This is a ``preproof'' accepted article for *The Journal of Symbolic Logic*.
This version may be subject to change during the production process.
DOI: 10.1017/jsl.2026.10182

INDEPENDENCE AND INDUCTION IN REVERSE MATHEMATICS

DAVID BELANGER, CHI CHONG, RUPERT HÖLZL, AND FRANK STEPHAN

Abstract. We continue the project of the study of reverse mathematics principles inspired by cardinal invariants. In this article in particular we focus on principles encapsulating the existence of large families of objects that are in some sense mutually independent. More precisely, we study the principle MAD stating that a maximal family of pairwise almost disjoint sets exists; and the principle MED expressing the existence of a maximal family of functions that are pairwise eventually different. We investigate characterisations of and relations between these principles and some of their variants. It turns out that induction strength at the levels of $\mathsf{B}\Sigma^0_2$ or $\mathsf{I}\Sigma^0_2$ is an essential parameter; for instance, over $\mathsf{B}\Sigma^0_2$, we show that $\neg$MAD is equivalent to the principle DOM expressing that every weakly represented family of functions is dominated by some other function.

1. Introduction

Reverse mathematics is the program of classifying the strength of mathematical statements relative to each other over a base system — see Simpson [9]. These mathematical statements are often referred to as "principles," as in the Ramsey-style "combinatorial principles" that have captured much attention since the mid-1990's.

Hölzl, Raghavan, Stephan and Zhang [4] introduced a number of principles modelled after notions of cardinal invariants from set theory. In this article we continue and extend this study with particular focus on principles that are in some sense related to the mutual independence of objects. More precisely, we use the notion of weakly represented families of objects introduced in the above article [4] to express the existence of large families of objects and then investigate from a reverse-mathematical point of view the strength of principles that assert the existence of maximal families of objects that are "independent" of each other in one sense or another. We begin by recalling some basic facts in reverse mathematics.

2. Preliminaries

We are interested in models in the language of second-order arithmetic. These take the form $\mathcal{M} = (M, \mathcal{S}, +, \cdot, 0, 1, \in)$ where M is the first-order part and $\mathcal{S}$ is a subset of the power set of M. Typically we suppress the operations and write simply $\mathcal{M} = (M, \mathcal{S})$. We call the elements of M *numbers*, with the idea that they stand in for the natural numbers ω and obey many of the same axioms. In this article we resist the convention of writing "sets" to mean the elements of $\mathcal{S}$, because we will often need to discuss sets $A \subseteq M$ which are possibly outside of $\mathcal{S}$. Call a formula a Σ^A_n *formula* if it is Σ_n while using the given set A as a second-order parameter (which we also call an *oracle*). Call a set $B \subseteq M$ a Σ^A_n *set* if B is definable by a Σ^A_n formula. Formulas with multiple oracles are allowed; call a formula or set Σ^0_n if it is $\Sigma^{A_1,\dots,A_k}_n$ for some $A_1, \dots, A_k \in \mathcal{S}$; this includes the case where $k = 0$ and the list is empty. Analogous definitions with "Π" or "Δ" in place of "Σ" can be given in the obvious ways.

Here are some of the axioms we will consider. Let P^- be the axioms of Peano arithmetic not including any induction schemes. Let $\mathsf{I}\Sigma^0_n$ denote the induction scheme for Σ^0_n formulas, and let $\mathsf{B}\Sigma^0_n$ denote the Σ^0_n-bounding scheme. Over P^-, Kirby and Paris [5] proved the following strict implications for every $n \geq 1$:

$$\mathsf{B}\Sigma^0_{n+1} \to \mathsf{I}\Sigma^0_n \quad \text{and} \quad \mathsf{I}\Sigma^0_n \to \mathsf{B}\Sigma^0_n$$

The investigators acknowledge the following partial support: F. Stephan's reserach was supported by Singapore Ministry of Education AcRF Tier 2 grant MOE-000538-01 and AcRF Tier 1 grants A-0008454-00-00 and A-0008494-00-00, the grant A-0008494-00-00 also supported the research visits of R. Hölzl to the National University of Singapore and the past employment of D. Belanger at the National University of Singapore. C. T. Chong's research was partially supported by NUS grant WBS E-146-00-0001-01. The views expressed in this article are those of its authors; they do not necessarily reflect those of the Deutsche Bundesbank or the Eurosystem.

https://doi.org/10.1017/jsl.2026.10182 Published online by Cambridge University Press

(In their account there were no second-order parameters, but the proof is the same.) The system RCA_0 consists of P^- together with $\mathsf{I}\Sigma^0_1$ and the comprehension scheme for Δ^0_1 sets. In particular, if $\mathcal{M} = (M, \mathcal{S})$ is a model of RCA_0, then S is closed under Turing reducibility.

We say a nonempty set $A \subseteq M$ is $\mathcal{M}$-*finite* if A is in $\mathcal{S}$ and A has a maximum element. (Our axioms are strong enough to support a number of equivalent definitions of $\mathcal{M}$-finite, but this one suits our purposes well enough.) We say $A \subseteq M$ is $\mathcal{M}$-*infinite* if there is an injection $f \in \mathcal{S}$ from M into A. If $\mathcal{M}$ is a model of RCA_0, then every $A \in \mathcal{S}$ is either $\mathcal{M}$-finite or $\mathcal{M}$-infinite; but for general $A \subseteq M$, it is possible to be neither $\mathcal{M}$-finite nor $\mathcal{M}$-infinite. We say that $A \subseteq \mathcal{M}$ is *regular* if for each $b \in M$, its initial segment $A \upharpoonright b = A \cap \{0, \ldots, b-1\}$ is $\mathcal{M}$-finite. If $\mathcal{M} \models \mathsf{RCA}_0$ then every element $A \in \mathcal{S}$ is regular, and in particular every $\mathcal{M}$-finite set is regular. Using the standard quadratic pairing function $\langle \cdot, \cdot \rangle$ to represent an ordered pair (and again, leaning on the strength of our axiom systems) we can define an appropriate notion of $\mathcal{M}$-finite strings, that is, sequences of length $m \in M$ and with elements taken from M, via $\sigma = \{\langle x, y\rangle : x < m \text{ and } \sigma(x) = y\}$, if this set is $\mathcal{M}$-finite. We also render this sequence as $\sigma = \langle \sigma(0), \ldots, \sigma(m-1)\rangle$, abusing notation in the case $m = 2$. We use $M^{<M}$ to denote the set of all $\mathcal{M}$-finite strings, and $2^{<M}$ to denote the set of all $\mathcal{M}$-finite strings which consist only of 0's and 1's, i.e. binary strings.

In this paper, we consider principles related to set theory's cardinal invariants, in the context of RCA_0. In particular, we are interested in the role played by the inductive strength of a model of RCA_0 in relation to these principles. This naturally requires an analysis of nonstandard models.

The families of functions and families of sets that we consider are the same as those considered by Hölzl, Raghavan, Stephan and Zhang [4], and informally speaking consist of those functions that can be written as the "rows" Ψ_e of some set Ψ, with the following details: we require that Ψ be Σ^0_1 (with parameters), as opposed to being an element of $\mathcal{S}$; we require that each row Ψ_e be well-defined as a function, but not necessarily total; and all rows Ψ_e which are not total are simply ignored and do not contribute to the family. More formally, we will work with the following definition.

