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One-dimensional self-consistent kinetic models as Vlasov equation solutions

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One-dimensional self-consistent kinetic models may describe some states of intense charged particle beams. In the collisionless approximation, which is appropriate for the short current pulse duration, the kinetic distribution function may be built as a function of the motion integrals. Two situations are considered corresponding to the wide charged particle flux with the sharp front and to the sheet continuous beam freely propagating in space. In both cases, one-dimensional Vlasov equation solutions are shown to exist, which are based on algebraic functions of the motion invariants.

Key words: intense particle beams

1. Introduction

Vlasov equation solutions are applicable for the description of the charged particle beam behaviour, if the beam intensity is fairly low or the beam pulse duration is rather small in comparison with the time of the beam particle–particle collisions. In the latter case, the Vlasov–Poisson system should be integrated, which may be rather difficult. Usually, to simplify such a problem solution various approximations are applied which allow us to find some kind of invariance in the task, for example, phase space volume conservation along with a waterbag distribution in Feix & Bertrand (2005), application of a quasi-linear approximation in Besse *et al.* (2011), a multistream model in Bertrand *et al.* (2005), Inglebert *et al.* (2011) and Ghizzo & Bertrand (2013), where the so-called 'multiring approach' is used and the invariance of canonical angular momentum is exploited. The kinetic distribution function built as a function of the particle motion integral (and, particularly, the Hamiltonian) automatically satisfies the Vlasov equation. One of such solutions was described in Kapchinsky & Vladimirsky (1959), where the invariant was built for a two-dimensional quasi-stationary charged particle beam with elliptical cross-section as a function of transverse beam coordinates and velocities. In Kapchinsky & Vladimirsky (1959) the kinetic distribution function, looking like a delta function of the transverse invariant, is time independent, but the beam transverse characteristics (namely, the envelopes) oscillate with time. Sometimes, such a distribution looks like a delta function with respect to transverse projectiles of six-dimensional (6-D) phase volume or so-called transverse emittances (where the 'emittance' represents the value proportional to the phase volume or to the phase

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square occupied by the beam particles in the corresponding 2-D, 4-D or 6-D phase space). This is useful under conditions when the transverse Hamiltonian is not a constant of the motion. Such a model allows us to obtain the envelope equations self-consistently. The distribution function found in Kapchinsky & Vladimirsky (1959) is called the Kapchinsky–Vladimirsky distribution (KV distribution) or the microcanonical one. When applied to the continuous beam in the accelerator channel or the facility it corresponds to the maximum Courant–Snyder invariant (Courant & Snyder 1958), widely used in accelerator physics (Reiser 2008). To prove the validity of KV distribution it is necessary to fulfil the next requirement – the fields acting on the beam particles both external and internal must be linear. If the beam propagates freely without external fields, this means that the particle density must be uniform in the beam cross-section in the case of a continuous beam or inside the area occupied by the particles in the case of the bunched beam. The linear equation for the independent beam particle oscillation has two independent integrals, whose combination or bilinear integral (or quadratic form with respect to the particle coordinates and velocities in the most general case) may be used as an invariant and be included into the distribution function. Similar invariants are developed for two-dimensional and three-dimensional configurations of the charged particle beams, one can see, for example, Yarkovoi (1966), Chikhachev (1984), Barminova & Chikhachev (1991), Danilov *et al.* (2003) and Barminova & Chikhachev (2013). For some physical tasks the application of a 1-D approximation simplifies the analysis, so a one-dimensional model appears to be more useful, for instance, when considering the problem of wide flow propagation in plasma diodes and ion diodes (Barminova & Chikhachev 2012), or the propagation of the ribbon beams born in the terrestrial magnetosphere or in the laboratory while extracted from the ion and plasma sources with the slit geometry of extractors (Barminova 2014). Such models may be useful to study the laser–plasma interactions too (Bertrand *et al.* 2005). The present paper considers some aspects of the development of one-dimensional self-consistent models and examples of their application to specific physical tasks.

2. One-dimensional models: the principle of development and some applications

2.1. *General principle of the model development*

Applicability of one-dimensional self-consistent models is valid for the class of tasks with specific geometry, i.e. for both the beam configuration and the external fields geometry. It is a relevant model in the case of one-direction oscillations of the beam particles, for example, when the beam propagates in a diode with large aspect ratio or when the external field is extremely strong and forbids the transverse oscillations (in high-voltage planar diodes and in long solenoids). Another example of one-dimensional model applicability is the case of a sheet geometry of the particle flow, when the beam has significant width in one of the transverse coordinate directions so the beam oscillations corresponding to this direction may be neglected. Such a model is supposed to be applicable for investigation of some regimes of plasma interaction with a short duration power laser pulse, laser-based particle acceleration and plasma wakefield acceleration. In all the cases where the particle oscillations may be described by a motion equation dependent from the only space coordinate, the appropriate invariant may be built. The requirement of the linearity of the external field as well as the beam self-fields results in the possibility of building the integral of the motion as a quadratic function of the particle coordinates and velocities, which may have a canonical (diagonal) form under specific initial

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conditions of the task. In contrast to the case of a two-dimensional problem, in which the kinetic distribution function looks like a Dirac or δ function with respect to the invariant I (Kapchinsky & Vladimirsky 1959):

$$f = \kappa\delta(1 - I), \tag{2.1}$$

the one-dimensional model should include the 'step' function to be fully self-consistent and to describe the beam with a sharp edge in the corresponding coordinate direction:

$$f = \kappa g(I)\sigma(1 - I). \tag{2.2}$$

Here σ is the Heaviside function, κ is the normalization constant, $g(I)$ – a specific algebraic function with respect to I, I – is the bilinear integral of the motion equation or the quadratic form with respect to the corresponding coordinate and its time derivative.

