A mechanical method for the solution of second-order linear differential equations. By E. C. BULLARD, B.A., Clare College, and P. B. MOON, B.A., Sidney Sussex College.

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1. *Introduction.*

The solution of second-order linear differential equations is frequently required in physical problems. Analytical methods, while always possible, are generally extremely tedious. The well-known mechanical method due to Kelvin* unfortunately requires the considerable additional labour of transforming the equation into the form

$$\frac{d}{dx}\left(P(x)\frac{dy}{dx}\right)=f(x).$$

We here describe a method directly applicable to the form

$$\frac{d^2y}{dx^2}+f(x)\,y=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

to which the standard form

$$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)\,y=0$$

may readily be reduced, $f(x)$, $p(x)$, and $q(x)$ being any real functions of x. The method depends on the equations of motion of a current-carrying coil, suspended in a magnetic field, being of the form (1). The current is varied with time in a manner determined by $f(x)$, and the motion of the system then represents (on a scale which we discuss below) a solution of equation (1).

2. *Method.*

Consider a system of magnetic moment μ suspended in a uniform magnetic field H, the axis of suspension being perpendicular both to the field and to the direction of the magnetic moment. Let $k\theta$ be the restoring couple due to the suspension, where θ is the angle between the field and the magnetic moment. Then, if θ be small, the equation of motion of the system is, to a first approximation,

$$I\frac{d^2\theta}{dt^2}+(\mu H+k)\,\theta=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

If we write

$$y=\alpha\theta, \quad x=\beta t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

* Sir W. Thomson, *Proc. Roy. Soc.* 24, p. 269 (1876).

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(1) and (2) become identical, if

$$\beta^2 f(\beta t) = (\mu H + k)/I \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Thus, if μ or H can be varied with time so that (4) is satisfied, the variation of θ with t gives, by the use of (3), a solution of the original equation (1).

3. The equation (4) may be realised in practice either by keeping μ constant (using a small permanent magnet) and varying H; or by keeping H constant and using as the suspended system

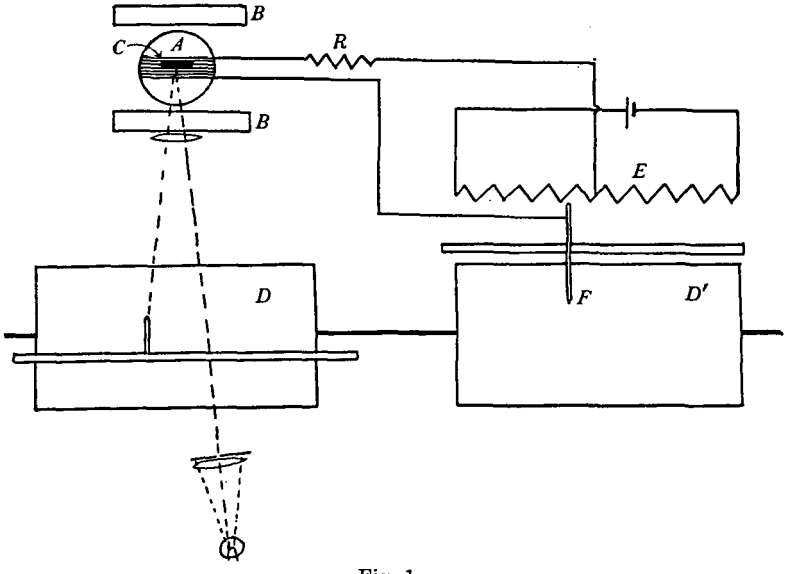


Fig. 1.

a small coil through which a variable current i may be passed. For the latter arrangement, which we have found more convenient, equation (2) becomes

$$I\frac{d^2\theta}{dt^2} + (nAiH + k)\theta = 0 \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

where n and A are the number and mean area of the turns.

The arrangement used is indicated in Fig. 1. A coil A consisting of 400 turns of No. 40 S.W.G. copper wound upon a brass tube 2·5 cm. long, and 3 cm. in diameter, is suspended by a phosphor-bronze strip, through which the current is led into the coil. The current leaves the coil through a silver spiral. The magnetic field is provided by a permanent magnet whose poles are shown at B and B. The suspended system carries a small

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mirror C which reflects a spot of light on to a rotating drum D. The motion of this spot is recorded upon the drum by the operator, who follows it with a pen movable in geometrical slides parallel to the axis of the drum.