Definition 1 (Weakly represented families)**.** *Let $\Psi \subseteq M^2 \to M$ be a Σ^0_1 (with set parameters) partial function, and for each e define $\Psi_e = \{\langle x, y\rangle : \Psi(e, x) = y\}$. View Ψ_e as a partial function $M \to M$. Define $\mathcal{F} = \{\Psi_e : \Psi_e \text{ is total}\}$. Then $\mathcal{F}$ is a family of total functions $M \to M$; we say that $\mathcal{F}$ is* weakly represented *by Ψ. In this case we call $\mathcal{F}$ a* weakly represented family*.*

By identifying a set A with its characteristic function, we can naturally speak of a family $\mathcal{F}$ of sets $A \subseteq M$ being weakly represented. We mention that these notions of weakly represented families also were used by Hölzl, Jain and Stephan [3] to study inductive inference in the setting of reverse mathematics.

If $\mathcal{F}$ is a weakly represented family (of functions or of sets), then each element $f \in \mathcal{F}$ is Δ^0_1 relative to sets in S and therefore is in $\mathcal{S}$. By considering the model $\mathcal{M}$'s version of the universal oracle Turing machine, we obtain the following result.

Lemma 2. *For each $A \in \mathcal{S}$ there exists a Σ^A_1 partial function $\Phi^A : M^2 \to M$ which is* universal *in the sense that for every Σ^A_1 partial function $\Psi^A : M^2 \to M$ there is a Δ^A_1 function f satisfying $\Psi^A_e = \Phi^A_{f(e)}$.*

We use $\mathcal{F}^A_{\mathrm{Tot}}$ to denote the family of functions weakly represented by Φ^A. Then every weakly represented family $\mathcal{F}$ is contained in $\mathcal{F}^A_{\mathrm{Tot}}$ for some $A \in \mathcal{S}$. Lastly, we fix a definition of *finite* that is suitable for weakly represented families.

Definition 3. *A weakly represented family $\mathcal{F}$ is called* finite *if there is a Ψ which weakly represents it, and an $\mathcal{M}$-finite set E such that $\mathcal{F} = \{\Psi_e : e \in E\}$. Otherwise, $\mathcal{F}$ is called* non-finite*.*

In the absence of stronger induction axioms, a family $\mathcal{F}$ which is *non-finite* in this sense does not satisfy certain intuitive properties of infinity. As an example: If $\neg\mathsf{I}\Sigma^0_2$ holds, then the family $\mathcal{F}$ conisting of functions $f_e(x) = e$ if $\Phi^A_e(x)\!\downarrow$ for $e < e_0$ is bounded, but in general not finite; this is because $\{e < e_0 : \Phi^A_e \text{is total}\}$ is in general not M-finite. We bring this up to explain the presence of the unusual term *non-finite*, as well as to point out that certain of our results below depend on this definition, and could end up different if some other notion of *finite* were used.

The rest of this paper is structured as follows. Section 3 deals with the principle DOM which expresses essentially that there is a function dominating every total function in a weakly represented family; we prove a version of Martin's theorem [7] characterizing such functions in terms of high Turing degrees. Section 4 is a technical section where we carry out a general $0''$ tree argument for use in later sections, to construct new

https://doi.org/10.1017/jsl.2026.10182 Published online by Cambridge University Press

weakly represented families. Section 5 deals with the principle MAD, which says that there is a maximal almost disjoint (MAD) family of sets. We obtain results comparing MAD with DOM, and with $\mathsf{I}\Sigma^0_2$. In Section 6, we investigate the principles MED, which says there is a maximal eventually different (MED) family of functions, as well as AVOID, which says essentially that for every weakly represented family there is a function eventually different (ED) from all functions in the family; we obtain results about the relationship between the two. Section 7 covers the principle HI, which says essentially that there is a hyperimmune Turing degree. Finally, Section 8 is concerned with the principle BI, which says that there is a bi-immune Turing degree; we use the existence of a low bi-immune Turing degree to obtain a conservation result.

3. Dominating families of functions

Statement 4 (Domination principle, DOM)**.** *Given any weakly represented family $\mathcal{F}$ of functions, there exists a function g such that g* dominates *$\mathcal{F}$, that is, for every $f \in \mathcal{F}$, we have $g(x) > f(x)$ for all sufficiently large x.*

In an ω-model — meaning a model $(M, \mathcal{S})$ in which M is the true natural numbers — if $X \in \mathcal{S}$ computes a function which dominates every total recursive function, then X is generalised high, that is, $X' \equiv_T (X \oplus \emptyset')'$. This was first proven by Martin [7]. The phenomenon is reflected in RCA_0 by the following proposition, proven essentially by Chong, Qian, Slaman and Yang [2, Lemma 2]:

Proposition 5 (Martin's Theorem, formalised)**.** *For a model $(M, \mathcal{S})$, the following are equivalent over RCA_0:*

(i) DOM: *For every $\mathcal{F}$ that is a weakly represented family in $\mathcal{S}$ there is a function $g \in \mathcal{S}$ which grows faster than every $f \in \mathcal{F}$.*
(ii) HIGH: *For every $A \in \mathcal{S}$ there is a $B \in \mathcal{S}$ such that every Σ^A_2 set $C \subseteq M$ is Δ^B_2.*
(iii) For each $A \in \mathcal{S}$ and every Σ^A_2 set $C \subseteq M$ there is a $B \in \mathcal{S}$ such that C is Δ^B_2.

Note that we do not assume $C \in \mathcal{S}$.

Proof. (i $\Rightarrow$ ii) Fix A and let f be a total function which dominates all total Δ^A_1 functions. Take any C which is Π^A_2, and let ψ^A be the Δ^A_0 formula satisfying

$$x \in C \iff (\forall y)(\exists z)[\psi^A(x, y, z)].$$

Define $g(x, y)$ to be the partial Σ^A_1 function mapping each pair $\langle x, y\rangle$ to the least z satisfying $\psi^A(x, y, z)$. Then for a given x, we have $x \in C$ iff the function $\lambda y.g(x, y)$ is total Δ^A_1. Hence,

$$x \in C \iff (\exists y_0)(\exists z_0)\left[\begin{array}{c}(\forall y > y_0)(\exists z < f(y))[\psi^A(x, y, z)]\\ \text{and } (\forall y \leq y_0)(\exists z < z_0)[\psi^A(x, y, z)]\end{array}\right],$$

meaning C is Σ^f_2 and hence Δ^f_2.

(ii $\Rightarrow$ iii) Immediate; the difference between (ii) and (iii) is that whereas in (ii) a single B is assigned to each A, in (iii) B is allowed to vary with each Σ^A_2-definable set.

(iii $\Rightarrow$ i) Suppose $\mathcal{F}$ is a weakly represented family of functions. Then $\mathcal{F} \subseteq \mathcal{F}^A_{\mathrm{Tot}}$ for some A; so it suffices to exhibit some g which dominates all $f \in F^A_{\mathrm{Tot}} = \{\Phi^A_e \colon \Phi^A_e \text{ is total}\}$.

