J. Plasma Physics (2010), *vol.* 76, *part* 5, *pp.* 795–807. © Cambridge University Press 2009
doi:10.1017/S0022377809990328

Evolution of complex amplitudes ratio in weakly anisotropic plasma

YURY A. KRAVTSOV[1,2] and BOHDAN BIEG[2]

[1]Space Research Institute, Profsoyuznaya St. 82/34, Moscow 117997, Russia
[2]Institute of Physics, Maritime University of Szczecin, 1–2 Waly Chrobrego, Szczecin 70–500, Poland
(b.bieg@am.szczecin.pl)

(Received 14 September 2009; accepted 22 September 2009, first published online 10 November 2009)

Abstract. The equation for evolution of the complex amplitudes ratio (CAR) $\zeta = E_y/E_x$ in weakly anisotropic inhomogeneous media is derived on the basis of quasi-isotropic approximation (QIA) of the geometrical optics method. This equation is convenient for the description of electromagnetic wave polarization in magnetized plasma of thermonuclear reactors like the ITER. The equation for the CAR is in agreement with other approaches, analyzing polarization evolution in weakly anisotropic media, in particular, with the equation for complex polarization angle and, via QIA equations, with the Segre equation for Stokes vector evolution. Simple analytical solutions for the CAR, which relates to normal mode propagation in homogeneous and weakly inhomogeneous plasma, are obtained. Besides, the equation for the CAR is solved numerically to describe the phenomenon of normal wave conversion in magnetized plasma in the vicinity of the orthogonality point between the ray and the static magnetic field. In distinction to the line-averaged measurements in traditional plasma polarimetry, the phenomenon of normal wave conversion opens the way for measuring the local plasma parameters near the orthogonality point.

1. Introduction

Complex amplitudes ratio (CAR)

$$\zeta = \frac{E_y}{E_x} \tag{1}$$

is a convenient characteristic of polarization of the electromagnetic wave $\mathbf{E}e^{ikz}$, propagating in the direction of the z axis with the vector amplitude $\mathbf{E} = E_x\mathbf{i_x} + E_y\mathbf{i_y}$. The CAR (1), analyzed in detail in [1], is widely used in modern optics along with other parameters, such as traditional angular parameters of the polarization ellipse and the components of the Stokes vector, to characterize polarization [2].

The main aim of the current paper is to derive the equation for CAR evolution in weakly anisotropic media, using quasi-isotropic approximation (QIA) of geometrical optics, developed in [3–7]. Another aim is to apply the equation for CAR evolution for plasma polarimetry, which embraces a wide range of frequencies in millimeter, submillimeter, and far infrared (FIR) bands of the electromagnetic spectrum.

https://doi.org/10.1017/S0022377809990328 Published online by Cambridge University Press

The paper is organized as follows: Basic equations of QIA are presented in Sec. 2 along with the general form of the evolution equation for the CAR. Section 3 outlines the anisotropy tensor of the electric permittivity for high-frequency electromagnetic waves in magnetized plasma and specifies QIA equations for plasma, improving simultaneously the results of the previous publications [3–7]. The equation for CAR evolution in plasma is studied in Sec. 4. In Sec. 5 we present approximate solution of this equation in weakly inhomogeneous and weakly anisotropic plasma.

Finally, Sec. 6 demonstrates the ability of the equation for the CAR to describe the phenomenon of normal wave conversion in the vicinity of the orthogonality point between the ray and the static magnetic field. It is pointed out that this phenomenon forms a basis for local plasma diagnostics, having important advantages as compared with the traditional Faraday and Cotton–Mouton plasma polarimetries, which deal with rather line-averaged, and not local, measurements.

2. Quasi-isotropic approximation (QIA) and equation for evolution of complex amplitudes ratio (CAR) in weakly anisotropic media

For weakly anisotropic media the anisotropy tensor

$$v_{\alpha\beta} = \varepsilon_{\alpha\beta} - \varepsilon_0 \delta_{\alpha\beta} \tag{2}$$

by definition is small as compared with the permittivity ε_0 of the background isotropic medium. Weakness of anisotropy can be characterized by "anisotropic" small parameter μ_A, defined as

$$\mu_A = \frac{\max |v_{\alpha\beta}|}{\varepsilon_0} \ll 1. \tag{3}$$

Electromagnetic wave propagation in weakly anisotropic media is adequately described by the QIA of geometrical optics method, which was suggested first in [3] and developed in depth in [4, 5]. A short outline of the QIA is presented in [6, 7] and in the recent papers [8, 9] as well.

QIA is based on asymptotic expansion of the wave field $\mathbf{E}$ in "anisotropic" small parameter μ_A (3) along with "geometric" small parameter

$$\mu_{\mathrm{GO}} = \frac{1}{k_0 L} = \frac{\lambda_0}{2\pi L}, \tag{4}$$

where k_0 and $\lambda_0 = 2\pi/k_0$ are correspondingly a wavenumber and a wavelength in a free space and L is a characteristic scale of the medium inhomogeneity.

At the lowest (zeroth) order in a combined small parameter

$$\mu = \max(\mu_A, \mu_{\mathrm{GO}}) \tag{5}$$

QIA provides asymptotic solution of the Maxwell equations of the form

$$\mathbf{E} = \Gamma A(\mathbf{r}) \exp[ik_0 \Psi(\mathbf{r})], \tag{6}$$

where $\Psi(\mathbf{r})$ and $A(\mathbf{r})$ are the eikonal and amplitude of the scalar wave field in the isotropic inhomogeneous medium with permittivity $\varepsilon_0(\mathbf{r})$. These values obey correspondingly the eikonal and transport equations [2, 6, 7]:

$$(\nabla\Psi)^2 = \varepsilon_0, \ \mathrm{div}(A^2 \nabla\Psi) = 0. \tag{7}$$

https://doi.org/10.1017/S0022377809990328 Published online by Cambridge University Press

The eikonal $\Psi(\mathbf{r})$ can be found by the integration of refractive index $n_0 = \sqrt{\varepsilon_0}$ along the ray,

$$\Psi = \int \sqrt{\varepsilon_0} d\sigma, \tag{8}$$

whereas the amplitude A obeys the energy conservation law in a ray tube.

