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Recursive modelling in dynamics of Delta parallel robot

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SUMMARY

Recursive matrix relations in kinematics and dynamics of a Delta parallel robot having three revolute actuators are established in this paper. The prototype of the manipulator is a three degrees-of-freedom space mechanism, which consists of a system of parallel closed kinematical chains connecting to the moving platform. Knowing the translation motion of the platform, we develop first the inverse kinematics problem and determine the position, velocity and acceleration of each robot's element. Further, the inverse dynamic problem is solved using an approach based on the fundamental principle of virtual work. Finally, a comparative study on time-history evolution of the torques of the three actuators is analysed.

KEYWORDS: Dynamics modelling; Kinematics; Parallel mechanism; Virtual work.

LIST OF SYMBOLS

$a_{k,k-1}$:	orthogonal transformation matrix
R:	general transformation matrix of moving platform
$\vec{u}_1,\ \vec{u}_2,\ \vec{u}_3$:	three orthogonal unit vectors
$\alpha_A,\ \alpha_B,\ \alpha_C$:	angles giving the position of three actuators
β:	initial inclination of four-bar parallelogram
$\varphi_{k,k-1}$:	relative rotation angle of T_k rigid body
$\vec{\omega}_{k,k-1}$:	relative angular velocity of T_k
$\vec{\omega}_{k0}$:	absolute angular velocity of T_k
$\tilde{\omega}_{k,k-1}$:	skew-symmetric matrix associated to the angular velocity $\vec{\omega}_{k,k-1}$
$\vec{\varepsilon}_{k,k-1}$:	relative angular acceleration of T_k
$\vec{\varepsilon}_{k0}$:	absolute angular acceleration of T_k
$\tilde{\varepsilon}_{k,k-1}$:	skew-symmetric matrix associated to the angular acceleration $\vec{\varepsilon}_{k,k-1}$
$\vec{r}^{\,A}_{k,k-1}$:	relative position vector of the centre of A_k joint
$\vec{v}^{\,A}_{k,k-1}$:	relative velocity of the centre A_k
$\vec{\gamma}^{\,A}_{k,k-1}$:	relative acceleration of the centre A_k
m_k:	mass of T_k rigid body
$\hat{J}_k$:	symmetric matrix of tensor of inertia of T_k about the link-frame $x_k y_k z_k$
$J_1,\ J_2$:	two Jacobian matrices of the manipulator
$m^A_{10},\ m^B_{10},\ m^C_{10}$:	torques of three actuators pointing about the $z^A_1,\ z^B_1,\ z^C_1$ directions

1. Introduction

Parallel robots are closed-loop mechanical structures presenting very good performances in terms of accuracy, rigidity and ability to manipulate large loads.

Generally, the mechanism of the manipulator has two platforms: one of them is attached to the fixed reference frame and the other one can perform arbitrary motions in its workspace. Some moving legs made up as serial robots connect the moving platform to the fixed platform. Spherical joints, revolute joints or prismatic joints connect the elements of the robot to one another.

Typically, a parallel mechanism is said to be *symmetrical* if it satisfies the following conditions: the number of legs is equal to the number of degrees of freedom of the moving platform, every limb is controlled by a single actuator and the location and number of actuated joints in all limbs are equal.[1]

For two decades, parallel manipulators had increasingly focused attention of many researchers that consider their design as a valuable alternative design for robotic mechanisms.[2,3] These complex architectures can be found in many technical applications where a high-speed displacement or orientation of a rigid body in space is desired. Since the robot is intended to operate on fragile objects, accuracy and precision in the execution of the tasks are essential, any error in the positioning of the tool could lead to costly damages.

Many parallel robots are equipped with hydraulic or pneumatic actuators. They have a robust construction and can move bodies of considerable mass and size with high speed. This is why mechanisms, which impart translation or spherical motion to a platform, are based on the concept of parallel manipulators.

Compared with serial robots, the parallel robots have specific characteristics: higher structural rigidity, better orientation accuracy, stable functioning, larger dynamic charge capacity and suitable position of the actuating systems. On the other hand, parallel kinematical machines offer essential advantages over their serial counterparts: lower moving mass, higher natural frequencies, simpler modular mechanical construction and possibility to mount all actuators on or near to the fixed base. However, most existing parallel manipulators have limited and complicated

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workspace volume with singularities, while being governed by highly non-isotropic input-output relations.[4,5]

Recently, many efforts have been devoted to the kinematics and dynamics analysis of fully parallel manipulators. Important companies such as Giddings & Lewis, Ingersoll, Hexel and others have developed them as high precision machining tools. The most known class of manipulators is the flight simulator with six degrees–of-freedom, which is in fact the Gough–Stewart platform (see refs. [6–9]). The Star parallel manipulator[10,11] and the Delta parallel robot[12–14] are all equipped with three motors, which have a parallel setting and push the end-effector in a three degrees-of-freedom general translation. Angeles, Gosselin, Gagné and Wang[15–17] developed the direct kinematics and dynamics of an Agile Wrist prototype that presents three concurrent rotations.

In 1988, Reymond Clavel developed the prototype of the Delta parallel robot at Lausanne Federal Polytechnic Institute. The parallel manipulator built at the University of Maryland[13] employs only revolute joints in contrast to ball-joints used in the Delta parallel robot[18] to constrain the motion of the platform to a pure translation.

The kinematics and dynamics of parallel structures has been studied extensively during the last two decades. When a good dynamic performance and a precise positioning of the moving platform under high load are required, the dynamic model of the robot is important for its automatic control.

The dynamics analysis of parallel robots is usually implemented through analytical methods from the classical mechanics,[19] in which projection and resolution of equations on the reference axes are written in a considerable number of cumbersome, scalar relations and the solutions are rendered by large scale computations implemented through time consuming computer codes.

Meanwhile, quite limited approaches have been directed to the dynamic modelling of parallel mechanism configurations.[20] Kane and Levinson[21] had obtained some vector recursive relations concerning the equilibrium of generalized forces that are applied to a serial manipulator. Sorli *et al.*[22] conducted the dynamics modelling for the Turin parallel manipulator, where the mechanism with three identical legs has just 6-DOFs. Geng *et al.*[23] developed Lagrange's equations of motion under some simplifying assumptions regarding the geometry and inertia distribution of the manipulator. Dasgupta and Mruthyunjaya[24] used the Newton–Euler approach to develop closed-form dynamics equations of Stewart platform, considering dynamic and gravity effects as well as viscous friction at joints.

A space parallel mechanism, which can be used in several applications including machining tools, is proposed in the paper. We focus our attention on an efficient and fast recursive matrix method, which is adopted to derive the kinematical model and the inverse dynamics equations of the three-degree-of-freedom Delta parallel robot with revolute actuators.