Since the predicate "Φ^A_e is total" is Π^A_2, there exists by (iii) a B and a Σ^B_0 formula ψ satisfying

$$\Phi^A_e \text{ is total} \iff (\exists s)(\forall t)[\psi^B(e, s, t)].$$

Then for every input x let $g(x)$ be the least $t_0 > x$ such that

$$(\forall e < x)\left[\text{either } \Phi^A_{e,t_0}(x)\!\downarrow \text{ or } (\forall s < x)(\exists t < t_0)[\neg\psi^B(e, s, t)]\right].$$

Then g is well-defined, $\Sigma^{A,B}_1$ and total. By definition it dominates each total Δ^A_1 function. □

This formalised version of Martin's Theorem allows for a short proof of the previously known fact that DOM and $\mathsf{B}\Sigma^0_2$ together imply full arithmetical induction and thus Peano Arithmetic (PA).

Theorem 6 (Hölzl, Raghavan, Stephan, Zhang [4])**.** $\mathsf{RCA}_0 + \mathsf{DOM} + \mathsf{B}\Sigma^0_2$ *implies* $\mathsf{I}\Sigma^0_n$ *for every $n \geq 1$. In short,* $\mathsf{RCA}_0 + \mathsf{DOM} + \mathsf{B}\Sigma^0_2 \vdash \mathsf{PA}$.

We will use the following well-known fact in the new proof.

https://doi.org/10.1017/jsl.2026.10182 Published online by Cambridge University Press

Lemma 7. *The following statements hold for every model $(M, \mathcal{S})$ of RCA_0:*

(1) $\mathsf{I}\Sigma^0_n$ iff every Σ^0_n set is regular iff every $A^{(n)}$ is regular for $A \in \mathcal{S}$,

(2) $\mathsf{B}\Sigma^0_n$ iff every Δ^0_n set is regular iff every $B \leq_T A^{(n-1)}$ is regular for $A \in \mathcal{S}$.

Proof of Theorem 6. Fix $(M, \mathcal{S}) \models \mathsf{RCA}_0 + \mathsf{DOM} + \mathsf{B}\Sigma^0_2$ and let $A_0 \in \mathcal{S}$. Use Proposition 5 to obtain a sequence $\{A_n : n \in \omega\}$ such that A_{n+1} is high relative to A_n. In particular, the double jump A_n'' is $\Delta_2^{A_{n+1}}$ and therefore regular by $\mathsf{B}\Sigma^0_2$ and Lemma 7. Since $A_0'' \leq_T A_1'$, we also have $A_0''' \leq_T A_1''$. Hence by transitivity ($\mathsf{B}\Sigma_1$ relative to A_2'), we have $A_0''' \leq_T A_2'$ and hence that A_0''' is regular. Iterating this argument for larger and larger n, we find that $A_0^{(n)} \leq_T A_{n-1}'$ and $A_0^{(n)}$ is regular for all $n \in \omega$. The result follows from Lemma 7 and the fact that A_0 was arbitrary. □

Thus it is all the more surprising that, by the following theorem, DOM does not imply $\mathsf{B}\Sigma^0_2$. The following theorem also gives an alternative (but similar) proof that HIGH does not imply ACA_0, by way of conservation in place of cone avoidance. Recall that a system T_1 is Π^1_1-conservative over a system T_2 if $T_1 \supset T_2$ and every Π^1_1-sentence provable in T_1 is provable in T_2. By a well-known observation of Harrington, one can establish Π^1_1-conservation by showing that every countable model $\mathcal{M} = (M, \mathcal{S})$ of T_2 may be extended to a model $\tilde{\mathcal{M}} = (M, \tilde{\mathcal{S}})$ of T_1 having the same first-order part M and such that $\mathcal{S} \subset \tilde{\mathcal{S}}$. We say that a model $\mathcal{M} = (M, \mathcal{S})$ is *topped* if there is a $B \in \mathcal{S}$ which computes every $A \in \mathcal{S}$. As noted in [1], to show that T_1 is Π^1_1-conservative over T_2, it is sufficient to show that every countable topped model of T_2 can be extended to a model of T_1 through expanding $\mathcal{S}$. This is a consequence of the following observation: If T_1 is not Π^1_1-conservative over T_2, then there is a Π^1_1-sentence $\forall X \varphi$ provable in T_1 but not T_2. Let $\mathcal{M}$ be countable such that $\mathcal{M} \models T_2 + \exists X \neg\varphi$. In $\mathcal{M}$ choose $A \in \mathcal{S}$ to witness $\exists X \neg\varphi$ and let $\mathcal{M}' \subseteq \mathcal{M}$ be the submodel topped by A. A contradiction is derived by showing that $\mathcal{M}'$ can be exrtended to a model of T_1. We will use this fact in the proof of the following theorem.

Theorem 8. $\mathsf{RCA}_0 + \mathsf{DOM}$ *is* Π^1_1*-conservative over* RCA_0.

Proof. Let $\mathcal{M}_0 = (M, \mathcal{S}_0)$ be a countable model of RCA_0 topped by a function $f_0 \in \mathcal{S}$; that is, every $A \in \mathcal{S}$ is Turing-reducible to f_0. We will construct a sequence of functions $(f_n)_{n<\omega}$, and with it a sequence of countable models $(\mathcal{M}_n)_{n<\omega}$ given by

$$\mathcal{M}_{n+1} = \mathcal{M}_n[f_{n+1}] = (M, \mathcal{S}_{n+1}),$$

where $\mathcal{S}_{n+1}$ is the closure of $\mathcal{S} \cup \{f_{n+1}\}$ under Δ^0_1 comprehension. We ensure that each f_{n+1} dominates $\mathcal{F}^{f_n}_{\mathrm{Tot}}$ and each $\mathcal{M}_n$ satisfies $\mathsf{I}\Sigma^0_1$ (and, in particular, $\mathcal{M}_n$ is topped by f_n). Then the limiting model $\mathcal{M}_\omega = (M, \bigcup_{n<\omega} \mathcal{S}_n)$ will satisfy $\mathsf{RCA}_0 + \mathsf{DOM}$, proving the theorem by Harrington's method.

For the inductive step, begin with the countable model $\mathcal{M}_n \models \mathsf{I}\Sigma^0_1$ topped by f_n, and consider the class $\mathcal{F}^{f_n}_{\mathrm{Tot}}$. Since $\mathcal{M}_n$ is countable, we may (externally to the model) let $(g_i)_{i<\omega}$ be a list of all $\Delta_1^{f_n}$ total functions; let $(\varphi_i^{f_n})_{i<\omega}$ be a list of all $\Sigma_1^{f_n}$ formulae; and let $(c_k)_{k<\omega}$ be an increasing sequence that is cofinal in M. We construct f_{n+1} by initial segments $\sigma_k \in M^{<M}$. Beginning with σ_0 as the empty string, for each $k \in \omega$ we choose the next σ_{k+1} to satisfy

(i) σ_{k+1} is a proper extension of σ_k,
(ii) $|\sigma_{k+1}| > c_k$,
(iii) $\sigma_{k+1}(x) \geq g_i(x)$ whenever $i < k$ and $|\sigma_k| < x \leq |\sigma_{k+1}|$, and
(iv) either $\mathcal{M}_n \models \varphi_k^{f_n}(\sigma_{k+1})$ or $\mathcal{M}_n \models (\forall \tau \supseteq \sigma_{k+1})[\neg\varphi_k^{f_n}(\tau)]$.