In the zeroth order in small parameter μ, a unit polarization vector Γ is orthogonal to the ray and can be presented as

$$\Gamma = \Gamma_1 \mathbf{e_1} + \Gamma_2 \mathbf{e_2}, \mathbf{e_1} \perp \mathbf{e_2} \perp \mathbf{l}, \tag{9}$$

where $\mathbf{l} = d\mathbf{r}/d\sigma$ is a unit vector, tangent to the ray trajectory $\mathbf{r}(\sigma)$; $d\sigma$ is an elementary arc length along the ray; and $\mathbf{e_1}$ and $\mathbf{e_2}$ are the unit vectors, orthogonal to the trajectory.

It is convenient to choose the unit vectors $\mathbf{e_{1,2}}$ as the basis vectors of the orthogonal curvilinear coordinate system, performing parallel transport of the vector wave field $\mathbf{E}$ along the ray (the so-named torsionless or rotationless coordinate system). Such vectors, introduced first by Popov [10], are widely used in the theory of open resonators [11] as well as in the theory of transverse elastic wave propagation [12]. According to [10–12], the unit vectors $\mathbf{e_{1,2}}$ obey the equations

$$\dot{\mathbf{e}}_\mathbf{1} = \mathbf{l}(\mathbf{e_1} \cdot \nabla \ln(n)), \dot{\mathbf{e}}_2 = \mathbf{l}(\mathbf{e}_2 \cdot \nabla \ln(n)), \tag{10}$$

where the overdot means derivation in arc length σ: $\dot{\mathbf{e}}_{\mathbf{1,2}} = d\mathbf{e}_{\mathbf{1,2}}/d\sigma$.

In Popov's orthogonal coordinate frame QIA equations for polarization vector Γ take the form

$$\left.\begin{aligned} \dot{\Gamma}_1 &= \frac{ik_0}{2\sqrt{\varepsilon_0}}[v_{11}\Gamma_1 + v_{12}\Gamma_2], \\ \dot{\Gamma}_2 &= \frac{ik_0}{2\sqrt{\varepsilon_0}}[v_{21}\Gamma_1 + v_{22}\Gamma_2], \end{aligned}\right\} \tag{11}$$

where $v_{\alpha\beta}$ are anisotropy vector components in Popov's coordinate system.

In distinction to original QIA equations, derived in [3–7] in the frame of non-orthogonal system, connected with the natural trihedral $(\mathbf{n}, \mathbf{b}, \mathbf{l})$, where $\mathbf{n}$, $\mathbf{b}$, and $\mathbf{l}$ are correspondingly normal, binormal, and tangent to the ray, (11) does not contain torsion of the ray because Popov's orthogonal system provides parallel, i.e. torsionless, transport of the electrical intensity vector along the ray.

For isotropic medium, where $v_{\alpha\beta} = 0$, QIA equations (11) take the simplest form:

$$\dot{\Gamma}_{1,2} = 0. \tag{12}$$

They describe parallel transport of the polarization vector along the ray in Popov's orthogonal torsionless frame. Equation (12) is equivalent to Rytov's law [13] (see also [6, 7]).

In the case of wave propagation in the inhomogeneous medium, it looks natural to define the CAR as

$$\zeta = \frac{\Gamma_2}{\Gamma_1}. \tag{13}$$

Differentiating (13) along the ray, one has

$$\dot{\zeta} = \frac{d}{d\sigma}\left(\frac{\Gamma_2}{\Gamma_1}\right) = \frac{\dot{\Gamma}_2\Gamma_1 - \dot{\Gamma}_1\Gamma_2}{\Gamma_1^2}. \tag{14}$$

https://doi.org/10.1017/S0022377809990328 Published online by Cambridge University Press

Using derivatives $\dot{\Gamma}_1$ and $\dot{\Gamma}_2$ from QIA equations (11), we arrive to the following equation, describing the evolution of the CAR in weakly anisotropic medium:

$$\dot{\zeta} = \frac{ik_0}{2\sqrt{\varepsilon_0}}[v_{21} + (v_{22} - v_{11})\zeta - v_{12}\zeta^2]. \tag{15}$$

This basic equation will be analyzed in detail below (Sec. 4).

The substitution $\zeta = \tan\gamma$ transforms (15) into the equation for complex polarization angle γ [3, 9], which in turn is equivalent to the equations for traditional angular parameters of polarization ellipse [9] and to the Segre equation for Stokes vector evolution [14]. The latter can also be derived in the frame of the QIA approach [8]. Thus, (15) for the CAR agrees with other approaches, describing polarization evolution in weakly anisotropic media.

