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Stable manifolds and the Perron–Irwin method

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Abstract. We establish rather general and simple theorems implying, among other things, the pseudo-(un)stable manifold theorem, Sternberg's theorem on smooth conjugacy between hyperbolic germs of maps or vector fields, and results of Fenichel, Hirsch, Pugh and Shub on existence, uniqueness and structural stability of stable or unstable manifolds at compact invariant manifolds. The Perron–Irwin approach *via* sequence spaces plays a crucial role.

0. *Introduction*

Recall that a *semi-conjugacy* between two maps $f : X \to X$ and $g : Y \to Y$ is a map $h : X \to Y$ such that $h \circ f = g \circ h$. The starting point of the present work is the remark that h is a semi-conjugacy if and only if its graph is invariant by $f \times g : (x, y) \mapsto (f(x), g(y))$. This remark induced in us the temptation to prove new results about invariant manifolds from which known and, hopefully, unknown (semi-)conjugacy results would follow.

The point is that, for each conjugacy result, the more abstract invariant manifold theorem(s) from which it can be deduced will have many more consequences, due to basic remarks like the following.

- Given a diffeomorphism h of a manifold M into itself and $\lambda : M \to \mathbb{R}$, the vector field X on M satisfies $h^*X = \lambda X$ if and only if $X(M) \subset TM$ is invariant by $h_1 : (x, v) \mapsto (h(x), \lambda(x)Dh(x)v)$.
- Given a vector field X on M with flow h^t and a q-form α on M, the q-form ω on M satisfies $\mathcal{L}_X\omega = \alpha$ if and only if its image $\omega(M) \subset \bigwedge^q T^*M$ is invariant by the flow $h_1^t : (x, p) \mapsto (h^t(x), (p + \int_0^t h^{s*}\alpha(x)\,ds) \circ (Dh(x)^{-1})^q)$.

Our first test was the following theorem, due to Sternberg [**19**]. Let M and N be two manifolds, then two smooth hyperbolic diffeomorphism germs $f : (N, b) \to (N, b)$ and $g : (P, c) \to (P, c)$ are smoothly conjugate if and only if they are formally conjugate.

In 1993, we were able to formulate and prove a generalization of the stable manifold theorem implying Sternberg's theorem when f is a contraction (or dilation) germ [**3**]:

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the very simple proof, inspired by Irwin's previous work [**13**], consisted of applying the ordinary implicit function theorem in an appropriate sequence space—we shall be a little more precise later.

We then turned to the truly hyperbolic case where f is neither a contraction nor a dilation. By a standard result—which can be deduced in a purely 'algebraic' way [**6, 4**] from the simple invariant manifold theorem in [**3**]—there is a smooth diffeomorphism germ $\varphi : (N, b) \to (P, c)$ such that $\varphi^{-1} \circ g \circ \varphi$ and f have infinite contact along the unstable manifold W^u of f: Sternberg's theorem then follows from the following more precise fact: there exists a smooth conjugacy between f and g which has infinite contact with φ along W^u.

In terms of invariant manifolds, here is the situation: the diffeomorphism germ $h := f \times g$ leaves invariant the submanifold germ $W_0 := \text{graph}(\varphi|_{W^u})$ of $M := N \times P$, the graph $\hat{W}$ of φ is formally h-invariant along W_0, and we want to find an h-invariant smooth submanifold germ W (the graph of the conjugacy) having infinite contact with $\hat{W}$ along W_0. Thus, Sternberg's result follows from the following theorem.

THEOREM. *Let M be a manifold and $h : (M, a) \to (M, a)$ a smooth map germ preserving a smooth submanifold germ W_0 at a, assume that $L := T_a h$ leaves invariant some vector subspace S of $E := T_a M$ containing $S_0 := T_a W_0$ and denote by A (respectively B, A_1) the endomorphism of S (respectively E/S, S/S_0) induced by L. Assume that:*

(i) *the endomorphisms A_1, B are invertible; and*

(ii) *the spectral radii of A and A_1 satisfy $\rho(A_1) < 1 < \rho(A)$.*

Then, for each germ $\hat{W}$ at a of a smooth submanifold of M containing W_0, satisfying $T_a \hat{W} = S$ and formally h-invariant along W_0, there exists an h-invariant germ W at a of a smooth submanifold of M having infinite contact with $\hat{W}$ along W_0.

This statement—established later as Theorem 3.4.1—is much more general than needed for Sternberg's theorem: for example, we do not assume h to be a product $f \times g$ nor even a cross-product as in the applications mentioned earlier. We were determined to prove it using the Perron–Irwin approach via sequence spaces, as we strongly suspected that the range of the method could be extended far beyond its known limits. The programme turned out to be more difficult than expected.

To convince ourselves that the theorem was true, we first introduced [**6**] a simple-minded blow-up to obtain the existence of a C^k invariant manifold W for every finite k as a corollary of the pseudo-(un)stable manifold theorem [**14, 10**]—see the proof of Theorem 3.4.1 later—hence a simple new proof of Sternberg's theorem in the C^r category, $r < \infty$. However, as pseudo-stable manifolds are not unique in general, there was no hope of passing to the limit when $k \to \infty$.

The invariant manifold W itself is far from unique anyway: for example, under the hypotheses of the general Sternberg theorem as previously stated, assuming f to be 'truly hyperbolic', the space of conjugacies between f and g is infinite dimensional since the centralizer of f is an infinite dimensional sub-group [**1**] of the group of smooth diffeomorphism germs $(N, b) \to (N, b)$.

To get uniqueness in our theorem, we have to make the problem more global: we may assume that M is the product $S_0 \times S_1 \times U$ of three vector spaces, that $a = 0$, $W_0 = S_0 \times \{0\}$,

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$\hat{W} = S_0 \times S_1 \times \{0\}$ and (using a bump function on S_0) that h is defined in an open 'tube' $S_0 \times B$ around W_0 and equal to L off some bounded neighbourhood of 0. Then, in some smaller (closed) tube $S_0 \times B'$, it can be shown [**4, 5**] that W exists, is unique and is the graph of a function $\varphi : S_0 \times B_1' \times \{0\} \to U$, where B_1' is a closed ball in S_1. In fact, the result holds under less restrictive hypotheses, yielding a version of the previous theorem where the fixed point a is replaced by an invariant torus.

Its proof follows Irwin's idea: if W exists, then, for each point $z_0 \in W$ close enough to W_0, the sequence $z_n = h^n(z_0)$ must lie in the Banach space $\mathcal{S}_{\lambda,\mu,\nu}$ of those sequences $(z_n) = (\theta_n, x_n, y_n)$ in M such that $\lambda^{-n}\theta_n$, $\mu^{-n}x_n$ and $\nu^{-n}y_n$ are bounded†, where λ, μ, ν are positive constants with $\lambda > 1$, $\mu < 1$ (slightly larger than $\rho(A_1)$) and ν arbitrarily small (because of the infinite contact). If we can show that, in some tube around $\mathcal{W}_0 := \{(x_n) = 0 \text{ and } (y_n) = 0\}$ in this sequence space, the solutions of the equation

$$\forall n \in \mathbb{N} \quad z_{n+1} = h(z_n) \tag{1}$$

form the graph of an implicit function $(y_0, (z_n)_{n>0}) = \Phi(\theta_0, x_0)$ whose y_0-component φ has infinite contact with 0 along $x_0 = 0$, then our result will be established: the graph of φ will be a manifold W having infinite contact with $\hat{W}$ along W_0 and it will be h-invariant because the sequence (z_{n+1}) obviously satisfies (1) when (z_n) does; moreover, it will be unique since any other W must be contained in this one by its very definition.

To prove that φ exists and is unique, we wanted to do the usual thing: denoting the points of M by (θ, x, y) and the components of h by e, f, g, invert g with respect to y, call the resulting function G and rewrite (1) as

$$(y_0, (z_n)_{n>0}) = (G(\theta_0, x_0, y_1), (e(z_{n-1}), f(z_{n-1}), G(\theta_n, x_n, y_{n+1}))_{n>0}). \tag{2}$$

Then came a real problem: except when $W_0 = \{0\}$ as in [**3**], the right-hand side of (2) does *not* define (for fixed (θ_0, x_0)) a contraction $\mathcal{A}_{\theta_0,x_0}$ of some tube in $(y_0, (z_n)_{n>0})$-space: to obtain a contraction, one should choose μ, ν greater than 1, in which case any neighbourhood of $\mathcal{W}_0$ contains sequences for which (x_n, y_n) is unbounded and $\mathcal{A}_{\theta_0,x_0}$ undefined. Trying to extend h was not a good idea, since our previous work [**1**] had convinced us that (semi-)local uniqueness is essential in such matters.

The solution of the dilemma took us quite a while, for it is somewhat subtle: one should indeed work in some closed tube $\mathcal{T}$ around $\mathcal{W}_0$ in the natural sequence space $\mathcal{S}_{\lambda,\mu,\nu}$ but *endowed with the norm of* $\mathcal{S}_{\lambda,\mu_1,\nu_1}$ *for suitable constants* $\mu_1 > 1$ *and* $\nu_1 > 1$. This approach is successful because $\mathcal{T}$ *is a closed subset of the Banach space* $\mathcal{S}_{\lambda,\mu_1,\nu_1}$ and, now, every $\mathcal{A}_{\theta_0,x_0}$ is a contraction of this closed subset if the radius of the tube is small enough.

Smoothness again was a slight problem since the interior of $\mathcal{T}$ in $\mathcal{S}_{\lambda,\mu_1,\nu_1}$ is empty, making ordinary differential calculus irrelevant. After some time, we were able to extend the work of de la Llave and Wayne on pseudo-stable manifolds [**10**] to get what we wanted.

The method was now almost applicable to all kinds of problems involving invariant manifolds. The missing test before the final state of the work [**5**] was normal hyperbolicity [**11, 12**]. The new difficulty was nonlinearity, as the maps were now defined on a fibre

† Endowed with the norm $|(z_n)| = \sup_n \max\{\lambda^{-n}|\theta_n|, \mu^{-n}|x_n|, \nu^{-n}|y_n|\}$ for suitable norms on S_0, S_1, U.

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bundle over an arbitrary compact manifold M. In particular, we wondered what was the appropriate space of sequences in M. The answer came when we discovered which statements to prove: the appropriate space is just the space of *all* sequences in M, endowed with various complete metrics (denoted later by d_κ, $\kappa > 1$), all of which define the product topology.

The aim of the present paper is not to rewrite [**12**] by Irwin's method (this will be done in [**5**]) but to state and prove general results—Theorems 1.2.1, 1.3.1 and 2.2.2—implying a substantial amount of the theory and going beyond normal hyperbolicity†. These results are rather strikingly simple and their Lipschitzian part holds in complete metric spaces. Even though they might not surprise a careful reader of [**12**] (at least in the bounded case), they can be considered as new. The proofs are anyway: they are given in detail for the first time (in [**4**], we ran out of steam after proving that the invariant manifold is C^1—and the problem was very different, both more specific and more complicated).

In the first section, we state and prove our results about stable‡ subspaces in the Lipschitz category. The second section, on smoothness, makes the Irwin–de la Llave–Wayne method work in a very general nonlinear setting§, yielding sharp regularity results. In the third section, we present a short list of consequences (including a slightly acrobatic proof of Theorem 3.4.1, stated earlier), to be completed elsewhere [**5**]. Its purpose is to show how the simple but extremely general results of the first two sections can be used in non-trivial concrete situations; for example, §§1.3 and 3.2 provide a really short and easy proof of the Hopf bifurcation phenomenon for maps.

1. *The Lipschitz bones of the theory*

1.1. *Hypotheses.* Given a complete metric space X and a complete subspace Y of a metric space F, we endow $X \times F$ and $Z := X \times Y$ with the product space distance $d((x, y), (x', y')) := \max\{d(x, x'), d(y, y')\}$ and consider a Lipschitzian map $h = (f, g) : Z \to X \times F$ with the following properties:

Hypothesis 1. There is a positive constant ρ_0^{-1} such that we have $d(g(x, y), g(x, y')) \geq \rho_0^{-1} d(y, y')$ for all x, y, y'. Thus, each $g_x : y \mapsto g(x, y)$ is injective.

Hypothesis 2. We have $Y \subset g_x(Y)$, hence the formula $G(x, y) := g_x^{-1}(y)$ defines a Lipschitzian mapping $G : Z \to Y$.

It follows that the Lipschitz constant ν_0 of G with respect to Y is at most ρ_0. We denote by μ_0 its Lipschitz constant with respect to X, by λ_0^{-1} its Lipschitz constant with respect to both variables and by κ_0 that of f.

A model situation and some comments. It might be good for the reader to have in mind the following typical situation: Z is the product $B^s \times M \times B^u$ of a closed ball B^s in a Euclidean

† Hence the titles 'pseudo-(un)stable manifolds' later.

‡ Although the word has another dynamical meaning, it would be better adapted to our results than 'invariant', for we never assume our maps to be invertible. By the way, this is what makes the proof of smoothness—inspired by [**10**]—a bit elaborate, as we could not find any nice trick avoiding higher-order derivatives in the general case.

§ It can be made into a theory of smoothness and implicit functions extending [**16**] to maps between countable products of manifolds, which may not be so uninteresting.

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space E^s, a complete Riemannian manifold M and a closed ball B^u in a Euclidean space E^u and either (in §1.2) $X = B^s \times M$ and $F = E^u$, or (in §1.3) $X = B^s$ and $F = M \times E^u$.

In the language of [**11**], Hypothesis 2 then means that h is 'outflowing' in the F direction: when M is compact, Z is a very primitive version of Conley's isolating blocks, as in [**15**] (see also [**9**] and the references therein).

1.2. *'Pseudo-stable manifolds'.*

1.2.1. *Statement of the main theorem.*

THEOREM 1.2.1. *Assume Y bounded and $\mu_0 + \nu_0 \max\{\kappa_0, 1\} < 1$ (respectively $\max\{\kappa_0, 1\} < \lambda_0$). Then, the points $z \in Z$ such that the sequence $(h^n(z))_{n\in\mathbb{N}}$ is well defined in Z form the graph W^s of a Lipschitzian function $\varphi : X \to Y$ whose Lipschitz constant satisfies*† $\mathrm{Lip}(\varphi) \le \mu_0/(1 - \nu_0\kappa_0) < 1$ *(respectively* $\mathrm{Lip}(\varphi) \le \lambda_0^{-1} < 1$*).*

Note. In set-theoretic language,

$$W^s = \bigcap_{n\in\mathbb{N}} h^{-n}(Z).$$

The boundedness of Y is essential: for instance, if $Y = F$ (which, under the other hypotheses of the theorem, must then be unbounded since each g_x expands it), then $W^s = Z$.

1.2.2. *Proof of Theorem 1.2.1 when X is bounded.*

The subspace W^s is the set of those $z_0 = (x_0, y_0) \in Z$ for which we can define a sequence $z_\bullet = (z_n)_{n\in\mathbb{N}} = (x_n, y_n)_{n\in\mathbb{N}}$ in Z by

$$\forall n \in \mathbb{N} \quad z_{n+1} = h(z_n). \tag{3}$$

Thus, what we wish to show is that the solutions of (3) in the sequence space $\mathcal{S} := Z^{\mathbb{N}}$ satisfy one of the following two equivalent conditions:

- their z_0-components form the graph of a Lipschitzian function φ satisfying the required estimates;
- they form the graph of a function $(y_0, (z_n)_{n>0}) = \Phi(x_0)$ whose y_0-component φ is Lipschitzian and satisfies the required estimates—of course, the z_n-component of $\Phi(x_0)$ is $h^n(x_0, \varphi(x_0))$.

Our hypotheses on h imply that (3) is equivalent to

$$(y_0, (z_n)_{n>0}) = \mathcal{A}(z_\bullet), \tag{4}$$

where

$$\mathcal{A}(z_\bullet) := (G(x_0, y_1), (f(z_{n-1}), G(x_n, y_{n+1}))_{n>0}). \tag{5}$$

Let $\mathcal{T} = Y \times Z^{\mathbb{N}^*}$ be the set of all $u_\bullet = (y_0, (z_n)_{n>0})$, $y_0 \in Y$, $z_n \in Z$. Clearly, (5) defines a map $\mathcal{A}$ of $\mathcal{S} = X \times \mathcal{T}$ into $\mathcal{T}$. In order to obtain Φ by the contraction principle, we endow $\mathcal{S}$ and $\mathcal{T}$ with suitable complete metrics:

† For example, if $\mu_0 = 0$, then G is a function of y alone which, as a contraction, has a unique fixed point c, and φ is the constant c.

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LEMMA 1. *For $\kappa \geq 1$, the set $\mathcal{S}$ is made into a complete metric space $\mathcal{S}_\kappa$ by the distance*

$$d_\kappa(z_\bullet, z'_\bullet) := \sup_{n\in\mathbb{N}} \kappa^{-n} d(z_n, z'_n).$$

Hence $\mathcal{T}$ is a complete metric space $\mathcal{T}_\kappa$ for the induced metric (viewing $\mathcal{T}$ as a fibre of the natural projection $\mathcal{S} = X \times \mathcal{T} \to X$), *namely*

$$d_\kappa(u_\bullet, u'_\bullet) := \max\left\{ d(y_0, y'_0), \sup_{n>0} \kappa^{-n} d(z_n, z'_n) \right\}.$$

Proof of Lemma 1. The function d_κ, well defined because Z is bounded and κ at least equal to 1, is obviously a distance on $\mathcal{S}$. If $(z^p_\bullet)_{p\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}_\kappa$, then each $(z^p_n)_{p\in\mathbb{N}}$ is a Cauchy sequence in the complete metric space Z and, therefore, has a limit $z_n \in Z$: by a quite standard argument, $z^p_\bullet$ tends to $z_\bullet := (z_n)$ in $\mathcal{S}_\kappa$ when $p \to \infty$. □

LEMMA 2. *There exists $\kappa \geq 1$ with $\mu_0 + \nu_0\kappa < 1$ and $\kappa_0 < \kappa$ (respectively with $\kappa_0 < \kappa < \lambda_0$); for any such κ, the Lipschitz constant of $\mathcal{A} : \mathcal{S}_\kappa = X \times \mathcal{T}_\kappa \to \mathcal{T}_\kappa$ is less than* 1.