Because k is always in ω, some suitable σ_{k+1} exists at every stage. This completes the construction.

The requirements (i) and (ii) together ensure that f is a well-defined, total function (rather than having a proper cut as its domain). Item (iii) ensures that f_{n+1} dominates each function in $\mathcal{M}_n$, and (iv) is a form of "forcing the jump" that ensures that $\mathcal{M}_n[f_{n+1}]$ models $\mathsf{I}\Sigma^0_1$. This completes the proof. □

4. Handling index sets

This technical section lists some methods for producing new weakly represented families $\mathcal{G}$ from known families $\mathcal{F}$, in models of $\mathsf{I}\Sigma^0_2$. These take the form of sufficient conditions on the set B in the definition $\mathcal{G} = \{\Psi_e : e \in B\}$, where $\mathcal{F}$ is weakly represented by Ψ. However, the constructions are not as simple as just restricting Ψ to the rows in B, because we would like B to be of higher complexity than that would allow. Instead, we use a conventional $0''$ infinite injury argument — reminiscent of, for instance, Yates [10] — to

https://doi.org/10.1017/jsl.2026.10182 Published online by Cambridge University Press

get a new Θ which weakly represents $\mathcal{G}$, and which can be viewed as a scrambled version of Ψ restricted to B.

We begin with an easy but helpful lemma; here, we let

$$\mathrm{rt}B = \{\sigma \in 2^{<M} : \sigma(b) = 0 \to b \in B\},$$

that is, $\mathrm{rt}B$ consists of the binary strings which are either substrings of the complement $\overline{B}$'s characteristic function, or lexicographically to the right of that characteristic function.

Lemma 9. *Let $(M, \mathcal{S})$ be a model of RCA_0 and assume that $B \subseteq M$ is Π^A_2 for some A. Then $\mathrm{rt}B$ is Π^A_2.*

Proof. Fix A such that B is Π^A_2. Now

$$\sigma \in \mathrm{rt}B \leftrightarrow \sigma \in 2^{<M} \ \wedge \ \forall b \leq |\sigma|(\sigma(b) = 0 \to b \in B),$$

showing that $\mathrm{rt}B$ is is Π^A_2.

□

Next we present three new methods of representing families of functions under different assumptions; in each case, the Θ we are constructing can be thought of as a restricted and scrambled version of the input representation Ψ.

Lemma 10 (Π^0_2 weak representation)**.** *Suppose $\mathcal{M} = (M, \mathcal{S})$ is a model of $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$ and Ψ is Σ^0_1. If $B \subseteq M$ is Π^0_2, then the family $\{\Psi_e : e \in B$ and Ψ_e is total$\}$ is weakly represented.*

Proof. Let $\mathrm{rt}B$ be as in Lemma 9. Since $\mathrm{rt}B$ is Π^0_2, it has a Δ^0_1 approximation $(\sigma_s)_{s \in M}$ satisfying $\sigma \in \mathrm{rt}B$ iff $(\exists^\infty s)[\sigma = \sigma_s]$. Begin the construction of a new Σ^0_1 set Θ by assigning to each $\sigma \in 2^{<M}$ a row Θ_σ, while leaving $\mathcal{M}$-infinitely many rows of Θ unassigned. Then proceed by stages as follows:

At even stages $2s$, for each τ strictly to the right of σ_s, abandon its currently assigned row Θ_τ and assign to it a new, fresh row (which now takes the name Θ_τ). Leave $\mathcal{M}$-infinitely many rows of Θ still unassigned.

At odd stage $2s + 1$, for each $e < |\sigma_s|$ for which $\sigma_s(e) = 0$, make $\Theta_{\sigma_s \upharpoonright e}$ imitate Ψ_e for s-many steps of computation.

This completes the construction. Clearly Θ is a partial function, and every total row of Θ is the row Θ_σ eventually assigned to some σ which is an initial segment of the complement $\overline{B}$'s characteristic function. And on the other hand, such a Θ_σ is total if and only if $\sigma(|\sigma| - 1) = 0$ (meaning $|\sigma| - 1 \in B$) and $\Psi_{|\sigma|-1}$ is total. Hence Θ weakly represents the desired family of functions. □

Lemma 11 (Σ^0_2 weak representation)**.** *As in Lemma 10, but with B being Σ^0_2 instead of Π^0_2.*

This can be seen by almost the same proof as above, but using $\overline{B}$ as the Π^0_2 set and accordingly replacing "$\sigma_s(e) = 0$" by "$\sigma_s(e) = 1$" in the odd stages.

Lemma 12 (Π^0_2 dependent weak representation)**.** *Suppose $\mathcal{M} = (M, \mathcal{S})$ models $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$ and that φ is a Π^0_2 predicate. Then*

$$\{\Psi_{e_i} : e_i \text{ is least such that } \varphi(\langle e_0, \ldots, e_{i-1}\rangle) \text{ holds}\}$$

is a weakly represented family.

In other words: e_0 is least such that $\varphi(\langle e_0\rangle)$ holds, then e_1 is least such that $\varphi(\langle e_0, e_1\rangle)$ holds, and so on. It is possible that e_i is defined for all $i \in M$; it is possible that there is no i at all, or a maximum i for which e_i is defined; and it is also possible that the set of i forms a proper cut.

Proof. Associate to each $\sigma \in 2^{<M}$ the sequence $\langle e^\sigma_0, \ldots, e^\sigma_k\rangle \in M^{<M}$ where

- e^σ_0 is the number of 1's occurring before the first 0 in σ, and
- e^σ_{i+1} is the number of 1's occurring between the $(i+1)$-th and the $(i+2)$-th 0 in σ; and
- for each σ, we use k to mean the largest number for which e^σ_k is defined.

Then the set of σ for which $\mathcal{M} \models \varphi(\langle e^\sigma_0, \ldots, e^\sigma_k\rangle)$ holds is closed under initial segment — in other words, it forms a binary tree — it is Π^0_2 definable, and its leftmost infinite path σ corresponds to the sequence $\langle e_0, e_1, \ldots\rangle$ described in the lemma's statement. By applying the argument of Lemma 10 to this tree and its e_i, we obtain the required Θ. □

https://doi.org/10.1017/jsl.2026.10182 Published online by Cambridge University Press

5. Maximal almost disjoint families of sets

Two sets $A, B \subseteq M$ are said to be *almost disjoint* if $A \cap B$ is $\mathcal{M}$-finite. A family $\mathcal{F}$ of subsets of M is *almost disjoint* if any two distinct elements of $\mathcal{F}$ are almost disjoint. $\mathcal{F}$ is called *maximal almost disjoint* if it is non-finite, almost disjoint and not properly contained in any almost disjoint family.

