the implication (xi) $\Rightarrow$ (v) is proven.

10. *Decay of correlations*

In this section we prove that (vi) $\Rightarrow$ (ii). Let $\bar{F}(\omega) = \int_0^{\tau(\omega)} F(\omega, s)\,ds$. Take such that $\bar{F}(\omega) = f(\omega) + G(\omega) - G(\sigma\omega) + Pr(F)\tau$ and $\mathcal{L}_f 1 = 1$. Let $\hat{A}(\omega,\xi)$ be the Laplace transform of A, $\hat{A}(\omega,\xi) = \int_0^{\tau(\omega)} F(\omega,s)e^{-\xi s}ds$. We use the following expression of the Laplace transform of $R_{A,B}(F,t)$ (see [**P2**]),

$$\hat{R}_{A,B}(F,\xi) = \frac{1}{\nu_f(\tau)}\int \hat{B}(\omega,\xi)[(1-\mathcal{L}_{f-\xi\tau})^{-1}\hat{A}(\omega,-\xi)]\,d\nu_f + \tilde{R}(\xi)$$

where $\tilde{R}$ is an entire function bounded in any strip $|\Re\xi| \le M$.

We have already noticed that (vi) $\Rightarrow$ (viii) and, moreover, for Σ^τ Lipschitz A, B we have the following inequality in the region $|b| > 1$, $a > -|b|^{-\alpha_2}$,

$$\hat{R}_{A,B}(F, a+ib) \le C_{23}\|A\|_1\|B\|_1|b|^{\alpha_{10}}.$$

Now consider the case when $A, B \in D(\Sigma^\tau)$. In this case, $R_{A,B}(F,t)$ is a smooth function and $(\partial/\partial t)R_{A,B}(F,t) = -R_{\partial_t A,B}$, ∂_t being the derivative along the orbits. Consider the Taylor expansion

$$R_{A,B}(t) = \sum_{j=0}^{N-1}\frac{R_{A,B}^{(j)}(0)}{j!}t^j + \int_0^t R_{\partial_t A,B}(s)\frac{(s-t)^N}{(N-1)!}\,ds.$$

The Laplace transform of the last term is analytic near the imaginary axis except the pole of the Nth order at zero and decays not slower than

$$C_{23}\|A\|_N\|B\|_1|b|^{\alpha_{10}-N}$$

in the region above. (Here $\|1_C(\omega)a(t)\|_N$ means $(1/\theta)^{l(C)}\|a\|_{C^N}$.)

Applying the inversion formula for the Laplace transform and moving the contour of the integration to $\{-a = \min(\varepsilon, |b|^{-\alpha_2})\}$ we get

$$\left|\sum_{j=0}^{N-1}\frac{R_{A,B}^{(j)}(0)}{j!}t^j + \int_0^t R_{\partial_t A,B}(s)\frac{(s-t)^N}{(N-1)!}\,ds\right|$$
$$\le C_{24}\|A\|_N\|B\|_1\int_1^\infty |b|^{\alpha_{10}-N}\exp(-t|b|^{-\alpha_2})\,db$$

which implies (ii).

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11. *Polefree regions*

In this section we prove that (vii) $\Rightarrow$ (ix). To this end, it is enough to show that if for some f (2(ii)) holds then the poles of $(1-\mathcal{L}_\xi)^{-1}$ accumulate to the imaginary axis faster than any power of ξ ($\mathcal{L}_\xi$ is a shorthand for $\mathcal{L}_{f+i\xi\tau}$). The idea is to establish the analyticity of the leading eigenvalue of $\mathcal{L}_\xi$ and then use Rouche's theorem to show that $1 \in Sp(\mathcal{L}_\xi)$. So assume that for some $\bar b$ there exists G, $|G| \equiv 1$, $\|G\|_\theta \le |\bar b|^{\alpha_3}$ such that $\mathcal{M}h = (1/G)\mathcal{L}_{ib_0}^{n(\beta,\bar b)}(Gh)$ satisfies

$$\|\mathcal{L}_f^{n(\beta,\bar b)} - \mathcal{M}\|_0 \le |\bar b|^{-\alpha}.$$

Denote $\mathcal{M}_\xi = (1/G)\mathcal{L}_\xi{}^{n(\beta,\bar b)}(Gh)$. $\mathcal{L}_\xi$ is invertible if and only if $\mathcal{M}_\xi$ is invertible. We show that $\mathcal{M}_\xi$ has 1 as an eigenvalue for ξ close to $i\bar b$. A direct calculation shows that

$$\|\mathcal{M} - \mathcal{L}_f^{n(\beta,\bar b)}\|_{(N)} \le \frac{1}{|\bar b|^\alpha} + \frac{C_{23}|\bar b|^{\tilde\alpha}}{N}$$

where $\tilde\alpha = 2\alpha_3 + 1$ (since $\|\bar b\tau\|_\theta = O(|\bar b|)$, $\|G\|_\theta = \|G^{-1}\|_\theta = O(|\bar b|^{\alpha_3})$). Take $N = |\bar b|^{\alpha+\tilde\alpha}$.