Why Lemma 2 implies Theorem 1.2.1 (except the precise estimates on $\operatorname{Lip}\varphi$). Here and often in the following, we shall use the following obvious result:

LEMMA 3. *Given two metric spaces $\Lambda, \mathcal{F}$, assume that $\Lambda\times\mathcal{F}$ is equipped with the product space distance and that $\mathcal{F}$ is complete. If the Lipschitz constant of $\mathcal{C} : \Lambda \times \mathcal{F} \to \mathcal{F}$ is less than* 1, *then the map Ω which to $\lambda \in \Lambda$ associates the unique fixed point of $F \mapsto \mathcal{C}(\lambda, F)$ satisfies* $\operatorname{Lip}\Omega \leq \operatorname{Lip}\mathcal{C} < 1$.

Proof of Lemma 3. The identity $\Omega(\lambda) = \mathcal{C}(\lambda, \Omega(\lambda))$ implies the inequality $\operatorname{Lip}\Omega \leq \operatorname{Lip}(\mathcal{C}) \max\{1, \operatorname{Lip}\Omega\}$ and $\operatorname{Lip}\Omega > 1$ would lead to the contradiction $0 < (1 - \operatorname{Lip}\mathcal{C})\ \operatorname{Lip}\Omega \leq 0$. □

By Lemma 2, we can apply Lemma 3 to $\Lambda = X$, $\mathcal{F} = \mathcal{T}_\kappa$ and $\mathcal{C} = \mathcal{A}$. Denoting by Φ the corresponding Ω, we see that the solutions of (4) do form the graph of a function Φ: since the Lipschitz constant of Φ as a map into $\mathcal{T}_\kappa$ is less than 1, so is the Lipschitz constant of its y_0-component φ. □

Proof of Lemma 2. The existence of κ follows at once from our hypotheses. We wish to find a constant $c < 1$ such that

$$\forall z_\bullet, z'_\bullet \in \mathcal{S}, \quad d_\kappa(\mathcal{A}(z_\bullet), \mathcal{A}(z'_\bullet)) \leq c\, d_\kappa(z_\bullet, z'_\bullet). \tag{6}$$

Now, for every positive integer n, we have

$$\begin{aligned} \kappa^{-n} d(f(z_{n-1}), f(z'_{n-1})) &\leq \kappa^{-n}\kappa_0\, d(z_{n-1}, z'_{n-1}) \\ &\leq \kappa^{-1}\kappa_0(\kappa^{-n+1} d(z_{n-1}, z'_{n-1})) \\ &\leq \kappa^{-1}\kappa_0\, d_\kappa(z_\bullet, z'_\bullet) \end{aligned}$$

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and similarly for every integer n, setting $v_n := (x_n, y_{n+1})$,

$$\begin{aligned}\kappa^{-n} d(G(v_n), G(v_n')) &\le \kappa^{-n}(\mu_0 d(x_n, x_n') + \nu_0 d(y_{n+1}, y_{n+1}'))\\ &\le \mu_0(\kappa^{-n} d(x_n, x_n')) + \nu_0\kappa(\kappa^{-n-1} d(y_{n+1}, y_{n+1}'))\\ &\le (\mu_0 + \nu_0\kappa)\max\{\kappa^{-n} d(x_n, x_n'), \kappa^{-n-1} d(y_{n+1}, y_{n+1}')\}\\ &\le (\mu_0 + \nu_0\kappa) d_\kappa(z_\bullet, z_\bullet')\end{aligned}$$

(respectively

$$\begin{aligned}\kappa^{-n} d(G(v_n), G(v_n')) &\le \kappa^{-n}(\lambda_0^{-1}\max\{d(x_n, x_n'), d(y_{n+1}, y_{n+1}')\})\\ &\le \kappa\lambda_0^{-1}\max\{\kappa^{-n} d(x_n, x_n'), \kappa^{-n-1} d(y_{n+1}, y_{n+1}')\}\\ &\le \kappa\lambda_0^{-1} d_\kappa(z_\bullet, z_\bullet')).\end{aligned}$$

Hence, (6) follows with

$$c = \max\{\kappa^{-1}\kappa_0, \mu_0 + \nu_0\kappa\} \quad (\text{respectively } c = \max\{\kappa^{-1}\kappa_0, \kappa\lambda_0^{-1}\}). \qquad \square$$

Proof of the estimates on $\operatorname{Lip}\varphi$. We already know that φ satisfies

$$\operatorname{Lip}\varphi < 1. \tag{7}$$

Case 1. We have $g(x, \varphi(x)) = \varphi(f(x, \varphi(x)))$, i.e. $\varphi(x) = G(x, \varphi(f(x, \varphi(x))))$; hence,

$$\begin{aligned}d(\varphi(x), \varphi(x')) &\le \mu_0\, d(x, x') + \nu_0 d(\varphi(f(x, \varphi(x))), \varphi(f(x', \varphi(x'))))\\ &\le \mu_0 d(x, x') + \nu_0 \operatorname{Lip}(\varphi) d(f(x, \varphi(x)), f(x', \varphi(x')))\\ &\le \mu_0 d(x, x') + \kappa_0\nu_0 \operatorname{Lip}(\varphi)\max\{d(x, x'), d(\varphi(x'), \varphi(x'))\}\\ &\le (\mu_0 + \kappa_0\nu_0 \operatorname{Lip}\varphi)\, d(x, x')\end{aligned}$$

by (7) and $\operatorname{Lip}\varphi \le \mu_0 + \kappa_0\nu_0 \operatorname{Lip}\varphi$, as required.

Case 2. Similarly, we have

$$\begin{aligned}d(\varphi(x), \varphi(x')) &\le \lambda_0^{-1}\max\{d(x, x'), d(\varphi(f(x, \varphi(x))), \varphi(f(x', \varphi(x'))))\}\\ &\le \lambda_0^{-1}\max\{d(x, x'), \operatorname{Lip}(\varphi) d(f(x, \varphi(x)), f(x', \varphi(x')))\}\\ &\le \lambda_0^{-1}\max\{d(x, x'), \kappa_0 \operatorname{Lip}(\varphi) d(x, x')\}\\ &\le \lambda_0^{-1}\max\{1, \kappa_0 \operatorname{Lip}\varphi\} d(x, x')\end{aligned}$$

and, therefore, $\operatorname{Lip}\varphi \le \lambda_0^{-1}\max\{1, \kappa_0 \operatorname{Lip}\varphi\}$: since the inequality $\kappa_0 \operatorname{Lip}\varphi > 1$ would yield $0 < (1 - \lambda_0^{-1}\kappa_0)\operatorname{Lip}\varphi \le 0$, we do have $\operatorname{Lip}\varphi \le \lambda_0^{-1}$. $\square$

1.2.3. *Proof of Theorem 1.2.1 for unbounded X.* What is nice in the previous proof is that we can work in the space $\mathcal{S}$ of *all* sequences in Z. When X is not bounded, this is not possible since d_κ is not defined on the whole of $\mathcal{S}^2$. However, the proof is exactly the same as before, taking this time for $\mathcal{S}_\kappa$ one of the 'natural' sequence spaces. We let the reader check the following analogue of Lemma 1 (we can keep the same notation since the $\mathcal{S}_\kappa$ defined here consists of *all* sequences in Z when X is bounded):

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LEMMA 4. *For $\kappa > 1$, the set $\mathcal{S}_\kappa$ of those $z_\bullet \in \mathcal{S}$ such that $(\kappa^{-n} d(a, z_n))_{n\in\mathbb{N}}$ is bounded for some (and therefore every) $a \in Z$ is made into a complete metric space $\mathcal{S}_\kappa$ by the distance*

$$d_\kappa(z_\bullet, z'_\bullet) := \sup_{n\in\mathbb{N}} \kappa^{-n} d(z_n, z'_n).$$

Hence, the set $\mathcal{T}_\kappa$ defined by $\mathcal{S}_\kappa = X \times \mathcal{T}_\kappa$ (which consists of those $u_\bullet \in \mathcal{T}$ such that the sequence $(\kappa^{-n} d(a, z_n))_{n>0}$ is bounded for some $a \in Z$) is a complete metric space for the induced metric

$$d_\kappa(u_\bullet, u'_\bullet) := \max\{d(y_0, y'_0), \sup_{n>0} \kappa^{-n} d(z_n, z'_n)\}.$$

LEMMA 5. *For $\kappa > \max\{1, \kappa_0\}$, every sequence $z_\bullet$ in Z satisfying (3) must lie in $\mathcal{S}_\kappa$.*

Proof. Let D denote the diameter of Y. The inequality

$$\begin{aligned} d(x_{n+1}, x_{n+2}) = d(f(z_n), f(z_{n+1})) &\le \kappa_0, d(z_n, z_{n+1}) \\ &\le \kappa \max\{d(x_n, x_{n+1}), D\} \end{aligned}$$

yields inductively $d(x_n, x_{n+1}) \le \kappa^n \max\{d(x_0, x_1), D\}$, from which we obtain

$$d(x_0, x_n) \le \frac{\kappa^n - 1}{\kappa - 1} \max\{d(x_0, x_1), D\}.$$

LEMMA 6. *There exists $\kappa > 1$ with $\mu_0 + \nu_0\kappa < 1$ and $\kappa_0 < \kappa$ (respectively with $\kappa_0 < \kappa < \lambda_0$): for any such κ, (5) defines a map $\mathcal{A} : \mathcal{S}_\kappa \to \mathcal{T}_\kappa$ whose Lipschitz constant is less than* 1.

This lemma leads to the same conclusion as Lemma 2. The only new feature in its proof is the inclusion $\mathcal{A}(\mathcal{S}_\kappa) \subset \mathcal{T}_\kappa$, which we let the reader check.

1.2.4. *A simple particular case: stable manifolds.* Here is a link with the simplest invariant manifold theorem:

THEOREM 1.2.2. *Assume Y to be bounded and $\kappa_0 < 1 < \lambda_0$* (which is the case for $\max\{\kappa_0, \mu_0 + \nu_0\} < 1$). *Then, the hypotheses of Theorem 1.2.1 are satisfied and W^s is the stable manifold of the unique fixed point a of h: for $\kappa_0 < \kappa_1 < 1$ and $z \in W^s$, the distance $d(h^n(z), a)$ is an $o(\kappa_1^n)$.*

Proof. The hypotheses of Theorem 1.2.1 clearly are satisfied. As the mapping $z \mapsto (f(z), G(z))$ of Z into itself is a strict contraction, h has a unique fixed point $a = (b, c)$. The rest follows at once from the following observation, which we let the reader check.

LEMMA 7. *For $1 < \kappa < \lambda_0$, the hypotheses of Lemma 6 are satisfied and, for each $x \in X$, the set $\{u_\bullet \in \mathcal{T} / \max\{d(y_0, c), \sup_{n>0} \kappa_1^{-n} d(z_n, a)\} \le d(x, b)\}$ is a* closed *subset of $\mathcal{T}_\kappa$, invariant by the strict contraction $u_\bullet \mapsto \mathcal{A}(x, u_\bullet)$ of $\mathcal{T}_\kappa$ and, therefore, containing its fixed point $\Phi(x)$.*

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1.3. *'Pseudo-unstable manifolds'.*

1.3.1. *Statement of the main theorem.*

THEOREM 1.3.1. *Assume X bounded, $\kappa_0 < 1$ and $\mu_0 + \nu_0\kappa_0 < 1$. Then, there is a Lipschitzian function $\psi : Y \to X$ with the following property: its 'graph' $x = \psi(y)$ is the set W^u of those $z = (x, y) \in Z$ such that there exists a sequence (z_n) in Z with $z_0 = z$ and $h(z_{n+1}) = z_n$ for every integer n. The Lipschitz constant of ψ satisfies*† $\operatorname{Lip}\psi \leq 2\kappa_0\nu_0/(1 + \sqrt{1 - 4\kappa_0\mu_0\nu_0}) < 1$. *The restriction of h to $W^u \cap h^{-1}(W^u)$ is a bijection h_u onto W^u; hence, the sequences (z_n) considered in the definition of W^u are of the form $z_n = h_u^{-n}(z)$.*

Note. When Z is compact,

$$W^u = \bigcap_{n\in\mathbb{N}} h^n(Z).$$

The boundedness of X is essential: for instance, if each $f^y : x \mapsto f(x, y)$ satisfies $f^y(X) = X$ (which, under the other hypotheses of the theorem, must then be unbounded since f^y contracts it), then $W^s = Z$.

1.3.2. *Proof of Theorem 1.3.1.* As it is very similar to the proof of Theorem 1.2.1, we shall give fewer details.

Proof when Y is bounded. The subspace W^u is defined as the set of those $z_0 = (x_0, y_0) \in Z$ for which there exists a sequence $z_\bullet = (z_n)_{n\in\mathbb{N}} = (x_n, y_n)_{n\in\mathbb{N}}$ in Z satisfying

$$\forall n \in \mathbb{N} \quad z_n = h(z_{n+1}). \tag{8}$$

We shall show that the solutions $z_\bullet \in \mathcal{S} = Z^{\mathbb{N}}$ of (8) form the 'graph' of a function $(x_0, (z_n)_{n>0}) = \Psi(y_0)$ whose x_0-component ψ is Lipschitzian and satisfies the required estimates. Then, W^u will be the 'graph' of ψ.

Moreover, the invertibility of h_u will follow. Indeed, if $z_\bullet \in \mathcal{S}$ satisfies (8), so does $(z_{n+1})_{n\in\mathbb{N}}$, hence $z_0 = h(z_1)$ belongs to $h(W^u)$. This proves that h_u is onto. If it were not injective, there would exist two different points $z', z'' \in W^u$ with $h(z') = h(z'') = z = (x, y) \in W^u$. Denoting by $(z'_n)_{n>0}$ and $(z''_n)_{n>0}$ two sequences with $z'_1 = z'$, $z''_1 = z''$, $h(z'_{n+1}) = z'_n$ and $h(z''_{n+1}) = z''_n$ for every positive integer n, we would get two different solutions $z_\bullet \in \mathcal{S} = Z^{\mathbb{N}}$ of (8) with $y_0 = y$, namely $(z, (z'_n)_{n>0})$ and $(z, (z''_n)_{n>0})$, contradicting the fact that Ψ takes only one value at y. □

Our hypotheses on h imply that (8) is equivalent to

$$(x_0, (z_n)_{n>0}) = \mathcal{B}(z_\bullet), \tag{9}$$

where

$$\mathcal{B}(z_\bullet) := (f(z_1), (f(z_{n+1}), G(x_n, y_{n-1}))_{n>0}). \tag{10}$$

Let $\mathcal{U} = X \times Z^{\mathbb{N}^*}$ be the set of all $u_\bullet = (x_0, (z_n)_{n>0})$, $x_0 \in X$, $z_n \in Z$. Clearly, (9) defines a map $\mathcal{B}$ of $\mathcal{S}$ into $\mathcal{U}$.

† For example, if $\kappa_0 = 0$, then f is a constant b and $\psi = b$.

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For $\nu \geq 1$, we let $\mathcal{U}_\nu$ denote $\mathcal{U}$, endowed with the complete metric induced by d_ν (viewing $\mathcal{U}$ as a(ny) fibre of the natural projection $\mathcal{S} \ni z_\bullet \mapsto y_0 \in Y$), namely

$$d_\nu(u_\bullet, u'_\bullet) := \max\left\{d(x_0, x'_0), \sup_{n>0} \nu^{-n} d(z_n, z'_n)\right\}.$$

By Lemma 3, applied to $\mathcal{C}(y_0, (x_0, (z_n)_{n>0})) = \mathcal{B}(z_\bullet)$, the following lemma does imply that the solutions of (8) form the 'graph' of an implicit function $(x_0, (z_n)_{n>0}) = \Psi(y_0)$ whose x_0-component ψ is Lipschitzian and satisfies

$$\operatorname{Lip}\psi < 1. \tag{11}$$

LEMMA 8. *For $\nu \geq 1$ and $\mu_0\nu + \nu_0 < \nu < \kappa_0^{-1}$, the Lipschitz constant of $\mathcal{B} : \mathcal{S}_\nu \to \mathcal{U}_\nu$ is less than* 1.