3. QIA and CAR equations for weakly anisotropic plasma

As is well known, at sufficiently high frequencies magnetoactive plasma manifests properties of weakly anisotropic medium. According to (2), the anisotropy tensor $v_{\alpha\beta}$ can be obtained by subtracting the isotropic part $\varepsilon_0\delta_{\alpha\beta}$ from the full electric permittivity tensor $\varepsilon_{\alpha\beta}$. The permittivity tensor for high-frequency electromagnetic waves in non-relativistic cold plasma is of the form [15]

$$\varepsilon_{11} = 1 - \frac{v}{1-u}(1 - u\sin^2\alpha_\perp\cos^2\alpha_{||}), \quad \varepsilon_{22} = 1 - \frac{v}{1-u}(1 - u\sin^2\alpha_\perp\sin^2\alpha_{||}),$$
$$\varepsilon_{12} = \varepsilon_{12}^* = i\frac{v\sqrt{u}}{1-u}\cos\alpha_\perp + \frac{vu}{1-u}\sin^2\alpha_\perp\sin\alpha_{||}\cos\alpha_{||}. \tag{16}$$

Here u and v are standard plasma parameters. "Magnetic" parameter

$$u = \frac{\omega_c^2}{\omega^2} = \left(\frac{eB_0}{mc\omega}\right)^2 \tag{17}$$

equals the squared ratio of the electron cyclotron frequency ω_c to the signal frequency ω. In FIR and submillimeter frequency bands, used for plasma diagnostics in the thermonuclear reactors, this parameter is small enough, $u \ll 1$, and can be used as small parameter μ_A in frame of QIA. In turn the parameter

$$v = \frac{\omega_p^2}{\omega^2} \equiv \frac{4\pi e^2 N_e}{m\omega^2} \tag{18}$$

is squared plasma frequency ω_p, normalized to the signal frequency ω. Plasma parameters u and v frequently appear as correspondingly Y^2 and X. Here e and m are electron charge and mass respectively; N_e is electron density. Furthermore, $\mathbf{B}_0$ is static magnetic field; $\alpha_{||}$ is an angle between the ray and the magnetic vector $\mathbf{B}_0$ (see Fig. 1); $\alpha_\perp$ is an angle between the unit vector $\mathbf{e_1}$ of Popov's coordinate system the and transverse (with respect to the ray) component $\mathbf{B}_{0\perp}$. Axes ξ_1 and ξ_2 correspond here to the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ of the curvilinear Popov orthogonal system.

Assuming the isotropic part ε_0 to be the permittivity of isotropic plasma,

$$\varepsilon_0 = 1 - v, \tag{19}$$

https://doi.org/10.1017/S0022377809990328 Published online by Cambridge University Press

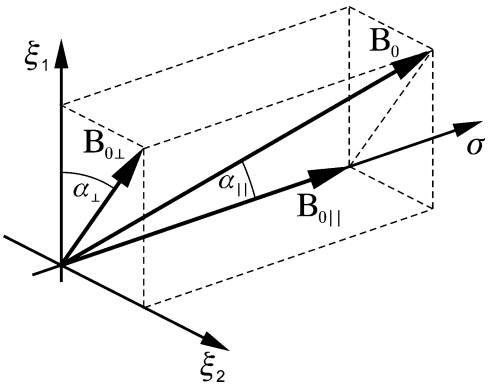


Figure 1. Vector of the static magnetic field $\mathbf{B}_0$ in the frame of Popov's orthogonal coordinate system.

and subtracting $\varepsilon_0\delta_{\alpha\beta}$ from the full dielectric tensor (16), we arrive at the following expressions for the components of the anisotropy tensor $v_{\alpha\beta}$:

$$v_{\alpha\beta} = V\begin{bmatrix} -u(\sin^2\alpha_{||}\sin^2\alpha_{\perp} + \cos^2\alpha_{||}) & i\sqrt{u}\cos\alpha_{||} + u\sin^2\alpha_{||}\sin\alpha_{\perp}\cos\alpha_{\perp} \\ -i\sqrt{u}\cos\alpha_{||} + u\sin^2\alpha_{||}\sin\alpha_{\perp}\cos\alpha_{\perp} & -u(\sin^2\alpha_{||}\cos^2\alpha_{\perp} + \cos^2\alpha_{||}) \end{bmatrix}, \tag{20}$$

where one more plasma parameter is involved, namely

$$V = \frac{v}{1-u} \approx v + uv. \tag{21}$$

It is convenient to write down QIA equations (11) via plasma parameters $\Omega_1, \Omega_2, \Omega_3$, involved first by Segre and widely used in plasma polarimetry (see [14] and the references therein):

$$\left.\begin{aligned} \Omega_1 &= \frac{k_0 uV}{2\sqrt{1-v}}\sin^2\alpha_{||}\cos 2\alpha_{\perp} = \Omega_{\perp}\cos 2\alpha_{\perp} = \Omega_{\perp 0}\sin^2\alpha_{||}\cos 2\alpha_{\perp}, \\ \Omega_2 &= \frac{k_0 uV}{2\sqrt{1-v}}\sin^2\alpha_{||}\sin 2\alpha_{\perp} = \Omega_{\perp}\sin 2\alpha_{\perp} = \Omega_{\perp 0}\sin^2\alpha_{||}\sin 2\alpha_{\perp}, \\ \Omega_3 &= \frac{k_0\sqrt{u}V}{\sqrt{1-v}}\cos\alpha_{||}. \end{aligned}\right\} \tag{22}$$

In addition to parameters (22), which form a vector $\Omega = (\Omega_1, \Omega_2, \Omega_3)$, the following combinations of plasma parameters are handy:

$$\Omega_{\perp} = \sqrt{\Omega_1^2 + \Omega_2^2} = \frac{k_0 uV}{2\sqrt{1-v}}\sin^2\alpha_{||} = \Omega_{\perp 0}\sin^2\alpha_{||}, \Omega_{\perp 0} = \frac{k_0 uV}{2\sqrt{1-v}} \tag{23}$$

and

$$\Omega = |\Omega| = \sqrt{\Omega_1^2 + \Omega_2^2 + \Omega_3^2} \equiv \sqrt{\Omega_{\perp}^2 + \Omega_3^2}. \tag{24}$$