2. Inverse Kinematics

The mechanism input of the robot is structured into three active revolute joints while the output body is connected to the fixed base through a set of three variable length legs with identical topology.

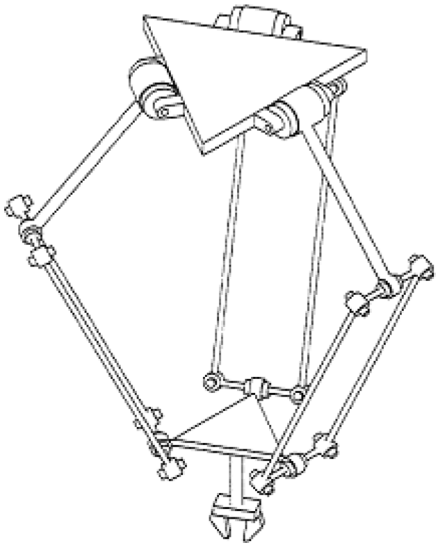


Fig. 1. Delta parallel robot with revolute actuators.

The architecture of one of the three kinematical closed chains of the manipulator is made of an electrical motor, an active revolute joint and an intermediate mechanism with four revolute joints that connect four bars parallel two by two, ending with a passive revolute joint connected to the moving platform (Fig. 1).

In the following we develop by inverse kinematics a means to determine the velocities and accelerations of all moving components of the robot, given the characteristics of the translation motion of the moving platform.

Let us locate a fixed Cartesian frame $x_0y_0z_0(T_0)$ at the centre O of the triangular base of side L about which the Delta parallel manipulator of known geometry moves. The elements of its legs have known size and given mass. To simplify the graphical image of the kinematical scheme of the mechanism, in what follows we will represent the intermediate reference systems by only two axes, so as one precedes in most of robotics papers.[1,4,7] The z_k axis is represented, of course, for each component element T_k. It is noted that the relative rotation with angle $\varphi_{k,k-1}$ or the relative translation of the body T_k with the displacement $\lambda_{k,k-1}$ must always be pointed along the direction of the z_k axis.

One of the three active elements of the robot is the input link of the first upside-down leg A (Fig. 2). This is a homogenous crank of length $A_1A_2 = l_1$ and mass m_1, which rotates about z_1^A axis with the angle φ_{10}^A, angular velocity ω_{10}^A and angular acceleration ε_{10}^A. The total tensor of inertia of the input link-rotor mounted on this link is $\hat{J}_1$. The centre of the transmission rod $A_3A_6 = l_2$ is denoted by A_2. This link is connected to the frame $x_2^A y_2^A z_2^A$ (called T_2^A) and has a relative rotation about axis z_2^A with the angle φ_{21}^A, so that $\omega_{21}^A = \dot{\varphi}_{21}^A$ and $\varepsilon_{21}^A = \ddot{\varphi}_{21}^A$. It has the mass m_2 and the central tensor of inertia $\hat{J}_2$.

Further on, two identical and parallel bars A_3A_4 and A_6A_7 with same length l_3 rotate about two axes z_3^A, z_6^A respectively, with the angle $\varphi_{32}^A = \varphi_{62}^A$. They have also the same mass m_3

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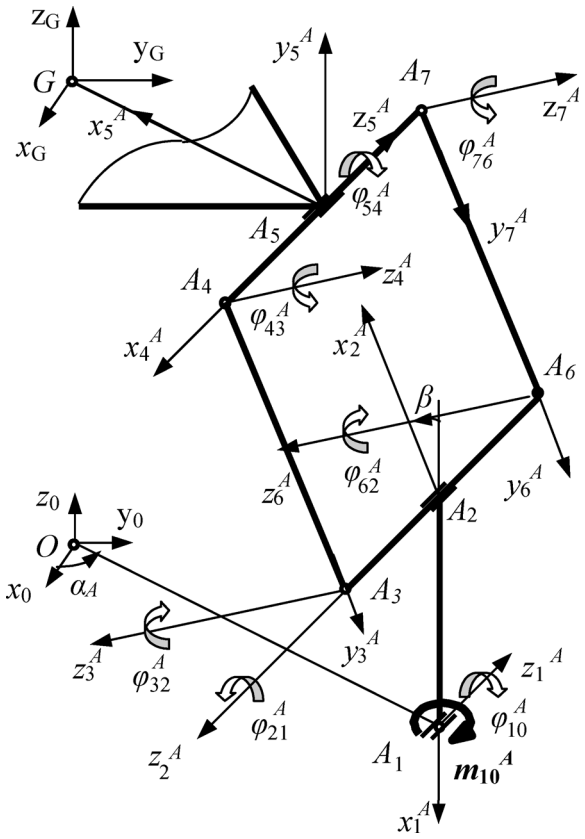


Fig. 2. Kinematical scheme of the leg A of upside-down mechanism.

and the same tensor of inertia $\hat{J}_3$. The four-bar parallelogram is closed by an element T_4^A identical with T_2^A, but its tensor of inertia is $\hat{J}_4$. This element rotates about z_4^A axis with the relative angle $\varphi_{43}^A = \varphi_{32}^A$.

The centre $A_5$of the interval between the two revolute joints connects the moving platform $x_5^A y_5^A z_5^A (T_5^A)$. The platform of the robot is an equilateral triangle of side $l = 0.5L - \sqrt{3}\, l_3 \sin\beta$, mass m_5, tensor of inertia $\hat{J}_5$ with respect to A_5, which rotates with the angle φ_{54}^A and the angular velocity $\omega_{54}^A = \dot{\varphi}_{54}^A$ about the neighbouring body T_4^A. Finally, another reference system $x_G y_G z_G$ is located just at the centre G of the moving platform.

The following angles give the initial position of the robot

$$\alpha_A = \frac{\pi}{3}, \quad \alpha_B = \pi, \quad \alpha_C = -\frac{\pi}{3},$$
$$\beta_A = \beta_B = \beta_C = \beta = \frac{\pi}{6}. \tag{1}$$

The angle α_A, for example, is measured from the fixed x_0 axis to the line which connects the centre O to the first actuated joint A_1, while the angle β_A denotes the initial inclination of the four-bar parallelogram, which is defined from the extended line of A_1A_2 to the median x_2^A axis.

Due to the special arrangement of the four-bar parallelograms and the position of three active revolute joints at points A_1, B_1, C_1, the mechanism has three translation degrees of freedom.