Proof of Lemma 8. Given $z_\bullet, z'_\bullet \in \mathcal{S}$, for every integer $n \in \mathbb{N}$, we have

$$\begin{aligned}\nu^{-n} d(f(z_{n+1}), f(z'_{n+1})) &\leq \nu^{-n}\kappa_0 d(z_{n+1}, z'_{n+1}) \\ &\leq \nu\kappa_0(\nu^{-n-1} d(z_{n+1}, z'_{n+1})) \\ &\leq \nu\kappa_0 d_\nu(z_\bullet, z'_\bullet)\end{aligned}$$

and similarly for every positive integer n, setting $w_n := (x_n, y_{n-1})$,

$$\begin{aligned}\nu^{-n} d(G(w_n), G(w'_n)) &\leq \nu^{-n}(\mu_0 d(x_n, x'_n) + \nu_0 d(y_{n-1}, y'_{n-1})) \\ &\leq \mu_0(\nu^{-n} d(x_n, x'_n)) + \nu_0\nu^{-1}(\nu^{-n+1} d(y_{n-1}, y'_{n-1})) \\ &\leq (\mu_0 + \nu_0\nu^{-1}) \max\{\nu^{-n} d(x_n, x'_n), \nu^{-n+1} d(y_{n-1}, y'_{n-1})\} \\ &\leq (\mu_0 + \nu_0\nu^{-1}) d_\nu(z_\bullet, z'_\bullet).\end{aligned}$$

Hence, we do get $\operatorname{Lip}\mathcal{B} \leq \max\{\nu\kappa_0, \mu_0 + \nu_0\nu^{-1}\} < 1$. □

Proof of the estimate on $\operatorname{Lip}\psi$. Given $y, y' \in Y$, let $u, u' \in Y$ be defined by $(\psi(u), u) := h_u^{-1}(\psi(y), y)$ and $(\psi(u'), u') := h_u^{-1}(\psi(y'), y')$. We have

$$\begin{aligned}d(\psi(y), \psi(y')) &= d(f(\psi(u), u), f(\psi(u'), u')) \\ &\leq \kappa_0 \max\{d(\psi(u), \psi(u')), d(u, u')\} = \kappa_0 d(u, u')\end{aligned}$$

by (11) and

$$\begin{aligned}d(u, u') &= d(G(\psi(u), y), G(\psi(u'), y')) \\ &\leq \mu_0 d(\psi(u), \psi(u')) + \nu_0 d(y, y') \\ &\leq \mu_0 \operatorname{Lip}(\psi) d(u, u') + \nu_0 d(y, y').\end{aligned}$$

Hence, $d(u, u') \leq \nu_0/(1 - \mu_0 \operatorname{Lip}\psi) d(y, y')$ (because the inequality $\mu_0 \leq \mu_0 + \nu_0 < 1$ and (11) yield $\mu_0 \operatorname{Lip}\psi < 1$) and, therefore,

$$d(\psi(y), \psi(y')) \leq \frac{\kappa_0\nu_0}{1 - \mu_0 \operatorname{Lip}(\psi)} d(y, y');$$

thus, we get

$$\operatorname{Lip}\psi \leq \frac{\kappa_0\nu_0}{1 - \mu_0 \operatorname{Lip}\psi} \leq \frac{\kappa_0\nu_0}{1 - \mu_0} < 1.$$

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The first inequality holds for

$$\operatorname{Lip}\psi \le \frac{2\kappa_0\nu_0}{1+\sqrt{1-4\kappa_0\mu_0\nu_0}} \quad \text{or} \quad \operatorname{Lip}\psi \ge \frac{1+\sqrt{1-4\kappa_0\mu_0\nu_0}}{2\mu_0}.$$

The latter is excluded since we have $\mu_0+\kappa_0\nu_0<1$, hence $1-4\kappa_0\mu_0\nu_0 \ge (1-2\mu_0)^2$ and

$$\frac{1+\sqrt{1-4\kappa_0\mu_0\nu_0}}{2\mu_0} \ge \frac{1+|1-2\mu_0|}{2\mu_0} \ge 1.$$

Thus, we have

$$\operatorname{Lip}\psi \le \frac{2\kappa_0\nu_0}{1+\sqrt{1-4\kappa_0\mu_0\nu_0}},$$

which is less than 1 because, for $\kappa_0\nu_0 \ne 0$, the inequality $\mu_0+\kappa_0\nu_0<1$ yields $1-4\kappa_0\mu_0\nu_0 > (1-2\kappa_0\nu_0)^2$, hence

$$\frac{2\kappa_0\nu_0}{1+\sqrt{1-4\kappa_0\mu_0\nu_0}} < \frac{2\kappa_0\nu_0}{1+|1-2\kappa_0\nu_0|} \le 1. \qquad \square$$

Proof for unbounded Y. The proof is the same as before with $\nu>1$, taking for $\mathcal{S}_\nu$ the space of those sequences $z_\bullet$ in Z such that $\nu^{-n}d(z_n,a)$ is bounded for some (hence any) $a\in Z$, on which d_ν is a complete metric. This works because every sequence $z_\bullet$ in Z satisfying $z_n = h(z_{n+1})$ for all n must lie in this new $\mathcal{S}_\nu$: indeed, denoting by D the diameter of X, we have

$$\begin{aligned} d(y_{n+1},y_{n+2}) &= d(G(x_{n+1},y_n),G(x_{n+2},y_{n+1})) \\ &\le \mu_0 d(x_{n+1},x_{n+2}) + \nu_0 d(y_n,y_{n+1}) \le \mu_0\, D + \nu d(y_n,y_{n+1}), \end{aligned}$$

which yields inductively

$$d(y_n,y_{n+1}) \le \frac{\nu^n-1}{\nu-1}\mu_0 D + \nu^n d(y_0,y_1)$$

and, therefore,

$$d(y_0,y_n) \le \frac{\nu^n-1}{\nu-1}\left(\frac{1}{\nu-1}\mu_0 D + d(y_0,y_1)\right).$$

1.3.3. *Link with hyperbolic fixed points.*

THEOREM 1.3.2. *Assume X, Y to be bounded, $\kappa_0<1$ and $\mu_0+\nu_0<1$. Then, the conclusions of both Theorems 1.2.1 and 1.3.1 hold. In fact, h has a unique fixed point a, which, of course, lies in $W^u\cap W^s$ and we have the following:*

(i) *For each $z\in W^s$, the sequence $z_n = h^n(z)$ tends to a; more precisely, for $\kappa_0<\kappa_1<1$, the distance $d(z_n,a)$ is an $o(\kappa_1^n)$.*

(ii) *For every $z\in W^u$, the sequence $z_n = h_u^{-n}(z)$ tends to a; more precisely, for $\mu_0\nu_1+\nu_0<\nu_1<1$, the distance $d(z_n,a)$ is an $o(\nu_1^n)$.*

(iii) *The intersection $W^u\cap W^s$ equals $\{a\}$.*

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Proof. The hypotheses of Theorems 1.2.1 and 1.3.1 are, of course, satisfied and we already know (Theorem 1.2.2) that h has a unique fixed point and that (i) holds.

Denoting by D the diameter of Z, the proof of (ii) is very similar: with the notation of 1.3.2, each $(x,(z_n)_{n>0}) \mapsto \mathcal{B}((x,y),(z_n)_{n>0})$ maps the *closed* subset $\mathcal{U}_{(\nu_1,a)} := \{(x,(z_n)_{n>0}) \,/\, \sup_n \nu_1^{-n} d(z_n,a) \le D\}$ of $\mathcal{U}_\nu$ into itself, hence its fixed point $\Psi(y)$ lies in $\mathcal{U}_{(\nu_1,a)}$.

To prove (iii), we should show that a is the only point in $W^u \cap W^s$, i.e. the only $(x,\varphi(x)) \in W^s$ such that $x = \psi(\varphi(x))$, which is clear: the strict contraction $\psi \circ \varphi$ has a unique fixed point b, hence $a = (b,\varphi(b))$.

1.4. *Link with more general invariant manifold theorems.* We first provide some further analysis of the situation.

1.4.1. *Pseudo-unstable manifolds are stable.*

THEOREM 1.4.1. *Under the hypotheses of Theorem 1.3.1 with* $\kappa_0 + \mu_0 \operatorname{Lip}\psi < 1$, *we have*

$$k_s := \frac{\kappa_0}{1-\mu_0 \operatorname{Lip}\psi} < 1$$

and, for $z = (x,y) \in h^{-1}(Z)$,

$$d(h(z), W^u) \le d(f(z), \psi(g(z))) \le k_s\, d(x,\psi(y)). \tag{12}$$

Hence, for every $z \in Z$, *the* (finite or infinite) *sequence* $z_n := h^n(z)$ *satisfies*

$$d(z_n, W^u) \le d(x_n, \psi(y_n)) \le k_s^n\, d(x_0,\psi(y_0)).$$

In particular, if h *maps* Z *into itself, then, for every* $z \in Z$, *the distance from* $h^n(z)$ *to* W^u *is a* $O(k_s^n)$.

Note. The additional hypothesis $\kappa_0 + \mu_0 \operatorname{Lip}\psi < 1$ is satisfied in the applications which I know since μ_0 is very small.

Proof of the theorem. The first inequality in (12) is clear since $d(f(z),\psi(g(z)))$ is the distance from $h(z)$ to $(\psi(g(z)), g(z))$, which belongs to W^u. In order to prove the second, let us introduce the point $z' = (\psi(y'), y') \in W^u$ such that $g(z') = g(z)$, namely $z' = h_u^{-1}(\psi(g(z)), g(z))$. The identities $y = G(x, g(z))$ and $y' = G(\psi(y'), g(z))$ yield

$$d(y,y') \le \mu_0 d(x,\psi(y')). \tag{13}$$

As $\psi(g(z)) = f(z')$, we have

$$d(f(z),\psi(g(z))) = d(f(z), f(z')) \le \kappa_0 d(z,z') = \kappa_0 d(x,\psi(y')) \tag{14}$$

since $d(z,z') = \max\{d(x,\psi(y')), d(y,y')\} = d(x,\psi(y'))$ by (13) and the inequality $\mu_0 \le \mu_0 + \nu_0\kappa_0 < 1$. Now, we have

$$\begin{aligned} d(x,\psi(y')) &\le d(x,\psi(y)) + d(\psi(y),\psi(y')) \le d(x,\psi(y)) + \operatorname{Lip}(\psi) d(y,y') \\ &\le d(x,\psi(y)) + \mu_0 \operatorname{Lip}(\psi) d(x,\psi(y')) \end{aligned}$$

by (13) and, therefore, $d(x,\psi(y')) \le 1/(1-\mu_0 \operatorname{Lip}(\psi)) d(x,\psi(y))$. Hence, (12) follows from (14).

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1.4.2. *Pseudo-stable manifolds are unstable.*

THEOREM 1.4.2. *Under the hypotheses of Theorem 1.2.1, assume* $\nu_0 + \kappa_0\nu_0 \operatorname{Lip}\varphi < 1$. *We then have* $k_u := \nu_0/(1 - \kappa_0\nu_0 \operatorname{Lip}\varphi) < 1$ *and, for* $z = (x, y) \in h^{-1}(Z)$,

$$d(z, W^s) \le d(y, \varphi(x)) \le k_u d(g(z), \varphi(f(z))). \tag{15}$$

Hence, every (finite or infinite) *sequence* (z_n) *in* Z *such that* $h(z_{n+1}) = z_n$ *for every* n *satisfies*

$$d(z_n, W^s) \le d(y_n, \varphi(x_n)) \le k_u^n d(y_0, \varphi(x_0)).$$

In particular, for $Z \subset h(Z)$, *every* $z \in Z$ *is the* z_0 *term of such an* infinite *sequence* (z_n) *and* $d(z_n, W^s)$ *is a* $O(k_u^n)$.

Note. As before, the additional hypothesis $\nu_0 + \kappa_0\nu_0 \operatorname{Lip}\varphi < 1$ is satisfied when μ_0 and, therefore, $\operatorname{Lip}\varphi$ is small.

Proof of the theorem. The first inequality in (15) is clear because $d(y, \varphi(x))$ is the distance from z to $(x, \varphi(x))$, which belongs to W^s. Let us prove the second: since $g(x, \varphi(x)) = \varphi(f(x, \varphi(x)))$, we have $\varphi(x) = G(x, \varphi(f(x, \varphi(x))))$ and, of course, $y = G(x, g(z))$, hence

$$\begin{aligned} d(y, \varphi(x)) &= d(G(x, g(z)), G(x, \varphi(f(x, \varphi(x))))) \\ &\le \nu_0 d(g(z), \varphi(f(x, \varphi(x)))). \end{aligned} \tag{16}$$

Now, the estimates

$$\begin{aligned} d(g(z), \varphi(f(x, \varphi(x)))) &\le d(g(z), \varphi(f(z))) + d(\varphi(f(z)), \varphi(f(x, \varphi(x)))) \\ &\le d(g(z), \varphi(f(z))) + \operatorname{Lip}(\varphi) d(f(z), f(x, \varphi(x))) \\ &\le d(g(z), \varphi(f(z))) + \kappa_0 \operatorname{Lip}(\varphi) d(y, \varphi(x)) \end{aligned}$$

and (16) yield $d(y, \varphi(x)) \le \nu_0 d(g(z), \varphi(f(z))) + \kappa_0\nu_0 \operatorname{Lip}(\varphi) d(y, \varphi(x))$, from which (15) follows at once.

1.4.3. *A first step in the world of normal hyperbolicity.*

THEOREM 1.4.3. *Given three metric spaces* Z^s, Z^c, F^u *and a subset* Z^u *of* F^u, *assume* Z^s, Z^c, Z^u *complete and let* h *be a map of* $Z := Z^s \times Z^c \times Z^u$ *into* $Z^s \times Z^c \times F^u$.

(i) *If the hypotheses of Theorem 1.2.1 are satisfied by* h, $X = Z^s \times Z^c$, $F = F^u$ *and* $Y = Z^u$, *then the set of those* $z \in Z$ *for which* $(h^n(z))_{n\in\mathbb{N}}$ *exists is the graph* W^s *of a Lipschitzian function* $\varphi : Z^s \times Z^c \to Z^u$ *with* $\operatorname{Lip}\varphi < 1$.

(ii) *If the hypotheses of Theorem 1.3.1 are satisfied by* h, $X = Z^s$, $F = Z^c \times F^u$ *and* $Y = Z^c \times Z^u$, *then the set of those* $z \in Z$ *for which there exists a sequence* $(z_n)_{n\in\mathbb{N}}$ *with* $h(z_{n+1}) = z_n$ *and* $z_0 = z$ *form the 'graph'* W^u *of a Lipschitzian function* $\psi : Z^c \times Z^u \to Z^s$ *with* $\operatorname{Lip}\psi < 1$.

(iii) *If the hypotheses of both (i) and (ii) are satisfied, the* h*-invariant set* $W^c := W^s \cap W^u$ *is the 'graph' of a function* $\chi = (\chi^s, \chi^u) : Z^c \to Z^s \times Z^u$ *satisfying* $\operatorname{Lip}\chi^s \le \operatorname{Lip}\psi$ *and* $\operatorname{Lip}\chi^u \le \operatorname{Lip}\varphi$.

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(iv) *If, moreover, the map h, viewed as in (i), satisfies the hypotheses of Theorem 1.4.1*†*, then* W^s *is the stable subspace of h at* W^c*, meaning that, for every* $z \in W^s$*, the distance* $d(h^n(z), W^c)$ *is a* $O(k_s^n)$.

(v) *If the hypotheses of both (i) and (ii) are satisfied and, moreover, the map h, viewed as in (ii), satisfies the hypotheses of Theorem 1.4.2, then* W^u *is the unstable subspace of h at* W^c*, meaning that, for every sequence* (z_n) *as in (ii), the distance* $d(z_n, W^c)$ *is a* $O(k_u^n)$.

Proof. (i), (ii), (iv) and (v) follow tautologically from Theorems 1.2.1, 1.3.1, 1.4.1 and 1.4.2 respectively.

Let us prove (iii). W^c is defined by the equations $x = \psi(\theta, \varphi(x, \theta))$ and $y = \varphi(\psi(\theta, y), \theta)$: applying Lemma 3 to the map $Z^c \times Z^u \ni (\theta, y) \mapsto \varphi(\psi(\theta, y), \theta)$ (respectively $Z^c \times Z^s \ni (\theta, x) \mapsto \psi(\theta, \varphi(x, \theta))$), whose Lipschitz constant is at most that of φ (respectively ψ), we do see that the unique fixed point $\chi^u(\theta)$ (respectively $\chi^s(\theta)$) of $y \mapsto \varphi(\psi(\theta, y), \theta)$ (respectively $x \mapsto \psi(\theta, \varphi(x, \theta))$) is a Lipschitzian function of θ satisfying $\operatorname{Lip} \chi^u \leq \operatorname{Lip} \varphi$ (respectively $\operatorname{Lip} \chi^s \leq \operatorname{Lip} \psi$). □

Note. In our model situation with M compact, the situations (iii)–(v) correspond to the 'absolutely normally hyperbolic' case of [**12**].

2. *Smoothness*

As in the previous section, we shall state and prove *very* general results, whose proof is not substantially easier in the model situation introduced in §1.1, except when M is a point, a vector space or a torus‡. The manifolds we consider are separable Banach manifolds (separability is not essential).

2.1. *Some notation and background.*

2.1.1. *Definition of integer and non-integer smoothness.* We write $r \in \widetilde{\mathbb{R}}_+$ if either $r \in [0, +\infty]$ or $r = k^-$ for some positive integer k, in which case we set $s < k^- < k$ for real $s < k$, $[k^-] := k - 1$, $k^- - [k^-] := 1$ and $x^{\pm k^-} := x^{\pm k}$ for positive x.

For finite $r \in \widetilde{\mathbb{R}}_+$, a map is called C^r if it is

- $[r]$ times differentiable, with an $[r]$th derivative which is locally Hölder continuous of exponent $r - [r]$, when r is not an integer and
- r times differentiable with locally *uniformly* continuous rth derivative§ when r is an integer.

2.1.2. *Higher-order derivatives in a nonlinear framework.* Given $r \geq 1$ in $\widetilde{\mathbb{R}}_+$, if f is a C^r map between two manifolds M, N equipped with C^r linear connections

† We are not totally sure that this additional hypothesis is needed but it is satisfied in the applications we know of. The same remark applies to the additional hypothesis in (v).

‡ In the first two cases, the exponential is very simple and there is no problem with its injectivity radius or the definition of higher-order derivatives: this last feature is common with the torus case, which can be made a little simpler by working in the universal cover.

§ On a finite dimensional space, this is the usual notion.

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(see Appendix B), then, for each positive integer $k \leq r$ and each $x \in M$, we can define a continuous symmetric k-linear map $D^k_x f : (T_x M)^k \to T_{f(x)} N$ by

$$D^k_x f := D^k f_x(0), \quad \text{where } f_x := \exp^{-1}_{f(x)} \circ f \circ \exp_x \quad \text{(local inverse)}.$$

This defines a section $D^k f$ of the vector bundle $L^k_s(TM, f^*TN)$ over M whose fibre over x is the space $L^k_s(T_x M, T_{f(x)}N)$ of continuous symmetric k-linear maps of $(T_x M)^k$ into $T_{f(x)}N$.

'Landau' notation. For finite integer (respectively non-integer) r, the notation $f(x,h) = \mathrm{O}_x(|h|^r)$ means that $\lim_{h\to 0} |h|^{-r} f(x,h) = 0$ (respectively that $|h|^{-r} f(x,h)$ is bounded for small enough h) *locally uniformly with respect to* x. The following result is proven in Appendix B (which provides a definition of Finsler manifolds).

