Math. Struct. in Comp. Science (2004), *vol.* 14, *pp.* 815–832. © 2004 Cambridge University Press
DOI: 10.1017/S0960129504004372 Printed in the United Kingdom

Domains, integration and 'positive analysis'

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Received 13 March 2002; revised 29 July 2003

This article surveys a variety of approaches to integration theory that have arisen in the context of continuous domain theory. The presentation is set in a broader context of 'positive' analysis, analysis where order-theoretic notions and notions of positivity play a key role.

1. Introduction

The theory of Riemann integration of real-valued functions was developed by Cauchy, Riemann, Stieltjes, Darboux and other mathematicians of the nineteenth century. With its elementary and constructive formulation and its computational power, primarily through the Fundamental Theorem of Calculus, it rapidly became foundational for the calculus, and ubiquitous throughout the physical and engineering sciences. However, its well-known limitations led to the introduction in the early twentieth century of the more versatile and mathematically more powerful Lebesgue integral. Nonetheless, the more constructive nature of the Riemann integral frequently makes its use advantageous in typical computational settings. In this paper we consider the problem of defining a 'Riemann-like' integral of $f : X \to \mathbb{R}$, where X is a general topological space, and survey the various solutions that have been proposed in the context of domain theory. One major motivation has been the desire to develop a constructive approach to integration in this general context. An advantage of such constructive approaches is that they may suggest algorithmic approaches for approximating the resulting integral.

There are certain characteristics or distinctions of the accompanying analysis that has arisen in the domain-theoretic approach to integration. Notable among these is the prominence of notions of order and positivity. In our survey of integration theory we attempt to highlight some of the prominent viewpoints, features and techniques of this broader 'positive' analysis that have had their origin in domain theory. An important role in this theory is played by the non-negative real numbers $\mathbb{R}_+ = [0, \infty)$ and their closure $\overline{\mathbb{R}}_+ = [0, \infty]$. We extend the operations of addition and multiplication from $\mathbb{R}_+$ to $\overline{\mathbb{R}}_+$ by defining $t + \infty = \infty + t = \infty$ for all $t \in \overline{\mathbb{R}}_+$, and defining $0 \cdot \infty = \infty \cdot 0 = 0$, and $\infty \cdot t = t \cdot \infty = \infty$ otherwise. Note that these extended operations are Scott-continuous.

2. Valuations

We assume throughout that the reader is familiar with the basic notions of domain theory as they may be found, for example, in Abramsky and Jung (1995) or Gierz *et al.* (2003).

For general integration theory, we must have some measure against which to integrate. We restrict our attention to the case of non-negative extended real-valued functions

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$f : X \to [0, \infty] = \overline{\mathbb{R}}_+$ defined on a topological space. From a computational viewpoint the open sets are typically computable subsets, so one is motivated to model a measure by its restriction to the open sets, and there also turn out to be good mathematical reasons for considering these 'reduced measures'.

Definition 2.1. A *valuation* on a lattice $(L, \vee, \wedge)$ is a function $v : L \to [0, \infty]$ satisfying:

(i) $v(a) + v(b) = v(a \vee b) + v(a \wedge b)$
(ii) $v(\perp) = 0$
(iii) $a \leqslant b \Rightarrow v(a) \leqslant v(b)$.

Property (i) is called the *modular* property or the *inclusion–exclusion principle*. Our case of primary interest will be the lattice $(\Omega X, \cup, \cap)$ of all open sets of a topological space X (although some results remain valid for more general lattices). In this case, we say that a valuation v is *continuous* if

$$v\left(\bigcup_{O \in \mathscr{D}}{}^{\uparrow} O\right) = \sup\{v(O) : O \in \mathscr{D}\}$$

for $\mathscr{D}$ a directed set (with respect to $\subseteq$) of open sets in X. Being continuous is precisely the same as being Scott-continuous as a function from the complete lattice ΩX to the continuous lattice $\overline{\mathbb{R}}_+ = [0, \infty]$.