In the following we apply the method of successive displacements to geometric analysis of closed-loop chains and we note that a joint variable is the displacement required to move a link from the initial location to the actual position. If every link is connected to at least two other links, the chain forms one or more independent closed-loops. The variable angles $\varphi_{k,k-1}$of rotation about the joint axes z_k are the parameters needed to bring the next link from a reference configuration to the next configuration. We call the matrix $a_{k,k-1}^{\varphi}$, e.g. the orthogonal transformation 3×3 matrix of relative rotation with the angle $\varphi_{k,k-1}^A$ of link T_k^A around z_k^A axis.

In the study of the kinematics of robot manipulators, we are interested in deriving a matrix equation relating the location of an arbitrary T_k body to the joint variables. When the change of coordinates is successively considered, the corresponding matrices are multiplied. So, starting from the reference origin O and pursuing the independent legs $A_1A_2A_3A_4A_5$, $B_1B_2B_3B_4B_5$ and $C_1C_2C_3C_4C_5$, one obtains the following transformation matrices

$$\begin{aligned} &q_{10} = q_{10}^{\varphi}\theta_1 a_{\alpha}^{j}, \quad q_{21} = q_{21}^{\varphi} a_{\beta}\theta_2 \\ &q_{32} = q_{32}^{\varphi}\theta_1, \quad q_{43} = q_{43}^{\varphi}\,\theta_2 \\ &q_{54} = q_{54}^{\varphi} a_{\beta}\,\theta_3, \quad q_{62} = q_{32} \\ &(q = a, b, c), \quad (j = A, B, C), \end{aligned} \tag{2}$$

where we denote:[25]

$$\theta_1 = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \theta_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\theta_3 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, a_{\alpha}^{j} = \begin{bmatrix} \cos\alpha_j & \sin\alpha_j & 0 \\ -\sin\alpha_j & \cos\alpha_j & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$a_{\beta} = \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

$$q_{k,k-1}^{\varphi} = \begin{bmatrix} \cos\varphi_{k,k-1}^{j} & \sin\varphi_{k,k-1}^{j} & 0 \\ -\sin\varphi_{k,k-1}^{j} & \cos\varphi_{k,k-1}^{j} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$q_{k0} = \prod_{s=1}^{k} q_{k-s+1,k-s}$$

$$(k = 1,\ 2, \ldots, 6). \quad .$$

The translation conditions concerning the absolute orientation of the moving platform are given by the following identities

$$a_{50}^{\circ T} a_{50} = b_{50}^{\circ T} b_{50} = c_{50}^{\circ T} c_{50} = R = I, \tag{4}$$

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and by the matrices

$$q_{50} = q_{54}q_{43}q_{32}q_{21}q_{10}, \quad (q = a, b, c) \tag{5}$$

$$a_{50}^{\circ T} = a_{50}^{T}(t=0) = a_z^{\frac{\pi}{2}-\alpha_A} a_y^{\frac{\pi}{2}} a_z^{\frac{\pi}{2}} = \frac{1}{2}\begin{bmatrix} -1 & 0 & -\sqrt{3} \\ -\sqrt{3} & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

$$b_{50}^{\circ T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad c_{50}^{\circ T} = \frac{1}{2}\begin{bmatrix} -1 & 0 & \sqrt{3} \\ \sqrt{3} & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix},$$

where $R = I$ is the diagonal identity matrix while the resulting matrix q_{50} is obtained by multiplying the five basic matrices. From these conditions one obtains the first relations between the angles of rotation

$$\varphi_{54}^{j} = \varphi_{21}^{j} - \varphi_{10}^{j}, \quad (j = A, B, C). \tag{6}$$

For the inverse geometric analysis, the position of an arbitrary end-point $P(x_0^P, y_0^P, z_0^P)$ is treated as known and the goal is to find the joint variables φ_{10}^A, φ_{10}^B, φ_{10}^C that yield the given location of the tool.

The three rotation angles φ_{10}^A, φ_{10}^B, φ_{10}^C of the actuators A_1, B_1, C_1 are the joint variables that give the input vector $\vec{\varphi}_{10} = [\varphi_{10}^A \quad \varphi_{10}^B \quad \varphi_{10}^C]^T$ of the instantaneous position of the mechanism. But, the objective of the inverse geometric problem is to find the vector $\vec{\varphi}_{10}$ and the position of the mechanism with given three absolute coordinates of the centre G of the platform: x_0^G, y_0^G, z_0^G.

Consider, for example, that during three seconds the moving platform remain in same orientation and the motion of mass centre G along a *rectilinear trajectory* is expressed in fixed frame $x_0y_0z_0$ through the following equations

$$\begin{aligned} \frac{x_0^G}{x_0^{G*}} &= \frac{y_0^G}{y_0^{G*}} = \frac{h - z_0^G}{z_0^{G*}} = 1 - \cos\frac{\pi}{3}t \\ h &= l_1 + l_3\cos\beta \\ \vec{r}_0^G &= \begin{bmatrix} x_0^G & y_0^G & z_0^G \end{bmatrix}^T. \end{aligned} \tag{7}$$

The input angles φ_{10}^j $(j = A, B, C)$ of the robot and the variables φ_{21}^j, $\varphi_{32}^j = \varphi_{43}^j$ and $\varphi_{54}^j = \varphi_{21}^j - \varphi_{10}^j$ will be obtained from the following geometrical conditions

$$\begin{aligned} &\vec{r}_{10}^A + \sum_{k=1}^{4} a_{k0}^T \vec{r}_{k+1,k}^A + a_{50}^T \vec{r}_5^{GA} \\ &= \vec{r}_{10}^B + \sum_{k=1}^{4} b_{k0}^T \vec{r}_{k+1,k}^B + b_{50}^T \vec{r}_5^{GB} \\ &= \vec{r}_{10}^C + \sum_{k=1}^{4} c_{k0}^T \vec{r}_{k+1,k}^C + c_{50}^T \vec{r}_5^{GC} = \vec{r}_0^G, \end{aligned} \tag{8}$$

where one denoted:

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \tilde{u}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \vec{r}_{10}^A &= \frac{L}{2\sqrt{3}} a_\alpha^{AT} \vec{u}_1, \quad \vec{r}_{21}^A = -l_1\vec{u}_1 \\ \vec{r}_{32}^A &= \frac{l_2}{2}\vec{u}_3, \quad \vec{r}_{43}^A = -l_3\vec{u}_2 \\ \vec{r}_{54}^A &= -\frac{l_2}{2}\vec{u}_1, \quad \vec{r}_5^{GA} = \frac{l}{\sqrt{3}}\vec{u}_1. \end{aligned} \tag{9}$$