LEMMA 9. *Let M, N be Finsler manifolds endowed with C^r linear connections, $r \in \widetilde{\mathbb{R}}_+$. For $r < \infty$, given a continuous map $\Psi : M \to N$ and locally bounded sections Ψ^j of $L^j_s(TM, \Psi^*TN)$, $1 \leq j \leq [r]$, the following two conditions are equivalent.*

(i) *The map Ψ is C^r and $D^j\Psi = j!\,\Psi^j$, $1 \leq j \leq [r]$.*

(ii) *We have $|\Psi_x(h) - \Psi^1_x h - \cdots - \Psi^{[r]}_x h^{[r]}| = \mathrm{O}_x(|h|^r)$.*

C^r functions on closed subsets. Under the hypotheses of Lemma 9, given a *closed* subset F of M, a continuous function $\Psi : F \to N$ will be called C^r when there exist bounded sections Ψ^j of $L^j_s(TM, \Psi^*TN)$ above F, $1 \leq j \leq [r]$, such that (ii) is satisfied. By Whitney's extension theorem, if M is finite dimensional, then such a Ψ can be extended to a C^r function on a neighbourhood of F.

2.2. *Hypotheses and statement of the theorem.*

2.2.1. *Basic additional hypotheses.*

In addition to the hypotheses stated in §1.1, we assume that, for some $r \geq 1$ in $\widetilde{\mathbb{R}}_+$, we have the following.

Hypothesis 3. The spaces X, Y are closures of open subsets† in complete C^{r+2} Finsler manifolds E, F respectively.

Hypothesis 4. The map h is C^r. More precisely, f and G are well defined, Lipschitzian and C^r in an open neighbourhood U of Z in $E \times F$.

2.2.2. *Statement of the smoothness theorem for compact Z.*

THEOREM 2.2.1. *Assume Hypotheses 1–4 are satisfied and Z is compact.*

- *Under the hypotheses of Theorem 1.2.1, for each $k \in \widetilde{\mathbb{R}}_+$ with $1 \leq k \leq r$ and $\mu_0 + \nu_0 \kappa_0^k < 1$ (respectively $\kappa_0^k < \lambda_0$), the function φ is C^k in the sense of Whitney‡; in particular, when X is a manifold with corners, W^s is a C^k manifold with corners whose (intrinsic) interior is the graph of φ restricted to the interior of X.*

† This hypothesis could be considerably weakened but, in the applications I know of, it is satisfied and X, Y are submanifolds with corners.

‡ As we assume $1 \leq r$ and $\mu_0 + \nu_0\kappa_0 < 1$ (respectively $\kappa_0 < \lambda_0$), φ always is C^1.

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- *Under the hypotheses of Theorem 1.3.1, if* $k \in \widetilde{\mathbb{R}}_+$ *satisfies* $1 \le k \le r$ *and* $\mu_0 + \nu_0 \kappa_0^{1/k} < 1$, *then* ψ *is* C^k *in the sense of Whitney*†; *in particular, when* Y *is a manifold with corners,* W^u *is a* C^k *manifold with corners whose (intrinsic) interior is the graph of* ψ *restricted to the interior of* Y.

2.2.3. *Additional hypotheses on* X, Y, F *in the non-compact case.* Endowing $E \times F$ with the Finsler metric

$$|\delta z| := \max\{|\delta x|, |\delta y|\}, \quad \delta z = (\delta x, \delta y) \in T_x E \times T_y F = T_z(E \times F),$$

we assume E,F equipped with C^r linear connections such that there exist positive constants $\varepsilon, \varepsilon', C, \alpha$ with the following properties.

Hypothesis 5. For each $z = (x, y) \in Z$, the exponential $\exp_z = \exp_x \times \exp_y$ is defined on the whole of the tangent space $T_z(E \times F) = T_x E \times T_y F$ and we have

$$d(x, \exp_x \delta x) \le C|\delta x| \quad \text{for } x \in X,\ \delta x \in T_x E \text{ and } |\delta x| \le \varepsilon,$$
$$d(y, \exp_y \delta y) \le C|\delta y| \quad \text{for } y \in Y,\ \delta y \in T_y F \text{ and } |\delta y| \le \varepsilon.$$

Hypothesis 6. Each $e_z := \exp_z |_{\{|\delta z| < \varepsilon\}}$ (i.e. $e_x \times e_y := \exp_x |_{\{|\delta x| < \varepsilon\}} \times \exp_y |_{\{|\delta y| < \varepsilon\}}$) with $z = (x, y) \in Z$ is a diffeomorphism onto an open neighbourhood $\Omega_z(\varepsilon) = \Omega_x(\varepsilon) \times \Omega_y(\varepsilon)$ of z in $E \times F$ *containing the open ball* $B_z(\varepsilon')$ and we have

$$|e_x^{-1}(x')| \le C d(x, x') \quad \text{for } x \in X \text{ and } x' \in \Omega_x(\varepsilon),$$
$$|e_y^{-1}(y')| \le C d(y, y') \quad \text{for } y \in Y \text{ and } y' \in \Omega_y(\varepsilon).$$

Hypothesis 7. The definition domain U of f and G contains every $\exp_z \delta z$ with $z \in Z$, $\delta z \in T_z(E \times F)$ and $|\delta z| < \alpha$.

LEMMA 10. *If Hypotheses 3 and 4 hold and* Z *is compact, then there exist linear connections and constants* $\varepsilon, \varepsilon', C, \alpha$ *satisfying Hypotheses 5–7.*

Proof. The Levi–Civita connections of arbitrary complete Riemannian metrics on E, F will do. □

Notes. To avoid unnecessarily strong hypotheses, the connection is only loosely related to the metric.

In the model situation introduced in §1.1, the reader may, nevertheless, assume M endowed with the Levi–Civita connection and $E^s \times M \times E^u$ with the product metric connection, for which $\exp_z \delta z = (z_s + \delta z_s, \exp_{z_c} \delta z_c, z_u + \delta z_u)$ (setting $z = (z_s, z_c, z_u)$ and $\delta z = (\delta z_s, \delta z_c, \delta z_u)$), hence $C = 1$ and $\varepsilon = \varepsilon'$.

2.2.4. *Additional hypothesis on* h *in the non-compact case.* As f, G are Lipschitzian, there exists $\alpha > 0$ (which we may assume to be that of Hypothesis 7) such that, for all

† As we assume $1 \le r$ and $\mu_0 + \nu_0 \kappa_0 < 1$, the function ψ always is C^1.

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$z, z' \in Z$ with $d(z, z') < C\alpha$, both $d(f(z), f(z'))$ and $d(G(z), G(z'))$ are less than ε'. Hence, for every $z \in Z$, we can define C^r maps

$$\begin{aligned} f_z &:= e_{f(z)}^{-1} \circ f \circ \exp_z : \{\delta z \in T_z(E \times F) : |\delta z| < \alpha\} \to T_{f(z)}X, \\ G_z &:= e_{G(z)}^{-1} \circ G \circ \exp_z : \{\delta z \in T_z(E \times F) : |\delta z| < \alpha\} \to T_{G(z)}Y, \end{aligned} \tag{17}$$

and (taking a smaller α if necessary) we assume the following.

Hypothesis 8. For every positive integer $j \le r$, we have

$$\gamma_j := \sup_{z \in Z, |\delta z| < \alpha} \max\{|D^j f_z(\delta z)|, |D^j G_z(\delta z)|\} < \infty$$

and, moreover:

- for integer r, continuity of $D^r f_z$ and $D^r G_z$ at 0 is uniform with respect to $z \in Z$; and
- for non-integer r, there exists a positive number β such that, for every $z \in Z$ and every $\delta z \in T_z(E \times F)$ with $|\delta z| < \alpha$, we have

$$\max\{|D^{[r]} f_z(\delta z) - D^{[r]} f_z(0)|, |D^{[r]} G_z(\delta z) - D^{[r]} G_z(0)|\} \le \beta |\delta z|^{r-[r]}.$$

Notes. When Z is compact, Hypothesis 8 follows automatically from Hypotheses 3 and 4 (together with Lemma 10, this proves that Theorem 2.2.1 is a particular case of Theorem 2.2.2). Otherwise, it is a very strong hypothesis, as shown in the model situation (1.1) when M is a vector space.

The derivatives of G can be computed from those of g as in linear spaces, since the usual expression of the derivatives of a composed map remains valid for the derivatives introduced in §2.1.2 [**1**]. It follows, for example, that, when the $D^j g$'s are bounded, so are the $D^j G$'s (which is the essence of Hypothesis 8).

2.2.5. *The smoothness theorem.*

THEOREM 2.2.2. *Assume Hypotheses 1–8 satisfied.*

- *Under the hypotheses of Theorem 1.2.1, for each $k \in \widetilde{\mathbb{R}}_+$ with $1 \le k \le r$ and $\mu_0 + \nu_0 \kappa_0^k < 1$ (respectively $\kappa_0^k < \lambda_0$), the function φ is C^k in the sense of Whitney†; in particular, when X is a manifold with corners, W^s is a C^k manifold with corners whose (intrinsic) interior is the graph of φ restricted to the interior of X.*
- *Under the hypotheses of Theorem 1.3.1, if $k \in \widetilde{\mathbb{R}}_+$ satisfies $1 \le k \le r$ and $\mu_0 + \nu_0 \kappa_0^{1/k} < 1$, then ψ is C^k in the sense of Whitney†; in particular, when Y is a manifold with corners, W^u is a C^k manifold with corners whose (intrinsic) interior is the graph of ψ restricted to the interior of Y.*

2.3. *Proof of Theorem 2.2.2.*

2.3.1. *Proof in the case of Theorem 1.2.1, second hypothesis.* We assume $\kappa_0 \ge 1$: otherwise, by Theorem 1.2.2, W^s is the stable manifold of the fixed point of h, a well-known situation [**13, 4**].

† For the same reason as in Theorem 2.2.1, it always is C^1.

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Notation and idea of the proof. We denote by $T_{z_\bullet}\mathcal{S}_\mu$, $z_\bullet \in \mathcal{S}_\mu$, $\mu \geq 1$, the Banach space of all $\delta z_\bullet = (\delta z_n)_{n\in\mathbb{N}}$ with $\delta z_n \in T_{z_n}(E \times F)$ and $|\delta z_\bullet|_\mu := \sup\{\mu^{-n}|\delta z_n|\} < \infty$. Similarly, we let $T_{u_\bullet}\mathcal{T}_\mu$, $u_\bullet \in \mathcal{T}_\mu$, be the Banach space of all $\delta u_\bullet = (\delta y_0, (\delta z_n)_{n>0})$ with $\delta y_0 \in T_{y_0}E$, $\delta z_n \in T_{z_n}(E \times F)$ for $n > 0$ and $|\delta u_\bullet|_\mu := \max\{|\delta y_0|, \sup_n \mu^{-n}|\delta z_n|\} < \infty$†.

For $z_\bullet \in \mathcal{S}$, we define the 'derivatives' $D^j\mathcal{A}(z_\bullet)$,$1 \leq j \leq r$, by componentwise differentiation: setting $v_n := (x_n, y_{n+1})$, we let

$$D^j\mathcal{A}(z_\bullet)\delta z_\bullet^j := (D^jG(v_0)\delta v_0^j, (D^jf(z_{n-1})\delta z_{n-1}^j, D^jG(v_n)\delta v_n^j)_{n>0})$$

for $\delta z_\bullet = (\delta z_n)$, $\delta z_n \in T_{z_n}(E \times F)$; similarly,

$$\exp_{z_\bullet}\delta z_\bullet := (\exp_{z_n}\delta z_n)_{n\in\mathbb{N}}.$$

Under the hypotheses and with the notation of Theorem 1.2.1 (second case), we shall show that the maps $\mathcal{A}$ and Φ considered in §1.2.1 are C^k as maps into $\mathcal{T}_\lambda$ for suitable κ, λ, in the sense of Lemma 9(ii); in particular, φ is C^k.

This forces us to consider $\mathcal{A}_{z_\bullet}$, Φ_x and, in particular, to *define* them since, for arbitrarily small $\delta z_\bullet \in T_{z_\bullet}\mathcal{S}_\kappa$, $z_\bullet \in \mathcal{S}_\kappa$, the distance between the z_n components of $\mathcal{A}(\exp_{z_\bullet}\delta z_\bullet)$ and $\mathcal{A}(z_\bullet)$ can be greater than the injectivity radius of the exponential. The crux of the proof is that this is not a real problem, as we can then define the δz_n-component of $\mathcal{A}_{z_\bullet}(\delta z_\bullet)$ to be (almost) *any* element of the tangent space, for example zero:

Extension of f_z, G_z and definition of $\mathcal{A}_{z_\bullet}$. For each $z \in Z$, we extend the two functions f_z and G_z defined by (17) to the whole of $T_z(E \times F)$ by letting‡

$$f_z(\delta z) = 0 \quad \text{and} \quad G_z(\delta z) = 0 \text{ for } |\delta z| \geq \alpha;$$

for $z_\bullet \in \mathcal{S}$, if $\delta z_\bullet = (\delta z_n)_{n\in\mathbb{N}}$ satisfies $\delta z_n \in T_{z_n}(E \times F)$ for every n, we then set

$$\mathcal{A}_{z_\bullet}(\delta z_\bullet) := (G_{v_0}(\delta v_0), (f_{z_{n-1}}(\delta z_{n-1}), G_{v_n}(\delta v_n))_{n>0}).$$

LEMMA 11. *If $\kappa, \lambda \in \mathbb{R}$ and $k \in \widetilde{\mathbb{R}}_+$ satisfy $1 \leq k \leq r$ and $1 < \kappa^k < \lambda$, then the function $\mathcal{A}$ is C^k as a map $\mathcal{S}_\kappa \to \mathcal{T}_\lambda$, in the following sense: for $z_\bullet \in \mathcal{S}_\kappa$ and $\delta z_\bullet \in T_{z_\bullet}\mathcal{S}_\kappa$, we have*

$$\left|\mathcal{A}_{z_\bullet}(\delta z_\bullet) - \sum_{j=1}^{[k]} \frac{1}{j!} D^j\mathcal{A}(z_\bullet)\delta z_\bullet^j\right|_\lambda = \mathrm{O}_{z_\bullet}(|\delta z_\bullet|_\kappa^k) \tag{18}$$

uniformly with respect to $z_\bullet$.

Proof. We shall provide an explicit treatment for the X components, the case of the Y components being entirely analogous.

(a) For $|\delta z_n| \geq \alpha$, Hypothesis 8 yields

$$\left|f_{z_n}(\delta z_n) - \sum_{j=1}^{[k]} \frac{1}{j!} D^j f(z_n)\delta z_n^j\right| = \left|\sum_{j=1}^{[k]} \frac{1}{j!} D^j f(z_n)\delta z_n^j\right| \leq \sum_{j=1}^{[k]} \frac{\gamma_j}{j!}|\delta z_n|^j; \tag{19}$$

† Note that, for $\mu > 1$, every neighbourhood of zero in these 'tangent spaces' contains sequences such that (δy_n) is unbounded; hence, $\mathcal{S}_\mu$, $\mathcal{T}_\mu$ are *not* Banach manifolds in general since Y at least is bounded (just consider the case where it is a ball).

‡ These brutal extensions do not rely on any special property of the Banach space $T_z(E \times F)$.

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moreover, we have $\kappa^n|\delta z_\bullet|_\kappa \geq |\delta z_n| \geq \alpha$. Hence, $n \geq (\log\alpha - \log|\delta z_\bullet|_\kappa)/\log\kappa$ and, therefore, by (19),

$$\begin{aligned}
&|\delta z_\bullet|_\kappa^{-k}\lambda^{-n}\left|f_{z_n}(\delta z_n) - \sum_{j=1}^{[k]}\frac{1}{j!}D^j f(z_n)\delta z_n^j\right| \\
&\quad\leq |\delta z_\bullet|_\kappa^{-k}\lambda^{-n}\sum_{j=1}^{[k]}\frac{\gamma_j}{j!}|\delta z_n|^j \leq \sum_{j=1}^{[k]}\frac{\gamma_j}{j!}|\delta z_\bullet|_\kappa^{j-k}(\lambda^{-1}\kappa^j)^n \\
&\quad\leq \left(\sum_{j=1}^{[k]}\frac{\gamma_j}{j!}\alpha^{j-k}\right)(\lambda^{-1}\kappa^k)^n \leq \left(\sum_{j=1}^{[k]}\frac{\gamma_j}{j!}\alpha^{j-k}\right)(\lambda^{-1}\kappa^k)^{(\log\alpha-\log|\delta z_\bullet|_\kappa)/\log\kappa},
\end{aligned}$$

which does tend uniformly to 0 when $|\delta z_\bullet|_\kappa \to 0$.

(b) For $|\delta z_n| < \alpha$, we consider two cases.