There are very close connections between continuous valuations and regular Borel measures via extensions and restrictions. (Recall that a Borel measure is a σ-additive measure on the Borel algebra, the smallest σ-algebra containing the open sets, and it is *regular* if $\mu(E)$ can be approximated from the inside by compact subsets and from the outside by open supersets for any Borel set E.) Of course, the easy direction is that of restriction.

Lemma 2.2. If μ is a regular Borel measure on any topological space, or simply a Borel measure on a space with a countable basis of open sets, then μ restricted to the open sets is a continuous valuation.

Proof. Properties (i), (ii) and (iii) are standard basic facts from measure theory. Let $\mathscr{D}$ be a directed family of open sets with union U. If μ is regular and $s < \mu(U)$, then there exists a compact set K such that $s < \mu(K) \leqslant \mu(U)$. Finitely many members of $\mathscr{D}$ cover K, and since $\mathscr{D}$ is directed, there is one single member $V \in \mathscr{D}$ such that $K \subseteq V$. Then $s < \mu(K) \leqslant \mu(V)$. Since s was arbitrary below $\mu(U)$, it follows that $\mu(U) = \sup\{\mu(V) : V \in \mathscr{D}\}$.

If the space has a countable basis of open sets, then for each $x \in U$, there exists B_x in the basis such that $x \in B_x \subseteq V$ for some $V \in \mathscr{D}$. Since the basis is countable, we can enumerate the set $\{B_x : x \in U\}$, say $\{B_1, B_2, B_3, \ldots\}$. Set $W_n = \bigcup_{i=1}^{n} B_i$. Then the collection $\{W_n : n \in \mathbb{N}\}$ is an increasing sequence with union U. Since the measure is σ-additive, for any $s < \mu(U)$, there exists W_n such that $s < \mu(W_n) \leqslant \mu(U)$. By the directedness of $\mathscr{D}$, there exists $V \in \mathscr{D}$ such that $W_n \subseteq V$. Then $s < \mu(V)$; it again follows that $\mu(U) = \sup\{\mu(V) : V \in \mathscr{D}\}$. □

Note that the second part of the preceding proof remains valid for spaces in which every open set is Lindelöf.

We shall return to the problem of extending a valuation to a measure, but only to the generated algebra, not the generated σ-algebra. There does exist a substantial body of

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results on extending continuous valuations to regular Borel measures, and one then obtains a theory of Lebesgue integration via classical measure theory. However, we do not survey these developments since our focus is on more constructive, Riemannian approaches. For general results on extending continuous valuations to Borel measures, see, for example, Alvarez-Manilla (2000), Alvarez-Manilla *et al.* (1999) and Lawson (1982).

3. Horizontal integrals

The problem of integration based on valuations sharpens to the following: let $f : X \to [0, \infty]$ be a function defined on a topological space X, and let μ be a (continuous) valuation on ΩX, the lattice of open sets. We will develop a theory of Riemann-like integration in this setting. But there are important variants of this problem that we shall consider later. For example, the measure may be known only through a sequence or net of approximating simpler measures.

The theory of domains suggests that we concentrate on the lower semicontinuous functions (when X is locally compact, or more generally core compact, the collection $\mathrm{LSC}(X, \overline{\mathbb{R}}_+)$ of lower semicontinuous functions with the pointwise order is a continuous lattice). However, there are reasons for developing this section in a more general framework that includes the particular case of lower semicontinuous functions as the motivating example, particularly since the more general approach does not involve additional work, so this will be our approach.

Let X be a set, let ΩX be a lattice of subsets of X that contains the empty set, and let $\mu : \Omega X \to \overline{\mathbb{R}}_+$ be a valuation. We consider the class of *upper measurable functions*, functions $f : X \to \overline{\mathbb{R}}_+$ such that $U_t := f^{-1}(]t, \infty]) \in \Omega X$ for all $t \in \mathbb{R}_+$. Note that the upper measurable functions are precisely the lower semicontinuous functions if ΩX is the lattice of open sets of a topological space X, so our framework does include the special case of open set lattices and lower semicontinuous functions.