Statement 13 (Maximal Almost Disjoint Family principle, MAD)**.** *There is a non-finite, weakly represented family of infinite sets that is maximal almost disjoint.*

Hölzl, Raghavan, Stephan and Zhang [4] showed that MAD is equivalent to ¬DOM in ω-models. We extend this result to all models of $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$.

Theorem 14. $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2 \vdash \mathsf{DOM} \leftrightarrow \neg\mathsf{MAD}$.

Proof. (DOM → ¬MAD) Fix a model $\mathcal{M} = (M, \mathcal{S}) \models \mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2 + \mathsf{DOM}$, and any non-finite almost disjoint family $\mathcal{F}$ which is weakly represented, say by Ψ^A for some $A \in \mathcal{S}$ (note that since $\mathsf{I}\Sigma^0_2$ holds in the model, $\mathcal{F}$ is in fact $\mathcal{M}$-infinite). The set

$$E = \{e \colon \Psi^A_e \text{ is total and (the characteristic function of) an } \mathcal{M}\text{-infinite set}\}$$

is Π^A_2, and hence by DOM and Proposition 5 it is Δ^f_2 for some $f \in \mathcal{S}$. Let $(E_s)_{s\in M}$ be a Δ^f_1 approximation to E.

Define a new set $B = \{b_0 < b_1 < \ldots\}$ as follows. Let b_0 be any number. To define b_{k+1}, search for a pair $x, s > b_k$ satisfying

$$(\forall e < k)[\text{either } \Psi^A_{e,s}(x)\!\downarrow = 0 \text{ or } e \notin E_s],$$

and let b_{k+1} be this x. Since B is being enumerated in increasing order and in a Σ^f_1 way, we know by $\mathsf{I}\Sigma^0_1$ that $B \in \mathcal{S}$ and (using the definition of E) that b_k is defined for every $k \in M$. And since $(E_s)_{s\in M}$ reaches a pointwise limit, $\mathcal{M} \models (\forall e \in E)(\forall^\infty k)[b_k \notin \Psi^A_e]$. Hence, if we let Θ equal Ψ prepended with a row for B's characteristic function, we see that $\mathcal{F} \cup \{B\}$ is a weakly represented family strictly larger than $\mathcal{F}$. Now fix any $C \in \mathcal{F}$; we claim that B and C are almost disjoint. Suppose for a contradiction that $B \cap C$ were $\mathcal{M}$-infinite, and let Ψ^A_e be C's characteristic function. Then from the definition of b_k we know that $e \notin E_s$ for unboundedly many s, and so in the limit we have $e \notin E$. But this contradicts the definition of E. Therefore $\mathcal{F}$ is contained in the strictly larger almost disjoint family $\mathcal{F} \cup \{B\}$, and in particular $\mathcal{F}$ is not a MAD family. Since $\mathcal{F}$ was arbitrary, we conclude ¬MAD holds.

(¬MAD → DOM) Fix any model $\mathcal{M} = (M, \mathcal{S}) \models \mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2 + \neg\mathsf{MAD}$, fix any $A \in \mathcal{S}$, and let Φ^A be universal as in Lemma 2. Consider the formula ψ which takes a tuple $\langle e_0, \ldots, e_k\rangle$ of indices as input and expresses that

(a) each $\Phi^A_{e_i}$ is the characteristic function of an $\mathcal{M}$-infinite set — call it C_i;
(b) these C_i's are pairwise disjoint;
(c) the union $D_k = \bigcup_{i<k} C_i$ is $\mathcal{M}$-coinfinite; and
(d) for all x and $e < i < k$, either Φ^A_e is not total or $\Phi^A_e(x)$ is less than the x-th smallest element of $\overline{D_i}$.

These statements are Π^0_2; for (a)–(c) that is straightforward; for (d) consider the formula

$$(\forall x)(\forall i < k)(\forall s)(\exists t > s)(\exists y)(\forall z < y)(\forall e < i)[z \in D_{i,t} \wedge (\Phi^A_{e,t}(x)\!\downarrow < y \vee e \in \mathrm{Tot}^A_t)],$$

where $(\mathrm{Tot}^A_t)_t$ is the usual A-computable approximation to $\mathrm{Tot}^A = \{e \in M \colon \Phi^A_e \text{ is total}\}$.

Thus, applying Lemma 12, we obtain some Θ weakly representing

$$\mathcal{F} = \{\Phi^A_{e_i} \colon e_i \in \langle e_0, e_1, \ldots\rangle\},$$

where $\langle e_0, e_1, \ldots\rangle$ is lexicographically least satisfying both ψ and the relation $e_0 < e_1 < \cdots$.

Now $\mathcal{F}$ forms a partition of M into $\mathcal{M}$-infinite sets, and hence is an almost disjoint family. Since $\mathcal{M} \models \neg\mathsf{MAD}$, there is a non-finite $B \in \mathcal{S}$ almost disjoint from each C_i. By $\mathsf{B}\Sigma^0_2$ this B is almost disjoint from each D_i, meaning all but $\mathcal{M}$-finitely much of B is contained in each complement $\overline{D_i}$. Hence the function p_B which maps x to the x-th element of B dominates all total Φ^A_e for $e < i$. By allowing i to vary, we see that p_B dominates all total Δ^A_1 functions. Since $A \in \mathcal{S}$ was arbitrary, we conclude that DOM holds. □

An alternate proof of ¬MAD → DOM might construct the partition directly using a $0''$ tree argument similar to the proof of Lemma 10.

https://doi.org/10.1017/jsl.2026.10182 Published online by Cambridge University Press

Proposition 15. $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \neg\mathsf{I}\Sigma^0_2 \vdash \mathsf{MAD}$.

Proof. Fix a model $\mathcal{M} = (M, \mathcal{S})$ and let $A \in \mathcal{S}$ be such that there is a Π^A_2 set B which is bounded from above but not $\mathcal{M}$-finite; such A, B are guaranteed to exist by $\neg\mathsf{I}\Sigma^0_2$ and Lemma 7. It is routine to use the Π^A_2 formula defining B and its upper bound $a \in M$ to define a partition $C_0, \ldots, C_{a-1} \in \mathcal{S}$ of M such that C_i is $\mathcal{M}$-infinite iff $i \in B$. Let $\Psi^A \colon M^2 \to M$ be the Σ^A_1 partial function

$$\Psi^A(i,s) = \begin{cases} 1 & \text{if } i < a \text{ and } s \in C_i, \\ 0 & \text{if } i < a \text{ and } s \notin C_i \text{ and } (\exists t > s)[t \in C_i], \\ \uparrow & \text{otherwise.} \end{cases}$$

Then Ψ^A weakly represents the family $\mathcal{F} = \{C_i \colon i \in B\}$. Since the C_i form a partition, $\mathcal{F}$ is almost disjoint; we claim that $\mathcal{F}$ is MAD. To see this, suppose for a contradiction that some $D \in \mathcal{S}$ is $\mathcal{M}$-infinite, but $D \cap C_i$ is $\mathcal{M}$-finite for every $i \in B$. Then $D \cap C_i$ is $\mathcal{M}$-finite for every $i < a$. That is, we have

$$(\forall i < a)(\exists s)(\forall t > s)[t \notin C_i \cap D].$$

Using $\mathsf{B}\Sigma^0_2$ to exchange the two leading quantifiers yields $(\exists s)(\forall i < a)(\forall t > s)[t \notin C_i \cap D]$. Since the C_i form a partition, this implies that D is $\mathcal{M}$-finite, a contradiction.