Parameter Ω_3 linearly depends on magnetic field B_0 and describes the Faraday effect. In turn the terms $\Omega_{1,2}$, $\Omega_{\perp}$, and $\Omega_{\perp 0}$ are quadratic in B_0 and correspond to the Cotton–Mouton effect [2, 14].

https://doi.org/10.1017/S0022377809990328 Published online by Cambridge University Press

Using plasma parameters (22)–(24) one can present QIA equations (11) as

$$\left.\begin{aligned}\dot{\Gamma}_1 &= -\frac{i}{2}(2\Omega_0 - \Omega_\perp - \Omega_1)\Gamma_1 + \frac{1}{2}(i\Omega_2 - \Omega_3)\Gamma_2,\\ \dot{\Gamma}_2 &= \frac{1}{2}(i\Omega_2 + \Omega_3)\Gamma_1 - \frac{i}{2}(2\Omega_0 - \Omega_\perp + \Omega_1)\Gamma_2.\end{aligned}\right\} \tag{25}$$

These equations differ from the original QIA equations in [3–7] in the more accurate description of the static magnetic field $\mathbf{B}_0$ in the orthogonal coordinate system $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{l})$.

It is worth noticing that QIA equations in [8] and [9] were derived for an alternative choice of the isotropic part ε_0, i.e. $\varepsilon_0 = 1 - V$ instead of $\varepsilon_0 = 1 - v$ above. Therefore they have a form slightly different from that in (25). Small difference uv between parameters v and V (see (21)) results in some distinction between the corresponding anisotropy tensors. However, in agreement with the general theory of wave propagation in weakly anisotropic media [3–5] the wave field (6) is not sensitive to the first-order variations in the isotropic part ε_0, so that QIA equations, in answer to $\varepsilon_0 = 1 - v$ and $\varepsilon_0 = 1 - V$, lead to the equivalent wave fields.

Equation (15) for the CAR, answering QIA equations (25), takes the form

$$\dot{\zeta} = \frac{1}{2}[(i\Omega_2 + \Omega_3) - 2i\Omega_1\zeta - (i\Omega_2 - \Omega_3)\zeta^2] \tag{26}$$

or, after regrouping the terms,

$$\dot{\zeta} = \frac{1}{2}[i\Omega_2(1 - \zeta^2) - 2i\Omega_1\zeta + \Omega_3(1 + \zeta^2)]. \tag{27}$$

4. Evolution of the CAR in the homogeneous weakly anisotropic plasma

Equation (27) admits the exact analytic solution in the case of homogenous plasma. Let us present this equation as

$$\dot{\zeta} = \frac{-(i\Omega_2 - \Omega_3)}{2}(\zeta - \zeta_+)(\zeta - \zeta_-), \tag{28}$$

where

$$\zeta_\pm = \frac{-\Omega_1 \pm \Omega}{\Omega_2 + i\Omega_3} \tag{29}$$

are the roots of quadratic polynomial on the right side of (27). In the homogeneous medium the roots $\zeta_\pm$ are invariants of (27) because

$$\dot{\zeta} = 0 \,\text{at}\, \zeta = \zeta_\pm. \tag{30}$$

These invariants correspond to the normal waves in homogeneous plasma

$$\Gamma_\pm = \mathbf{e}_\pm \exp(iN_\pm\sigma), \tag{31}$$

where

$$N_\pm = \frac{1}{2}(\Omega_\perp \pm \Omega) - \Omega_{\perp 0} \tag{32}$$

are propagation constants; $\mathbf{e}_\pm$ are complex eigenvectors,

$$\mathbf{e}_\pm = \frac{(\mathbf{e}_1 + \zeta_\pm\mathbf{e}_2)}{\sqrt{1 + |\zeta_\pm|^2}} = p_\pm(\mathbf{e}_1 + \zeta_\pm\mathbf{e}_2); \tag{33}$$

https://doi.org/10.1017/S0022377809990328 Published online by Cambridge University Press

and $p_{\pm} = 1/\sqrt{1 + |\zeta_{\pm}|^2}$ are normalizing factors, providing $|\mathbf{e}_{\pm}| = 1$. Here and in the rest of the paper the symbol $\pm$ means nothing more than the signs before Ω in the characteristic equation (29).

One can make sure that the CARs $\zeta_{\pm}$ satisfy the condition

$$\zeta_+\zeta_-^* = -1, \tag{34}$$

which corresponds to the condition of orthogonality of normal modes (31):

$$(\Gamma_+ \cdot \Gamma_-^*) \propto (\mathbf{e}_+ \cdot \mathbf{e}_-^*) \propto (\mathbf{e}_1 + \zeta_+\mathbf{e}_2)(\mathbf{e}_1 + \zeta_-^*\mathbf{e}_2) = 1 + \zeta_+\zeta_-^* = 1 - 1 = 0. \tag{35}$$

This property of the CAR, which was established in [1], can be illustrated by two examples. In the case of pure Faraday effect, when the Faraday term Ω_3 exceeds the Cotton–Mouton term $\Omega_\perp$, $\Omega_3 \gg \Omega_\perp$, one has $\zeta_{\pm} = \mp i$, which means that normal waves have circular polarizations. In contrast with the Faraday effect, pure Cotton–Mouton effect occurs at $\Omega_\perp \gg \Omega_3$. In this case the CAR takes the real values, answering linear polarization: for $\zeta_+ = \mathrm{tg}\alpha_\perp$ the electrical vector is parallel to the static magnetic field, $\mathbf{E}_0 || \mathbf{B}_0$, and for $\zeta_- = -\mathrm{ctg}\alpha_\perp$ it is directed perpendicularly to $\mathbf{B}_0$, $\mathbf{E} \perp \mathbf{B}_0$. For both the Faraday and Cotton–Mouton effects the CAR satisfies the condition of orthogonality (34).