Actually, these equations mean that there is only one inverse geometric solution for the manipulator:

$$\begin{aligned} &l_3 \sin\varphi_{32}^j = y_0^G\cos\alpha_j - x_0^G\sin\alpha_j \\ &l_1\sin\varphi_{10}^j + l_3\sin\left(\varphi_{10}^j - \varphi_{21}^j - \beta\right)\cos\varphi_{32}^j \\ &\quad = x_0^G\cos\alpha_j + y_0^G\sin\alpha_j - l_3\sin\beta \\ &l_1\cos\varphi_{10}^j + l_3\cos\left(\varphi_{10}^j - \varphi_{21}^j - \beta\right)\cos\varphi_{32}^j = z_0^G \\ &\quad (j = A, B, C). \end{aligned} \tag{10}$$

The motions of the component elements of each leg (e.g., the leg A) are characterized by the following skew-symmetric matrices:[26]

$$\tilde{\omega}_{k0}^A = a_{k,k-1}\tilde{\omega}_{k-1,0}^A a_{k,k-1}^T + \omega_{k,k-1}^A \tilde{u}_3, \tag{11}$$

which are *associated* to the absolute angular velocities given by the recursive relations

$$\vec{\omega}_{k0}^A = a_{k,k-1}\vec{\omega}_{k-1,0}^A + \omega_{k,k-1}^A\vec{u}_3, \quad \omega_{k,k-1}^A = \dot{\varphi}_{k,k-1}^A. \tag{12}$$

Following relations give the absolute velocities $\vec{v}_{k0}^A$ of centres of the joints A_k

$$\begin{aligned} \vec{v}_{k0}^A &= a_{k,k-1}\vec{v}_{k-1,0}^A + a_{k,k-1}\tilde{\omega}_{k-1,0}^A\vec{r}_{k,k-1}^A + v_{k,k-1}^A\vec{u}_3 \\ v_{k,k-1}^A &= 0, \quad (k = 1, 2, \ldots, 6). \end{aligned} \tag{13}$$

Equations of geometrical constraints (4) and (8) can be derivate with respect to time to obtain the following *matrix conditions of connectivity* for the relative angular velocities:[27]

$$\begin{aligned} &\omega_{10}^A\vec{u}_i^T a_{10}^T\vec{u}_3 + \omega_{21}^A\vec{u}_i^T a_{20}^T\vec{u}_3 + \omega_{54}^A\vec{u}_i^T a_{50}^T\vec{u}_3 = 0 \\ &\omega_{10}^A\left(l_1\vec{u}_i^T a_{10}^T\tilde{u}_3\vec{u}_1 + l_3\vec{u}_i^T a_{10}^T\tilde{u}_3 a_{21}^T a_{32}^T\vec{u}_2\right) \\ &\quad + \omega_{21}^A l_3\vec{u}_i^T a_{20}^T\tilde{u}_3 a_{32}^T\vec{u}_2 + \omega_{32}^A l_3\vec{u}_i^T a_{30}^T\tilde{u}_3\vec{u}_2 \\ &\quad = -\vec{u}_i^T\dot{\vec{r}}_0^G, \quad (i = 1, 2, 3), \end{aligned} \tag{14}$$

where $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{u}_3$ are skew-symmetric matrices associated to three orthogonal unit vectors $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$. If the other two kinematical chains of the robot are pursued, analogous relations can be easily obtained.

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Relative angular velocities ω_{10}^{A}, ω_{21}^{A}, $\omega_{32}^{A}=\omega_{43}^{A}$ and $\omega_{54}^{A}=\omega_{21}^{A}-\omega_{10}^{A}$ result from above Eqs. (14) as functions of the platform's *translation velocity*. The same relations (14) give immediately the *complete* Jacobian matrix of the manipulator. This matrix is a fundamental element for the analysis of the robot workspace and the particular configurations of singularities where the manipulator becomes uncontrollable.[30,32] Rearranging, the above nine constraint Eqs. (10) of the Delta robot can be written as follows

$$\begin{aligned}&\left(x_0^G\cos\alpha_j+y_0^G\sin\alpha_j-l_1\sin\varphi_{10}^j-l_3\sin\beta\right)^2\\&+\left(z_0^G-l_1\cos\varphi_{10}^j\right)^2+\left(y_0^G\cos\alpha_j-x_0^G\sin\alpha_j\right)^2=l_3^2\quad(15)\\&(j=A,B,C),\end{aligned}$$

where the 'zero' position $\vec{r}_0^{0G}=[0\quad 0\quad h]^T$ corresponds to the joint variables $\vec{\varphi}_{10}^{0}=[0\quad 0\quad 0]^T$. The derivative with respect to time of above conditions (15) leads to the matrix equation

$$J_1\dot{\vec{\varphi}}_{10}=J_2\dot{\vec{r}}_0^G.\qquad(16)$$

The matrices J_1 and J_2 are, respectively, the inverse and forward Jacobian of the manipulator and can be expressed as

$$\begin{aligned}J_1&=diag\{\delta_A\quad\delta_B\quad\delta_C\}\\J_2&=\begin{bmatrix}\beta_1^A&\beta_2^A&\beta_3^A\\\beta_1^B&\beta_2^B&\beta_3^B\\\beta_1^C&\beta_2^C&\beta_3^C\end{bmatrix},\end{aligned}\qquad(17)$$

with

$$\begin{aligned}\delta_j&=l_1\big[\left(x_0^G\cos\alpha_j+y_0^G\sin\alpha_j\right.\\&\quad\left.-l_3\sin\beta\right)\cos\varphi_{10}^j-z_0^G\sin\varphi_{10}^j\big]\\\beta_1^j&=x_0^G-\left(l_1\sin\varphi_{10}^j+l_3\sin\beta\right)\cos\alpha_j\qquad(18)\\\beta_2^j&=y_0^G-\left(l_1\sin\varphi_{10}^j+l_3\sin\beta\right)\sin\alpha_j\\\beta_3^j&=z_0^G-l_1\cos\varphi_{10}^j\quad(j=A,B,C).\end{aligned}$$

The three kinds of singularities of the three closed-loop kinematical chains can be determined through the analysis of two Jacobian matrices J_1 and J_2.

Based on the Eqs. (16) and the conditions of connectivity (14), in the direct kinematical problem we calculate the velocities of all links in terms of input velocities ω_{10}^{i}. The idea of expressing the velocities and accelerations of all the moving components in terms of those of the actuators was explicitly discussed by Xi *et al.*[31] using the natural orthogonal complement method, where various Jacobian matrices were derived to relate the velocities and accelerations of each body to those of the actuators for numeric computation of the inverse dynamics.