(b_1) If we have $k < [r]$, then, setting $\ell := [k] + 1$, Hypothesis 8 yields

$$\begin{aligned}
\left|f_{z_n}(\delta z_n) - \sum_{j=1}^{[k]}\frac{1}{j!}D^j f(z_n)\delta z_n^j\right| &= \left|\int_0^1 \frac{(1-t)^{[k]}}{[k]!}D^\ell f_{z_n}(t\delta z_n)\delta z_n^\ell\, dt\right| \\
&\leq \frac{\gamma_\ell}{\ell!}|\delta z_n|^\ell,
\end{aligned}$$

hence

$$\begin{aligned}
|\delta z_\bullet|_\kappa^{-k}\lambda^{-n}\left|f_{z_n}(\delta z_n) - \sum_{j=1}^{[k]}\frac{1}{j!}D^j f(z_n)\delta z_n^j\right| &\leq |\delta z_\bullet|_\kappa^{-k}\lambda^{-n}\frac{\gamma_\ell}{\ell!}|\delta z_n|^\ell \\
&\leq (\lambda^{-1}\kappa^k)^n\frac{\gamma_\ell}{\ell!}|\delta z_n|^{[k]+1-k}.
\end{aligned}$$

- If k is not an integer, we just note that the right-hand side is at most $(\gamma_\ell/\ell!)\alpha^{[k]+1-k}$, which provides a uniform bound and (together with what has been done in (a) and the analogous estimates for G) yields our lemma.
- The case of integer k is a bit subtler: the previous estimate and the inequalities $|\delta z_n| < \alpha$, $|\delta z_n| \leq \kappa^n|\delta z_\bullet|_\kappa$ yield

  $$|\delta z_\bullet|_\kappa^{-k}\lambda^{-n}\left|f_{z_n}(\delta z_n) - \sum_{j=1}^{k}\frac{1}{j!}D^j f(z_n)\delta z_n^j\right| \leq (\lambda^{-1}\kappa^k)^n\frac{\gamma_\ell}{\ell!}\min\{\alpha, \kappa^n|\delta z_\bullet|_\kappa\}$$

  whose right-hand side does tend to 0 *uniformly with respect to* n when $|\delta z_\bullet|_\kappa \to 0$ since $(\lambda^{-1}\kappa^k)^n \to 0$ when $n \to \infty$.

(b_2) For $[r] \leq k \leq r$, setting $\ell := [r] = [k]$, we have

$$f_{z_n}(\delta z_n) - \sum_{j=1}^{[k]}\frac{1}{j!}D^j f(z_n)\delta z_n^j = \int_0^1 \frac{(1-t)^{\ell-1}}{(\ell-1)!}(D^\ell f_{z_n}(t\delta z_n) - D^\ell f_{z_n}(0))\delta z_n^\ell\, dt.$$

- If r is not an integer, Hypothesis 8 yields

  $$|\delta z_\bullet|_\kappa^{-k}\lambda^{-n}\left|f_{z_n}(\delta z_n) - \sum_{j=1}^{[k]}\frac{1}{j!}D^j f(z_n)\delta z_n^j\right| \leq (\lambda^{-1}\kappa^k)^n\frac{\beta}{\ell!}|\delta z_n|^{r-k},$$

  from which we conclude as before.

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- Otherwise, we have $k = r = \ell$ and

$$|\delta z_\bullet|_\kappa^{-k}\lambda^{-n}\left|f_{z_n}(\delta z_n) - \sum_{j=1}^{[k]}\frac{1}{j!}D^j f(z_n)\delta z_n^j\right| \le \frac{(\lambda^{-1}\kappa^k)^n}{k!}\max_{0\le t\le 1}|D^r f_{z_n}(t\delta z_n) - D^r f_{z_n}(0)|,$$

whose right-hand side does tend to 0 *uniformly with respect to* n *and* $z_\bullet$ when $|\delta z_\bullet|_\kappa \to 0$ for the following three reasons:

— we have that $(\lambda^{-1}\kappa^k)^n \to 0$ when $n \to \infty$;

— Hypothesis 8 yields $\max_{0\le t\le 1}|D^r f_{z_n}(t\delta z_n) - D^r f_{z_n}(0)| \le 2\gamma_r$;

— again by Hypothesis 8, $|D^r f_{z_n}(t\delta z_n) - D^r f_{z_n}(0)| \to 0$ when $|\delta z_n| \to 0$, *uniformly with respect to* $t \in [0,1]$ *and* $z_n \in Z$. □

LEMMA 12. *Let* $z_\bullet = (x, u_\bullet) \in \mathcal{S} = X \times \mathcal{T}$.

(i) *For* $\mu \ge 1$, *the partial* $\partial_1\mathcal{A}(z_\bullet)$ *with respect to* x *defines a continuous linear map* $\partial_1\mathcal{A}(z_\bullet)_\mu : T_x E \to T_{\mathcal{A}(z_\bullet)}\mathcal{T}_\mu$ *and we have* $\sup_{z_\bullet}|\partial_1\mathcal{A}(z_\bullet)_\mu| < \infty$.

(ii) *For* $\kappa_0 < \mu < \lambda_0$ *the partial* $\partial_2\mathcal{A}(z_\bullet)$ *with respect to* $\mathcal{T}$ *defines a linear strict contraction* $\partial_2\mathcal{A}(z_\bullet)_\mu$ *of* $T_{\mathcal{A}(z_\bullet)}\mathcal{T}_\mu$ *and we have* $\sup_{z_\bullet}|\partial_2\mathcal{A}(z_\bullet)_\mu| < 1$.

Proof. (i) is obvious because the variable x_0 appears only in the first two components of $\mathcal{A}$. To prove (ii), we use the estimate $|Df(z)| \le \operatorname{Lip} f$ (see Appendix B): given $z_\bullet \in \mathcal{S}$ and $\delta z_\bullet \in T_{z_\bullet}\mathcal{S}_\mu$, we have

$$\mu^{-n}|Df(z_{n-1})\delta z_{n-1}| \le \mu^{-1}|Df(z_{n-1})|(\mu^{-n+1}|\delta z_{n-1}|) \le \mu^{-1}\kappa_0|\delta z_\bullet|_\mu$$

and, setting $v_n := (x_n, y_{n+1})$,

$$\mu^{-n}|DG(v_n)\delta v_n| \le \mu|DG(v_n)|(\mu^{-n-1}|\delta v_n|) \le \mu\lambda_0^{-1}|\delta z_\bullet|_\mu,$$

hence $|\partial_2\mathcal{A}(z_\bullet)_\mu| \le \max\{\mu^{-1}\kappa_0, \mu\lambda_0^{-1}\} < 1$. □

We can consider in the same fashion higher-order derivatives.

LEMMA 13. *If* $\mu = (\mu_1, \dots, \mu_j) \in \mathbb{R}^j$, $1 < j \le r$, *satisfies* $\mu_i \ge 1$, $1 \le i \le j$, *then every* $D^j\mathcal{A}(z_\bullet)$, $z_\bullet \in \mathcal{S}$, *defines a* j*-linear continuous map* $D^j\mathcal{A}(z_\bullet)_\mu$ *of* $T_{z_\bullet}\mathcal{S}_{\mu_1} \times \cdots \times T_{z_\bullet}\mathcal{S}_{\mu_j}$ *into* $T_{\mathcal{A}(z_\bullet)}\mathcal{T}_{\mu_1\cdots\mu_j}$, *whose norm is a bounded function of* $z_\bullet$.

Proof. For $(\delta z_\bullet^1, \dots, \delta z_\bullet^j) \in T_{z_\bullet}\mathcal{S}_{\mu_1} \times \cdots \times T_{z_\bullet}\mathcal{S}_{\mu_j}$, we have

$$\begin{aligned}(\mu_1\cdots\mu_j)^{-n}&|D^j f(z_{n-1})(\delta z_{n-1}^1, \dots, \delta z_{n-1}^j)| \\ &\le (\mu_1\cdots\mu_j)^{-1}|D^j f(z_{n-1})|(\mu_1^{-(n-1)}|\delta z_{n-1}^1|)\cdots(\mu_j^{-(n-1)}|\delta z_{n-1}^j|) \\ &\le (\mu_1\cdots\mu_j)^{-1}\gamma_j|\delta z_\bullet^1|_{\mu_1}\cdots|\delta z_\bullet^j|_{\mu_j}\end{aligned}$$

(since $D^j f(z) = D^j f_z(0)$ by definition) and, as the μ_i's are ≥ 1,

$$\begin{aligned}(\mu_1\cdots\mu_j)^{-n}&|D^j G(v_n)(\delta v_n^1, \dots, \delta v_n^j)| \\ &\le \mu_1\cdots\mu_j|D^j G(v_n)|(\mu_1^{-(n+1)}|\delta v_n^1|)\cdots(\mu_j^{-(n+1)}|\delta v_n^j|) \\ &\le \mu_1\cdots\mu_j\gamma_j|\delta z_\bullet^1|_{\mu_1}\cdots|\delta z_\bullet^j|_{\mu_j}.\end{aligned}$$

□

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Definition of $\Phi_x : \exp_x^{-1}(X) \to T_{\Phi(x)}\mathcal{T}_\kappa$, $x \in X$, $\kappa \geq 1$. Setting

$$\Phi(x) = (y_0(x), (z_n(x))_{n>0}) = (y_0(x), (x_n(x), y_n(x))_{n>0}) \text{ (hence } y_0(x) = \varphi(x)),$$

we define

$$\delta\Phi(x) = \Phi_x(\delta x) = (\delta y_0(x), (\delta z_n(x))_{n>0}) = (\delta y_0(x), (\delta x_n(x), \delta y_n(x))_{n>0})$$

by

$$\delta x_n(x) := \begin{cases} e_{x_n(x)}^{-1}(x_n(\exp_x \delta x)) & \text{for } d(x_n(x), x_n(\exp_x \delta x)) < \varepsilon', \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta y_n(x) := \begin{cases} e_{y_n(x)}^{-1}(y_n(\exp_x \delta x)) & \text{for } d(y_n(x), y_n(\exp_x \delta x)) < \varepsilon', \\ 0 & \text{otherwise.} \end{cases}$$

The following lemma is crucial.

LEMMA 14. *Set* $F(x) := (x, \Phi(x))$ *and* $\delta F(x) := (\delta x, \delta\Phi(x))$. *If* $\lambda \in \mathbb{R}$ *and* $k \in \widetilde{\mathbb{R}}_+$ *satisfy* $1 \leq k \leq r$ *and* $\kappa_0^k < \lambda < \lambda_0$, *then we have*

$$\left| \delta\Phi(x) - \sum_{j=1}^{[k]} \frac{1}{j!} D^j \mathcal{A}(F(x)) \delta F(x)^j \right|_\lambda = \mathrm{O}_x(|\delta x|^k) \tag{20}$$

uniformly with respect to x.

Proof. Let us choose κ with $\kappa_0^k < \kappa^k < \lambda$.

CLAIM 1. *For* $|\delta x| \leq \varepsilon$, *we have* $|\delta F(x)|_\kappa \leq C^2 \operatorname{Lip}(F)|\delta x|$.

Indeed, the definition of δz_n and Hypothesis 6 yield $|\delta F(x)|_\kappa \leq C d_\kappa(F(x), F(\exp_x \delta x))$, hence $|\delta F(x)|_\kappa \leq C \operatorname{Lip}(F) d(x, \exp_x \delta x) \leq C^2 \operatorname{Lip}(F)|\delta x|$ by Hypothesis 5. □

Applying (18) with $z_\bullet = F(x)$ and $\delta z_\bullet = \delta F(x)$, and using Claim 1, we get

$$\left| \mathcal{A}_{F(x)}(\delta F(x)) - \sum_{j=1}^{[k]} \frac{1}{j!} D^j \mathcal{A}(F(x)) \delta F(x)^j \right|_\lambda = \mathrm{O}_{F(x)}(|\delta F(x)|_\kappa^k) = \mathrm{O}_x(|\delta x|^k)$$

uniformly with respect to x: we should, therefore, prove that

$$|\delta\Phi(x) - \mathcal{A}_{F(x)}(\delta F(x))|_\lambda = \mathrm{O}_x(|\delta x|^k) \quad \text{uniformly with respect to } x. \tag{21}$$

Set $x_0(x) := x$, $\delta x_0(x) := \delta x$ and $v_n := (x_n, y_{n+1})$. The relation $\mathcal{A} \circ F = \Phi$ means that $x_n = f \circ z_{n-1}$ $(n > 0)$ and $y_n = G \circ v_n$; therefore, the x_n-component $\delta x_n(x) - f_{x_n(x)}(\delta z_{n-1}(x))$ of $\delta\Phi(x) - \mathcal{A}_{F(x)}(\delta F(x))$ is

$$\begin{cases} \delta x_n(x) & \text{for } d(z_{n-1}(x), z_{n-1}(\exp_x \delta x)) \geq \alpha \text{ and } d(x_n(x), x_n(\exp_x \delta x)) < \varepsilon', \\ 0 & \text{otherwise} \end{cases}$$

and its y_n-component $\delta y_n(x) - G_{y_n(x)}(\delta v_n(x))$ is

$$\begin{cases} \delta y_n(x) & \text{for } d(v_n(x), v_n(\exp_x \delta x)) \geq \alpha \text{ and } d(y_n(x), y_n(\exp_x \delta x)) < \varepsilon', \\ 0 & \text{otherwise.} \end{cases}$$

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For $d(z_{n-1}(x), z_{n-1}(\exp_x \delta x)) \geq \alpha$ and $d(x_n(x), x_n(\exp_x \delta x)) < \varepsilon'$, we have

$$\begin{cases} \alpha \leq d(z_{n-1}(x), z_{n-1}(\exp_x \delta x)) \leq \kappa^{n-1} \operatorname{Lip} \Phi d(x, \exp_x \delta x) \leq \kappa^{n-1} C \operatorname{Lip} \Phi |\delta x| \\ |\delta x_n(x)| \leq C\, d(x_n(x), x_n(\exp_x \delta x)) < C\varepsilon'; \end{cases}$$

hence, $|\delta x|^{-k} \leq (\kappa^{n-1} C\, \alpha^{-1} \operatorname{Lip} \Phi)^k$ and

$$\begin{cases} n \geq 1 + (\log \alpha - \log |\delta x| - \log(C \operatorname{Lip} \Phi))/\log \kappa, \\ |\delta x|^{-k} \lambda^{-n} |\delta x_n(x)| < C^{k+1} \varepsilon' (\kappa^{-1} \alpha^{-1} \operatorname{Lip} \Phi)^k (\lambda^{-1} \kappa^k)^n; \end{cases}$$

similarly, for $d(v_n(x), v_n(\exp_x \delta x)) \geq \alpha$ and $d(y_n(x), y_n(\exp_x \delta x)) < \varepsilon'$, we get

$$\begin{cases} n \geq -1 + (\log \alpha - \log |\delta x| - \log(C \operatorname{Lip} \Phi))/\log \kappa, \\ |\delta x|^{-k} \lambda^{-n} |\delta y_n(x)| < C^{k+1} \varepsilon' (\kappa \alpha^{-1} \operatorname{Lip} \Phi)^k (\lambda^{-1} \kappa^k)^n. \end{cases}$$

From these two sets of estimates, (21) follows at once. □

The following is the last step of the proof†.

LEMMA 15. *If $\lambda \in \mathbb{R}$ and $k \in \widetilde{\mathbb{R}}_+$ satisfy $1 \leq k \leq r$ and $\kappa_0^k < \lambda < \lambda_0$, then the function Φ is C^k as a map of X into $\mathcal{T}_\lambda$, in the following sense: there exist bounded $D^j \Phi(x) \in L_s^j(X, T_{\Phi(x)} \mathcal{T}_\lambda)$, $1 \leq j \leq [k]$, $x \in X$, such that*

$$\left| \Phi_x(\delta x) - \sum_{j=1}^{[k]} \frac{1}{j!} D^j \Phi(x) \delta x^j \right|_\lambda = \mathrm{O}_x(|\delta x|^k)$$

uniformly with respect to x. In particular, the first component φ of Φ is C^k, as required.

Proof. Estimate (20) reads

$$\left| (\mathrm{Id} - \partial_2 \mathcal{A}(F(x))_\lambda) \delta\Phi(x) - \partial_1 \mathcal{A}(F(x)) \delta x - \sum_{2 \leq j \leq [k]} \frac{1}{j!} D^j \mathcal{A}(F(x)) \delta F(x)^j \right|_\lambda = \mathrm{O}_x(|\delta x|^k),$$

which, by Lemma 12(ii), is equivalent to

$$\left| \delta\Phi(x) - \Delta^1 \Phi(x) \delta x - \sum_{2 \leq j \leq [k]} \frac{1}{j!} B^j(x) \delta F(x)^j \right|_\lambda = \mathrm{O}_x(|\delta x|^k), \tag{22}$$

where

$$\begin{cases} B^j(x) := (\mathrm{Id} - \partial_2 \mathcal{A}(F(x))_\lambda)^{-1} \circ D^j \mathcal{A}(F(x)), \\ \Delta^1 \Phi(x) := (\mathrm{Id} - \partial_2 \mathcal{A}(F(x))_\lambda)^{-1} \circ \partial_1 \mathcal{A}(F(x)). \end{cases}$$

CLAIM 2. *Lemma 15 is true for $k \leq 2^-$ and $D^1 \Phi = \Delta^1 \Phi$.*

Indeed, (22) reads $|\delta\Phi(x) - \Delta^1 \Phi(x) \delta x|_\lambda = \mathrm{O}_x(|\delta x|^k)$. □

† Compare with [**10**].

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Estimate (22) will enable us to construct the derivatives of Φ, given by the same induction formula as in the normal situation of calculus in Banach spaces: setting $\delta^i\Phi(x) := (1/i!)D^i\Phi(x)\delta x^i$, $\delta^1 F(x) := (\delta x, \delta^1\Phi(x))$ and $\delta^i F(x) := (0, \delta^i\Phi(x))$ for $\delta x \in T_x X$, we should have

$$\frac{1}{\ell!}D^\ell\Phi(x)\delta x^\ell = \sum_{j=1}^{\ell}\sum_{m}\frac{1}{m_1!\cdots m_{\ell-1}!}B^j(x)\delta^1F(x)^{m_1}\cdots\delta^{\ell-1}F(x)^{m_{\ell-1}}, \tag{23}$$

where the second sum is taken over those $m \in \mathbb{N}^{\ell-1}$ such that $\sum m_i = j$ and $\sum i m_i = \ell$. The following statement will be proved by induction.