LEMMA 8. *There exist constants C_{26}, C_{27} such that uniformly in $\bar b, \xi$ such that $|\xi - i\bar b| \le |\bar b|^{-1}$*

$$\left\|\frac{\partial}{\partial\xi}\mathcal{M}\right\|_{(N)} \le C_{26}\ln|\bar b|,$$
$$\left\|\frac{\partial^2}{\partial\xi^2}\mathcal{M}\right\|_{(N)} \le C_{27}\ln^2|\bar b|.$$

Proof. We prove only the first estimate, the proof of the second is similar. Since multiplications by G, G^{-1} are (uniformly) bounded operators, it is enough to bound $\|(\partial/\partial\xi)\mathcal{L}_\xi^{n(\beta,\bar b)}\|_{(N)}$:

$$\left\|\frac{\partial}{\partial\xi}\mathcal{L}_\xi^{n(\beta,\bar b)}\right\|_{(N)} = \left\|\sum_{j<n(\beta,\bar b)} \mathcal{L}_\xi^j\left(\frac{\partial}{\partial\xi}\mathcal{L}_\xi\right)\mathcal{L}_\xi^{n(\beta,\bar b)-j-1}\right\|_{(N)}.$$

Applying Proposition 2 to $\mathcal{L}_\xi/\exp[Pr(f+\Re\xi\tau)]$ and using the bound $Pr(f+\Re(\xi)\tau) \le C_{28}|\Re\xi|$ we get

$$\|\mathcal{L}_\xi^j\|_{(N)} \le C_{29}n(\beta,b)$$

and we have completed the proof. □

Let γ be the circle $\{|z| = (1+\delta)/2\}$, where δ is the constant from §6. By Lemma 2 for $\lambda \in \gamma$,

$$\|(\lambda - \mathcal{L}_f^{n(\beta,\bar b)})^{-1}\|_{(N)} \le C_{30}\ln|\bar b|$$

and so

$$\|(\lambda - \mathcal{M})^{-1}\|_{(N)} \le C_{31}\ln|\bar b|.$$

From Lemma 8 it follows that in $|\xi - i\bar b| \le |\bar b|^{-1}$, $\mathcal{M}_\xi$ has only one eigenvalue $\rho(\xi)$, this eigenvalue is simple and the corresponding eigenfunction is

$$g(\xi,\omega) = 1 - \frac{1}{2\pi i}\int_\gamma (z - \mathcal{M}_\xi)^{-1}1\,dz.$$

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Moreover, $\rho(\xi)$ is analytic in $\{|\xi - i\bar{b}| < 1/|b|\}$. By standard perturbation theory

$$|\rho(i\bar{b}) - 1| \le C_{32}|\bar{b}|^{-\alpha} \ln|\bar{b}|,$$
$$\left|\frac{\partial}{\partial\xi}\rho(\xi)\right| \le C_{33} \ln^2|\bar{b}|,$$
$$\left|\frac{\partial^2}{\partial\xi^2}\rho(\xi)\right| \le C_{34} \ln^3|\bar{b}|,$$
$$\|g(i\bar{b}, \omega) - 1\|_0 \le C_{35}|\bar{b}|^{-\alpha} \ln|\bar{b}|$$

and

$$\left\|\frac{\partial}{\partial\xi} g(\xi, \omega)\right\|_0 \le C_{36} \ln|\bar{b}|.$$