To complete the argument, we claim that $\mathcal{F}$ is a non-finite family. Assume for a contradiction that it is finite, that is, that there exists some other Θ weakly representing $\mathcal{F}$ such that there is an $\mathcal{M}$-finite set E with $\mathcal{F} = \{\Theta^{\tilde{A}}_e \colon e \in E\}$ for some $\tilde{A} \in \mathcal{S}$. Then, for each $e \in E$, we can find one $w_e \in \Theta^{\tilde{A}}_e$ and using A as an oracle the unique i such that $w_e \in C_i$. But this gives an effective procedure (relative to the parameters A and $\tilde{A}$) for determining which $i < a$ are elements of B. Thus B would be $\mathcal{M}$-finite, contradicting our assumption. □

As a corollary, we obtain an extension of Theorem 14.

Corollary 16. $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 \vdash \mathsf{DOM} \leftrightarrow \neg\mathsf{MAD}$.

Proof. As $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \mathsf{DOM} \vdash \mathsf{I}\Sigma^0_2$, no model of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \neg\mathsf{I}\Sigma^0_2$ satisfies DOM. But by Theorem 14 each such model satisfies MAD. □

6. Avoidance and eventual difference

Two functions are *eventually different* if there is an upper bound on the inputs where they are equal; they are *eventually equal* if there is an upper bound on the inputs where they are different.

Statement 17 (Avoidance principle, AVOID)**.** *Given any weakly represented family of functions $\mathcal{F}$, there exists a function g that is eventually different from all $f \in \mathcal{F}$.*

Statement 18 (Maximal Eventually Different Family principle, MED)**.** *There is a weakly represented family $\mathcal{F}$ of functions such that*

- *any two functions in $\mathcal{F}$ are either equal or eventually different, and*
- *every function coincides infinitely often with some function in $\mathcal{F}$.*

Theorem 19. $\mathsf{RCA}_0 \vdash \mathsf{MED} \to \neg\mathsf{AVOID}$.

Proof. Clearly, any family witnessing MED is a counterexample to AVOID. □

The following proposition is a reformulation of a previously known result in the terminology of reverse mathematics; we include our own proof since as far as we know it is more direct than others that have appeared in print.

Proposition 20 (Kjos-Hanssen, Merkle, Stephan [6])**.** $\mathsf{RCA}_0 \vdash (\mathsf{DOM} \vee \mathsf{DNR}) \leftrightarrow \mathsf{AVOID}$.

Proof. That DOM implies AVOID is immediate from the definitions. To see that DNR implies AVOID, first suppose that f is a DNR function relative to a given $A \in \mathcal{S}$, that is, that $(\forall e)[\Phi^A_e(e) \neq f(e)]$. Let Φ^A be universal in the sense of Lemma 2, and for each e let p_e be the e-th prime number, starting with $p_0 = 2$

https://doi.org/10.1017/jsl.2026.10182 Published online by Cambridge University Press

and possibly continuing into the nonstandard numbers. Define a function $g \leq_T f$ as follows. Given e, find a sequence $i_0, \ldots, i_e$ of indices of partial functions, for each j satisfying

$$\Phi^A_{i_j}(i_j) = \begin{cases} 0 & \text{if } \Phi^A_j(e) = 0, \\ k & \text{if } k \text{ is greatest such that } p_j^k \text{ divides } \Phi^A_j(e), \end{cases}$$

and let $g(e) = p_0^{f(i_0)} p_1^{f(i_1)} \cdots p_e^{f(i_e)}$. This ensures that $g(e) \neq \Phi^A_j(e)$ for every $j \leq e$; and thus for every $j \in M$, we have that g is eventually different from Φ^A_j.

For the other direction, we need to show that AVOID implies DOM$\vee$DNR. So suppose that AVOID$+\neg$DOM holds relative to some $A \in \mathcal{S}$; that is, speaking more formally, suppose that there is no $g \in \mathcal{S}$ which dominates $\mathcal{F}^A_{\mathrm{Tot}}$ and that there is an $f \in \mathcal{S}$ which is eventually different from every function in $\mathcal{F}^A_{\mathrm{Tot}}$. Notice that $\mathcal{F}^A_{\mathrm{Tot}} \subseteq \mathcal{F}^{A \oplus f}_{\mathrm{Tot}}$, which implies that $\neg$DOM also holds relative to $A \oplus f$. Suppose for a contradiction that $f(e) = \Phi^A_e(e)$ for $\mathcal{M}$-infinitely many e, and define an $A \oplus f$-computable function s by, for every n, letting $s(n)$ be the least number t such that for at least n many $e < t$ we have $\Phi^A_{e,t}(e)\downarrow = f(e)$. Because $\neg$DOM holds relative to $A \oplus f$, there is a function $g \in \mathcal{S}$ such that $s(n) \leq g(n)$ $\mathcal{M}$-infinitely often. But then the function $h \in \mathcal{S}$ defined for all e via

$$h(e) = \begin{cases} \Phi^A_{e,g(e)}(e) & \text{if } \Phi^A_{e,g(e)}(e)\downarrow, \\ 0 & \text{otherwise,} \end{cases}$$

is total and $\mathcal{M}$-infinitely equals f. This contradiction concludes the proof. □

To conclude this section, we will consider the role of MED in models of $\neg\mathsf{I}\Sigma^0_2$. In the presence of $\mathsf{B}\Sigma^0_2$, we obtain the below result; however, as was the case for Proposition 15, this could be seen partly as a consequence of our definition of *finite* for weakly represented families. What the situation looks like in the absence of $\mathsf{B}\Sigma^0_2$ is left as an open question. We begin with a lemma. It is essentially about purely first-order models, but we phrase it in terms of topped models.

Lemma 21. *Suppose $\mathcal{M} \models \mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \neg\mathsf{I}\Sigma^0_2$ is topped by A. Let Φ^A be universal in the sense of Lemma 2, suppose $c \in M$ is large enough that $\{e < c \colon \Phi^A_e$ is not total$\}$ is not $\mathcal{M}$-finite, and define a partial function $u(e, x)$ via*

$$u(e, x) = \min\{s \colon (\forall y < x)\Phi^A_{e,s}(y)\downarrow\}.$$

Then for each total Φ^A_e there is an $i < c$ such that $u(e, x) < u(i, x)$ $\mathcal{M}$-infinitely often.