In the general case, (27) can be presented in the form

$$\frac{d\zeta}{(\zeta - \zeta_+)(\zeta - \zeta_-)} = \frac{-(i\Omega_2 - \Omega_3)}{2} d\sigma \tag{36}$$

or

$$A\left(\frac{1}{\zeta - \zeta_+} - \frac{1}{\zeta - \zeta_-}\right) d\zeta = -\frac{i}{2}(\Omega_2 + i\Omega_3) d\sigma, \tag{37}$$

where

$$A = \frac{1}{\zeta_+ - \zeta_-} = \frac{(\Omega_2 + i\Omega_3)}{2\Omega}. \tag{38}$$

It follows from (37) that

$$\ln\left(\frac{\zeta - \zeta_+}{\zeta - \zeta_-}\right) = \ln\left(\frac{\zeta_0 - \zeta_+}{\zeta_0 - \zeta_-}\right) e^{-i\Omega\sigma} \tag{39}$$

or

$$\zeta = \frac{\zeta_+ - \zeta_-[(\zeta_0 - \zeta_+)/(\zeta_0 - \zeta_-)]e^{-i\Omega\sigma}}{1 - [(\zeta_0 - \zeta_+)/(\zeta_0 - \zeta_-)]e^{-i\Omega\sigma}}. \tag{40}$$

According to (40), the CAR oscillates with the characteristic spatial frequency Ω, which is the difference between two eigenvalues (32). This frequency coincides with Ω_3 for pure Faraday effect and with $\Omega_\perp$ for pure Cotton–Mouton effect. Solution (40) can also be obtained from the representation of the polarization vector Γ as a superposition of two normal modes:

$$\Gamma = a_+\Gamma_+ + a_-\Gamma_-. \tag{41}$$

By virtue of (31) and (41), the complex polarization ratio (13) takes the form

$$\zeta = \frac{a_+p_+\zeta_+e^{i\Omega_+\sigma} + a_-p_-\zeta_-e^{i\Omega_-\sigma}}{a_+p_+e^{i\Omega_+\sigma} + a_-p_-e^{i\Omega_-\sigma}} = \frac{\zeta_+ + \zeta_-Se^{-i\Omega\sigma}}{1 + Se^{-i\Omega\sigma}}, \tag{42}$$

https://doi.org/10.1017/S0022377809990328 Published online by Cambridge University Press

where

$$S = \frac{a_- p_-}{a_+ p_+} = \frac{a_-}{a_+} \frac{\sqrt{1 + |\zeta_+|^2}}{\sqrt{1 + |\zeta_-|^2}}. \tag{43}$$

Solutions (40) and (42) are equivalent to each other: substitution of the initial condition $\zeta_0 = (\zeta_+ + \zeta_- S)/(1 + S)$ into (40) leads to $S = -(\zeta_0 - \zeta_+)/(\zeta_0 - \zeta_-)$ and thereby converts (40) into (42).

5. Evolution of the CAR in weakly inhomogeneous and weakly anisotropic plasma

In a weakly inhomogeneous plasma, parameters Ω_j slowly change and normal waves start interacting. This interaction happens to be negligibly weak if the characteristic scale L of plasma inhomogeneity is large enough as compared with the spatial "beating" length $l_b \sim 2\pi/\Omega$ between normal modes [6, 7]:

$$L \gg l_b \text{ or } L\Omega \gg 2\pi. \tag{44}$$

In fact, inequality (44) requires the phase difference between modes, $\Delta\varphi = \int_0^L \Omega d\sigma$, to be large at distance L. In these conditions one can speak about *non-interacting* (independent) normal modes. Amplitudes $a_\pm$ of such modes remain practically the same as for homogeneous plasma. Therefore in the case of independent normal modes in the inhomogeneous plasma the CAR is given by the following relation, which is similar to (43), with only the phase difference $\Delta\varphi = \Omega\sigma$ being substituted by $\Delta\varphi = \int_0^\sigma \Omega d\sigma$:

$$\zeta = \frac{\zeta_+ + \zeta_- S e^{-i\int \Omega d\sigma}}{1 + S e^{-i\int \Omega d\sigma}}. \tag{45}$$

6. Interaction of normal modes near the orthogonality point

Circularly polarized normal modes in weakly anisotropic plasma effectively interact with each other near the orthogonality point $\sigma_\perp$, where the ray is perpendicular to the magnetic line: $\alpha_{||} = \pi/2$. In the vicinity of the orthogonality point the propagation constants (32) of normal modes are sufficiently small and close to each other. Interaction of circularly polarized normal modes near the orthogonality point was studied in [16, 17] on the basis of QIA equations, written on circular basis. Analysis, performed in [16, 17] and developed in depth in [4, 5], resulted in analytical expression for the coefficient of conversion. Here we describe the phenomenon of normal mode conversion on the basis of (27) for the CAR.