CLAIM 3. *For every integer $\ell \in [2, r]$ and every pair $(k_\ell, \mu_\ell) \in \widetilde{\mathbb{R}}_+ \times \mathbb{R}$ with $\ell \le k_\ell < \ell + 1$, $k_\ell \le r$ and $\kappa_0^{k_\ell} < \mu_\ell < \lambda_0$, the function Φ is C^{k_ℓ} as a map of X into $\mathcal{T}_{\mu_\ell}$ in the sense of Lemma 15 and $D^\ell\Phi$ is given by (23). More precisely, the Taylor remainder $\rho_\ell(x, \delta x) := \delta\Phi(x) - \sum_1^\ell \delta^i\Phi(x)$ satisfies*

$$|\rho_\ell(x, \delta x)|_{\mu_\ell} = \mathrm{O}_x(|\delta x|^{k_\ell}). \tag{24}$$

Proof. Assuming (24) true for smaller ℓ's (a reasonable induction hypothesis by Claim 2) and taking $k := k_\ell$, $\lambda := \mu_\ell$ in (22), we get

$$\left|\rho_{\ell-1}(x, \delta x) - \sum_{j=1}^{\ell}\sum\frac{1}{m_1!\cdots m_{\ell-1}!}B^j(x)R_0^{m_1}\cdots R_{\ell-2}^{m_{\ell-1}}\right|_{\mu_\ell} = \mathrm{O}_x(|\delta x|^{k_\ell}),$$

where the second sum is taken over those $m \in \mathbb{N}^{\ell-1}$ such that $\sum m_i = j$ and $\sum i m_i = \ell$ and

$$R_i = R_i(x, \delta x) := \begin{cases}(\delta x, \rho_0(x, \delta x)) & \text{if } i = 0,\\ (0, \rho_i(x, \delta x)) & \text{otherwise.}\end{cases}$$

Since $\rho_{i-1}(x, \delta x) = \delta^i\Phi(x) + \rho_i(x, \delta x)$, $1 \le i \le \ell$, we shall have established (24) if we can prove that

$$|B^j(x)R_0^{m_1}\cdots R_{n-1}^{m_n-1}R_nR_n^{m_{n+1}}\cdots R_{\ell-2}^{m_{\ell-1}}|_{\mu_\ell} = \mathrm{O}_x(|\delta x|^{k_\ell}) \tag{25}$$

for $m_n \ne 0$, $1 \le n < \ell$, $\sum m_i = j$ and $\sum i m_i = \ell$. Now, by our induction hypothesis, we know that

$$|R_0(x, \delta x)|_{\mu_0} = \mathrm{O}_x(|\delta x|) \text{ for } \kappa_0 < \mu_0 < \lambda_0,$$
$$|R_i(x, \delta x)|_{\mu_i} = \mathrm{O}_x(|\delta x|^{k_i}) \text{ for } \kappa_0^{k_i} < \mu_i < \lambda_0 \text{ and } 1 \le i \le k_i < i + 1 \le \ell.$$

Case 1. If we have $\ell < k_\ell < (\ell + 1)^-$, then, by Lemma 13, (25) will hold if (setting $k_0 := 1$) we can choose such $\mu_0, \ldots, \mu_{\ell-1}$ and $k_1, \ldots, k_{\ell-1}$ with

$$\begin{gathered} m_1 + m_2k_1 + \cdots + m_{\ell-1}k_{\ell-2} - k_{n-1} + k_n = k_\ell,\\ \mu_0^{m_1}\cdots\mu_{\ell-2}^{m_{\ell-1}}\mu_{n-1}^{-1}\mu_n = \mu_\ell.\end{gathered} \tag{26}$$

If we look for a solution such that $k_i - i$ is the same δk for $1 \le i \le \ell - 1$, the first equation reads

$$k_\ell = \begin{cases} m_1 + (m_2 + \cdots + m_{\ell-1} + 1)\delta k + \sum_{1\le i\le\ell-1}(i-1)m_i & \text{if } n = 1,\\ m_1 + (m_2 + \cdots + m_{\ell-1})\delta k + \sum_{1\le i\le\ell-1}(i-1)m_i + 1 & \text{otherwise;}\end{cases}$$

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hence,

$$\delta k = \begin{cases} \dfrac{j-m_1+k_\ell-\ell}{j-m_1+1} = 1+\dfrac{k_\ell-\ell-1}{j-m_1+1} & \text{if } n=1, \\ \dfrac{j-m_1+k_\ell-\ell-1}{j-m_1} = 1+\dfrac{k_\ell-\ell-1}{j-m_1} & \text{otherwise.} \end{cases}$$

As we have $\ell < k_\ell < (\ell+1)^-$, we do get $0 < \delta k < 1^-$. If we take

$$\mu_0 = \mu_\ell^{1/k_\ell} \quad \text{and} \quad \mu_i = \mu_\ell^{k_i/k_\ell}, \quad 0 < i < \ell,$$

(26) is satisfied and the estimates $1 \le \kappa_0^{k_\ell} < \mu_\ell < \lambda_0$ yield $\kappa_0 < \mu_0 < \lambda_0$ and $\kappa_0^{k_i} < \mu_i < \lambda_0$, $0 < i < \ell$, hence (25).

Case 2. If $k_\ell = (\ell+1)^-$, everything works as in Case 1, taking $k_i = (i+1)^-$ and $\mu_i = \mu_\ell^{(i+1)/(\ell+1)}$, $0 \le i < \ell$.

Case 3. When $k_\ell = \ell$, we can do exactly the same as in Case 1 if $m_1 = j$, $n = 1$ or if $m_1 = j-1$, $1 < n < \ell$; indeed, we get that $\delta k = 0$; hence, $k_i = i$, $0 < i < \ell$, and the very special meaning of $\mathrm{O}_x(|\delta x|^\ell)$ for integer ℓ is not a problem. Otherwise, we modify the strategy of case 1 a tiny bit: instead of solving (26), we solve

$$m_1 + m_2 k_1 + \cdots + m_{\ell-1}k_{\ell-2} - k_{n-1} + k_n = k'_\ell,$$
$$\mu_0^{m_1} \cdots \mu_{\ell-2}^{m_{\ell-1}} \mu_{n-1}^{-1} \mu_n = \mu_\ell$$

for some k'_ℓ slightly greater than ℓ. The same arguments as in Case 1 yield

$$|B^j(x) R_0^{m_1} \cdots R_{n-1}^{m_n-1} R_n R_n^{m_{n+1}} \cdots R_{\ell-2}^{m_{\ell-1}}|_{\mu_\ell} = \mathrm{O}_x(|\delta x|^{k'_\ell}) = \mathrm{O}_x(|\delta x|^\ell). \qquad \square$$

2.3.2. *Proof of smoothness in the other cases.*

The case of Theorem 1.2.1, first hypothesis. Still assuming $\kappa_0 \ge 1$, Lemmas 11 and 13 are unchanged, and Lemma 12 is true provided the condition $\kappa_0 < \mu < \lambda_0$ is replaced by $\kappa_0 < \mu$ and $\mu_0 + \nu_0\mu < 1$. We can then state and prove the following revised lemma.

LEMMA 15. *If $\lambda \in \mathbb{R}$ and $k \in \widetilde{\mathbb{R}}_+$ satisfy $1 \le k \le r$, $\kappa_0^k < \lambda$ and $\mu_0 + \nu_0\lambda < 1$, then the function Φ is C^k as a map of X into $\mathcal{T}_\lambda$, in the previous sense.*

Proof. We follow the same lines as before: by Lemma 11, we obtain (18) for $1 \le k \le r$ and $1 < \kappa^k < \lambda$; to get (20), we need Φ to be Lipschitzian into $\mathcal{T}_\kappa$, which, by Lemma 2, is the case for $\kappa_0 < \kappa$ and $\mu_0 + \nu_0\kappa < 1$; by the new version of Lemma 12(ii), we can deduce (22) from (20) provided we have $\kappa_0 < \lambda$ and $\mu_0 + \nu_0\lambda < 1$. These three conditions can be realized if and only if we have $\mu_0 + \nu_0\kappa_0^k < 1$, $1 \le k \le r$, in which case the rest of the proof is unchanged. $\square$

The case of Theorem 1.3.1. It is very much the same, *mutatis mutandis,* i.e. replacing $\mathcal{A}, \kappa, \Phi$ by $\mathcal{B}, \nu, \Psi$ respectively. We assume $\mu_0 + \nu_0 \ge 1$: otherwise, by Theorem 1.3.2, W^u is the unstable manifold of the fixed point of h.

Again, Lemmas 11 and 13 are unchanged and Lemma 12 is true under the condition $\mu_0\nu + \nu_0 < \nu < \kappa_0^{-1}$, i.e. $\nu_0/(1-\mu_0) < \nu < \kappa_0^{-1}$. We can then state and prove the following re-revised lemma.

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LEMMA 15. *If $\lambda \in \mathbb{R}$ and $k \in \widetilde{\mathbb{R}}_+$ satisfy $1 \le k \le r$ and $(\nu_0/(1-\mu_0))^k < \lambda < \kappa_0^{-1}$, then the function Ψ is C^k as a map of Y into $\mathcal{U}_\lambda$, in the previous sense.*

Proof. We follow again the same lines: by Lemma 11, we obtain the revised (18) for $1 \le k \le r$ and $1 < \nu^k < \lambda$; to get the revised (20), we need Ψ to be Lipschitzian into $\mathcal{U}_\nu$, which, by Lemma 8, is the case for $\nu_0/(1-\mu_0) < \nu < \kappa_0^{-1}$; by the new version of Lemma 12(ii), we can deduce (22) from (20) (revised versions) provided we have $\nu_0/(1-\mu_0) < \lambda < \kappa_0^{-1}$. The estimate $\nu_0/(1-\mu_0) \ge 1$ implies that these three conditions can be realised if and only if we have $(\nu_0/(1-\mu_0))^k < \kappa_0^{-1}$, i.e. $\mu_0 + \nu_0 \kappa_0^{1/k} < 1$, in which case the rest of the proof goes as before. □

3. *Examples and consequences*

3.1. *The 'saddle–node–saddle' bifurcation.*

Consider the family

$$\xi_u(x, y) = (u_1 + u_2 x - x^3)\partial_x + y\partial_y,\ u \in \mathbb{R}^2,$$

of vector fields on $\mathbb{R}^2$. Clearly, for fixed small positive r, r', the hypotheses of Theorem 1.2.1 are satisfied with $X = [-r, r]$ and $Y = [-r', r']$ by the time one (or any positive time) h_u of the flow of ξ_u for every small enough $u \in \mathbb{R}^2$ and, of course, W^s is $\{y = 0\}$. When the polynomial map $x \mapsto u_1 + u_2 x - x^3$ has only one real root, we get the stable manifold (in the given rectangle) of the corresponding saddle point of ξ_u; when it has three real roots, W^s is the union of the node of ξ_u and the stable manifolds of its two saddles.

The hypotheses of Theorem 1.2.1 are satisfied for positive time by the flow† of the *unfolding* associated to the family, i.e. the vector field $\tilde{\xi}$ on $\mathbb{R}^4$ defined by $\tilde{\xi}(u, x, y) = (u_1 + u_2 x - x^3)\partial_x + y\partial_y$, with $X := \{(u, x) \in \mathbb{R}^3 / |u| \le \varepsilon, |x| \le r\}$ and $Y := [-r', r']$. Moreover, given such X, Y, this remains true for any two-parameter family η_u of vector fields on the plane, C^1-close enough to ξ_u (indeed, transversality of each ξ_u with $|u| \le \varepsilon$ to $\{|x| = r\}$ and $\{|y| = r'\}$ at every boundary point of $[-r, r] \times [-r', r']$ is a C^0-open condition). The W^s we get for $\tilde{\eta}$ has the following properties.

- It contains (by its very definition) every $(u, x) \in Z$ with $\eta_u(x) = 0$.
- By a standard argument, it is the same for every positive time of the flow, hence $\tilde{\eta}$ is tangent to it.

Its smoothness (which depends on how small ε, r, r' are and how C^1-close $\tilde{\eta}$ is to $\tilde{\xi}$) tells us that the $W^s \subset [-r, r] \times [-r', r']$ associated to each η_u depends nicely on u. Of course, if $\tilde{\eta}$ is C^4-close enough to $\tilde{\xi}$, the same 'saddle–node–saddle' bifurcation will occur as in the model case.

These results can be deduced from the centre manifold theorem (see §3.3); however, as centre manifolds are not locally unique in general, uniqueness requires additional arguments, whereas Theorem 1.2.1 implies it at once.

3.2. *Hopf and Hopf-like bifurcations for maps.*

Let F_u be a smooth (enough) one-parameter family of maps of the plane into itself‡ near 0; assume that $F_0(0) = 0$ and

† Which leaves invariant each 'slice' $u =$ constant, on which it is the flow of ξ_u.

‡ By the centre manifold theorem given later, such a family may be obtained as the restriction to a centre manifold of a family G_u defined on a manifold of finite or infinite dimension not less than 2, near a fixed point of G_0.

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that the eigenvalues $\alpha_0, \overline{\alpha_0}$ of $DF_0(0)$ have modulus 1 but do not satisfy $\alpha_0^p = 1$ for $p = 1, 2, 3, 4$. As we have $\alpha_0 \neq 1$, the solutions (u, z) of the equation $F_u(z) = z$ form near 0 the graph of a smooth implicit function χ; hence, the change of variables $(u, z) \mapsto (u, z - \chi(u))$ allows us to assume that $F_u(0) = 0$ near $u = 0$.

Since the eigenvalue α_0 of $DF_0(0)$ is simple, there exists a smooth map $\alpha : (\mathbb{R}, 0) \to (\mathbb{C}, \alpha_0)$ such that the eigenvalues of $DF_u(0)$ are $\alpha(u), \overline{\alpha(u)}$ for small u. We make the (generic) hypothesis $d/du|_{u=0}|\alpha(u)|^2 \neq 0$, which, after a smooth change of parameter u, becomes $|\alpha(u)|^2 = 1 + u$.

By normal form theory, we may assume that $\mathbb{R}^2 = \mathbb{C}$ and that, for some smooth complex function β, each $z \mapsto F_u(z) - z(\alpha(u) + \beta(u)|z|^2)$ with u small vanishes at order 3 at 0. Hence, $|F_u(re^{i\theta})|^2 = r^2(1 + u - c(u)r^2) + R_u(re^{i\theta})$, where $c(u) := -2 \operatorname{Re} e(\alpha(u)\overline{\beta(u)})$ and R_u is a smooth family of maps vanishing at order 4 at $z = 0$. Assuming† $c(0) > 0$, we can conjugate each h_u by a homothety depending smoothly on u so that $c(u) \equiv 1$; hence, finally,

$$\begin{aligned} |F_u(re^{i\theta})|^2 &= r^2(1 + u - r^2 + r^3\varphi(r, \theta, u), \\ \frac{F_u(re^{i\theta})}{|F_u(re^{i\theta})|} &= \exp i(\theta + \gamma(u) + \delta(u)r^2 + r^3\psi(r, \theta, u)), \end{aligned} \tag{27}$$

where $\varphi, \psi, \gamma, \delta$ are smooth real functions.

If we had $\varphi = 0$, then F_u would leave the circle $|z|^2 = u$ invariant for positive u. To investigate the behaviour of a more general F_u near this circle, we introduce a small positive parameter ε and write

$$r^2 = \varepsilon^4(U + x), \quad \theta = \varepsilon^{-1}t \quad \text{and} \quad u = \varepsilon^4 U$$

with $(x, t, U) \in [-1, 1] \times (\mathbb{R}/2\pi\varepsilon\mathbb{Z}) \times [4, 5]$.

LEMMA 16. *In these new coordinates, the 'unfolding' $(z, u) \mapsto (F_u(z), u)$ is expressed as a map h_ε which, for small enough ε, satisfies the hypotheses of Theorems 1.3.1 and 1.4.1 with $X = [-1, 1]$ and $F = Y = (\mathbb{R}/2\pi\varepsilon\mathbb{Z}) \times [4, 5]$; moreover, if the family F_u if C^∞, the smoothness of the map ψ associated to $h = h_\varepsilon$ by Theorem 1.3.1 tends to infinity when $\varepsilon \to 0$.*

Proof. By (27), the X-component $f_\varepsilon(x, t, U)$ of h_ε is

$$x(1 - \varepsilon^4(U + x)) + \varepsilon^6(U + x)^{3/2}\varphi(\varepsilon^2(U + x)^{1/2}, \varepsilon^{-1}t, \varepsilon^4 U)$$

and its Y-component $g_\varepsilon(x, t, U)$

$$(t + \varepsilon\gamma(\varepsilon^4 U) + \varepsilon^5(U + x)\delta(\varepsilon^4 U) + \varepsilon^7(U + x)^{3/2}\psi(\varepsilon^2(U + x)^{1/2}, \varepsilon^{-1}t, \varepsilon^4 U), U).$$

It follows that h_ε maps $Z = X \times Y$ into itself for small ε and satisfies Hypothesis 1.1 with $\kappa_0 = 1 - \varepsilon^4 + \mathrm{O}(\varepsilon^5)$, $\mu_0 = \mathrm{O}(\varepsilon^5)$ and $\nu_0 = 1 + \mathrm{O}(\varepsilon^5)$ and, hence, our lemma by Theorems 1.3.1, 1.4.1 and 2.2.1. □

† The other generic case $c(0) < 0$ (pessimistic version of the Hopf bifurcation) can be obtained from this one by replacing the original F_u by F_u^{-1}.

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Combining this lemma with (27), we see that there exist positive numbers η, ζ such that, for $|u| \leq \zeta$ and $0 < |z| \leq \eta$, the sequence $h_u^n(z)$ tends to 0 for $u \leq 0$ and to an embedded circle C_u close to $|z| = u$ for $u > 0$. Moreover, if the family F_u if C^∞, the smoothness of the surface $S_{u_0} := \bigcup_{0<u<u_0} C_u \times \{u\}$ tends to infinity when $u_0 \to 0$ (but the smoothness at $(0, 0)$ of the 'paraboloid' $S_{u_0} \cup \{(0, 0)\}$ depends on the smallest positive integer p for which $\alpha_0^p = 1$).