We define an integral directly from a valuation μ by

$$\fint_X f \, d\mu := \int_0^\infty f^\mu(t)\, dt, \text{ where } f^\mu(t) := \mu(f^{-1}(]t, \infty])).$$

Since f^μ is immediately seen to be monotone decreasing and non-negative, we have that its improper Riemann integral $\int_0^\infty f^\mu(t)\, dt$ exists and is either convergent to some finite value or is divergent to ∞. It is a straightforward exercise in the theory of the improper Riemann integral to verify that

$$\fint_X f \, d\mu = \int_0^\infty f^\mu(t)\, dt = \lim_{n\to\infty} \sum_{i=1}^{n2^n} \frac{1}{2^n} f^\mu\left(\frac{i}{2^n}\right). \tag{3.1}$$

Indeed, note that $s_n := \sum_{i=1}^{n2^n} (1/2^n) f^\mu(i/2^n)$ is the (lower) Riemann sum of f^μ over the interval $[0, n]$ partitioned into subintervals of length $1/2^n$ with the selection of the right-hand endpoint in each subinterval. Hence the sequence $\{s_n\}$ is monotone increasing, and thus convergent in $\overline{\mathbb{R}}_+$.

We call the integral defined in this way the *horizontal integral*. *Monotonicity* of the horizontal integral, namely $f(x) \leqslant g(x)$ for all x implies $\fint_X f \, d\mu \leqslant \fint_X g \, d\mu$, follows directly

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from the definition – see König (1997, Chapter IV) for a more extensive treatment of the horizontal integral. The treatment of the horizontal integral in the context of domain theory was worked out by R. Tix in Tix (1995). We note that the concept of the horizontal integral resembles the basic idea of Lebesgue for the formation of an integral, but that the formula (3.1) also gives it computational content.

We now turn to an alternative characterisation of the horizontal integral in terms of *upper measurable simple functions* – functions $f : X \to \overline{\mathbb{R}}_+$ that are upper measurable and assume only finitely many values, all of which are finite. For each n, define $\delta_n : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ by

$$\delta_n(t) = \begin{cases} \frac{i-1}{n2^n}, & \text{if } (i-1)/2^n < t \leqslant i/2^n \text{ for some } i \in \mathbb{N},\ 1 \leqslant i \leqslant n2^n; \\ n, & \text{for } n < t. \end{cases}$$

Note that the monotone increasing step function δ_n is Scott-continuous (equal upper measurable for the Scott topology) as a mapping from $\overline{\mathbb{R}}_+$ to $\overline{\mathbb{R}}_+$ and has finite image contained in $\mathbb{R}_+$. Furthermore, the sequence δ_n is monotonically increasing with (pointwise) supremum the identity map on $\overline{\mathbb{R}}_+$, that is, $\bigvee^{\uparrow} \delta_n = \mathrm{id}_{\overline{\mathbb{R}}_+}$. Hence, if $f : X \to \overline{\mathbb{R}}_+$ is upper measurable, then $f_n := \delta_n f$ is upper measurable, has finite image contained in $\mathbb{R}_+$, and $\bigvee^{\uparrow} f_n = f$. The sequence $\{f_n\}$ is called the *canonical sequence of simple functions converging to* f.

One verifies directly that the sequence f_n has an alternative description given by

$$f_n = \sum_{i=1}^{n2^n} \frac{1}{2^n} \chi_{f^{-1}(]i/2^n, \infty])}, \tag{3.2}$$

where χ_A is the characteristic function for $A \subseteq X$. Since for $(i-1)/2^n \leqslant t < i/2^n \leqslant n$, we have

$$f_n^{\mu}(t) = \mu((f_n)^{-1}(]t, \infty])) = \mu\left(f^{-1}\left(\left]\frac{i}{2^n}, \infty\right]\right)\right) = f^{\mu}\left(\frac{i}{2^n}\right), \tag{3.3}$$

and $f_n^{\mu}(t) = 0$ for $t \geqslant n$, we have

$$\fint_X f_n \, d\mu = \int_0^{\infty} f_n^{\mu}(t)\, dt = \sum_{i=1}^{n2^n} \frac{1}{2^n} f^{\mu}\left(\frac{i}{2^n}\right). \tag{3.4}$$

