Math. Struct. in Comp. Science (2008), *vol.* 18, *pp.* 1005–1016. © 2008 Cambridge University Press
doi:10.1017/S0960129508006786 First published online 28 May 2008 Printed in the United Kingdom

Exponentiable morphisms of domains

F. CAGLIARI[†] and S. MANTOVANI[‡]

[†]*Dipartimento di Matematica, Università di Bologna,*
Piazza di Porta S. Donato 5, I-40127 Bologna, Italy
Email: cagliari@dm.unibo.it
[‡]*Dipartimento di Matematica, Università di Milano,*
Via Saldini 50, I-20133 Milano, Italy
Email: Sandra.Mantovani@mat.unimi.it

Received 9 May 2007; revised 1 February 2008

Given a map f in the category ω-**Cpo** of ω-complete posets, exponentiability of f in ω-**Cpo** easily implies exponentiability of f in the category **Pos** of posets, while the converse is not true. We investigate the extra conditions needed on f exponentiable in **Pos** to be exponentiable in ω-**Cpo** by showing the existence of partial products of the two-point ordered set $S = \{0 < 1\}$ (Theorem 2.8). Using this characterisation and the embedding through the Scott topology of ω-**Cpo** in the category **Top** of topological spaces, we compare exponentiability in each setting and find that a morphism in ω-**Cpo** that is exponentiable in both **Top** and **Pos** is exponentiable in ω-**Cpo** also. Furthermore, we show that the exponentiability in **Top** and **Pos** are independent of each other.

1. Introduction

It is a very useful fact for the application of domains to logic and computing that the category ω-**Cpo** of ω-complete posets with continuous functions is cartesian closed (Gierz *et al.* 2003). This means that the poset Y^X of continuous maps between two ω-cpos X and Y is an ω-cpo also, and that this construction gives rise to a functor $-^X$ that is right adjoint to $- \times X$. Unfortunately, this very important property is lost under 'slicing', that is, the categories ω-**Cpo**$/B$ of ω-cpos over a fixed base B are not always cartesian closed, as we will see soon. Hence, it makes sense to investigate the nature of maps f that are exponentiable in ω-**Cpo**, that is, those f for which the functor $(-) \times f$ has a right adjoint $(-)^f$. This property of exponentiability has been well investigated in the category **Pos** of partial order sets and monotone maps. In this case, exponentiability is characterised by a sort of interpolation property, a weakened version of the Giraud–Conduché result on exponentiable morphisms in the category **Cat** (see, for example, Giraud (1964), Niefield (2001) and Tholen (2000).) It is easy to see that exponentiable maps in ω-**Cpo** are exponentiable in **Pos**, but our recent characterisation of exponentiable monomorphisms in ω-**Cpo** shows that the converse is not true (Cagliari and Mantovani 2007). We now obtain, in Theorem 2.8, the extra conditions needed for f exponentiable in **Pos** to be exponentiable in ω-**Cpo**, using as a main tool the notion of a partial product (Dyckhoff and Tholen 1987).

In the category **Top** of topological spaces and continuous maps, exponentiability of morphisms is a rather complicated property, which has been well studied and characterised

https://doi.org/10.1017/S0960129508006786 Published online by Cambridge University Press

by means of many approaches: for example, by Niefield and Richter from different topological points of view (Niefield 1982; Richter 2002), and by Clementino, Hoffman and Tholen through an ultrafilter-interpolation property (Clementino *et al.* '03).

Using the Scott topology (Scott 1972), we can consider any poset and any ω-cpo as topological spaces, hence it may be interesting to compare exponentiable continuous maps in **Pos**, in ω-**Cpo** and in **Top**. Since there are posets and ω-cpos that are not core compact and, hence, not exponentiable in **Top**, it is easy to deduce that, in general, exponentiable continuous maps in **Pos** and in ω-**Cpo** are not exponentiable in **Top**. This remains true not only for objects, but also for monomorphisms, as of Cagliari and Mantovani (2007, Example 3.3) shows. On the other hand, using our characterisation theorem (Theorem 2.8), we show that if we have a continuous map f in ω-**Cpo** that is exponentiable in both **Top** and **Pos**, then f is exponentiable in ω-**Cpo** also. Furthermore, we show that exponentiability in **Pos** and **Top** are independent of each other by exhibiting an example of a continuous map between posets that is exponentiable in **Top** but not in **Pos**.

We would like to thank the anonymous referees for very helpful comments and valuable suggestions on a preliminary version of the paper.

2. Exponentiable objects in ω-**Cpo**$/B$

We are going to consider the category ω-**Cpo** of ω-chain complete posets and continuous maps (see, for example, Abramsky and Jung (1994), Markowsky (1976) and Markowsky and Rosen (1976)). We shall need some definitions and standard results about them.

Definition 2.1. A poset X in which every ω-chain has a supremum is called an ω-chain complete poset (or ω-cpo for short).