Generalizations. The Hopf bifurcation of Poincaré first return maps means the bifurcation of an attracting periodic orbit of a flow into an attracting quasi-periodic invariant 2-torus, a phenomenon which has drawn much attention [**17**]. Generically, it is not true [**8**] that such a 2-torus can bifurcate into an attracting invariant 3-torus, etc. However, families of invariant tori of every dimension $\leq n$ (and other interesting compact invariant manifolds) can appear locally in generic n-parameter families F_u of maps of a space of dimension (at least) $2n$ into itself (part of this is work in progress by Mathilde Kammerer). The difference with $n = 1$ is that complicated phenomena as in [**7**] can occur in the (closed) complementary subset of the set of values of the parameter for which there are attracting or normally hyperbolic invariant manifolds.

3.3. *Local pseudo-stable manifolds and stable foliations.*

3.3.1. *Hypotheses.* We consider a local C^r map $h : (V, 0) \to (V, 0)$, $r \geq 1$, where V is a Banach space. We assume that 0 is a μ-hyperbolic fixed point, meaning that the spectrum of $L := Dh(0)$ does not meet the circle of radius $\mu > 0$ centred at 0. Hence, we may assume that V is the product $E \times F$ of two Banach spaces and that $L = A \times B$, where A (respectively B) is an endomorphism of E (respectively F) with spectral radius $\rho(A) < \mu$ (respectively with $\rho(B^{-1})^{-1} > \mu$).

3.3.2. *Local pseudo-(un)stable manifolds.* For $\mu < 1$ (respectively > 1), it is easy [**13, 4**] to prove that there is a unique h-invariant manifold germ W^s (respectively W^u) with tangent space $E \times \{0\}$ (respectively $\{0\} \times F$) at 0, called the local *μ-stable* (respectively *μ-unstable*) manifold of h, and that it is C^r. In the remaining cases, let us prove the following theorem.

THEOREM 3.3.1. *For $\rho(A) \geq 1$, if E admits a C^r bump function† θ, then, for every finite $k \in \widetilde{\mathbb{R}}_+$ with $\rho(A)^k < \rho(B^{-1})^{-1}$ and $1 \leq k \leq r$, there is an h-invariant C^k submanifold germ W^s whose tangent space at 0 is $E \times \{0\}$.*

For $\rho(B^{-1})^{-1} \leq 1$, if F admits a C^r bump function, then, for every finite $k \in \widetilde{\mathbb{R}}_+$ with $\rho(B^{-1})^k < \rho(A)^{-1}$ and $1 \leq k \leq r$, there is an h-invariant C^k submanifold germ W^u whose tangent space at 0 is $\{0\} \times F$.

Proof. We sketch the proof of the first half, leaving the second to the reader. If we choose norms defining the topology of E and F so that the norms of A and B are close to their

† That is a C^r function θ with values in $[0, 1]$, equal to 1 near 0 and to 0 off the unit ball, all of whose derivatives are bounded, such that $D^{[r]}\theta$ is uniformly continuous for $r \in \mathbb{N}$ and Hölderian of exponent $r - [r]$ for non-integer $r < \infty$. Such a θ exists if E is a finite dimensional or Hilbert space, for example.

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spectral radii, then, for small enough positive ε, the hypotheses of Theorems 1.2.1 and 2.2.2† are satisfied by $X = E$, $Y = B_\varepsilon(0) \subset F$ and $h := h_\varepsilon$ given by

$$h_\varepsilon(x, y) = \begin{cases} L(x, y) & \text{for } |x| \geq \varepsilon, \\ L(x, y) + \theta(\varepsilon^{-1}x)(h - L)(x, y) & \text{otherwise,} \end{cases}$$

with κ_0, λ_0 close to $|A|, |B^{-1}|^{-1}$ respectively. Smoothness increases (but the size of the part of W^s which is invariant by the original h decreases) when $\varepsilon \to 0$. The W^s we get for h_ε contains the origin by its very definition. Its tangent space at 0 is the only L-invariant subspace complementary to $\{0\} \times F$, namely $E \times \{0\}$. □

Some definitions and comments. The manifold W^s (respectively W^u) is called a (μ)-*pseudo-stable* (respectively *pseudo-unstable*) *manifold* of h at 0. They are not unique in general. For $\rho(A) = 1$ (respectively $\rho(B^{-1})^{-1} = 1$), W^s (respectively W^u) is called a *centre-stable* (respectively *centre-unstable*) *manifold.* For $r = \infty$, they are not C^∞ in general.

If (and only if) both a centre-stable and a centre-unstable manifold exist, we can write V as the product $V^s \times V^c \times V^u$ of three L-invariant closed subspaces, corresponding respectively to the parts of the spectrum of L lying inside, on and outside the unit circle. Any centre-stable manifold is tangent to $V^s \times V^c \times \{0\}$ and any centre-unstable manifold, to $\{0\} \times V^c \times V^u$. Their intersection, tangent to $\{0\} \times V^c \times \{0\}$, is called a *centre manifold.*

With the notation of the previous proof, for fixed small enough ε, every C^1-small enough perturbation of h_ε satisfies the hypotheses of Theorems 1.2.1 and 2.2.2; the example of the saddle–node bifurcation shows that the invariant manifold we get does not necessarily contain any fixed point or invariant compact subset.

As far as I know, the technical novelty in Theorem 3.3.1 is that we do not assume the existence of a bump function on V itself. The pseudo-unstable case might be of interest in the study of parabolic semi-flows near a stationary point.

3.3.3. *Local stable foliations.* The proof of the following consequence of Theorem 3.3.1 provides a transition to the next section (the result itself can be improved [**5**] by avoiding blow-up, and we state neither the entirely analogous result for unstable foliations nor the obvious hypotheses about bump functions):

COROLLARY 3.3.2. *Assume A invertible and $r \geq 2$. Then, for every finite $k \in \widetilde{\mathbb{R}}_+$ with $1 \leq k \leq r - 1$, $\rho(A)\rho(B^{-1})^{k+1} < 1$ and $\rho(A)\rho(B)^k\rho(B^{-1}) < 1$, there exists an h-invariant C^k local foliation whose (h-invariant) leaf through 0 is tangent to $E \times \{0\}$. In particular, if h has a centre manifold, then, for every finite $k \in \widetilde{\mathbb{R}}_+$ with $1 \leq k \leq r - 1$, there are C^{k+1} centre-stable manifolds of h, and each one of them admits an h-invariant C^k foliation whose leaf through 0 is the stable manifold.*

Proof. The second statement is obtained by applying the first to the restriction of h to a C^{k+1} centre-stable manifold (which exists by Theorem 3.3.1 since we can find a decomposition $L = A \times B$ with $\rho(A) = 1 < \rho(B^{-1})^{-1} \leq \rho(B)$).

† With plain ordinary derivatives, $\exp_z \delta z = z + \delta z$ and $C = 1$.

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To prove the first statement, note that h has a C^{k+1} pseudo-unstable manifold W^u since we have $\rho(B^{-1})^{k+1} < \rho(A)^{-1}$: conjugating h by a C^{k+1} local diffeomorphism of the form $(x, y) \mapsto (x - \psi(y), y)$, we may assume that $r = k+1$ and $W^u = \{0\} \times F$. We shall define our local invariant foliation by a local C^k fibration $\pi : E \times F \to F = W^u$, tangent at 0 to the canonical projection. First, we make two basic remarks.

- The foliation defined by the fibres of π is h-invariant if and only if π is a *semi-conjugacy* between h and $g_0 = h|_{W^u}$, which means that $\pi \circ h = g_0 \circ \pi$: denoting by f, g the components of h, we see that $g_0(y) = g(0, y)$.
- A map π is a semi-conjugacy between h and g_0 if and only if its graph is invariant by $h \times g_0 : (x, y, z) \mapsto (f(x, y), g(x, y), g_0(y))$.

Hence, we are looking for a $h \times g_0$-invariant germ of a C^k submanifold (the graph of π) whose tangent space at 0 is the graph $\{(x, y, z) : z = y\}$ of the canonical projection $E \times F \to F$.

Conjugating $h \times g_0$ by the change of variables $(x, y, z) \mapsto (x, y, z - y)$, this amounts to looking for a germ W of a C^k submanifold tangent to $\{z = 0\}$ at 0 and invariant by $h_1 : (x, y, z) \mapsto (h(x, y), g(0, y + z) - g(x, y))$. Now, the differential of h_1 at 0 being $A \times B \times B$, such a W is *not* a (pseudo-)stable or (pseudo-)unstable manifold. In order to get it as a consequence of Theorem 3.3.1, we have to do something, namely a blow-up e which will 'chase the spectrum of the second B away from the origin'.

LEMMA 17. *Let $\widetilde{F} := L(E, F)$, and let e be the map $(x, y, \tilde{z}) \mapsto (x, y, \tilde{z}x)$ of $E \times F \times \widetilde{F}$ into $E \times F \times F$. Then there exists a C^k diffeomorphism germ $\tilde{h}_1 : (E \times F \times \widetilde{F}, 0) \to (E \times F \times \widetilde{F}, 0)$ admitting a C^k pseudo-stable manifold $\widetilde{W}$ tangent at 0 to a complementary subspace of $e^{-1}(0)$ and such that $h_1 \circ e = e \circ \tilde{h}_1$.*

Corollary 3.3.2 follows immediately: since $\widetilde{W}$ is the graph $\tilde{z} = \tilde{\varphi}(x, y)$ of a C^k map vanishing at 0, the h_1-invariant set $W = e(\widetilde{W})$ is the graph $z = \tilde{\varphi}(x, y)x$ of a C^k map tangent to 0 at 0.

Proof of Lemma 17. The first and last components of $h_1 \circ e(x, y, \tilde{z})$ are

$$a(x, y)x, \quad \text{where } a(x, y) := \int_0^1 \partial_1 f(tx, y)\, dt \in L(E, E),$$

and

$$b(x, y, \tilde{z})x, \quad \text{where } b(x, y, \tilde{z}) := \int_0^1 (-\partial_1 g(tx, y) + \partial_2 g(0, y + t\tilde{z}x) \circ \tilde{z})\, dt \in L(E, F).$$

As $a(0) = A$ is invertible, it follows that we can define $\tilde{h}_1$ near 0 by

$$\tilde{h}_1(x, y, \tilde{z}) := (h(x, y), b(x, y, \tilde{z}) \circ a(x, y)^{-1});$$

hence,

$$D\tilde{h}_1(0) = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ * & * & \tilde{B} \end{pmatrix},$$

where $\tilde{B} \in L(\widetilde{F}, \widetilde{F})$ is given by $\tilde{B}\tilde{z} = B \circ \tilde{z} \circ A^{-1}$. Since we have $\rho(\tilde{B}^{-1})^{-1} = \rho(B^{-1})^{-1}\rho(A)^{-1} > \rho(B)^k = \rho(A \times B)^k$, we can apply Theorem 3.3.1 with $h := \tilde{h}_1$, $A := A \times B$ and $B := \tilde{B}$ to get $\widetilde{W}$. □

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3.4. *Conjugacies and such.* We now come to the situation considered in the introduction.

3.4.1. *An exotic way to Sternberg's theorem.* We explained in the introduction how Sternberg's theorem can be deduced from the following result, which we shall now prove using Theorems 1.2.1 and 2.2.2:

THEOREM 3.4.1. *Given a Banach manifold M, let $h : (M, a) \to (M, a)$ be a smooth map germ preserving a smooth submanifold germ W_0 at a. Assume that $L := T_a h$ leaves invariant some factor subspace S of $E := T_a M$ containing $S_0 := T_a W_0$, and denote by A (respectively B, A_1) the endomorphism of S (respectively E/S, S/S_0) induced by L. Assume that:*
(i) *the space S_0 admits a smooth bump function;*
(ii) *the endomorphisms A_1, B are invertible; and*
(iii) *we have $\rho(A_1) < 1$ and $\rho(A) > 1$.*
Then, for each germ $\hat{W}$ at a of a smooth submanifold of M containing W_0, satisfying $T_a\hat{W} = S$ and formally an h-invariant along W_0, there exists a h-invariant germ W at a of a smooth submanifold of M having infinite contact with $\hat{W}$ along W_0.

Proof. In [**6**], we deduce from the pseudo-stable manifold theorem that, for each finite $k \geq 1$, there exists an h-invariant germ W_k at a of a C^k submanifold of M having kth-order contact with $\hat{W}$ along W_0. The proof goes as follows. We may assume $(M, \hat{W}, W_0, a) = (E, S, S_0, 0)$, choose a complementary subspace U (respectively S_1) of S (respectively S_0) in E (respectively S) and identify E to $S \times U = S_0 \times S_1 \times U$. For each positive integer ℓ, let $\widetilde{U}_\ell := L_s^\ell(S_1, U)$ and $\widetilde{E}_\ell := V_1 \times \widetilde{U}_\ell$. If the 'blow-up' $e_\ell : \widetilde{E}_\ell \to E$ is defined by $e_\ell(\theta, x, \tilde{y}) := (\theta, x, \tilde{y}x^\ell)$, we get as in the proof of Corollary 3.3.2 a smooth germ $h_\ell : (\widetilde{E}_\ell, 0) \to (\widetilde{E}_\ell, 0)$ such that $e_\ell \circ h_\ell = h \circ e_\ell$: indeed, denoting the components of h by α, β, γ, we have $(\beta, \gamma)(\theta, 0, 0) \equiv 0$ (invariance of W_0) and $\partial_2^j \gamma(\theta, 0, 0) \equiv 0$ for every integer j (formal invariance of $\hat{W}$); hence, $\beta \circ e_\ell(\theta, x, \tilde{y}) = b_\ell(\theta, x, \tilde{y})x$ and $\gamma \circ e_\ell(\theta, x, \tilde{y}) = c_\ell(\theta, x, \tilde{y})x^\ell$, where†

$$\begin{aligned} b_\ell(\theta, x, \tilde{y}) &= \int_0^1 (\partial_2\beta(\theta, tx, t\tilde{y}x^\ell) + \partial_3\beta(\theta, tx, t\tilde{y}x^\ell)\tilde{y}x^{\ell-1})\, dt \in L(S_1), \\ c_\ell(\theta, x, \tilde{y}) &= \int_0^1 \left(\frac{(1-t)^\ell}{\ell!} \partial_2^{\ell+1}\gamma(\theta, tx, 0)x + \partial_3\gamma(\theta, x, t\tilde{y}x^\ell)\tilde{y} \right) dt \in \widetilde{U}_\ell. \end{aligned} \tag{28}$$

As $b_\ell(0) = A_1$ is invertible, it follows that we can take

$$h_\ell(\theta, x, \tilde{y}) = (\alpha(\theta, x, \tilde{y}x^\ell), \beta(\theta, x, \tilde{y}x^\ell), c_\ell(\theta, x, \tilde{y}) \circ (b_\ell(\theta, x, \tilde{y})^{-1})^\ell). \tag{29}$$

Since $Dh_\ell(0) = A \times \widetilde{B}_\ell$ with $\widetilde{B}_\ell \tilde{y} = B \circ \tilde{y} \circ (A_1^{-1})^\ell$, we have $\rho(A)^k < \rho(\widetilde{B}_\ell^{-1})^{-1} = \rho(B^{-1})^{-1}\rho(A_1)^{-\ell}$ for large enough $\ell \geq k$, in which case, by Theorem 3.3.1, there exists an h_ℓ-invariant germ $\widetilde{W}_\ell$ at $0 \in \widetilde{E}_\ell$ of a C^k submanifold whose tangent space at 0 is $S \times \{0\}$. As $\widetilde{W}_\ell$ is the graph of a C^k germ $\tilde{\varphi}_\ell : (S, 0) \to (\widetilde{U}, 0)$, the h-invariant germ $W_k := e_\ell(\widetilde{W}_\ell)$,

† For $0 \leq m \leq \ell$, $x \in S_1$ and $\tilde{y} \in \widetilde{U}_\ell$, we denote by $\tilde{y}x^m$ the $(\ell - m)$-linear map $(x_1, \ldots, x_{\ell-m}) \mapsto y(x^m, x_1, \ldots, x_{\ell-m})$.

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graph of $\varphi : (\theta, x) \mapsto \tilde{\varphi}_\ell(\theta, x)x^\ell$, is the germ of a C^k manifold having the required contact with S along S_0.

Unfortunately, as pseudo-stable manifolds are not unique, we cannot 'pass to the limit' when $k \to \infty$: this proof does not yield a smooth W but only a C^k one for every finite k. We shall now see that the problem can be solved using Theorems 1.2.1 and 2.2.2 instead of Theorem 3.3.1.

For small positive ε, we define a smooth $\bar{h}_\varepsilon : S_0 \times \{|x| < \varepsilon\} \times \{|y| < \varepsilon\} \to E$ extending h by the formula

$$\bar{h}_\varepsilon(\theta, x, y) = \begin{cases} L(\theta, x, y) & \text{for } |\theta| \geq \varepsilon, \\ L(\theta, x, y) + u(\varepsilon^{-1}\theta)(h - L)(\theta, x, y) & \text{otherwise,} \end{cases}$$

where u is a smooth bump function on S_0. Clearly, the C^1 norm of $\bar{h}_\varepsilon - L$ tends to 0 when $\varepsilon \to 0$, all the derivatives of $\bar{h}_\varepsilon$ are bounded for small enough ε, the subspace $\bar{W}_0 = S_0 \times \{(0, 0)\}$ is $\bar{h}_\varepsilon$-invariant and the subspace $\bar{W} = S \times \{0\}$ is formally $\bar{h}_\varepsilon$-invariant along $\bar{W}_0$. The following lemma is not difficult to check.