Integration of the distribution function with respect to the velocity space must give us the expression for the particle beam density corresponding to the initial assumption of the uniformity of the beam density (the linearity of the self-forces). The distribution of the charge density has a sharp boundary, which automatically arises from the integration of the distribution function (2.2) over the velocity space (see, for example, Barminova & Chikhachev 2012). Such results serve as confirmation of the model self-consistency.

The distribution function (2.2) depends on the Heaviside function and looks like the waterbag distribution. If $g(I) = 1$, the phase space is filled uniformly, similarly the case of the waterbag function. But the real waterbag distribution in the case of a charge-dominated beam gives a non-uniform charge density in coordinate space. Contrary to this, function (2.2) can give a uniform charge distribution and, hence, linear self-fields, depending on the type of function $g(I)$, which in turn depends on the initial conditions of the problem. This is demonstrated in §§ 2.2 and 2.3.

One should note here that $g(I) = \sqrt{I_z - I_{z0}}$ formally corresponds to the Neuffer distribution (Neuffer 1979), which was first applied to describe the self-consistent independent longitudinal dynamics of a 3-D high current ion bunch. In the Neuffer distribution the integrals I_z, I_{z0} characterize the longitudinal particle motion. Neuffer distribution gives the linear self-field as is the case with the KV distribution.

The class of the distribution function considered above is less idealistic than the KV distribution. The KV distribution (2.1) describes only the shell of a hyperellipsoid, corresponding to the beam particles in the phase space, due to the delta function applied. The distributions (2.2) is relevant to non-thermal beams also, but the spectrum of accessible energies is more than in the case of distribution (2.1), because with (2.2) the phase volume of the beam is not empty but is filled uniformly. Both distributions (2.1) and (2.2) have a peculiarity in describing the beam with the sharp boundary in the coordinate space.

2.2. *Model for a wide flow with a sharp front*

The model corresponding to the one-dimensional flow of the particles accelerated in a strong non-steady-state electric field between two infinite planes was proposed in Barminova & Chikhachev (1992), where the extraction of the electron beam from the plasma layer formed near the hot cathode was investigated. The self-consistent motion of the beam particles inside the accelerating interval is described by:

$$\ddot{z} = \omega^2(t)(z + z_0), \tag{2.3}$$

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where $z(t)$ is longitudinal coordinate, t the time, $\omega(t)$ is the plasma frequency, corresponding to the particle with charge e and mass m, z_0 is the position of the additional plane, which is characterized by zero field, i.e. the coordinate where the gradient of self-consistent potential is equal to zero. This value is time dependent up to the time when the beam front leaves the area of the acceleration.

The invariant for this task may be written as

$$I = \frac{(\dot{R}_1 z_1 - R_1 \dot{z}_1)^2}{\varepsilon_0^2} + \frac{z_1^2}{R_1^2}. \tag{2.4}$$

Here ε_0 is constant, R_1 the coordinate of the front. The values with low index '1' correspond to the values describing the flow in the coordinate system where the motion equation is ordinary:

$$\ddot{z}_1 = \omega^2(t) z_1. \tag{2.5}$$

The distribution function for such a task may be written as

$$f = \frac{\kappa\sigma(1-I)}{\sqrt{(1-I)}}. \tag{2.6}$$

Using the property of I conservation and the beam particle motion equation in self-consistent field one can find the function R_1 as a solution of the equation:

$$\ddot{R}_1(t) = \omega^2(t) R_1(t) + \frac{\varepsilon_0^2}{R_1^3(t)}. \tag{2.7}$$

Integration of expression (2.6) over the velocity space gives the expression for the flow density:

$$n = \frac{\pi\kappa\varepsilon_0\sigma(R_1 - z_1)}{R_1}, \tag{2.8}$$

from which is clear that the time-dependent linear density of the flow is uniform throughout the area occupied by the particles. Such a situation may exist while the accelerating field has a specific time dependency, decreasing with time approximately as $1/t^4$.

The particles of the flow considered above occupy the specific area in phase space which can be represented as a part of the ellipse. The evolution of the phase portrait of the particle ensemble with longitudinal coordinate or with time is shown on probation in figure 1. Note here that the phase curve $\dot{R}_1(R_1)$ has the shape of an ellipse, which is evident from the first integral of (2.7).