Scott (1972) tells us that posets can be considered as topological spaces when endowed with the so called Scott topology, where C is closed in X if it is a lower set closed under existing suprema of directed sets. In a similar way, we can consider the ω-Scott topology on posets (and, in particular, on ω-cpos), where C is closed in X if it is a lower set closed under existing suprema of ω-chains. Also in this topology, the closure of x in X is given by $\downarrow x = \{y \in X | y \leqslant x\}$.

— A map $f : Y \to X$ between posets (ω-cpos) is continuous with respect to the ω-Scott topologies if and only if f preserves existing suprema of ω-chains.
— A map $f : Y \to X$ between ω-cpos is a regular monomorphism in ω-**Cpo** if and only if it is a continuous order embedding, that is, a continuous map such that $x \leqslant y$ if and only if $f(x) \leqslant f(y)$.

Definition 2.2. If $f : Y \to X$ is a regular mono in ω-**Cpo**, then Y is said to be a sub-ω-cpo of X.

If X is an ω-cpo and $Y \subseteq X$, we use Y^* to denote the smallest sub-ω-cpo of X containing Y, that is, the intersection of all sub-ω-cpos of X containing Y (see, for example, Fiech (1996)).

https://doi.org/10.1017/S0960129508006786 Published online by Cambridge University Press

We will now introduce a property of Y^* that we will need later.

Lemma 2.3. If X is an ω-cpo and $Y \subseteq X$, then Y^* is uniquely determined among all sub-ω-cpos Z of X by the following properties:

1 $Y \subseteq Z$.
2 Given $h,k : Z \rightrightarrows T$, if $h = k$ on Y, then $h = k$.

Proof. Y^* has these properties by definition as the subset on which two continuous functions agree is a sub-ω-cpo (Fiech 1996). Now, let Z fulfill Properties 1 and 2. So $Y^* \subseteq Z$, and we can consider the cokernel pair $(q_1, q_2) : Z \rightrightarrows Z \bigcup_{Y^*} Z$ in ω-**Cpo** of the inclusion of Y^* in Z. Since it is obvious that $q_1 = q_2$ on Y^*, we have, by Property 2, that $q_1 = q_2$. This means that $Z = Y^*$. □

The category ω-**Cpo** is cartesian closed since for any object X, the functor $- \times X$ has a right adjoint, which we will denote by $(-)^X$, that assigns to any Y the ω-cpo Y^X of the continuous maps from X to Y with the pointwise order (Gierz *et al.* 2003). This property is related to the fact that the category **Pos** of partially order sets and monotone maps is itself cartesian closed. This is no longer true when we consider the category **Pos**/B of partially order sets over a fixed base poset B, since not every map is *exponentiable*, where exponentiable is defined as follows.

Definition 2.4. A morphism $f : X \to B$ is *exponentiable* in a category **C** with finite limits if the functor $(-) \times f : \mathbf{C}/B \to \mathbf{C}/B$ has a right adjoint $(-)^f$.

The characterisation of exponentiable morphisms in **Pos** as *convex* (or *interpolation-lifting*) monotone maps has been known for a long time as a weakened version of the Giraud–Conduché result on exponentiable morphisms in the category **Cat** (see, for example, Giraud (1964), Niefield (2001) and Tholen (2000)) where convexity is defined as follows.

Definition 2.5. A map $f : X \to B$ in **Pos** is *convex* if for $x \leqslant z$ in X and for any b with $f(x) < b < f(z)$, there exists $y \in X$ such that $x < y < z$ and $f(y) = b$.

Using similar arguments to those in Niefield (2001), we can prove the following proposition.

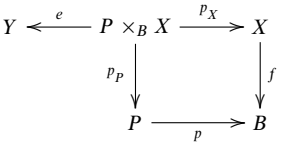
Proposition 2.6. Every exponentiable morphism in ω-**Cpo** is convex.

The condition of convexity is not sufficient. If N is the poset of natural numbers with the natural order and $\infty = \bigvee N$, the inclusion of ∞ in $N \cup \{\infty\}$ is convex, but it does not fulfill the necessary condition for exponentiability of monomorphisms given for Cagliari and Mantovani (2007, Theorem 1.10).

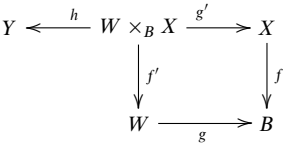
We need then to find further conditions in order to characterise the exponentiable maps amongst the convex ones. The main tool we use to obtain our result is the notion of a partial product (Dyckhoff and Tholen 1987).

https://doi.org/10.1017/S0960129508006786 Published online by Cambridge University Press