Let us assume that the robot has a virtual motion determined by the angular velocities $\omega_{10a}^{Av}=1$, $\omega_{10a}^{Bv}=0$, $\omega_{10a}^{Cv}=0$. The characteristic virtual velocities are expressed as functions of the position of the mechanism by the kinematical constraints Eqs. (14). Some other relations of connectivity can be obtained if one considers successively that $\omega_{10b}^{Bv}=1$, $\omega_{10b}^{Cv}=0$, $\omega_{10b}^{Av}=0$ and $\omega_{10c}^{Cv}=1$, $\omega_{10c}^{Av}=0$, $\omega_{10c}^{Bv}=0$.

As for the relative angular accelerations ε_{10}^{A}, ε_{21}^{A}, $\varepsilon_{32}^{A}=\varepsilon_{43}^{A}$, ε_{54}^{A} of the manipulator, the derivatives with respect to time of the Eqs. (14) give other following conditions of connectivity

$$\begin{aligned}&\varepsilon_{10}^{A}\vec{u}_i^T a_{10}^T\vec{u}_3+\varepsilon_{21}^{A}\vec{u}_i^T a_{20}^T\vec{u}_3+\varepsilon_{54}^{A}\vec{u}_i^T a_{50}^T\vec{u}_3=0\\&\varepsilon_{10}^{A}\left(l_1\vec{u}_i^T a_{10}^T\tilde{u}_3\vec{u}_1+l_3\vec{u}_i^T a_{10}^T\tilde{u}_3 a_{21}^T a_{32}^T\vec{u}_2\right)\\&\quad+\varepsilon_{21}^{A}l_3\vec{u}_i^T a_{20}^T\tilde{u}_3 a_{32}^T\vec{u}_2+\varepsilon_{32}^{A}l_3\vec{u}_i^T a_{30}^T\tilde{u}_3\vec{u}_2\\&\quad=-\vec{u}_i^T\ddot{\vec{r}}_0^G-\omega_{10}^{A}\omega_{10}^{A}\left(l_1\vec{u}_i^T a_{10}^T\tilde{u}_3\tilde{u}_3\vec{u}_1\right.\\&\quad\left.+l_3\vec{u}_i^T a_{10}^T\tilde{u}_3\tilde{u}_3 a_{21}^T a_{32}^T\vec{u}_2\right)-\omega_{21}^{A}\omega_{21}^{A}l_3\vec{u}_i^T a_{20}^T\tilde{u}_3\tilde{u}_3 a_{32}^T\vec{u}_2\\&\quad-\omega_{32}^{A}\omega_{32}^{A}l_3\vec{u}_i^T a_{30}^T\tilde{u}_3\tilde{u}_3\vec{u}_2-2\omega_{10}^{A}\omega_{21}^{A}l_3\vec{u}_i^T a_{10}^T\tilde{u}_3 a_{21}^T\tilde{u}_3 a_{32}^T\vec{u}_2\\&\quad-2\omega_{10}^{A}\omega_{32}^{A}l_3\vec{u}_i^T a_{10}^T\tilde{u}_3 a_{21}^T a_{32}^T\tilde{u}_3\vec{u}_2\\&\quad-2\omega_{21}^{A}\omega_{32}^{A}l_3\vec{u}_i^T a_{20}^T\tilde{u}_3 a_{32}^T\tilde{u}_3\vec{u}_2,\quad(i=1,2,3).\end{aligned}\qquad(19)$$

The following recursive relations give the angular accelerations $\vec{\varepsilon}_{k0}^{A}$ and the accelerations $\vec{\gamma}_{k0}^{A}$ of joints A_k

$$\begin{aligned}\vec{\varepsilon}_{k0}^{A}&=a_{k,k-1}\vec{\varepsilon}_{k-1,0}^{A}+\varepsilon_{k,k-1}^{A}\vec{u}_3\\&\quad+\omega_{k,k-1}^{A}a_{k,k-1}\tilde{\omega}_{k-1,0}^{A}a_{k,k-1}^T\vec{u}_3\\\tilde{\omega}_{k0}^{A}\tilde{\omega}_{k0}^{A}+\tilde{\varepsilon}_{k0}^{A}&=a_{k,k-1}\left(\tilde{\omega}_{k-1,0}^{A}\tilde{\omega}_{k-1,0}^{A}+\tilde{\varepsilon}_{k-1,0}^{A}\right)a_{k,k-1}^T\\&\quad+\omega_{k,k-1}^{A}\omega_{k,k-1}^{A}\tilde{u}_3\tilde{u}_3+\varepsilon_{k,k-1}^{A}\tilde{u}_3\\&\quad+2\omega_{k,k-1}^{A}a_{k,k-1}\tilde{\omega}_{k-1,0}^{A}a_{k,k-1}^T\tilde{u}_3\qquad(20)\\\vec{\gamma}_{k0}^{A}&=a_{k,k-1}\vec{\gamma}_{k-1,0}^{A}+a_{k,k-1}\{\tilde{\omega}_{k-1,0}^{A}\tilde{\omega}_{k-1.0}^{A}+\tilde{\varepsilon}_{k-1,0}^{A}\}\vec{r}_{k,k-1}^{A}\\&\quad+2v_{k,k-1}^{A}a_{k,k-1}\tilde{\omega}_{k-1,0}^{A}a_{k,k-1}^T\vec{u}_3+\gamma_{k,k-1}^{A}\vec{u}_3\\\gamma_{k,k-1}^{A}&=0\quad(k=1,2,\ldots,6).\end{aligned}$$

The relations (14), (19) represent the *inverse kinematics model* of the Delta parallel robot.

3. Dynamics Equations

In the context of the real-time control, neglecting the friction forces and considering the gravitational effects, the relevant objective of the dynamics is to determine the input torques, which must be exerted by the actuators in order to produce a given trajectory of the effector. There are three methods, which can provide the same result concerning these actuating moments. The first one is using the Newton–Euler classic procedure,[24,28,34] the second one uses the Lagrange's equations and multipliers formalism[23,33] and the third one is based on the principle of virtual work.[1,4,25,29,35,36]

Within the inverse dynamic problem, in the present paper the principle of virtual work is applied in order to establish some recursive matrix relations for the torques of the three active systems.

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Three independent electric motors A_1, B_1, C_1 that generate three moments $\vec{m}_{10}^{A} = m_{10}^{A}\vec{u}_3$, $\vec{m}_{10}^{B} = m_{10}^{B}\vec{u}_3$, $\vec{m}_{10}^{C} = m_{10}^{C}\vec{u}_3$, which have fixed directions of the axes z_1^A, z_1^B, z_1^C, control the motion of the platform $A_5B_5C_5$.