Proof. Suppose for a contradiction there is a Φ^A_e which is total and such that

$$(\forall i < c)(\exists d)(\forall x > d)[\text{either } u(i, x)\uparrow \text{ or } u(i, x) < u(e, x)].$$

Using $\mathsf{B}\Sigma^0_2$ to exchange the first two quantifiers, it follows that

$$(\exists d)(\forall i < c)(\forall x > d)[\text{either } u(i, x)\uparrow \text{ or } u(i, x) < u(e, x)].$$

But this means, with knowledge of c, d and e, the set $\{i < c \colon \Phi^A_i$ is total$\}$ is Π^0_1 and hence $\mathcal{M}$-finite, a contradiction. □

The next proposition shows that for topped models satisfying $\mathsf{B}\Sigma^0_2 + \neg\mathsf{I}\Sigma^0_2$, the axiom $\neg$AVOID coincides with MED. Topped models satisfy that there is an oracle A in the second order part such that every function is A-recursive. Then the A-recursive functions form a weakly represented family and thus all functions are in this family, thus this family witnesses $\neg$AVOID. This motivates the following proposition which shows that under $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \neg\mathsf{I}\Sigma^0_2$ topped models satisfy MED.

Proposition 22. *Every topped model of $\mathsf{RCA}_0 + \mathsf{B}\Sigma^0_2 + \neg\mathsf{I}\Sigma^0_2$ satisfies* MED.

Proof. Fix $\mathcal{M}$, A, Φ, u, c as in the Lemma. Assume (perturbing the definition if necessary) that u is injective, so that we can list all its values in strictly ascending order:

$$u(e_0, x_0) < u(e_1, x_1) < u(e_2, x_2) < \cdots.$$

We define by stages a set $(g_i)_{i<c}$ of partial functions; at stage $s = u(e_i, x_i)$, we define $g_i(x_i)$. Begin by forming the set $E = \{e < c \colon \Phi^A_{e,s}(x)\downarrow$ and $(\forall j \neq i) g_j(x) \neq \Phi^A_{e,s}(x)\}$. If E is empty, set $g_i(x)$ to equal some number that has not been seen previously in the construction. Otherwise, select an e which minimises $e + |\{y < x \colon \Phi_{e,s}(y) = g_i(y)\}|$. This completes the construction.

https://doi.org/10.1017/jsl.2026.10182 Published online by Cambridge University Press

We make four claims about $(g_i)_{i<c}$. The first is that if $i \neq j$, then $g_i(x) \neq g_j(x)$ for all x on which both are defined; this is direct from the construction. The second is that those g_i which are total are eventually different; this claim follows *a fortiori* from the first. The third is that g_i is total iff the domain of g_i is unbounded in M iff Φ_e^A is total; this is direct from the construction and the definition of $u(i,x)$. The fourth claim is that these g_i form a maximal such family. For a contradiction, suppose the fourth claim is false, which since the model is topped means that there is some Φ_e^A which is total and eventually different from each total g_i. That is, formally,

$$(\forall i < c)(\exists d)(\forall x > d)[\text{either } g_i(x)\!\uparrow \text{ or } g_i(x) \neq \Phi_e^A(x)].$$

Using $\mathsf{B}\Sigma^0_2$ to exchange the leading quantifiers, this becomes

$$(\exists d)(\forall i < c)(\forall x > d)[\text{either } g_i(x)\!\uparrow \text{ or } g_i(x) \neq \Phi_e^A(x)].$$

Keeping e fixed, let i be as provided by the Lemma. Consider $n(x) = e + |\{y < x \colon \Phi^A_{e,s}(y) = g_i(y)\}|$, the function used during the construction of g_i. Clearly n is nondecreasing; and the equation above tells us is that $n(x)$ reaches some $\mathcal{M}$-finite limit N, after some stage $s = \langle i, x\rangle$ of the construction. Then at each stage $t = \langle i, x'\rangle$ that is greater than s, we know that $\Phi_e^A(x')$ was not chosen to equal $g_i(x')$. Since i is given by the Lemma, we know that $u(e,x) < u(i,x)$ for $\mathcal{M}$-infinitely many x; hence at $\mathcal{M}$-infinitely many stages, there was another $e' \in E$ with

$$e' + |\{y < x \colon \Phi_{e',s}(y) = g_i(y)\}| < N.$$

But this is possible only if $e' < N$, and thus no more than $N - e'$ times. This contradicts the pigeonhole principle, and proves the fourth claim. Therefore $(g_i)_{i<c}$ is a MED family, as required. □

7. Hyperimmunity in Reverse Mathematics

The following principle is a reverse mathematics formalisation of the existences of hyperimmune degrees using weakly represented families.

Statement 23 (Hyperimmunity principle, HI)**.** *Given any weakly represented family $\mathcal{F}$ of functions, there exists a function g such that for all $f \in \mathcal{F}$, there are unboundedly many $x \in M$ such that $g(x) > f(x)$; in other words, there is no $f \in \mathcal{F}$ such that g is almost everywhere less than f.*

The following statement is immediate.

Proposition 24. $\mathsf{RCA}_0 \vdash \mathsf{DOM} \to \mathsf{HI}$. □

The reverse direction does not hold, as this example shows.

Example 25. *Consider the ω-model whose second-order part $\mathcal{S}$ consists of all second-order objects whose Turing degree is low for Martin-Löf random. These oracles form a Turing ideal not containing any high Turing degree,*[1] *thus* DOM *does not hold in this ω-model. Furthermore, every set in $\mathcal{S}$ is Δ_2 and $\mathcal{S}$ is not topped. Thus for every $A \in \mathcal{S}$ there is a $B \in \mathcal{S}$ with $B \geq_T A$ hyperimmune relative to A. Thus* HI *holds.*

The following theorem confirms the separation of these two notions with an alternative proof; it separates DOM from HI by showing that, whereas $\mathsf{DOM} + \mathsf{B}\Sigma^0_2$ implies full arithmetical induction as observed in Theorem 6, $\mathsf{HI} + \mathsf{I}\Sigma^0_2$ does not even imply $\mathsf{I}\Sigma^0_3$.

Theorem 26. $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2 + \mathsf{HI}$ *is Π^1_1-conservative over* $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$.

Proof. Fix a model $\mathcal{M} = (M, \mathcal{S})$ of $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$ topped by a set $A_0 \in \mathcal{S}$. Applying the Low Basis Theorem to an M-infinite A_0-computable binary tree without A_0-computable paths we obtain a set $A_1 \in \Delta_2^{A_0} \setminus \Delta_1^{A_0}$ that is low relative to A_0 and such that $A_0 \in \Delta_1^{A_1}$.

Given any A_0-computable approximation $(A_{1,s})_{s\in M}$ to A_1, its *settling-time* f, given by

$$f(x) = \min\{s \colon (\forall s' > s)[A_{1,s'}(x) = A_{1,s}(x)]\}$$

for all x, is not dominated by any total $\Delta_1^{A_0}$ function (as otherwise $A_1 \in \Delta_1^{A_0}$ after all). Furthermore, since A_1 is low relative to A_0, all $\Sigma_2^{A_1}$ sets are $\Sigma_1^{A'_0}$ and hence regular by Lemma 7. Therefore, $\mathcal{M}_1 = \mathcal{M}_0[A_1]$ is a model of $\mathsf{I}\Sigma^0_2$.