12. *Derivative of the eigenvalue*

It remains to estimate $(\partial/\partial\xi)\rho(\xi)$ from below. Differentiating

$$\frac{1}{G(\omega)} \sum_{\sigma^{n(\beta,\bar{b})}\varpi=\omega} e^{f_{n(\beta,\bar{b})}(\varpi)+\xi\tau_{n(\beta,\bar{b})}(\varpi)} G(\varpi)g(\xi, \varpi) = \rho(\xi)g(\xi, \omega)$$

at $\xi = i\bar{b}$, we obtain

$$\mathcal{M}_{i\bar{b}}(\tau_{n(\beta,\bar{b})} \circ \sigma^{n(\beta,\bar{b})} g) + \mathcal{M}_{i\bar{b}}\left(\frac{\partial}{\partial\xi} g\right) = \left(\frac{\partial}{\partial\xi}\rho(i\bar{b})\right) g + \rho(i\bar{b})\frac{\partial}{\partial\xi} g.$$

Integrate this identity against ν_f:

$$\nu_f\left(\mathcal{M}_{i\bar{b}}\frac{\partial}{\partial\xi} g\right) = \nu_f\left(\mathcal{L}_f^{n(\beta,\bar{b})}\frac{\partial}{\partial\xi} g\right) + O\left(\frac{\ln|\bar{b}|}{|\bar{b}|^\alpha}\right) = \nu_f\left(\frac{\partial}{\partial\xi} g\right) + O\left(\frac{\ln|\bar{b}|}{|\bar{b}|^\alpha}\right),$$
$$\rho(i\bar{b})\nu_f\left(\frac{\partial}{\partial\xi} g\right) = \nu_f\left(\frac{\partial}{\partial\xi} g\right) + O\left(\frac{\ln^2|\bar{b}|}{|\bar{b}|^\alpha}\right),$$
$$\frac{\partial}{\partial\xi}\rho(i\bar{b})\nu_f(g) = 1 + O\left(\frac{\ln^2|\bar{b}|}{|\bar{b}|^\alpha}\right),$$

$$\begin{aligned}\nu_f(\mathcal{M}_{i\bar{b}}(\tau_{n(\beta,\bar{b})} \circ \sigma^{n(\beta,\bar{b})} g)) &= \nu_f(\mathcal{L}_{i\bar{b}}^{n(\beta,\bar{b})}(\tau_{n(\beta,\bar{b})} \circ \sigma^n(\beta, \bar{b})g)) + O\left(\frac{\ln^2|\bar{b}|}{|\bar{b}|^\alpha}\right)\\ &= \nu_f(\tau_{n(\beta,\bar{b})} \circ \sigma^{n(\beta,\bar{b})}) + O\left(\frac{\ln^2|\bar{b}|}{|\bar{b}|^\alpha}\right)\\ &= n(\beta, \bar{b})\nu_f(\tau) + O\left(\frac{\ln^2|\bar{b}|}{|\bar{b}|^\alpha}\right).\end{aligned}$$

Hence,

$$\frac{\partial}{\partial\xi}\rho(i\bar{b}) = n(\beta, \bar{b})\nu_f(\tau) + O\left(\frac{\ln^2|\bar{b}|}{|\bar{b}|^\alpha}\right).$$

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Remark. Actually, we have shown that

$$\frac{\partial}{\partial\xi}\bigg|_{\xi=i\bar b}\rho(\mathcal{M}_\xi)=\frac{\partial}{\partial\xi}\bigg|_{\xi=0}(\rho^{n(\beta,b)}(\mathcal{L}_\xi))+O\left(\frac{\ln^2|\bar b|}{|\bar b|^\alpha}\right).$$

Taking into account the bound on the second derivative of $\rho(\xi)$ we conclude by Rouche's theorem that in $\{|\xi-\bar b|\le|\bar b|^{-1}\}$ the equations $\rho(\xi)=1$ and $\rho(i\bar b)+(\xi-i\bar b)\nu(\tau)=1$ have the same number of solutions. That is, both have one solution. □

13. *Proof of Theorem 3*

In this section we show that the conditions of Theorem 3 imply (xii). Indeed, let H be a $(1,|b|^{\alpha_5},|b|^{-\alpha})$-approximate eigenfunction of $\mathcal{V}_b^{n(\beta,b)}$, that is

$$\left|\frac{e^{ib\tau_{n(\beta,b)}(\omega)}H(\sigma^{n(\beta,b)}\omega)}{H(\omega)}\right|\le|b|^{-\alpha}.$$

Substituting $\omega=\omega_1$ we get

$$|e^{ibn(\beta,b)l_1}-1|\le|b|^{-\alpha}.$$

So $bn(\beta,b)l_1=2\pi m_1+O(|b|^{-\alpha})$ for some $m_1\in\mathbb{Z}$. Similarly, $bn(\beta,b)l_2=2\pi m_2+O(|b|^{-\alpha})$. Hence,

$$\frac{l_1}{l_2}=\frac{m_1}{m_2}+O(|b|^{-\alpha})=\frac{m_1}{m_2}+O(|m_2|^{-\alpha}).$$