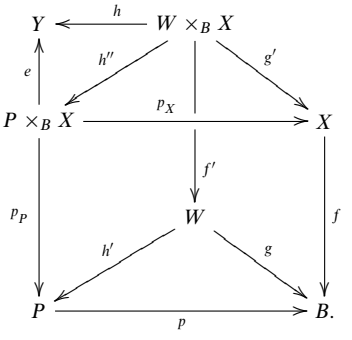
Definition 2.7. Given $f : X \to B$ and Y in a category $\mathbf{C}$ with finite limits, the *partial product* $P(f, Y)$ of Y on f is defined (when it exists) as a morphism $p : P \to B$ equipped with an 'evaluation' $e : P \times_B X \to Y$ such that the square in



is a pullback and, given a pullback diagram on f and a map $h : W \times_B X \to Y$,



there is a unique $h' : W \to P$ with $g = ph'$ and $h = eh''$, where $h'' : W \times_B X \to P \times_B X$ is given by the universal property of the pullback



The existence of partial products on f of every object Y in $\mathbf{C}$ is equivalent to exponentiability of f in $\mathbf{C}$ (Dyckhoff and Tholen 1987, Lemma 2.1).

In the category ω-**Cpo**, it is sufficient to prove the existence of partial products on f of the two-point ordered set $S = \{0 < 1\}$, since any object may be obtained as a regular subobject of a product of copies of S. In fact this is also the case in the category $\mathbf{Top_0}$ of T_0-topological spaces (Adamek *et al.* 1990), where ω-**Cpo** fully embeds. The result follows, since any topological product of copies of the continuous lattice S coincides with the product in ω-**Cpo** (Gierz *et al.* 2003), and topological embeddings between ω-cpos are regular monomorphisms in ω-**Cpo** (while the converse is not true, see, for example, the example due to Moggi in Taylor (2002)).

The extra conditions needed for f convex to be exponentiable in ω-**Cpo** are given in the next theorem, where an ω-chain $(b_i)_{i\in\mathbb{N}}$ is not eventually constant in B if, for any i, there exists $j > i$, with $b_j > b_i$.

https://doi.org/10.1017/S0960129508006786 Published online by Cambridge University Press

Theorem 2.8. Given $f : X \to B$ convex in ω-**Cpo**, the following are equivalent:

1 f is exponentiable in ω-**Cpo**.

2 Given an ω-chain $(b_i)_{i\in\mathbb{N}}$ that is not eventually constant in B, with $b = \bigvee b_i$, $\Omega = \bigcup_i \{b_i\} \cup \{b\}$ and $Z = f^{-1}(\Omega \setminus \{b\})$:

(a) $Z^* = f^{-1}(\Omega)$.

(b) Given an ω-chain $(x_n)_{n\in\mathbb{N}}$ in X, with $\bigvee x_n = x$, $f(x_n) \in \Omega$ and $f(x) = b$, the ω-Scott-closure C in Z of $\left(\bigcup \downarrow x_n\right) \bigcap Z$ coincides with $(\downarrow x) \cap Z$.

3 Given an ω-chain $(b_i)_{i\in\mathbb{N}}$ that is not eventually constant in B, with $b = \bigvee b_i$, $\Omega = \bigcup_i \{b_i\} \cup \{b\}$ and $Z = f^{-1}(\Omega \setminus \{b\})$:

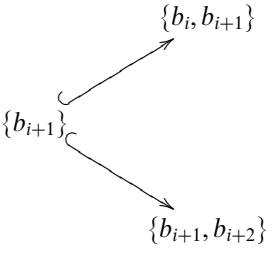
(a) $Z^* = f^{-1}(\Omega)$.

(b) If Z and Z^* are provided with the ω-Scott topology, the inclusion $j : Z \to Z^*$ is a topological embedding.

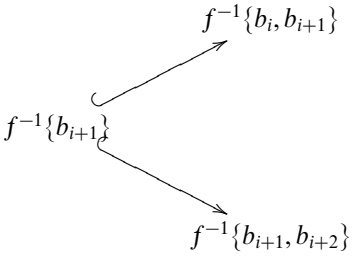
Proof.

(1 $\Rightarrow$ 3)

(a) Let f be exponentiable in ω-**Cpo**. Then Ω can be viewed as a colimit in ω-**Cpo** of the diagram



Since f is exponentiable, pulling back along f preserves colimits, so $f^{-1}(\Omega)$ turns out to be a colimit of



Given $h, k : f^{-1}(\Omega) \rightrightarrows T$ that coincide on $f^{-1}\{b_i, b_{i+1}\}$ for any i, by the universal property of colimits, we have $h = k$, so $Z^* = f^{-1}(\Omega)$ by Lemma 2.3.