Following [16, 17] and [4, 5] let us linearize $\cos\alpha_{||}$ near the orthogonality point $\sigma_\perp = 0$, where $\alpha_{||} = \pi/2$:

$$\cos\alpha_{||}(\sigma) \approx -\frac{\sigma}{\rho} + O\left(\frac{\sigma}{\rho}\right)^3. \tag{46}$$

The minus sign in (46) means that $\cos\alpha_{||}$ is positive at $\sigma < 0$, that is before the orthogonality point $\sigma_\perp = 0$, and becomes negative at $\sigma > 0$, that is after crossing the orthogonality point. Parameter ρ in (46) characterizes spatial scale of the static

https://doi.org/10.1017/S0022377809990328 Published online by Cambridge University Press

magnetic field variations. Correspondingly $\sin\alpha_{||}$ in the vicinity of the orthogonality point $\sigma_\perp = 0$ is close to unit:

$$\sin\alpha_{||}(\sigma) \approx 1 - O\left(\frac{\sigma}{\rho}\right)^2. \tag{47}$$

Besides, in analogy with [4, 5] we introduce a new parameter in (27), namly

$$G = \frac{k_0 v\sqrt{u}}{2\sqrt{1-v}}, \tag{48}$$

which is connected with plasma parameters Ω_3 and $\Omega_\perp$ by relations $\Omega_3 = 2G\cos\alpha_{||}$ and $\Omega_\perp = G\sqrt{u}$.

In frame of linear approximation (46), the equation for the CAR (27) takes the form

$$\dot{\zeta} = \frac{G}{2}\left[i(\sqrt{u}\sin 2\alpha_\perp)(1-\zeta^2) - 2i(\sqrt{u}\cos 2\alpha_\perp)\zeta - 2\frac{\sigma}{\rho}(1+\zeta^2)\right]. \tag{49}$$

Similarly [4, 5] we involve the dimensionless parameter of interaction, $p = Gu\rho$, and the dimensionless arc length $\xi = \sigma\sqrt{G/\rho}$ (we hope that the readers will be able to distinguish the dimensionless arc length ξ from the dimensional coordinates ξ_1 and ξ_2 given in Fig. 1). Then (49) can be rewritten in the dimensionless form,

$$\frac{\partial\zeta}{\partial\xi} = \frac{1}{2}\left[i(\sqrt{p}\sin 2\alpha_\perp)(1-\zeta^2) - 2i(\sqrt{p}\cos 2\alpha_\perp)\zeta - 2\xi(1+\zeta^2)\right]. \tag{50}$$

The first two terms in (49) and (50) have constant coefficients, whereas coefficients before the third terms linearly depend on the dimensionless arc length, ξ. It is this term that is responsible for normal mode conversion. All the terms in (49) and (50) become comparable in value at a distance $\sigma_{\text{int}} \approx -\rho\sqrt{u}$ from the orthogonality point. The scale $|\sigma_{\text{int}}| \approx \rho\sqrt{u}$ characterizes the interval of effective interaction of normal modes in the dimensional variable σ. The corresponding interval in the dimensionless variable ξ is equal to $\sqrt{p}$, i.e. $\xi_{\text{int}} = \sqrt{p}$.

Equations (49) and (50) should be solved with the initial value $\zeta_0 = \zeta(\sigma_0)$ at the starting point σ_0, which should lay outside the interaction zone $|\sigma_{\text{int}}| \approx \rho\sqrt{u}$; for instance, $|\sigma_0|$ should be as large as $|\sigma_0| \approx 10\rho\sqrt{u}$. The corresponding dimensionless starting point will be $\xi_0 = \sigma_0\sqrt{G/\rho} = 10\sqrt{p}$. It is reasonable to choose the final (output) point symmetrical to the input point σ_0: $\sigma_{\text{out}} = |\sigma_0| = 10|\sigma_{\text{int}}| = 10\rho\sqrt{u}$ or $\xi_{\text{out}} = |\xi_0| = 10|\xi_{\text{int}}| = 10\rho\sqrt{u}$.

Let the incident wave be of right-hand circular polarization, that is $\zeta_0 = \zeta(\sigma_0) = i$. According to (45), the current value of the CAR beyond the interaction zone is formed by the superposition of waves of right-hand and left-hand circular polarizations, whose eigenvalues are correspondingly $\zeta_+ = i$ and $\zeta_- = -i$. Therefore

$$\zeta = \frac{\zeta_+ + \zeta_- S e^{-i\int\Omega d\sigma}}{1 + S e^{-i\int\Omega d\sigma}} = i\frac{1 - S_c e^{-i\int\Omega d\sigma}}{1 + S_c e^{-i\int\Omega d\sigma}}, \tag{51}$$

where S_c is an amplitude ratio for circular polarizations:

$$S_c = \frac{a_-}{a_+}. \tag{52}$$

From (51) we have

$$S_c e^{-i\int\Omega d\sigma} = \frac{i-\zeta}{i+\zeta}. \tag{53}$$

https://doi.org/10.1017/S0022377809990328 Published online by Cambridge University Press

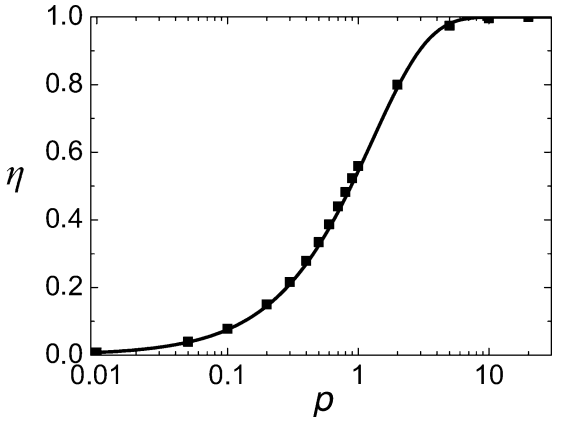


Figure 2. Dependence of the conversion coefficient η on the interaction parameter $p = Gu\rho$ for the linearized model (46). The black points correspond to the results of numerical simulations, and the continuous line answers the exponential function (57).