If $|b|$ is large enough this implies $\alpha\le\alpha_6$ where α_6 is the Diophantine exponent of l_1/l_2. □

14. *Concluding remarks*

As we have seen above, topological power mixing plays quite an important role for suspension flows over a subshift of finite type. Now several examples of dynamical systems satisfying this condition are known. We mention just a few of them: any topologically mixing Anosov flow, some maps on the boundary of Anosov diffeomorphisms, certain Henon maps etc. We expect that some convenient set of axioms could be formulated which would cover all these cases. Of course, in the general case one cannot get anything better than power-like correlation decay.

Another problem is whether generic Axiom A flow is actually exponentially mixing. The same question may also be asked for symbolic flows. Even if the answers to these two cases are the same, the proofs would probably be different. We expect that the question about the exponential bound is of topological rather than arithmetic nature.

It is also interesting to know to what extent Theorem 1 can be generalized to deal with exponential mixing. More specifically, we would like to ask two questions. Is it true that topological exponential mixing implies exponential correlation decay for any Gibbs measure? In particular, is it true that if correlations decay exponentially for some Gibbs measure then the same holds for any potential? Of course, the conditions of this theorem if true would be even more difficult to verify than for power mixing.

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Appendix. Dimension and rapid mixing

A.1. *Two-sided subshifts.* Above we gave several necessary and sufficient conditions for a symbolic flow to mix faster than any power of time. We also described an arithmetic property (incommensurability of periods) which implies these conditions. Here we present a topological mechanism for rapid mixing.

Here we let (Σ, σ) be a two-sided subshift of finite type since to formulate our condition for a one-sided subshift we need to pass to its natural extension. If $\omega \in \Sigma$, ω^- and ω^+ stand for $\{\omega_j\}_{j\le 0}$ and $\{\omega_j\}_{j\ge 0}$, respectively. $C_\theta^-(\Sigma)$ and $C_\theta^+(\Sigma)$ are subspaces of $C_\theta(\Sigma)$ depending only on ω^- (ω^+). As before (Σ^τ, g^t) means the suspension flow with the roof function τ.

Recall that any symbolic flow has a certain geometric structure, namely the structure of a Smale space [**R1**]. For the suspension (Σ^τ, g^t) of the two-sided subshift of finite type (Σ, σ) with roof function $\tau \in C_\theta(\Sigma)$, the Smale structure consists of the following objects:

- the unstable manifold $W^u(\omega, t) = \{(\bar\omega, \bar t) : \bar\omega^+ = \omega^+\}$;
- the stable manifold $W^s(\omega, t) = \{(\bar\omega, \bar t) : \bar\omega^- = \omega^-\}$;
- the strong unstable manifold $W^{su} = \{(\bar\omega, \bar t) \in W^u(\omega, t) : t - \bar t = \Delta^-(\bar\omega, \omega)\}$ where

$$\Delta^-(\bar\omega, \omega) = \sum_{j=1}^{\infty} [\tau(\sigma^{-j}\bar\omega) - \tau(\sigma^{-j}\omega)];$$

- the strong stable manifold $W^{ss} = \{(\bar\omega, \bar t) \in W^s(\omega, t) : t - \bar t = \Delta^+(\bar\omega, \omega)\}$ where

$$\Delta^-(\bar\omega, \omega) = \sum_{j=0}^{\infty} [\tau(\sigma^{j}\bar\omega) - \tau(\sigma^{j}\omega)];$$

- the local product structure: if $\omega_0 = \bar\omega_0$ then $[\omega, \bar\omega]$ is defined as follows:

$$([\omega, \bar\omega])_j = \begin{cases} \omega_j, & \text{if } j \le 0 \\ \bar\omega_j, & \text{if } j \ge 0; \end{cases}$$

- the temporal distance function (see [**Ch**])

$$\varphi(\omega, \bar\omega) = \Delta^+([\bar\omega, \omega], \omega) + \Delta^-([\bar\omega, \omega], \bar\omega) + \Delta^-([\omega, \bar\omega], \omega) + \Delta^+([\omega, \bar\omega], \bar\omega).$$