(b) Let A be an ω-Scott open set of Z and let $A_i = A \cap f^{-1}\{b_i\}$. By the universal property of colimits, the characteristic functions k_i of the open sets $A_i \cup A_{i+1}$ of $f^{-1}\{b_i, b_{i+1}\}$ determine a unique $k : Z^* \to S$, which is the characteristic function of a unique open set $\hat{A}$ of Z^* such that $\hat{A} \cap Z = A$.

https://doi.org/10.1017/S0960129508006786 Published online by Cambridge University Press

(2 ⇒ 3)

We want to prove that any ω-Scott closed set C of Z has an extension to an ω-Scott closed C' of Z^* (which turns out to be the closure of C in Z^*).
We define $C' = \{t \in Z^* \mid (\downarrow t) \bigcap Z \subseteq C\}$.
C' is down closed: if $t' \leqslant t \in C'$, $(\downarrow t') \bigcap Z \subseteq (\downarrow t) \bigcap Z \subseteq C$.
Condition (b) tells us that C' is closed under suprema of ω-chains: given an ω-chain (x_n) of C', if $\bigvee x_n = x \in Z$, then $x_n \in Z$, so $x_n \in (\downarrow x_n) \bigcap Z \subseteq C$. This means that $x \in C \subseteq C'$ also, as C is closed in Z. On the other hand, if $f(x) = b$, then, since $(\downarrow x_n) \bigcap Z \subseteq C$, we have $Cl_Z \left(\bigcup (\downarrow x_n) \bigcap Z\right) = (\downarrow x) \bigcap Z \subseteq C$ by Condition (b), therefore $x \in C'$.

(3 ⇒ 2)

First, observe that Condition 3(a) implies that if an ω-Scott closed set C of Z has an extension to an ω-Scott closed $\hat{C}$ of $f^{-1}(\Omega) = Z^*$, this extension is unique, so it coincides with the closure of C in Z^*. In fact, if C' is a closed set with $C' \cap Z = C$, then $\hat{C} \cup C'$, $\hat{C} \cap C'$ are also closed sets of Z^* with the same property. If $K = (\hat{C} \cup C') \setminus (\hat{C} \cap C')$, then, since $Z^* \setminus K \subseteq Z^*$ is an ω-cpo containing Z, we have $K = \varnothing$ and $\hat{C} = C'$.
Now let x_n be an ω-chain in X with $\bigvee x_n = x$, $x_n \in f^{-1}(\Omega) = Z^*$ and $f(x) = b$. For any x_n, $(\downarrow x_n) \cap Z$ is closed in Z, and its closure in Z^* is $(\downarrow x_n) \cap Z^*$ by Condition 3(b) and the observation above. Furthermore, if $C = Cl_Z \left(\bigcup (\downarrow x_n) \bigcap Z\right)$, then C has an extension to an ω-Scott closed $\hat{C}$ of Z^*. Since $(\downarrow x_n) \cap Z \subseteq C$, we have $(\downarrow x_n) \cap Z^* \subseteq \hat{C}$, so any $x_n \in \hat{C}$, so we also have $x \in \hat{C}$ and $(\downarrow x) \cap Z^* \subseteq \hat{C}$.
On the other hand, $\hat{C} \subseteq (\downarrow x) \cap Z^*$, since $x_n \leqslant x$, so

$$(\downarrow x) \cap Z = (\downarrow x) \cap Z^* \cap Z = \hat{C} \cap Z = C\,.$$

(2 ⇒ 1)

We want to prove that f is exponentiable by showing the existence of the partial product P of S on f. As a set, $P = \{(\sigma, b) | \sigma : f^{-1}(b) \to S, \sigma \text{ continuous}\}$. We endow P with the relation $(\sigma, b) \leqslant (\sigma', b')$ given by:

(a) $b \leqslant b'$;

(b) if $x \leqslant x'$ in X with $f(x) = b, f(x') = b'$, then $\sigma(x) \leqslant \sigma'(x')$.

It is trivial to show that this relation is reflexive and symmetric, while the transitivity depends on the convexity of f. Now we want to prove that any ω-chain (σ_i, b_i) in P has a supremum. Let $b = \vee b_i$, and $\sigma : f^{-1}(b) \to S$ be such that

$$\sigma(x) = 0 \Leftrightarrow \forall\, a_i \in f^{-1}(b_i) \text{ with } a_i \leqslant x, \quad \sigma_i(a_i) = 0\,.$$

Clearly, σ is monotone and $(\sigma, b) = \vee(\sigma_i, b_i)$. We are going to prove that σ is continuous by showing that $\sigma^{-1}(0)$ is ω-Scott closed in $f^{-1}(b)$. If b_i is eventually constant, there exists $\bar{i}$ with $b_j = b_{\bar{i}}$ for all $j > \bar{i}$, so $\sigma^{-1}(0) = \bigcap_{i \geqslant \bar{i}} \sigma_i^{-1}(0)$.
So, let b_i not be eventually constant: without a substantial loss of generality, we can suppose that b_i is strictly increasing. Let x_n be an ω-chain in $f^{-1}(b)$ with

https://doi.org/10.1017/S0960129508006786 Published online by Cambridge University Press

$\sigma(x_n) = 0$. Then $(\downarrow x_n) \cap f^{-1}(b_i) \subseteq {\sigma_i}^{-1}(0)$ for any n and any i, so

$$\bigcup_n (\downarrow x_n) \cap \bigcup_i f^{-1}(b_i) \subseteq \bigcup_i {\sigma_i}^{-1}(0).$$