The current to the coil is varied in the required manner by a potentiometer E, of 20 ohms resistance, whose variable contact carries a pointer F, the motion of the pointer being parallel to a second drum D' rotating synchronously with the first, and of the same diameter. A second observer follows with this pointer a curve traced upon the drum D'. In practice it is convenient to insert in series with the coil a resistance R, large compared with the potentiometer resistance. The current through the coil, and thus also the expression

$$(nAiH + k)/I,$$

is a linear function of the coordinate of the pointer.

Let $$\lambda\xi = (nAiH + k)/I \quad \text{.....(6).}$$

If now the function

$$\xi = \frac{\beta^2}{\lambda} f\left(\frac{\beta}{v}\eta\right) \quad \text{.....(7)}$$

is plotted on the drum D', where v is the peripheral velocity, and η is a coordinate measured round the drum, the quantity $\frac{2d}{\alpha} y$ will be traced on the drum D as a function of $\frac{v}{\beta} x$. The zero of ξ in (5) must be placed so that when the pointer is at $\xi = 0$, the suspended system has an indefinitely long period for a small displacement. It is of course necessary, in order that (2) may hold, that the elastic restoring force shall vanish when the plane of the coil is perpendicular to the magnetic field. In order to minimise the accuracy with which this adjustment must be made, it is desirable that the elastic restoring couple should be small in comparison with the electromagnetic torque. In practice the value of λ is most conveniently obtained by observation of the periodic time τ and the relation

$$\lambda\xi = \frac{4\pi^2}{\tau^2} \quad \text{.....(8).}$$

4. *Errors.*

For the purposes for which it is proposed to use the instrument, the solutions are usually oscillatory, and the accuracy aimed at has been a few per cent. in amplitude and phase; there would be little difficulty in reducing the errors considerably, should this be necessary. The principal sources of error are:

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(*a*) Failure of the second observer to follow the spot of light exactly. This error produces a random fluctuation of the curve, which can be much reduced by smoothing by eye.

(*b*) Failure to follow exactly the curve with the potentiometer pointer. This produces cumulative errors. These errors are much reduced by a little practice, and by running the drums slowly (about 1 cm. a sec. has been found satisfactory; a complete curve being about 50 cm. long).

(*c*) Damping caused by the air, or electromagnetically. With the coil described, the amplitude decays by less than one per cent. per vibration; the effect on the phase is quite negligible.

(*d*) The amplitude of oscillation is not indefinitely small. With a 5 cm. semi-amplitude at 15 cm. from the coil, this produces less than 0·2 per cent. change in period.

(*e*) The relation (6) will not be exactly fulfilled unless the resistance of the coil and its series resistance is indefinitely large compared with that of the potentiometer. In the instrument described, a 20 ohm potentiometer is used, and the departure from (6) is never more than one per cent.

The effects of (*c*), (*d*), and various other possible sources of error, such as non-uniformity of the field, and induced E.M.F.'s, may be tested by plotting the position of the potentiometer against the square of the reciprocal of the time of oscillation of the spot of light, when a straight line should be obtained.

5. *Boundary Conditions.*

In the preceding sections we have shown that the movement of the suspended system provides, to an approximation whose accuracy we have discussed, a solution of the equation (1).

In the solution of physical problems involving such an equation, we shall usually have to find a solution which satisfies two further conditions corresponding to the two arbitrary constants in the general solution of (1); it is clear that this may be done by suitable choice of the position θ and the angular velocity $\dot{\theta}$ of the suspended system at the instant $t = 0$. It might, therefore, be thought that to obtain a solution satisfying the given conditions we should need to find experimentally the correct initial values of θ and $\dot{\theta}$. We must, however, remember that (1) is homogeneous in y, so that if $y = F(x)$ is a solution, so also is $y = AF(x)$, where A is any constant whatever. $F(x)$ can, therefore, contain only one arbitrary constant apart from a multiplying factor which may be merged into A. The practical consequence of this is that we may satisfy our two conditions by:

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(1) Finding (by trial and error, or otherwise) the proper initial value of $\theta/\dot{\theta}$, their separate values being immaterial. (In the case of a sinusoidal solution, this is equivalent to fixing the initial phase, and choosing any convenient amplitude.)

(2) Multiplying the experimental solution by the proper constant quantity, real or imaginary. If the final solution is complex, the real and imaginary parts (which must separately satisfy (1)) may be subject to different "boundary conditions", in which case we find separately the real and imaginary solutions and take their sum.