The geometric meaning of the temporal distance is the following. Let $p_*^{(\bar\omega, \bar t)}$ denote the projection to $W^*(\bar\omega, \bar t)$ (along the leaves of the complimentary foliation). Then

$$p_s^{(\omega,0)} \circ p_u^{(\omega,0)} \circ p_s^{(\bar\omega,0)} \circ p_u^{(\bar\omega,0)}(\omega, t) = (\omega, t + \varphi(\omega, \bar\omega)).$$

Since φ is defined geometrically it is clear that it is invariant in the sense that it depends only on the cohomology class of τ. This can also be seen directly since the replacement of $\tau(\omega)$ by $\tau(\omega) + T(\omega) - T(\sigma\omega)$ alter $\Delta^+(\bar\omega, \omega)$ by $T(\bar\omega) - T(\omega)$ and $\Delta^-(\bar\omega, \omega)$ by $T(\omega) - T(\bar\omega)$. If $\tau \in C^+(\omega)$ then Δ^+ vanishes and

$$\varphi(\omega, \bar\omega) = \Delta^-([\bar\omega, \omega], \bar\omega) + \Delta^-([\omega, \bar\omega], \omega).$$

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A.2. *Local integrability.*

Definition. $\{g^t\}$ is called locally integrable if $\varphi \equiv 0$.

The following statement essentially goes back to Anosov (cf. [**A**]).

PROPOSITION 4. (Anosov alternative for symbolic flows) *If* $\{g^t\}$ *is locally integrable,* $\tau \sim \bar{\tau}$ *where* $\bar{\tau}$ *depends only on* ω_0.

Before giving the proof recall some standard results about the cohomology of subshifts of finite type. First, note that $\Delta^-([\bar{\omega}, \omega], \bar{\omega})$ does not depend on ω^- so we can write $\Delta^-([\bar{\omega}, \omega], \bar{\omega}) = \delta^-(\omega^+, \bar{\omega}^+, \bar{\omega}^-)$. Similarly, $\Delta^+([\bar{\omega}, \omega], \omega) = \delta^-(\bar{\omega}^-, \omega^-, \bar{\omega}^+)$.

PROPOSITION 5. [**S**] *There is a constant* C_{37} *such that*

$$|\delta^-(\omega_1^+, \omega_2^+, \omega_3^-) - \delta^-(\bar{\omega}_1^+, \bar{\omega}_2^+, \bar{\omega}_3^-)| \le C_{37}\sqrt{d_\theta(\omega_1^+, \bar{\omega}_1^+) + d_\theta(\omega_2^+, \bar{\omega}_2^+) + d_\theta(\omega_3^-, \bar{\omega}_3^-)}$$

and

$$|\delta^-(\omega_1^-, \omega_2^-, \omega_3^+) - \delta^-(\bar{\omega}_1^-, \bar{\omega}_2^-, \bar{\omega}_3^+)| \le C_{37}\sqrt{d_\theta(\omega_1^-, \bar{\omega}_1^-) + d_\theta(\omega_2^-, \bar{\omega}_2^-) + d_\theta(\omega_3^+, \bar{\omega}_3^+)}.$$

COROLLARY 5. [**S**] *Choose for any* i *a sequence* $\omega(i)$ *such that* $(\omega(i))_0 = i$. *Set* $T_+(\omega) = \delta^+(\omega(i)^-, \omega^-, \omega(i)^+)$, $\tau^+(\omega) = \tau(\omega) + T_+(\omega) - T_+(\sigma\omega)$, $T_+(\omega) = \delta^+(\omega^+, \omega(i)^+, \omega(i)^-)$, $\tau^-(\omega) = \tau(\omega) + T_-(\omega) - T_-(\sigma\omega)$. *Then* $\tau^+ \in C^+_{\sqrt{\theta}}(\Sigma)$, $\tau^- \in C^-_{\sqrt{\theta}}(\Sigma)$.

Proof of Proposition 4. By the discussion above we may assume that $\tau \in C^+_\theta(\Sigma)$. Consider $\tau^-(\omega) = \tau(\omega) - T_-(\omega) + T_-(\sigma\omega)$. τ^- does not depend on ω_j, $j > 0$, by Corollary 5. On the other hand,

$$\varphi(\omega, \bar{\omega}) = \delta^-(\omega^+, \bar{\omega}^+, \bar{\omega}^-) - \delta^-(\omega^+, \bar{\omega}^+, \omega^-). \tag{6}$$

Thus $\varphi \equiv 0$ just means that $\delta^-(\omega_1^+, \omega_2^+, \omega_3^-)$ does not depend on the third variable. Hence τ^- does not depend on ω_j, $j < 0$, as well and the proof is complete. □

A.3. *The result.* The aim of this appendix is to prove the following result.

