101.19 Approximate quadrature of the circle using a set square

In 1961, *The Mathematical Gazette* published Note 2978 under the title: 'Approximate quadrature of the circle' [1]. Its author, Louis Loynes, presented an approximate quadrature of the circle. The core of the described method is a right triangle with sides in the proportion 23:44. At the bottom of the short description of the geometrical construction was added the following Editorial Note: 'The Editor would welcome information on the history of this construction.' Yes, indeed this approach was originally proposed by a Russian engineer Edward Bing [2, 3]. His set square (Bing's Circular Square) was exhibited on Special Loan Collection of Scientific Apparatus (Science Museum, South Kensington, London) and was described in the 3rd edition of the associated catalogue (1876) [4]. Loynes is not alone in re-discovering this tool. At least two other people, in 1914 and 1918, respectively, published their 'invention' in the *Nature* [2, 5, 6].

Bing's circular square is based on the following relations. Consider the situation presented in Figure 1. For a given circle of radius r (and diameter $d = 2r$) it is possible to determine the side of square x, such that this square approximates the circle.

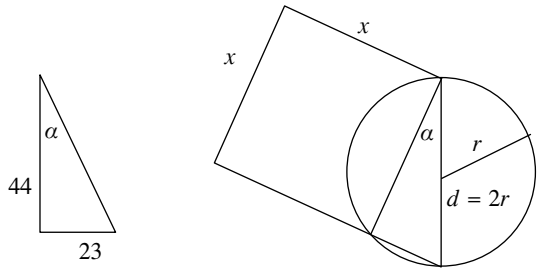


FIGURE 1: The basis of Bing's construction: a set square with sides in the ratio 23:44

The following relation holds $x = 2r\cos\alpha = d\cos\alpha$. Here the angle α is unknown. The condition on equal areas of two figures has the form $x^2 = \pi r^2$. From this relation $4r^2\cos^2\alpha = \pi r^2$ and finally $\cos\alpha = \frac{1}{2}\sqrt{\pi}$. The approximate value (estimated by Bing) for the angle is $\alpha \approx 27.6°$. In practice, the value of this angle is not needed, as the main goal is to construct a set square. For this purpose, the tangent of this angle is determined as follows:

$$\cos^2\alpha = \frac{\pi}{4}, \qquad \sin^2\alpha = \frac{4-\pi}{4},$$

and the tangent is given by the relation

$$\tan^2\alpha = \frac{4-\pi}{\pi}. \tag{1}$$

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Another derivation (proposed by an anonymous reviewer of this note) is based on the Pythagorean identity:

$$\sec^2 \alpha = 1 + \tan^2 \alpha$$

which gives (1). The approximate value is $\tan \alpha = 0.522723200877063\ldots$. This value can be approximated by a continued fraction. Nine first terms of it give [0; 1, 1, 10, 1, 1, 126, 3, 45] with the following convergents;

<numerator: 0, 1, 1, 11, 12, 23, 2910, 8753, 396795>,

<denominator: 1, 1, 2, 21, 23, 44, 5567, 16745, 759092>,

and the corresponding values are as follows:

0, 1, 0.5, 0.523809523809524, 0.521739130434783,

$\left(\frac{23}{24} =\right)$ 0.522727272727273, 0.52272319022813,

0.522723200955509, 0.522723200876837.

As a consequence, the set square with sides in the ratio 23 : 44 gives a relatively good approximate quadrature of the circle $\pi \approx 3.1415821501042$. Using Bing's set square it is possible to perform a few other geometrical tasks [3], among them to determine approximations to π, $\sqrt{\pi}$, π^2 and squaring the ellipse.

References

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