Repeating the argument, we can obtain $\mathcal{M}_n$ for all $n < \omega$, and finally we can let $\mathcal{M}_\omega$ be the model obtained in the limit. Then $\mathcal{M}_\omega \models \mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2 + \mathsf{HI}$, as desired. □

[1] See, for instance, Nies [8, Chapter 5].

https://doi.org/10.1017/jsl.2026.10182 Published online by Cambridge University Press

8. Bi-immunity

Statement 27 (Bi-immunity principle, BI)**.** *For every weakly represented family $\mathcal{F}$ of infinite sets there is a set B such that there is no $C \in \mathcal{F}$ with $C \subseteq B$ or $C \subseteq \overline{B}$.*

We can rephrase this statement in terms of the universal sets of Lemma 2 as follows: BI states that, for every A, there exists a B for which there is no infinite Δ_1^A set $C = \Phi_e^A$ such that either $C \subseteq B$ or $C \subseteq \overline{B}$.

The following is a formalization of the well-known fact that every hyperimmune set computes a bi-immune set. Note that, in combination with our previous results, this creates a connection between BI on the one hand and DOM, MAD and induction strength on the other.

Proposition 28. $\mathsf{RCA}_0 + \mathsf{HI} \vdash \mathsf{BI}$.

Proof. Let $\mathcal{M} = (M, \mathcal{S})$ be a model of HI, and fix any $A \in \mathcal{S}$. It is sufficient to construct a set B such that no $C = \Phi_e^A$ is contained in B or $\overline{B}$.

Consider the family of use functions of the functionals Φ_e^A for every e, and let f witness HI with respect to that family. In order to decide which elements we want to put into B, we proceed as follows for each y in ascending order: Check if there exists an $e < y$ such that $\Phi^A_{e,f(y)}(y)\!\downarrow$ but such that for all $x < y$,

$$\text{either} \quad \Phi^A_{e,f(y)}(x)\!\uparrow \quad \text{or} \quad \Phi^A_{e,f(y)}(x)\!\downarrow = B(x).$$

If no such e's exist, then let $y \notin B$; and if they do exist, then pick the least such e and put y into B or $\overline{B}$ in such way as to ensure $B(y) \neq \Phi_e^A(y)$. By $\mathsf{I}\Sigma^0_1$, every total Φ_e^A eventually gets treated in this way. □

The converse result does not hold. This can either be shown directly by constructing a model of RCA_0 that satisfies BI but not HI, or by employing the following conservation result.

Theorem 29. BI *is Π^1_1-conservative over* RCA_0.

The proof is an imitation of that used by Harrington [9, IX.2] in proving that WKL_0 is Π^1_1-conservative over RCA_0.

Proof. Suppose that $\mathcal{M} = (M, \mathcal{S})$ is countable model of RCA_0 topped by $A \in \mathcal{S}$. If we can construct an $A_1 \subseteq M$ such that $\mathcal{M}_1 = \mathcal{M}[A_1] \models \mathsf{I}\Sigma^0_1$ and such that A_1 is bi-immune relative to A (in the sense that for no e is Φ_e^A the characteristic function of an $\mathcal{M}$-infinite set which is either a subset of A_1, or a subset of $\overline{A_1}$), then we can iterate the construction to obtain $\mathcal{M}_2 = \mathcal{M}_1[A_2]$, and so on, until we obtain in the limit a model $\mathcal{M}_\omega$ of $\mathsf{RCA}_0 + \mathsf{BI}$.

We construct A_1 as the limit of a sequence of initial segments $(\sigma_i)_i$, as follows. To begin, let $(B_i)_{i<\omega}$ be an enumeration of all $\mathcal{M}$-infinite A-computable sets, and let $(c_i)_{i<\omega}$ be a cofinal sequence in M, both ordered by the true natural numbers ω.

- At stage 0, let σ_0 be the empty string.
- At stage $s+1 = 2i+1$ for some i, let σ_{s+1} be the shortest extension of σ_s such that σ_{s+1} disagrees with (the characteristic function of) B_i.
- At stage $s+1 = 2i+2$ for some i, let σ_{s+1} be the shortest extension of σ_s which maximises the size of $\{e < c_i : \Phi_e^A(e)\!\downarrow\}$.

The odd stages ensure bi-immunity, and the even stages guarantee that $\mathsf{I}\Sigma^0_1$ holds. □

References

[1] Peter A. Cholak, Carl G. Jockusch and Theodore A. Slaman. *On the strength of Ramsey's theorem for pairs*, Journal of Symbolic Logic, 66(1):1–55, 2001.

[2] Chi Tat Chong, Lei Qian, Theodore A. Slaman, and Yue Yang. Σ_2 *induction and infinite injury priority arguments: III. Prompt sets, minimal pairs and Shoenfield's conjecture*, Israel Journal of Mathematics, 121(1):1–28, 2001.

[3] Rupert Hölzl, Sanjay Jain and Frank Stephan, *Inductive inference and reverse mathematics*, Annals of Pure and Applied Logic, 167:1242–1266, 2016.

[4] Rupert Hölzl, Dilip Raghavan, Frank Stephan, and Jing Zhang. *Weakly represented families in reverse mathematics*, in *Computability and Complexity*, Lecture Notes in Computer Science 10010, pages 160–187, Springer, 2017

[5] Laurie A. Kirby and Jeff B. Paris. Σ_n*-collection schemas in arithmetic*, in *Proceedings of the Logic Colloquium '77, Wrocław*, pages 199–209, North Holland, 1978.

https://doi.org/10.1017/jsl.2026.10182 Published online by Cambridge University Press

[6] Bjørn Kjos-Hanssen, Wolfgang Merkle, and Frank Stephan. *Kolmogorov complexity and the recursion theorem*, in *Proceedings of the 23rd Symposium on Theoretical Aspects of Computer Science, Marseille*, Lecture Notes in Computer Science 3884, pages 149-161, Springer, 2006.
[7] Donald A. Martin. *Classes of recursively enumerable sets and degrees of unsolvability*, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, 12:295-310, Wiley, 1966.
[8] André Nies, *Computability and Randomness*, Oxford University Press, 2009.
[9] Stephen G. Simpson. *Subsystems of Second Order Arithmetic (second edition)*, Cambridge University Press, 2006.
[10] C. E. M. Yates. *On the degrees of index sets II*, Transactions of the American Mathematical Society, 135:249–266, American Mathematical Society, 1969.

School of Physical and Mathematical Sciences, Nanyang Technological University; Singapore 637371
E-mail address: david.r.belanger@gmail.com

Department of Mathematics, National University of Singapore, Singapore 119076
E-mail address: chongct@math.nus.edu.sg
URL: https://blog.nus.edu.sg/chongct/

Deutsche Bundesbank, Bavarian Regional Headquarters, Ludwigstrae 13, 80539 Munich, Germany
E-mail address: r@hoelzl.fr
URL: https://hoelzl.fr

Department of Mathematics, National University of Singapore; Singapore 119076
E-mail address: fstephan@nus.edu.sg
URL: https://www.comp.nus.edu.sg/~fstephan/

https://doi.org/10.1017/jsl.2026.10182 Published online by Cambridge University Press