The force of inertia of an arbitrary rigid body T_k^A, for example,

$$\vec{f}_{k0}^{inA} = -m_k^A\left[\vec{\gamma}_{k0}^{A} + \left(\tilde{\omega}_{k0}^{A}\tilde{\omega}_{k0}^{A} + \tilde{\varepsilon}_{k0}^{A}\right)\vec{r}_k^{CA}\right] \tag{21}$$

and the resulting moment of the forces of inertia

$$\vec{m}_{k0}^{inA} = -\left[m_k^A\tilde{r}_k^{CA}\vec{\gamma}_{k0}^{A} + \hat{J}_k^A\vec{\varepsilon}_{k0}^{A} + \tilde{\omega}_{k0}^{A}\hat{J}_k^A\vec{\omega}_{k0}^{A}\right], \tag{22}$$

are determined with respect to the centre of joint A_k. On the other hand, the wrench of two vectors $\vec{f}_k^{*A}$ and $\vec{m}_k^{*A}$ evaluates the influence of the action of the weight $m_k^A\vec{g}$ and of other external and internal forces applied to the same element T_k^A of the manipulator, for example:

$$\begin{aligned}\vec{f}_k^{*A} &= 9.81 m_k^A a_{k0}\vec{u}_3\\ \vec{m}_k^{*A} &= 9.81 m_k^A\tilde{r}_k^{CA}a_{k0}\vec{u}_3 \quad (k = 1, 2, \ldots, 6).\end{aligned} \tag{23}$$

Knowing the position and kinematics state of each link as well as the external forces acting on the robot, in the following one applies the principle of virtual work for an inverse dynamic problem. The active torques required in a given motion of the moving platform will easily be computed using a recursive procedure.

The fundamental principle of the virtual work states that a mechanism is under dynamic equilibrium if and only if the virtual work developed by all external, internal and inertia forces vanish during any general virtual displacement, which is compatible with the constraints imposed on the mechanism. Assuming that frictional forces at the joints are negligible, the virtual work produced by the forces of constraint at the joints is zero.

Applying *the fundamental equations of the parallel robots dynamics* established by Staicu,[37] the following compact matrix relation for the first active torque results

$$\begin{aligned}m_{10}^{A} = \vec{u}_3^{T}\big[&\vec{M}_1^{A} + \omega_{54a}^{Av}\vec{M}_5^{A}\\ &+ \omega_{21a}^{Av}\vec{M}_2^{A} + \omega_{32a}^{Av}\left(\vec{M}_3^{A} + \vec{M}_4^{A} + \vec{M}_6^{A}\right)\\ &+ \omega_{21a}^{Bv}\vec{M}_2^{B} + \omega_{32a}^{Bv}\left(\vec{M}_3^{B} + \vec{M}_4^{B} + \vec{M}_6^{B}\right)\\ &+ \omega_{21a}^{Cv}\vec{M}_2^{C} + \omega_{32a}^{Cv}\left(\vec{M}_3^{C} + \vec{M}_4^{C} + \vec{M}_6^{C}\right)\big],\end{aligned} \tag{24}$$

where the relative virtual displacements $\delta_{k,k-1}^{Av}$ are expressed using the independent virtual velocities $\omega_{k,k-1}^{Av}$.

Two recursive relations generate the vectors

$$\begin{aligned}\vec{F}_k^{A} &= \vec{F}_{k0}^{A} + a_{k+1,k}^{T}\vec{F}_{k+1}^{A}\\ \vec{M}_k^{A} &= \vec{M}_{k0}^{A} + a_{k+1,k}^{T}\vec{M}_{k+1}^{A} + \tilde{r}_{k+1,k}^{A}a_{k+1,k}^{T}\vec{F}_{k+1}^{A},\end{aligned} \tag{25}$$

where one denoted

$$\vec{F}_{k0}^{A} = -\vec{f}_{k0}^{inA} - \vec{f}_k^{*A}, \quad \vec{M}_{k0}^{A} = -\vec{m}_{k0}^{inA} - \vec{m}_k^{*A} \quad (k = 1, 2, \ldots, 6). \tag{26}$$

The relations (24) and (25) represent just the inverse *dynamics model* of Delta parallel robot. The various dynamical effects, including the Coriolis and centrifugal forces coupling and the gravitational actions are also considered in these explicit equations.

As application let us consider a robot which has the following characteristics:

$$\begin{aligned}&x_0^{G*} = 0.04\,m, \quad y_0^{G*} = 0.06\,m, \quad z_0^{G*} = 0.08\,m\\ &\Delta t = 3\,s, \quad L = 1.25\,m, \quad l_1 = 0.25\,m\\ &l_2 = l_4 = 0.05\,m, \quad l_3 = l_6 = 0.55\,m\\ &m_1 = 1.5\,kg, \quad m_2 = m_4 = 0.05\,kg\\ &m_3 = m_6 = 1\,kg, \quad m_5 = 5\,kg, \quad J_m = 0.02\,kgm^2\end{aligned} \tag{27}$$

$$\hat{J}_1 = \begin{bmatrix} J_m/2 & & \\ & m_1 l_1^2/3 + J_m/2 & \\ & & m_1 l_1^2/3 + J_m\end{bmatrix},$$

$$\hat{J}_2 = \begin{bmatrix} 0.0002 & & \\ & 0.0002 & \\ & & 0.0001\end{bmatrix}, \quad \hat{J}_3 = \begin{bmatrix} 0.15 & & \\ & 0.02 & \\ & & 0.15\end{bmatrix}$$

$$\hat{J}_4 = \begin{bmatrix} 0.0001 & & \\ & 0.0004 & \\ & & 0.0004\end{bmatrix}, \quad \hat{J}_5 = \begin{bmatrix} 0.35 & & \\ & 0.50 & \\ & & 0.15\end{bmatrix}.$$

4. Simulation Procedure

The procedure for solving the inverse dynamics of Delta parallel robot by using the principle of virtual work can be summarised in several basic steps.