Let $R = \mathrm{Re}\zeta_{\mathrm{out}}$ and $I = \mathrm{Im}\zeta_{\mathrm{out}}$, where ζ_{out} is output value of the CAR. Then

$$\left|\frac{a_-}{a_+}\right|^2 = \left|S_c e^{-i\int \Omega d\sigma}\right|^2 = \left|\frac{i-\zeta_{\mathrm{out}}}{i+\zeta_{\mathrm{out}}}\right|^2 = \frac{(1-I)^2+R^2}{(1+I)^2+R^2}. \tag{54}$$

The coefficient of the conversion of the right-hand into the left-hand circular polarization is defined as

$$\eta = \frac{|a_-|^2}{|a_-|^2+|a_+|^2}. \tag{55}$$

According to (54) the conversion coefficient (55) is equal to

$$\eta = \frac{(1-I)^2+R^2}{2(1+I^2+R^2)} = \frac{1}{2} - \frac{I}{(1+I^2+R^2)}. \tag{56}$$

Equation (50) for the CAR was solved by numerical simulations, and the conversion coefficient η was determined from expression (56). Dependence of the conversion coefficient η on the interaction parameter p is presented in Fig. 2 for the case $\alpha_\perp = 0$, when the static magnetic field is directed along the axis ξ_1 of Popov's orthogonal coordinate system (Fig. 1). According to Fig. 2, the conversion coefficient η is small at small values of p, then experiences continuous growth in the interval $0.05 < p < 5$, and at last saturates at $p > 10$.

Numerical results, presented in Fig. 2 by the black points, satisfactorily agree with the exponential function,

$$\eta(p) = 1 - \exp(-\pi p/4), \tag{57}$$

which was obtained analytically in [16, 17] for the linearized model (46) (see also [4, 5]) and is shown by the solid line in Fig. 2. Exponential dependence (57) also stems from the results, obtained earlier by Cohen [18] and Zheleznyakov and Zlotnik [19], in the frame of the phase integral method.

Furthermore, we have performed calculations for the shear angles $\alpha_\perp = \pm\pi/12$, $\pm\pi/6$, $\pm\pi/4$, $\pm\pi/2$. The corresponding values of the conversion coefficient $\eta(p)$ practically are not distinguishable from the values $\eta(p)$ for $\alpha_\perp = 0$.

https://doi.org/10.1017/S0022377809990328 Published online by Cambridge University Press

The fact that the conversion coefficient $\eta(p)$ is independent on the shear angle $\alpha_\perp$ only for circular polarization of the incident wave is worthy of being noticed. Unlike circular polarization, the output vector $\mathbf{E}_{\text{out}}$ in the case of linearly polarized incident wave E_0 will be sensitive to mutual orientation of the vectors $\mathbf{E}_0$ and $\mathbf{B}_0$.

The exponential dependence (57) of the conversion coefficient $\eta(p)$ on the interaction parameter p was confirmed recently in [20] by direct numerical solution of QIA equations. Numerical simulations of the normal mode conversion were illustrated in [20] also by the analysis of Stokes vector behavior near the orthogonality point.

Equation (57) opens the way for the measurement of local plasma parameters on the basis of the phenomenon of normal mode conversion. The conversion coefficient η being measured, the local parameter p can be determined from the simple relation

$$p|_{\text{near orthogonal point}} = -\frac{4}{\pi}\ln(1-\eta). \tag{58}$$

Thus, in distinction to line-averaged measurements in traditional plasma polarimetry, the phenomenon of normal wave conversion allows the measurement of the local value of the combination of plasma parameters, $p = Gu\rho$, in the vicinity of the orthogonality point. Applicability of localized polarimetric measurements in the frame of the ITER project will be discussed in a forecoming paper [21].

The linear model (46) can be readily modified to describe the saturation of $|\cos\alpha_{||}(\sigma)|$ at $\sigma \geqslant \rho$:

$$\cos\alpha_{||}(\sigma) \approx -\frac{\sigma}{\sqrt{\rho^2+\sigma^2}}. \tag{59}$$

Correspondingly

$$\sin^2\alpha_{||}(\sigma) \approx \frac{\rho^2}{\rho^2+\sigma^2}. \tag{60}$$

For the model ((59) and (60)) of saturated growth of $\cos\alpha_{||}(\sigma)$, (27) takes the form

$$\dot{\zeta} = \frac{G}{2}\left[\left(\frac{i\rho^2\sqrt{u}}{\rho^2+\sigma^2}\right)\left[(1-\zeta^2)\sin 2\alpha_\perp - 2\zeta\cos 2\alpha_\perp\right] - 2\frac{\sigma}{\sqrt{\rho^2+\sigma^2}}(1+\zeta^2)\right]. \tag{61}$$

Figure 3 presents the results of numerical simulations for variations of the conversion coefficient η along the ray at $\alpha_\perp = 0$ and for the interaction parameter $p = Gu\rho = 0.82$, in answer to $G = 305$, $u = 0.002$, and $\rho = 5$m. According to Fig. 3, the conversion takes place in the narrow interval $\Delta\sigma$, which is comparable to the interaction interval $\sigma_{\text{int}} = \sqrt{u}\rho$, amounting to 0.23 m for the above-mentioned plasma parameters. Thus, smooth variations of plasma parameters beyond the interaction interval $\sigma_{\text{int}} = \sqrt{u}\rho$ practically do not influence normal mode conversion.