This implies that the function $\bar{\sigma} : \bigcup_i f^{-1}(b_i) = Z \to S$ defined by

$$\bar{\sigma}(x) = \sigma_{i_x}(x), \text{ where } b_{i_x} = f(x)$$

has value 0 on $\bigcup_n (\downarrow x_n) \bigcap Z$. Since $\bar{\sigma}$ is trivially continuous, $\bar{\sigma}$ also has value 0 on its closure

$$Cl_Z \left(\bigcup_n (\downarrow x_n) \cap Z \right) = (\downarrow x) \cap Z,$$

by Condition 2(b). Consequently, for any $a_i \in f^{-1}(b_i)$ with $a_i \leqslant x$, we have $\bar{\sigma}(a_i) = 0 = \sigma_i(a_i)$. This means that $\sigma(x) = 0$, that is, that σ is continuous.

Now we are going to prove that the evaluation map $e : P \times_B X \to X$ is continuous. If (σ_i, b_i, x_i) is an ω-chain in $P \times_B X$, then $\vee(\sigma_i, b_i, x_i) = (\vee\sigma_i, \vee b_i, \vee x_i) = (\sigma, b, x)$ and $e(\vee(\sigma_i, b_i, x_i)) = \sigma(x)$, while $\vee e(\sigma_i, b_i, x_i) = \vee\sigma_i(x_i)$. If (b_i) is eventually constant, the ω-chain (x_i) is eventually in $f^{-1}(b)$, so $e(\sigma, b, x) = \vee\sigma_i(x_i)$, since any fibre is cartesian. Let us now suppose that $\sigma(x) = 1$ and $\vee\sigma_i(x_i) = 0$, that is, $\exists\, a_{\bar{i}} \in f^{-1}(b_{\bar{i}})$ with $a_{\bar{i}} \leqslant x$, $\sigma_{\bar{i}}(a_{\bar{i}}) = 1$, while $\sigma_i(x_i) = 0$, for any i. Such an $a_{\bar{i}}$ is in $(\downarrow x) \cap Z$. Since $\sigma_i(x_i) = 0$, we have $(\downarrow x_i) \cap f^{-1}(b_i) \subseteq {\sigma_i}^{-1}(0)$ and, as before, $\bar{\sigma}$ has value 0 on

$$Cl_Z \left(\bigcup_i (\downarrow x_i) \cap Z \right) = (\downarrow x) \cap Z.$$

This means that $\bar{\sigma}(a_{\bar{i}}) = 0$, but this is impossible, since $\sigma_{\bar{i}}(a_{\bar{i}}) = 1$.

The final thing we have to prove is the universal property of the partial product. Given a pullback diagram on f and a map $\alpha : U' \to S$

$$\begin{array}{ccccc} S & \xleftarrow{\alpha} & U' & \xrightarrow{g'} & X \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ & & U & \xrightarrow[g]{} & B \end{array}$$

for any $u \in U$, we have ${f'}^{-1}(u) \cong f^{-1}(g(u))$, hence we can consider the restriction $\alpha_{|u}$ of α to the fibre of f on $g(u)$. We can then define $\tilde{\alpha} : U \to P$ by $\tilde{\alpha}(u) = \big(\alpha_{|u}, g(u)\big)$, and we trivially have that $p\tilde{\alpha} = g$. We have to prove that $\tilde{\alpha}$ is continuous. So let u_i be an ω-chain in U with $\vee u_i = u$. If $g(u_i)$ is eventually constant in B, there exists $\bar{i}$ such that $g(u_i) = g(u_{\bar{i}}) = b$, for any $i \geqslant \bar{i}$. Consequently, $\tilde{\alpha}(u_i)$ for $i \geqslant \bar{i}$ is an ω-chain in the power object $S^{f^{-1}(b)}$ whose supremum is $\tilde{\alpha}(u) = \alpha_{|{f'}^{-1}(u) = f^{-1}(b)}$ by the universal property of the exponentiation. Without loss of generality, we can then suppose that $g(u_i) = b_i$ is strictly monotone. If $\Omega = \bigcup\{b_i\} \cup \{b\}$, then $Z = f^{-1}(\Omega \setminus \{b\})$, and $Z^* = f^{-1}(\Omega)$ can be seen to be subsets of U'.

https://doi.org/10.1017/S0960129508006786 Published online by Cambridge University Press