1. For a period of $\Delta t = 3$s, it is assumed that the time history of the moving platform is specified in terms of the position of the centre G from Eqs. (7). The relations (6), (10) give the evolution of joint variables φ_{10}^{j}, φ_{21}^{j}, $\varphi_{32}^{j} = \varphi_{43}^{j}$, φ_{54}^{j}, $(j = A, B, C)$.
2. Using the relations (2), (3), we compute the transformation matrices of three legs A, B, C: q_{10}^{j}, q_{21}^{j}, q_{32}^{j}, q_{43}^{j}, q_{54}^{j} and $q_{20}^{j} = q_{21}^{j}q_{10}^{j}$, $q_{30}^{j} = q_{32}^{j}q_{20}^{j}$, $q_{40}^{j} = q_{43}^{j}q_{30}^{j}$, $q_{50}^{j} = q_{54}^{j}q_{40}^{j}$, $(q = a, b, c)$.
3. Determine the velocity and acceleration of all link by performing the inverse kinematics analysis in terms of prescribed velocities $\dot{x}_0^G$, $\dot{y}_0^G$, $\dot{z}_0^G$ and accelerations $\ddot{x}_0^G$, $\ddot{y}_0^G$, $\ddot{z}_0^G$ of the moving platform. Specifically, for each leg, from the conditions of connectivity (14), (19) we compute the relative angular velocities ω_{10}^{j}, ω_{21}^{j}, ω_{32}^{j}, ω_{43}^{j}, ω_{54}^{j} and the relative angular accelerations ε_{10}^{j}, ε_{21}^{j}, ε_{32}^{j}, ε_{43}^{j}, ε_{54}^{j}.
4. The input velocities ω_{10}^{j} could be easily verified through the relation (16) based on the inverse and forward conventional Jacobian matrices of the manipulator.
5. Using same Eqs. (14), where we introduce $\omega_{10a}^{Av} = 1$, $\omega_{10a}^{Bv} = 0$, $\omega_{10a}^{Cv} = 0$, e.g., we compute the virtual characteristic velocities of each element of the robot. Other sets of virtual velocities are obtained if we consider successively that $\omega_{10b}^{Bv} = 1$, $\omega_{10c}^{Cv} = 1$.

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6. We compute the velocity and the acceleration of centre of mass as well as the angular velocity and the angular acceleration of each link using the Eqs. (12), (13) (20).
7. Decompose artificially the manipulator into several open-loop chains by cutting open at the moving revolute joints. More complicated system is the first open tree serial chain including the active link and ending with the moving platform.
8. For each link and platform we determine the inertia force (21) and the resulting force (excluding the actuator force) exerted to the rigid body T_k, from recursive Eqs. (25).
9. For each link and platform we determine the moment of inertia forces (22) and the resulting moment (excluding the actuator torque) exerted at the joint A_k, from recursive Eqs. (25).
10. We find the actuator torque m_{10}^A from the Eqs. (24). Analogous relations give the torques m_{10}^B, m_{10}^C of the other two actuators.

Based on the algorithm from above procedure, a computer program was developed to solve the inverse dynamics of the Delta parallel robot, using MATLAB software. To illustrate the algorithm, it is assumed that the platform starts at rest from a central configuration and moves simultaneously along the three orthogonal directions for a period of three seconds. Furthermore, at the initial location, the moving platform is assumed to be located 0.725 m lower the fixed base, namely $t = 0$: $x_0^G = 0,\ y_0^G = 0,\ z_0^G = 0.725\ m$.

Assuming that there is no external force and moment acting on the moving platform, the time-history of the torques m_{10}^A, m_{10}^B, m_{10}^C of the three actuators are shown during the platform's evolution.

The following examples are solved to illustrate the algorithm. For the first example, the moving platform moves along the z_0 *vertical direction* with variable acceleration while all the other positional parameters are held equal to zero. As can be seen from Fig. 3, it is proved to be true that all actuating torques are permanently equal to one another.

If the platform's centre G moves along a *rectilinear trajectory*, the torques exerted by the actuators A_1, B_1, C_1 are calculated by the program and plotted vs. time as follows: m_{10}^A (Fig. 4), m_{10}^B (Fig. 5), m_{10}^C (Fig. 6). When the moving platform is going to the fixed base, the limbs become more horizontally oriented, therefore, increasing the actuating torques.

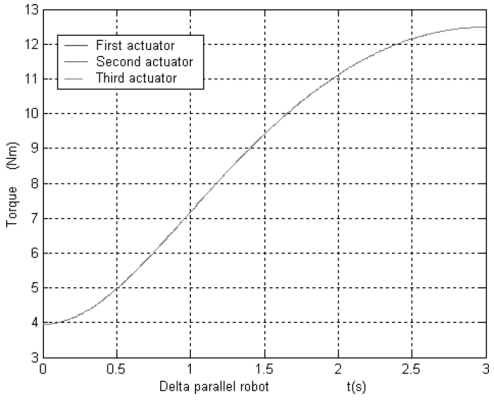


Fig. 3. Torques of three actuators.

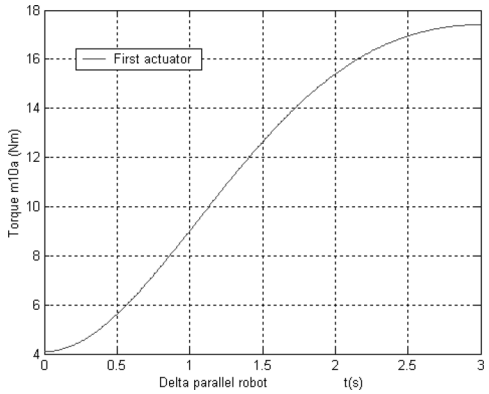


Fig. 4 Torque m_{10}^A of first actuator.

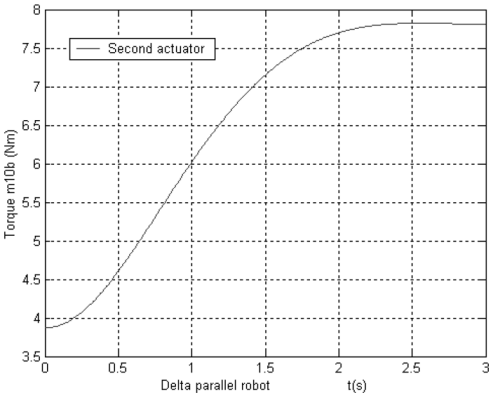


Fig. 5 Torque m_{10}^B of second actuator.

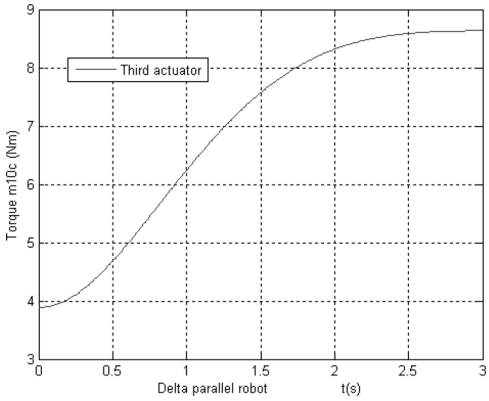


Fig. 6 Torque m_{10}^C of third actuator.