LEMMA 18. *If we fix a small enough $\varepsilon \leq 1$, set $h := \bar{h}_\varepsilon$ and denote its components by α, β, γ, then there exists a non-increasing sequence (η_ℓ) of positive numbers $\leq \varepsilon$ and a sequence (k_ℓ) of positive numbers < 1 such that the following properties hold for every large enough ℓ.*

(i) *Theorems 1.2.1 and 2.2.2 applies to the map h_ℓ defined by (28) and (29) with $X = X_\eta := \{(\theta, x) \in S : |x| \leq \eta\}$ and $Y = Y_{\eta,\ell} := \{\tilde{y}_\ell \in \tilde{U}_\ell : |\tilde{y}_\ell| \leq \eta\}$ for each positive $\eta \leq \eta_\ell$. This yields a* unique *h_ℓ-invariant manifold W_ℓ, graph of the restriction to X_η of a function $\varphi_\ell : X_{\eta_\ell} \to Y_{\eta_\ell,\ell}$ whose smoothness tends to infinity with ℓ.*

(ii) *Each sequence $(\theta_n, x_n, y_n) = h_\ell^n(\theta, x, \varphi_\ell(\theta, x))$ satisfies $|x_n| \leq k_\ell^n|x|$.*

Now, for large $m \leq \ell$, the 'blow-up' $e_\ell^m : X_{\eta_\ell} \times Y_{\eta_\ell,\ell} \to X_{\eta_\ell} \times Y_{\eta_\ell,m}$ defined by $e_\ell^m(\theta, x, \tilde{y}_\ell) := (\theta, x, \tilde{y}_\ell x^{\ell-m})$ is such that $h_m \circ e_\ell^m = e_\ell^m \circ h_\ell$; hence, $\varphi_m(\theta, x) = \varphi_\ell(\theta, x)x^{\ell-m}$ in X_{η_ℓ} by uniqueness. It follows that φ_m is as smooth as φ_ℓ in X_{η_ℓ} and, therefore, in the whole of X_{η_m} by (ii). Since the smoothness of φ_ℓ tends to infinity with ℓ, we conclude that each φ_m with m large enough is C^∞. Hence, the h-invariant manifold $W = e_m(W_m)$, graph of $x \mapsto \varphi_m(\theta, x)x^m$, is C^∞ and satisfies all our requirements. □

The manifold W we get at the end of the proof is unique [**5**] but this requires a more classical blow-up or a more direct proof (needed anyway [**4, 5**] to get a C^k invariant W for a C^r map without requiring r to be too large).

3.4.2. *Some comments on this approach.* Even though the version of Theorem 2.2.2 needed in this case is much easier than the general one, it is not clear that the proof of Theorem 3.4.1 is simpler than the usual proofs of Sternberg's theorem! However, using the basic remarks at the beginning of the introduction, Theorem 3.4.1 can easily be shown to imply that every formal first integral of a hyperbolic germ of a diffeomorphism or vector field is the jet of at least one smooth first integral, every element of the formal centralizer of the germ is the jet of at least one element of its smooth centralizer, etc.

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The reader may wonder why such a general result as Theorems 1.2.1 and 2.2.2 is used here. An answer is that what we have just done for fixed points applies to invariant manifolds. For example, we can prove along the same lines the following result [**5**]. Let V be a compact submanifold of a manifold M and let $h_0 : (M, V) \to (M, V)$ be a smooth germ of a diffeomorphism, 'normally hyperbolic' in the following *very* weak sense: as usual, its differential leaves invariant the direct sums of TV with a stable and with an unstable bundle, one of which is contracted and the other expanded but there is *no* comparison with what happens on V. Then, every smooth germ $h_1 : (M, V) \to (M, V)$ formally conjugate to h_0 (along V) is smoothly conjugate to it. This is proven in [**1**, (4.2.3), Theorem 2] for trivial bundles. Such results are essential in the local classification of $\mathbb{Z}^\ell \times \mathbb{R}^m$-actions [**1, 2**], yielding, for example, [**1, 5**] the following generalization of Sternberg's theorem: given mutually commuting germs $h_1, \ldots, h_\ell, X_1, \ldots, X_m$ vanishing at $0 \in \mathbb{R}^n$ of diffeomorphisms and vector fields whose differentials at 0 are in a general position, they are simultaneously conjugate to another such family $h'_1, \ldots, h'_\ell, X'_1, \ldots, X'_m$ if and only if the two families are formally (simultaneously) conjugate.

3.5. *Twisted bundles and some further work.* How could a result like Theorem 1.2.1, about maps defined on a trivial bundle and satisfying a resolutely non-intrinsic hypothesis as far as bundles go, be applied in twisted bundles? The answer is at the beginning of the proof of the main result (Theorem 4.1) in [**12**]: if $h : (M, V) \to (M, V)$ is normally hyperbolic at V, then M can be viewed near V as a vector bundle over the direct sum F of TV and the unstable bundle (for example). This vector bundle $M \to F$ can be made into a sub-vector bundle of a trivial vector bundle $E \times F \to F$ to which h can be extended trivially (composing the original h with the orthogonal projection), yielding a map which satisfies the hypotheses of Theorem 1.3.1 in the 'absolutely normally hyperbolic' case.

The general normally hyperbolic case, where the 'contraction' of h in the X direction and its 'expansion' in the Y direction are compared only in each fibre of the projection $(x, y) \mapsto x$, can be treated by a refinement of the methods described here [**5**].

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A. *Appendix. A converse of Taylor's formula*

LEMMA A.1. *Let E, F be Banach spaces and r a finite element of $\widetilde{\mathbb{R}}_+$. Given $\Phi : (E, x_0) \to (F, y_0)$ and $\Phi^j : (E, x_0) \to L^j_s(E, F)$, $1 \le j \le [r]$, the following two conditions are equivalent.*

(i) *The germ Φ is C^r and $D^j\Phi = j!\,\Phi^j$, $1 \le j \le [r]$.*

(ii) *We have that $\Phi(x + h) - \Phi(x) - \Phi^1(x)h - \cdots - \Phi^{[r]}(x)h^{[r]} = \mathrm{O}_x(|h|^r)$ and the Φ^j's are germs of bounded functions.*

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Proof. First observe that, for $0 \le r \le 1^-$, there is nothing to prove. Otherwise, setting $p := [r]$, (i) yields

$$\Phi(x+h) - \Phi(x) - \sum_1^p \Phi^j(x)h^j = \int_0^1 p(1-t)^{p-1}(\Phi^p(x+th) - \Phi^p(x))h^p\,dt;$$

hence, (ii) (recall that $\Phi^p = (1/p!)D^p\Phi$ is assumed uniformly continuous if $r = p$).

To prove that (ii) implies (i), let us first assume $p = 1$. Then, (ii) implies that Φ is differentiable, that $\Phi^1 = D\Phi$ and that, if $v \in E$ satisfies $|v| = 1$,

$$|h|^{-1}(\Phi(x+h+|h|v) - \Phi(x+h)) = \Phi^1(x+h)v + \mathrm{O}_x(|h|^{r-1})$$

uniformly with respect to v; now, again by (ii), the left-hand side also reads as

$$\begin{aligned}|h|^{-1}&(\Phi(x+h+|h|v) - \Phi(x+h))\\ &= |h|^{-1}(\Phi(x) + \Phi^1(x)(h+|h|v) - (\Phi(x) + \Phi^1(x)h)) + \mathrm{O}_x(|h|^{r-1})\\ &= \Phi^1(x)v + \mathrm{O}_x(|h|^{r-1})\end{aligned}$$

uniformly with respect to v. Subtracting these two estimates of the same thing, we see that $(\Phi^1(x+h) - \Phi^1(x))v = \mathrm{O}_x(|h|^{r-1})$ uniformly with respect to v and, therefore, $\Phi^1(x+h) - \Phi^1(x) = \mathrm{O}_x(|h|^{r-1})$, which proves that Φ^1 is C^{r-1}.

For $p > 1$, we can make the induction hypothesis that our lemma is true for $[r] = p-1$; thus, if (ii) is satisfied, as Φ^p is bounded, we know that Φ is C^{p-1} and that $\Phi^j = (1/j!)D^j\Phi$ for $1 \le j \le p-1$. Hence, we just have to prove that Φ^{p-1} is C^{r-p+1} and that $D\Phi^{p-1} = p\Phi^p$. The proof is very much the same as before: setting $\Delta_w\Phi(x) = \Phi(x+w) - \Phi(x)$, we shall give two different estimates of

$$|h|^{-p+1}(\Delta_{|h|v})^{p-1}\Phi(x+h) = |h|^{-p+1}\sum_{j=0}^{p-1}(-1)^{p-1-j}\binom{p-1}{j}\Phi(x+h+j|h|v)$$

for $|v| = 1$: by (ii), setting $\Phi^0 := \Phi$,

$$\begin{aligned}&|h|^{-p+1}(\Delta_{|h|v})^{p-1}\Phi(x+h)\\ &\quad= |h|^{-p+1}\sum_{j=0}^{p-1}(-1)^{p-1-j}\binom{p-1}{j}\sum_{k=0}^{p}\Phi^k(x+h)(j|h|v)^k + \mathrm{O}_x(|h|^{r-p+1})\\ &\quad= |h|^{-p+1}\sum_{k=0}^{p}\Phi^k(x+h)(|h|v)^k\sum_{j=0}^{p-1}(-1)^{p-j}\binom{p-1}{j}j^k + \mathrm{O}_x(|h|^{r-p+1})\\ &\quad= (p-1)!\Phi^{p-1}(x+h)v^{p-1} + \mathrm{O}_x(|h|^{r-p+1}) \qquad \text{(A.1)}\end{aligned}$$

(uniformly in v) because†

$$\sum_{j=0}^{p-1}(-1)^{p-1-j}\binom{p-1}{j}j^k = (p-1)!\delta^k_{p-1}. \qquad \text{(A.2)}$$

† As $\binom{p}{j}j^k = p\binom{p-1}{j-1}j^{k-1}$ for positive j, p, k.

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Now, (ii) also yields

$$\begin{aligned}
&|h|^{-p+1}(\Delta_{|h|v})^{p-1}\Phi(x+h)\\
&\quad= |h|^{-p+1}\sum_{j=0}^{p-1}(-1)^{p-1-j}\binom{p-1}{j}\sum_{k=0}^{p}\Phi^k(x)(h+j|h|v)^k+\mathrm{O}_x(|h|^{r-p+1})\\
&\quad= |h|^{-p+1}\sum_{j=0}^{p-1}(-1)^{p-1-j}\binom{p-1}{j}\sum_{k=0}^{p}\sum_{\ell=0}^{k}\binom{k}{\ell}\Phi^k(x)h^{k-\ell}(j|h|v)^\ell\\
&\qquad+\mathrm{O}_x(|h|^{r-p+1})\\
&\quad= |h|^{-p+1}\sum_{k=0}^{p}\sum_{\ell=0}^{k}\binom{k}{\ell}\Phi^k(x)h^{k-\ell}(|h|v)^\ell\sum_{j=0}^{p-1}(-1)^{p-1-j}\binom{p-1}{j}j^\ell\\
&\qquad+\mathrm{O}_x(|h|^{r-p+1})\\
&\quad= (p-1)!\Phi^{p-1}(x)v^{p-1}+p!\Phi^p(x)hv^{p-1}+\mathrm{O}_x(|h|^{r-p+1}) \qquad (A.3)
\end{aligned}$$

(uniformly in v) by (A.2). Subtracting (A.3) from (A.1), we get that

$$(p-1)!(\Phi^{p-1}(x+h)-\Phi^{p-1}(x)-p\Phi^p(x)h)v^{p-1}=\mathrm{O}_x(|h|^{r-p+1})$$

uniformly with respect to v and, therefore,

$$\Phi^{p-1}(x+h)-\Phi^{p-1}(x)-p\Phi^p(x)h=\mathrm{O}_x(|h|^{r-p+1}),$$

which enables us to conclude, applying the first step to Φ^{p-1}. □

B. *Appendix. Finsler manifolds, connections and Lemma 9*

B.1. *Finsler manifolds.* A C^r Finsler manifold, $r\geq 1$, is a connected C^r Banach manifold M endowed with a Finsler structure, which means the following: each tangent space T_xM is endowed with a norm $|\cdot|_x$ defining the original topology on T_xM, and the map $TM\ni(x,v)\mapsto|v|_x$ is continuous: more precisely, for each chart χ of M and each $a\in M$, we have that $\lim_{x\to a}|(T_x\chi)^{-1}T_a\chi(v)|_x=|v|_a$ uniformly with respect to $v\in\{|v|_a\leq 1\}$. Every piecewise C^1 continuous arc $\gamma:[0,1]\to M$ has a *length* $\ell(\gamma):=\int_0^1|\gamma'(t)|_{\gamma(t)}\,dt$, and we can define the (geodesic) *distance* between two points $a,b\in M$ to be $d(a,b)=\inf\ell(\gamma)$, the infimum being taken on all piecewise C^1 continuous arcs $\gamma:[0,1]\to M$ such that $\gamma(0)=a$ and $\gamma(1)=b$. This does make M into a metric space.

In the proof of Lemma 12, we used the fact that every C^1 map f between Finsler manifolds satisfies $\mathrm{Lip}\,f\geq\sup|T_xf|$. The proof goes as follows: for every C^1 arc γ in a Finsler manifold, $\lim_{\varepsilon\to0^+}\varepsilon^{-1}d(\gamma(0),\gamma(\varepsilon))=|\gamma'(0)|_{\gamma(0)}$; therefore, for $|v|_x=1$, $\gamma(0)=x$ and $\gamma'(0)=v$, we get

$$|T_xf(v)|=\lim_{\varepsilon\to0^+}\varepsilon^{-1}d(f(\gamma(0)),f(\gamma(\varepsilon)))\leq\mathrm{Lip}\,f\lim_{\varepsilon\to0^+}\varepsilon^{-1}d(\gamma(0),\gamma(\varepsilon))=|v|\,\mathrm{Lip}\,f.$$

B.2. *Connections.* Recall that a C^r connection C, $r\geq1$, on a C^{r+2} manifold M modelled on a Banach space E is a C^r field $TM\ni x\mapsto H_x$ of horizontal subspaces:

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each H_x is a complementary subspace in $T_x(TM)$ of the *vertical space* V_x, kernel of the differential at x of the canonical projection $\tau : TM \to M$; hence, we may identify the connection to the field of projections $T_x(TM) \to V_x$ with kernel H_x; now, since V_x is the tangent space at x of the fiber $T_{\tau(x)}M$, it is canonically isomorphic to it and a connection C can be viewed as a field of linear maps $C_x : T_x(TM) \to T_{\tau(x)}M$ such that each $C_x|_{V_x}$ is the canonical isomorphism. A C^1 path c in TM is called *horizontal* or *parallel* if it satisfies $c'(t) \in H_{c(t)}$ or, equivalently, $Dc/dt(t) := C_{c(t)}(c'(t)) = 0$ for every t. A C^2 path γ in M is called a *geodesic* of C if $\dot\gamma : t \mapsto (\gamma(t), \gamma'(t)) \in TM$ is parallel.

In a chart of M, the points of TM can be written $(x, v) \in E^2$ and each horizontal space is the graph of a continuous linear map $-\Gamma_{(x,v)} : E \to E$, the connexion being the field of linear maps $(\delta x, \delta v) \mapsto \delta v + \Gamma_{(x,v)}\delta x$. The connexion is called *linear* when $\Gamma_{(x,v)}$ is a linear function of v, in which case we write $\Gamma_x(v, \delta x) := \Gamma_{(x,v)}\delta x$; thus, a linear connexion is described in a chart χ by its 'Christoffel map' Γ, which to each $x \in \operatorname{Im}\chi \subset E$ associates the continuous bilinear map Γ_x of $E \times E$ into E. A path γ in M is a geodesic if and only if, for each chart χ of M at some point of $\operatorname{Im}\gamma$, the function $x(t) := \chi(\gamma(t))$ satisfies $x''(t) + \Gamma_{x(t)}x'(t)^2 = 0$. As this is a second-order equation, there is an open neighbourhood V of the zero section in TM such that, for each $(a, v) \in V$, there exists a unique geodesic $\gamma : [0, 1] \to M$ with $\gamma(0) = a$ and $\gamma'(0) = v$. Setting $\exp_a tv := \gamma(t)$, the map $V \ni (a, v) \mapsto \exp_a v$ is C^r: it satisfies $\exp_a 0 = a$ and (by its very definition) $T_0 \exp_a = \mathrm{Id}_{T_aM}$. Hence, there exists an open 'tube' $W \subset V$ around the zero section such that the map $W \ni (a, v) \mapsto (a, \exp_a v)$ is a C^r-diffeomorphism onto an open neighbourhood of the diagonal in $M \times M$.

B.3. *Proof of Lemma 9.* Clearly, Ψ is C^r near a if and only if Ψ_a is C^r near 0. Near each $a \in M$, we may identify M, N to $E := T_aM$ and $F := T_{\Psi(a)}N$ by $\exp_a, \exp_{\Psi(a)}$ and replace the Finsler structures by the fixed norms $|\cdot|_a, |\cdot|_{\Psi(a)}$, which, in our identification, are uniformly equivalent to the Finsler norms nearby.

Assuming that $\Phi := \Psi_a$ is C^r, for each x near $a = 0$ and each small $h \in E$, we have (since $\Psi = \Phi$ in our identification)

$$\Psi_x(h) = \exp^{-1}_{\Phi(x)} \Phi(\exp_x h) = \sum_{j=1}^{[r]} \frac{1}{j!} D^j(\exp^{-1}_{\Phi(x)} \circ \Phi \circ \exp_x)(0)h^j + \mathrm{O}_x(|h|^r)$$

$$= \sum_{j=1}^{[r]} \frac{1}{j!} D^j_x \Psi h^j + \mathrm{O}_x(|h|^r),$$

hence (ii). Conversely, if (ii) holds, then, setting $v := \exp^{-1}_x(x + h)$,

$$\Phi(x + h) - \Phi(x) = \exp_{\Phi(x)} \Psi_x(v) - \exp_{\Phi(x)} 0$$

$$= \exp_{\Phi(x)} \left(\sum_{j=1}^{[r]} \Psi^j_x v^j + \mathrm{O}_x(|v|^r) \right) - \exp_{\Phi(x)} 0;$$

as we can write Taylor expansions $v = \sum_1^{[r]} v_k(x)h^k + \mathrm{O}_x(|h|^r)$ and (since Φ is continuous) $\exp_{\Phi(x)} w - \exp_{\Phi(x)} 0 = \sum_1^{[r]} e_k(x)w^k + \mathrm{O}_x(|w|^r)$, it does follow that Φ fulfils criterion (ii) of Lemma A.1. □

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