Let $\bar{\alpha} : \bigcup_i f^{-1}(b_i) = Z \to S$ be defined by

$$\bar{\alpha}(x) = \alpha_{|u_{i_x}}(x), \text{ where } b_{i_x} = f(x).$$

Then, as before, $\bar{\alpha} : Z \to S$ is continuous, so $C = (\bar{\alpha})^{-1}(0)$ is a closed set of Z. Since $2 \Rightarrow 3$, we know that C has a unique extension to a closed set $\hat{C}$ of Z^* (the closure of C in Z^*). If $\tilde{\alpha}(u) > \vee\tilde{\alpha}(u_i)$, there would be $z \in f^{-1}(b)$ such that

$$\alpha_{|u}(z) = 1 \text{ and } \left(\vee\alpha_{|u_i}\right)(z) = 0.$$

$V = \alpha^{-1}(1) \cap Z^*$ is an open set such that $z \in V$ and $V \cap C = \varnothing$ since $\alpha_{|C} = 0$, so $z \notin \hat{C}$. But $z \in \left(\vee\alpha_{|u_i}\right)^{-1}(0) \cap Z^* = C'$, which is then a closed set with $C' \cap Z = C$ with $C' \neq \hat{C}$. But this is impossible, since we know that such an extension of C must be unique. This means that $\tilde{\alpha}(u) = \vee\tilde{\alpha}(u_i)$, so $\tilde{\alpha}$ is continuous and f is proved to be exponentiable. □

3. Comparing exponentiability in ω-Cpo, in Pos and in Top

Let $f : X \to B$ be a morphism in ω-**Cpo**, and thus also a morphism in **Pos** and **Top** through the ω-Scott topology. It may be worth comparing exponentiability of f in ω-**Cpo**, **Pos** and **Top**. If we take f exponentiable in ω-**Cpo**, we have already observed in Proposition 2.6 that f is exponentiable in **Pos**, while, in general, f is not exponentiable in **Top** (see Cagliari and Mantovani (2007, Example 3.3)).

Theorem 3.1. Let $f : X \to B$ be a morphism in ω-**Cpo**. If f is exponentiable in both **Pos** and **Top**, then f is exponentiable in ω-**Cpo** also.

Proof. Let $f : X \to B$ be exponentiable in both **Pos** and **Top**. Since f is then convex, we can apply Theorem 2.8 once we have shown that f fulfills Conditions 3 (a) and (b).

Given $(b_i)_{i\in\mathbf{N}}$ not eventually constant in B with $b = \bigvee b_i$, $\Omega = \bigcup_i \{b_i\} \cup \{b\}$ and $Z = f^{-1}(\Omega \setminus \{b\})$, we have to prove that:

(a) $Z^* = f^{-1}(\Omega)$.

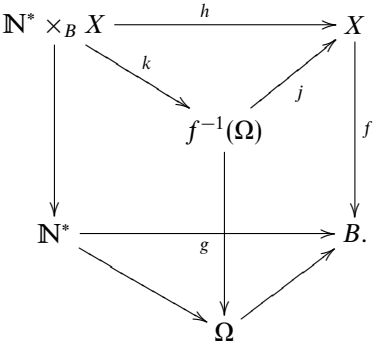
If this were not the case, there would exist $\bar{x} \in f^{-1}(\Omega) \setminus Z^*$. Consider in ω-**Cpo** the cokernel pair $f_1, f_2 : f^{-1}(\Omega) \rightrightarrows f^{-1}(\Omega) \bigcup_{Z^*} f^{-1}(\Omega)$ of the inclusion i of Z^* in $f^{-1}(\Omega)$:

$$\begin{array}{ccc} Z^* & \xrightarrow{\ i\ } & f^{-1}(\Omega) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f_2} \\ f^{-1}(\Omega) & \xrightarrow[\ f_1\ ]{} & f^{-1}(\Omega) \bigcup_{Z^*} f^{-1}(\Omega) \end{array}$$

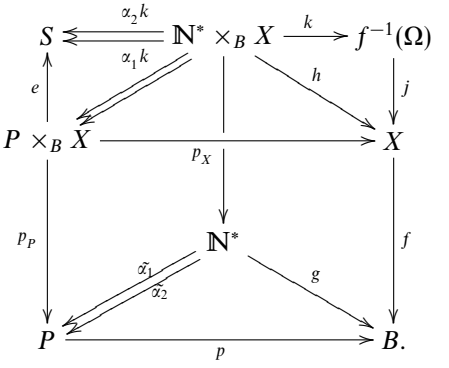
The map i is the equaliser of f_1, f_2, so $f_1(x) = f_2(x)$ holds for any $x \in Z^*$, while $f_1(\bar{x}) \neq f_2(\bar{x})$. Thus, since $f^{-1}(\Omega) \bigcup_{Z^*} f^{-1}(\Omega)$ is T_0, there exists a map $\gamma : f^{-1}(\Omega) \bigcup_{Z^*} f^{-1}(\Omega) \to S$ with $\gamma(f_1(\bar{x})) \neq \gamma(f_2(\bar{x}))$. Using α_1 and α_2 to denote γf_1 and γf_2, respectively, we have that $\alpha_1(x) = \alpha_2(x)$ for any $x \in Z^*$ and $\alpha_1(\bar{x}) \neq \alpha_2(\bar{x})$.

https://doi.org/10.1017/S0960129508006786 Published online by Cambridge University Press