7. Conclusion

The analysis performed above has evidenced advantages of the CAR $\zeta = \Gamma_2/\Gamma_1$ for the description of electromagnetic wave polarization in weakly anisotropic media, first of all – in weakly anisotropic magnetized plasma. It is shown that the CAR satisfactorily describes the evolution of electromagnetic wave polarization in both homogeneous and weakly inhomogeneous plasmas under the conditions of independent (non-interacting) propagation of normal modes. Numerical simulation of CAR evolution has demonstrated its ability to describe normal mode conversion

https://doi.org/10.1017/S0022377809990328 Published online by Cambridge University Press

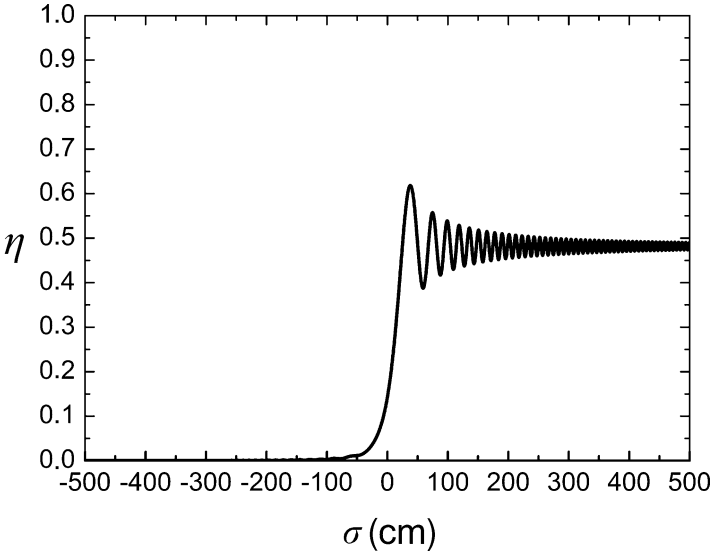


Figure 3. Evolution of the conversion coefficient η along the ray for "saturated" model (59) with the interaction parameter $p = 0.82$.

near the orthogonality point. In distinction to traditional plasma polarimetry, which deals with the line-averaged plasma parameters, conversion phenomenon promises to carry out polarimetric measurements of local plasma parameters in the vicinity of the orthogonality point.

Acknowledgements

This work, supported by the European Communities under the contract of Association EURATOM with IPPLM was carried out within the framework of the European Fusion Development Agreement. This work is supported also by grant no. 202 249535 of the Polish Ministry of Science and Higher Education.

References

[1] Huard, S. 1997 *Polarization of Light*. Paris: John Willey/Masson.
[2] Born, M. and Wolf, E. *Principles of Optics*. Oxford: Pergamon.
[3] Kravtsov, Yu A. 1969 *Sov. Phys.- Doklady* **13**, 1125.
[4] Kravtsov, Yu A., Naida, O. N. and Fuki, A. A. 1996 *Phys.-Uspekhi* **39**, 129.
[5] Fuki, A. A., Kravtsov, Yu A. and Naida, O. N. 1997 *Geometrical Optics of Weakly Anisotropic Media*. London and New York: Gordon & Breach.
[6] Kravtsov, Yu A. and Orlov, Yu I. 1990 *Geometrical Optics of Inhomogeneous Media*. Berlin: Springer.
[7] Kravtsov, Yu A. 2005 *Geometrical Optics in Engineering Physics*. Harrow, Middlesex, UK: Alpha Science International
[8] Kravtsov, Yu A., Bieg, B. and Bliokh, K. Yu 2007 *J. Opt. Soc. Am.* A **24**, 10 3388.
[9] Czyz, Z. H., Bieg, B. and Kravtsov, Yu A. 2007 *Phys. Lett*. A **368**, 101.
[10] Popov, M. M. 1969 *Vest. Leningr. Univ.* (*Bull. Leningr. Univ.*) **22**, 44.

https://doi.org/10.1017/S0022377809990328 Published online by Cambridge University Press

[11] Babich, V. M. and Buldyrev, V. S. 1990 *Short-Wavelength Diffraction Theory: Asymptotic Methods.* Berlin: Springer. (Original Russian edition in 1972 by Nauka, Moscow).

[12] Cerveny, V. 2001 *Seismic Ray Theory*. Cambridge, New York, Melbourne: Cambridge University Press.

[13] Rytov, S. M. 1938 *Dokl. Akad. Nauk SSSR* **18**, 263. (English translation in: *Topological Phase in Quantum Theory*, ed. B.I. Markovski and S.I. Vinitsky, 1989, World Scientific, Singapore).

[14] Segre, S. E. 1999 *Plasma Phys. Control. Fusion* **41**, R57.

[15] Ginzburg, V. I. 1970 *Propagation of Electromagnetic waves in Plasma.* New York: Gordon & Breach.

[16] Kravtsov, Yu A. and Naida, O. N. 1976 *Sov. Phys.-JETP* **44**, 122.

[17] Kravtsov, Yu A. and Naida, O. N. 2000 *J.Tech. Phys.* **41**, 155.

[18] Cohen, M. H. 1960 *Astrophys. J.* **131**, 664.

[19] Zheleznyakov, V. V. and Zlotnik, E.Ya. 1964 *Soviet Astron. J. AJ* **7**, 485.

[20] Kravtsov, Yu A. and Bieg, B. 2008 *Proc. SPIE* **7141**, 71410K-1, doi: 10.1117/12.822365.

[21] Kravtsov, Yu A. and Bieg, B. 2009 *Plasma Phys. Control. Fusion* (submitted).

https://doi.org/10.1017/S0022377809990328 Published online by Cambridge University Press