In figure 1 $v_z = \dot{z}$, v_T the characteristic thermal velocity for the case of an electron beam, formed in the diode with the thermocathode, is shown. The beam behaviour demonstrated in figure 1 is valid in the case of the beam extraction from the ion source too, but the initial thermal velocity of the electron should be replaced by the Bohm velocity.

2.3. *Model for a sheet flow with uniform linear density*

Let us consider another physical situation which may be described by means of a one-dimensional Vlasov equation solution. The dynamics of a charged particle continuous beam with a sheet geometry will be fully determined by the oscillations of the beam

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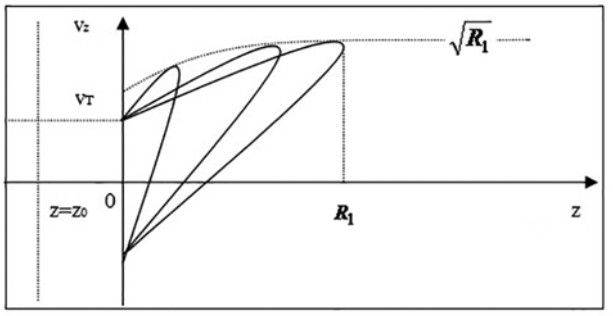


FIGURE 1. Beam phase volume evolution with longitudinal coordinate (time).

envelope, corresponding to the smallest beam size. The particle motion equation is an equation of a harmonic oscillator if one assumes that there are no external nonlinear fields in the area of the beam propagation, the beam charge density is uniform, the own space charge of the beam is neutralized and the beam current is far from Alfvén limit. For such a beam one can introduce the next distribution function f:

$$f = \kappa\sigma(1 - I), \tag{2.9}$$

where κ is a time-independent normalization constant, σ the Heaviside function, I the invariant of the task, which looks similar to integral (2.4) with z replaced by x.

Integrating the function (2.9) over the velocity space one can obtain for the linear density of the beam:

$$n = \frac{\kappa\varepsilon^2}{2R}\sigma(R - |x|), \tag{2.10}$$

where R is the radius of the beam envelope corresponding to the smallest size of the sheet beam, x is the coordinate direction, corresponding to the smallest size of the beam, ε is the partial emittance (phase volume) corresponding to the x-coordinate.

The behaviour of the beam radius R, corresponding the beam oscillations in the x-direction, is described by the equation

$$\ddot{R} + \omega^2(t) = \frac{\varepsilon^2}{R^3}, \tag{2.11}$$

where ω is the plasma frequency: $\omega^2 = 4\pi e^2 n(t)/m$, e and m are the charge and the mass of the particle respectively.

The expression (2.10) satisfies the initial assumption of the linearity of the fields acting on the particles inside the beam, so the distribution function satisfies the requirement of self-consistency.

A similar model was shown to be relevant for the description of a sheet beam with non-uniform charge distribution, i.e. when the self-forces and hence the oscillations inside the beam are nonlinear. In the nonlinear case some additional assumptions should be made. For this aim, the modified one-dimensional invariant should be used (Barminova 2014), which allows us to study beams with both strong and weak nonlinearity. In figure 2 the case of a weak self-field nonlinearity, caused by the deviation of the charge distribution from uniformity, is illustrated. In figure 2 the dimensionless longitudinal coordinate z is used instead of the time. This is valid if

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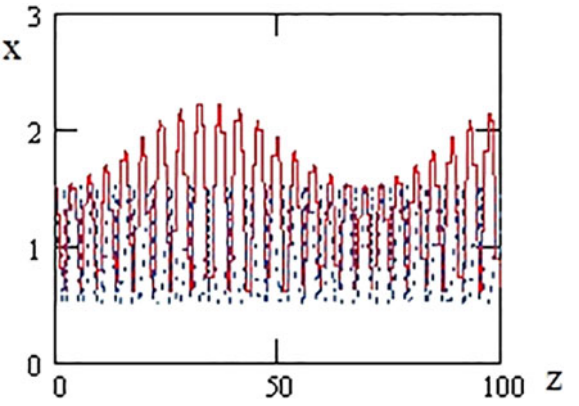


FIGURE 2. Two frequency radius oscillations of a sheet beam in the case of a weak deviation of the field from linearity (x and z are dimensionless transverse and longitudinal coordinates respectively).

the drift velocity of the particles is a constant value. The oscillations of the envelope are demonstrated as having two characteristic frequencies. The envelope oscillations are obtained with the help of the kinetic distribution function of the same type as (2.9).

3. Conclusions

One-dimensional models as Vlasov equation solutions may be applicable for the description of some states of charged particle beams. In the collisionless approximation, which is fulfilled when the beam pulse duration is less than the time between the particle collisions, the series of the distribution functions may be built which allows us to predict the behaviour of the particle flows far from thermodynamical equilibrium. The only assumption of the linearity of the fields acting the particles gives the possibility of building the simple integrals of the motion equations and to describe the particle flows with the most physical generality and scalability, without complicated numerical simulation. In some particular cases it is possible to apply the obtained models to nonlinear one-dimensional systems.

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