We can define a continuous map $g : \mathbb{N}^* \to B$ defined as $g(i) = b_i$, $g(\infty) = b$, so that g factorises along the embedding of Ω into B.
Now consider the pullback of $g : \mathbb{N}^* \to B$ along f:



If we consider g restricted to its image Ω, by the universal property of the pullback, there is a unique map $k : \mathbb{N}^* \times_B X \to f^{-1}(\Omega)$. Now consider in **Top** the partial product $P = P(f, S)$ and the two maps $\alpha_1 k, \alpha_2 k : \mathbb{N}^* \times_B X \to S$. By the universal property of the partial product, we get two morphisms $\tilde{\alpha}_1, \tilde{\alpha}_2 : \mathbb{N}^* \to P$

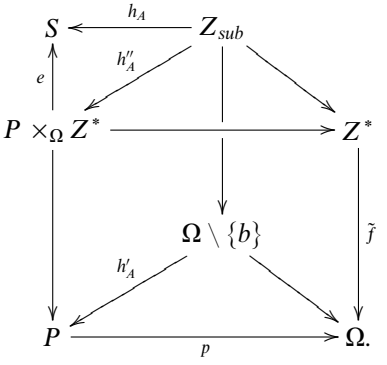


such that $\tilde{\alpha}_1(\infty) \neq \tilde{\alpha}_2(\infty)$, while $\tilde{\alpha}_1(i) = \tilde{\alpha}_2(i)$ for any i.
Any open set containing $\tilde{\alpha}_1(i) = \tilde{\alpha}_2(i)$ must also contain $\tilde{\alpha}_1(\infty)$ and $\tilde{\alpha}_2(\infty)$, since otherwise there would be an open set in $\mathbb{N}^*$ containing n, but not ∞; on the other hand, any open set containing $\tilde{\alpha}_1(\infty)$ must also contain some $\tilde{\alpha}_1(i)$, since the sequence $\tilde{\alpha}_1(i)$ converges to $\tilde{\alpha}_1(\infty)$. Therefore, any open set containing $\tilde{\alpha}_1(\infty)$ must also contain $\tilde{\alpha}_2(\infty)$ and, by the same arguments, any open set containing $\tilde{\alpha}_2(\infty)$ must also contain $\tilde{\alpha}_1(\infty)$; but this is impossible, since P is a T_0 space.

(b) Considering also on Z the ω-Scott topology, we want to prove that the inclusion $j : Z \to Z^*$ is a topological embedding.
Using Z_{sub} to denote the topological space obtained by considering on Z the topology induced by Z^*, which is weaker than the Scott topology, it is sufficient to prove that every ω-Scott open of Z is open in Z_{sub}.
Now, since the restriction $\tilde{f} : Z^* \to \Omega$ of f to Z^* is again exponentiable in **Top** (Niefield 1982), we can consider the partial product $P = P(\tilde{f}, S)$ of S on $\tilde{f}$ and the pullback of the topological inclusion of $\Omega \setminus \{b\}$ in Ω along $\tilde{f}$.

https://doi.org/10.1017/S0960129508006786 Published online by Cambridge University Press

Given an ω-Scott open A in Z, we want to show that the characteristic map $h_A : Z_{sub} \to S$ of A is continuous. So we are going to prove that the map $h'_A : \Omega \setminus \{b\} \to P$ given by $h'_A(b_i) = (b_i, h_A|A \cap \tilde{f}^{-1}(b_i))$ is continuous, and that $h_A = eh''_A$, where $h''_A : Z_{sub} \to P \times_\Omega Z^*$ is obtained by pullback:



h'_A is well defined since the ω-Scott topology coincides with the induced topology on every fibre $\tilde{f}^{-1}(b_i)$.

$\Omega \setminus \{b\}$ has the final topology induced by the inclusions of $\{b_i, b_{i+1}\}$ into $\Omega \setminus \{b\}$, so h'_A is continuous if and only if any restriction of h'_A to $\{b_i, b_{i+1}\}$ is continuous. By the universal property of the partial product, it is then sufficient to prove that $h_A|\tilde{f}^{-1}(b_i, b_{i+1}) : \tilde{f}^{-1}(b_i, b_{i+1}) \to S$ is continuous. But this is true, since on each pair of fibres the ω-Scott topology coincides with the induced topology. □

Remark 3.2. In the previous theorem, the condition of exponentiability in both **Top** and **Pos** was essential for obtaining exponentiability in ω-**Cpo**. In fact, the next example will show that exponentiability in **Top** and **Pos** are independent of each other by showing a continuous map between posets that is exponentiable in **Top** but not convex, and thus not exponentiable in **Pos**.

Example 3.3. Let $f : X \to T$ be the function defined by

$$X_A = (0, 1], \quad X_B = (0, 1), \quad X = X_A \cup X_B \cup \{c_1, c_2\},$$

$$T = \{a, b, c\}, \ a < b < c;$$

$$f(X_A) = a, f(X_B) = b, f\{c_1, c_2\} = c.$$

We have to define a partial order $\preceq$ on X. In X_A and X_B, the relation $\preceq$ is the natural one. Moreover, for any $x \in (0, 1)$,

$$x_A \preceq x_B \preceq c_i,$$

so $1_A \preceq c_i$ also.