THEOREM 6. *If* g^t *is not rapidly mixing then the lower box counting dimension of the range of* φ *vanishes:*

$$\underline{\mathrm{BD}}(\mathrm{Range}(\varphi)) = 0.$$

Proof. By the discussion in §A.1 we may assume that $\tau \in C^+_\theta(\Sigma)$. By Theorem 2 we can find sequences $\{b_k\}$ and $\{h_k\}$ satisfying conditions (2(v)). Let N be some large natural number which we assume to be a multiple of $n(\beta, b_k)$. Take some $\omega = \omega^-\omega^+$, $\bar{\omega} = \bar{\omega}^-\bar{\omega}^+$ and, moreover, let $\omega^- = \omega^\Omega\xi_N$, $\bar{\omega}^- = \bar{\omega}^\Omega\bar{\xi}_N$ where ξ_N and $\bar{\xi}_N$ are words of length N. Inequalities

$$|e^{-ib_k\tau_N(\xi_N\omega^+)}h_k(\xi_N\omega^+) - h_k(\omega^+)| \le N|b_k|^{-\alpha},$$
$$|e^{-ib_k\tau_N(\bar{\xi}_N\omega^+)}h_k(\bar{\xi}_N\omega^+) - h_k(\omega^+)| \le N|b_k|^{-\alpha}$$

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imply that

$$|e^{-ib_k\tau_N(\xi_N\omega^+)}h_k(\xi_N\omega^+) - e^{-ib_k\tau_N(\bar{\xi}_N\omega^+)}h_k(\bar{\xi}_N\omega^+)| \le 2N|b_k|^{-\alpha}.$$

Using (2(v)) once more we get

$$\left|\exp(ib_k[\tau_N(\xi_N\omega^+) - \tau(\bar{\xi}_N\omega^+)]) - \frac{h_k(\bar{\xi}_N)}{h_k(\xi_N)}\right| \le C_{38}(N|b_k|^{-\alpha} + |b_k|^{\alpha_5}\theta^N).$$

Similarly,

$$\left|\exp(ib_k[\tau_N(\xi_N\bar{\omega}^+) - \tau(\bar{\xi}_N\bar{\omega}^+)]) - \frac{h_k(\bar{\xi}_N)}{h_k(\xi_N)}\right| \le C_{38}(N|b_k|^{-\alpha} + |b_k|^{\alpha_5}\theta^N).$$

Thus,

$$|\exp(ib_k[\tau_N(\xi_N\omega^+) - \tau_N(\bar{\xi}_N\omega^+) - \tau_N(\xi_N\bar{\omega}^+) + \tau_N(\bar{\xi}_N\bar{\omega}^+)])| \le C_{39}(N|b_k|^{-\alpha} + |b_k|^{\alpha_5}\theta^N).$$

In view of (6) this gives

$$|\exp(ib_k\varphi(\omega,\bar{\omega})) - 1| \le C_{39}(N|b_k|^{-\alpha} + |b_k|^{\alpha_5}\theta^N).$$

Choose $N \sim \alpha \ln|b_k|/\ln(\theta^{-1})$, then

$$|\exp(ib_k\varphi(\omega,\bar{\omega})) - 1| \le C_{40}|b_k|^{\alpha_5-\alpha}.$$

If also $|\varphi| \le M$ then

$$\mathrm{Range}(\varphi) \subset \bigcup_{|m|\le M|b_k|/2\pi}\left[\frac{2\pi m}{|b_k|} - C_{41}|b_k|^{\alpha_5-\alpha}, \frac{2\pi m}{|b_k|} + C_{41}|b_k|^{\alpha_5-\alpha}\right],$$

which implies $\underline{\mathrm{BD}}(\mathrm{Range}(\varphi)) \le (\alpha-\alpha_5)^{-1}$. Since α is arbitrary, Theorem 6 is proven. □

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Note added in proof. Some of the questions of §14 have been answered in the following papers: W. Parry and M. Pollicott, Stability of mixing for toral extensions of hyberbolic systems, *Proc. Steklov Inst. Math.* **216** (1997), 350–360; L. S. Young, Recurrence times and the rates of mixing, *Preprint*; D. Dolgopyat, Prevalence of rapid mixing in hyperbolic flows-2, *Preprint*.

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