The simulation through the MATLAB program certify that one of the major advantages of the current matrix recursive formulation is a reduced number of additions or multiplications and consequently a smaller processing time of numerical computation in comparison with the approach based on the Lagrange equations. Also, the proposed method can be applied to various types of complex manipulators of higher degrees of freedom, when the number of components of the mechanism is increased.

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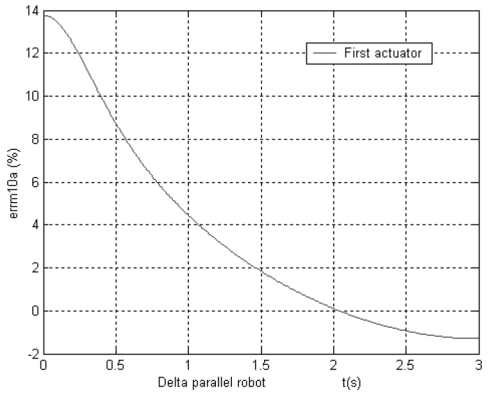


Fig. 7. Relative error-coefficient $err\, m_{10}^{A}$ of first actuator.

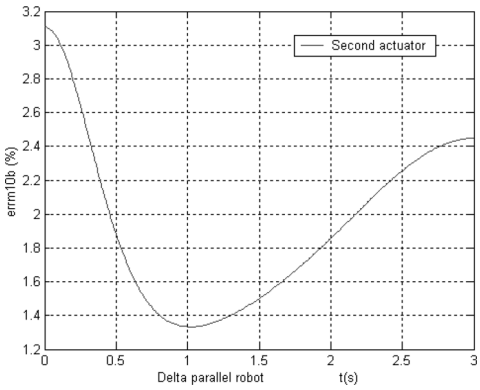


Fig. 8. Relative error-coefficient $err\, m_{10}^{B}$ of second actuator.

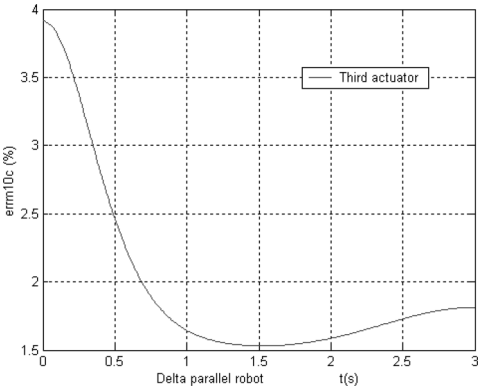


Fig. 9. Relative error-coefficient $err\, m_{10}^{C}$ of third actuator.

5. Comparative Analysis

The analytical inverse dynamics of the University of Maryland manipulator is solved by Tsai and Stamper[1,13] using the Lagrange equations of second kind, which are written in terms of a set of three redundant coordinates x_0^G, y_0^G, z_0^G. The number of generalized coordinates x_0^G, y_0^G, z_0^G, φ_{10}^A, φ_{10}^B, φ_{10}^C exceeds the number of degrees of freedom of the moving platform. Three scalar constraint equations containing only the above six coordinates are easily obtained from the general vector conditions (8).

However, this formalism generally leads to a cumbersome expression for the Lagrange's equations, due to the complex kinematics of the manipulator.

To *simplify* the analysis, one assumed that the mass $m_3$of each connecting rod in the upper arm assembly is divided evenly and concentrated at its two endpoints. The masses m_2 and $m_4$of two transmission bars are supposed as negligible with respect to the mass of other links.

Within these *simplifying assumptions*, one obtains a set of six equations [1] from which the three Lagrange multipliers $\lambda_1, \lambda_2, \lambda_3$ and the actuator torques $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{\tau}_3$ are determined.

A comparative study of two sets of torques m_{10}^A, m_{10}^B, m_{10}^C, obtained in the present paper using the principle of virtual work and $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, $\boldsymbol{\tau}_3$ in applying Lagrange's equations of second kind is illustrated graphically through the relative error-coefficients:

$$
\begin{aligned}
err\, m_{10}^{A} &= \frac{m_{10}^{A} - \tau_1}{m_{10}^{A}} \times 100 \\
err\, m_{10}^{B} &= \frac{m_{10}^{B} - \tau_2}{m_{10}^{B}} \times 100 \\
err\, m_{10}^{C} &= \frac{m_{10}^{C} - \tau_3}{m_{10}^{C}} \times 100.
\end{aligned}
\tag{28}
$$

Finally, one obtains the graphs of the three relative error-coefficients $err\, m_{10}^{A}$ (Fig. 7), $err\, m_{10}^{B}$ (Fig. 8), $err\, m_{10}^{C}$ (Fig. 9).

The simplified analysis is the consequence of expressing of the geometrical constraint conditions in the reduced form of Eqs. (15), which are obtained from the remark that the distance between A_3 and A_4 is always equal to the length l_3 of the connecting rod of the upper arm.

The conventional Jacobian matrix can relate only the input velocities ω_{10}^i to the absolute velocity of the moving platform and therefore avoid the presence of the velocities ω_{21}^i, ω_{32}^i, ω_{43}^i, ω_{54}^i. Moreover, the transformation between the end-effector output forces and the joint torques is governed, in the *absence* of gravitational effects, by this conventional Jacobian matrix.

6. Conlusions on the Advantage of the Present Method

Most of the dynamical models based on the Lagrange formalism neglect the weight of intermediate bodies and take into consideration the active forces or moments only and the wrench of applied forces on the moving platform. The number of relations given by this approach is equal to the total number of the position variables and Lagrange multipliers inclusive. Also, the analytical calculations involved in these equations are very tedious, thus presenting an elevated risk of errors.

The commonly known Newton–Euler method, which takes into account the free-body-diagrams of the mechanism, leads to a large number of equations with unknowns including also the connecting forces in the joints. Thus, in dynamics of the Delta robot a one hundred and eight equations system, that must be solved, would result. Finally, the actuating torques could be obtained.

Within the inverse kinematics analysis some exact relations that give in real-time the position, velocity and

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acceleration of each element of the parallel robot have been established in the present paper. The dynamics model takes into consideration the mass, the tensor of inertia and the action of weight and inertia force introduced by each element of the manipulator.

Based on the principle of virtual work, the new approach is far more efficient, can eliminate all forces of internal joints and establishes a direct determination of the time-history evolution of torques and powers required by the three actuators. Also, the method described above is quite available in forward and inverse mechanics of serial and parallel mechanisms, the platform of which behaves in translation, spherical evolution or more general six-degree-of-freedom motion.

The recursive matrix relations (24), (25) represent a set of explicit equations of the dynamic simulation and, in a context of automatic command, can easily be transformed into a robust model for computerized control of the Delta parallel robot.

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