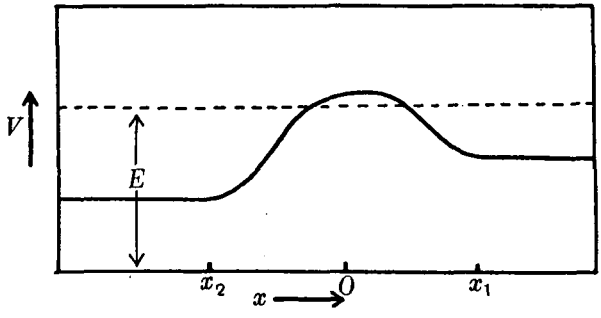


Fig. 2.

We shall now show, as an illustration of the preceding discussion, how the method may be applied to the problem of the motion of charged particles in a longitudinal electric field. Let us consider the now familiar "potential barrier problem", in which we require to find the transmission coefficient for particles moving parallel to a coordinate axis x through a field whose potential is a function of x only, and varies only near $x=0$. Let the potential energy V of the particle be given as a function of x by a curve such as Fig. 2. If E be the total energy of the particle, the appropriate wave function is $\Psi = \psi e^{2\pi i\nu t}$, where ψ does not involve the time and satisfies

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m}{h^2}(E-V)\psi = 0 \quad \text{.....(9).}$$

If V is constant when $x < x_2$ the solution in this region must be harmonic, and can be written

$$\psi_2 = A e^{2\pi i(lx+\alpha)} + Be^{-2\pi i(lx+\beta)},$$

where A and B are real.

Multiplying by $e^{2\pi i\nu t}$ we get

$$\psi_2 = A e^{2\pi i(lx+\nu t+\alpha)} + Be^{2\pi i(-lx+\nu t-\beta)}.$$

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The two terms represent respectively waves travelling in the negative and positive directions of x; if the electrons are incident from $+\infty$ there will be no electrons on the negative side of the barrier travelling towards $+\infty$; hence $B = 0$ and

$$\psi_2 = Ae^{2\pi i(lx+\alpha)}$$
$$= A\,[\cos 2\pi\,(lx+\alpha) + i \sin 2\pi\,(lx+\alpha)].$$

It follows that the required solution of (9) is obtained by finding two real solutions which, in the region of constant potential to the left of the barrier, have equal amplitudes but phases differing by $\pi/2$, multiplying either solution by i and adding it to the remaining solution.

The amplitudes of the two experimental real solutions will not in general be equal on the right of the barrier; if they are given by X and Y, then we have

$$\frac{\text{Particle density at } -\infty}{\text{Particle density at } +\infty} = \frac{2A^2}{X^2+Y^2},$$

and it is a simple matter to calculate the transmission coefficient T which is given by

$$\frac{2A^2}{X^2+Y^2} = \frac{T}{2-T}\sqrt{\frac{E-V_+}{E-V_-}} \quad \text{.....(10),}$$

where V_+ and V_- are the potential energies in the two constant potential regions to the right and left of the barrier.

The machine is employed for the purpose in the following manner:

The given potential function is drawn upon the drum D'. The coil is brought to rest by the use of a tapping key controlling a subsidiary magnetic field at right angles to the steady field. The apparatus is then set going, a convenient amplitude being given by a contact-maker rotating with D', which at the appropriate instant temporarily applies the transverse magnetic field. The procedure is then repeated with the contact-maker moved into a fresh position such that the initial impulse is given a quarter of a period later. The two traces may not have the same amplitude even in their initial stages, for the contact-maker cannot be relied upon to give a perfectly reproducible initial amplitude. This, however, merely involves the multiplication of one of them by a constant factor to make it comparable with the other. The transmission is then calculated from the expression (10).

In many problems (e.g. the scattering of electrons by a radial field) the initial displacement is zero. We may, then, simply set the coil at rest, give a convenient initial impulse by the contact-maker and auxiliary coils, and obtain at once the required solution.

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6. *Applications.*

Lord Kelvin, in the paper describing the application of the ball disc and cone integrator to this equation, mentions its importance in connexion with "vibrations of a non-uniform stretched cord, of a hanging chain, of water in a canal of non-uniform breadth and depth, of air in a pipe of non-uniform sectional area, conduction of heat along a bar of non-uniform section or non-uniform conductivity, Laplace's differential equation of the tides, etc. etc." To these may now be added the scattering of electrons in radially symmetrical (atomic) electric fields, the transmission of electrons through potential barriers and the calculation of atomic fields by Hartree's method.

It may be remarked that, by providing a third drum and a second potentiometer controlling the current through the auxiliary coils (which produce a field at right angles to the steady field), the equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y = F(x)$$

may be solved.

7. *Summary.*

A mechanical method of integrating a second-order differential equation, with any boundary conditions, is described and its applications are discussed.

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