Endowing X and T with the ω-Scott topology, f is continuous, but f is not convex, since $1_A \preceq c_1$, $f(1_A) = a < b < c = f(c_1)$, and there is no element in X between 1_A and c_1.

https://doi.org/10.1017/S0960129508006786 Published online by Cambridge University Press

We are going to show that f is exponentiable in **Top** as it fulfills the conditions proved by Niefield in Niefield (1982). In fact, for any $x \in X$ and any neighbourhood U of x in the fibre of $f(x)$, we are able to exhibit a family $\mathscr{H} = \mathscr{H}_a \cup \mathscr{H}_b \cup \mathscr{H}_c$ of open subsets of fibres that easily satisfies:

1 $U \in \mathscr{H}_{f(x)}$;
2 any $\mathscr{H}_t$ is Scott-open, for $t \in T$;
3 for any open set V in X, the set $\{t \in T | V_t \in \mathscr{H}_t\}$ is open in T;
4 $\cap\mathscr{H} = \cup_t (\cap\mathscr{H}_t)$ is a neighbourhood of x in X.

Consider $x_0 \in X_A$. If U is a neighbourhood of x_0 in X_A, there exists $0 < x'_A < x_0$ with $x'_A \in U$, and we can define:

— $\mathscr{H}_a = \{V \text{ open in } X_A | x'_A \in V\}$: this is ω-Scott open since it is the set of open sets containing a fixed compact set.
— $\mathscr{H}_b = \{U \text{ open in } X_B | x'_B \in U\}$: *idem.*
— $\mathscr{H}_c = \{\{c_1, c_2\}\}$.

If $x_0 \in X_B$ and U is a neighbourhood of x_0 in X_B, there exists $0 < x'_B < x_0$ with $x'_B \in U$, so we define:

— $\mathscr{H}_a = \varnothing$;
— $\mathscr{H}_b = \{U \text{ open in } X_B | x'_B \in U\}$;
— $\mathscr{H}_c = \{\{c_1, c_2\}\}$;

For c_i, we take $\mathscr{H}_a = \mathscr{H}_b = \varnothing$ and $\mathscr{H}_c = \{\{c_1, c_2\}\}$.

References

Abramsky, S. and Jung, A. (1994) Domain theory. *Handbook of logic in computer science* **3**, Oxford University Press 1–168.

Adámek, J., Herrlich, H. and Strecker, G. (1990) *Abstract and Concrete Categories*, Wiley.

Clementino, M. M., Hofmann, D. and Tholen, W. (2003) The convergence approach to exponentiable maps. *Port. Math. (N.S.)* **60** (2) 139–160.

Cagliari, F. and Mantovani, S. (2007) Exponentiable monomorphisms in categories of domains. *J. Pure Appl. Algebra* **211** (2) 404–413

Dyckhoff, R. and Tholen, W. (1987) Exponentiable morphisms, partial products and pullback complements. *J. Pure Appl. Algebra* **49** 103–106.

Fiech, A. (1996) Colimits in the category DCPO. *Mathematical Structures in Computer Science* **6** (5) 455–468.

Gierz, G., Hofmann, K. H., Keimel,K., Lawson, J. D., Mislove, M. and Scott, D. S. (2003) Continuous Lattices and Domains. *Encyclopedia of Mathematics and its Applications*, Cambridge University Press.

Giraud, J. (1964) Methode de la descente. *Bull. Soc. Math. France, Memoire* **2**.

Markowsky, G. (1976) Chain-complete posets and directed sets with applications. *Algebra Universalis* **6** (1) 53–68.

Markowsky, G. and Rosen, B. K. (1976) Bases for chain-complete posets. *IBM J. Res. Develop.* **20** (2) 138–147.

Niefield, S. (1982) Cartesianness: topological spaces, uniform spaces and affine schemes. *J. Pure and Appl. Algebra* **23** 147–167.

https://doi.org/10.1017/S0960129508006786 Published online by Cambridge University Press

Niefield, S. (2001) Exponentiable morphisms: posets, spaces, locales, and Grothendieck toposes. *Theory and Applications of Categories* **8** 16–32.

Richter, G. (2002) Exponentiable maps and triquotients in Top. *J. Pure Appl. Algebra* **168** 99–105.

Scott, D. S. (1972) Continuous lattices. *Springer-Verlag Lecture Notes in Mathematics* **274** 97–137.

Taylor, P. (2002) Subspaces In Abstract Stone Duality. *Theory and Applications of Categories* **10** (13) 301–368.

Tholen, W. (2000) Injectives, exponentials, and model categories. In: Abstracts of the Int. Conf. on Category Theory (Como, Italy, 2000) 183–190.

https://doi.org/10.1017/S0960129508006786 Published online by Cambridge University Press