Applying (3.1) and (3.4), we conclude

$$\begin{aligned} \fint_X f \, d\mu = \int_0^{\infty} f^{\mu}(t)\, dt &= \lim_{n\to\infty} \sum_{i=1}^{n2^n} \frac{1}{2^n} f^{\mu}\left(\frac{i}{2^n}\right) \\ &= \lim_{n\to\infty} \int_0^{\infty} f_n^{\mu}(t)\, dt = \lim_{n\to\infty} \fint_X f_n \, d\mu. \end{aligned} \tag{3.5}$$

A function $f : X \to \overline{\mathbb{R}}_+$ is said to *μ-vanish at infinity* if $\mu(f^{-1}(]t, \infty])) < \infty$ for all $t > 0$. Note that f μ-vanishes at ∞ if and only if $f^{\mu}(t) < \infty$ for all $t > 0$, and that otherwise $f^{\mu}(t) = \infty$ on some interval $[0, \epsilon]$ for some $\epsilon > 0$. It thus follows from (3.4) that f μ-vanishes at ∞ if and only if $\int_X f_n \, d\mu < \infty$ for all n.

Proposition 3.1. Let $f : X \to \overline{\mathbb{R}}_+$ be upper measurable, and let μ be a valuation on ΩX. The integral $\fint_X f \, d\mu = \lim_n \fint_X f_n \, d\mu$, where f_n is the canonical sequence of simple functions

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converging to f. The sequence on the right is monotonically increasing to $\fint_X f\,d\mu$, and assumes only finite values if and only if f μ-vanishes at ∞. In particular, $\fint_X f\,d\mu$ is the supremum of all $\fint_X \sigma\,d\mu$, as σ ranges over all upper measurable simple functions that are bounded above by f.

Proof. The first assertion follows from (3.5), the second from the monotonicity of the integral and the remarks just before the proposition, and the last again from the monotonicity of the integral and the first assertion. □

Remarks. It follows easily from the definition of the horizontal integral that $\fint_X \chi_U\,d\mu = \mu(U)$ for $U \in \Omega X$, and thus the horizontal integral may be viewed as an extension of the valuation μ from the lattice ΩX to the upper measurable functions, or from the lattice of open sets to the lower semicontinuous functions in the topological case.

The last assertion of the previous proposition gives an alternative characterisation of the horizontal integral, and was the one used by O. Kirch in his treatment of domain integration (Kirch 1993), although he took a different route to the evaluation of the integral of simple functions. We have not derived the linearity of the horizontal integral; for this one needs some rather careful analysis with the modular property. We prefer to derive it via an alternative approach in the next section.

4. Riemann–Stieltjes integration

In this section we define the integral $\int_X f\,d\mu$ for a valuation μ on the lattice of open sets ΩX of X by extending the valuation to a finitely additive measure on the algebra of sets generated by ΩX. The theory so closely resembles that of classical Riemann integration that we only sketch the development. Since Stieltjes suggested the extension of the Riemann integral to more general measures or masses (on the real line), it seems appropriate to call these Riemann–Stieltjes integrals for algebras of sets. Details may be found in Rao and Rao (1983, Chapter 4.5), where such integrals are called S-integrals for short (a terminology we also adopt), or in König (1997), where they are called *vertical integrals*.

Let X be a topological space and let μ be a *finite* valuation on ΩX, a valuation such that $\mu(X) < \infty$. Sets of the form $U \setminus V$, where U and V are open, are known in topology as *locally closed* sets; we refer to them more briefly as *crescents*. We may always assume without loss of generality that $V \subseteq U$ by replacing V by $U \cap V$ if necessary. By appropriate manipulation of the modular law, one deduces that $\mu(U) - \mu(V)$ is independent of U and V as long as $V \subseteq U$ and $U \setminus V$ gives some fixed crescent set. Thus μ extends to all crescents. The set of crescents forms a *semialgebra*, a non-empty collection of subsets closed under finite intersections and having the property that for each $S \in \mathscr{S}$, the complement S^c is a finite disjoint union of members of $\mathscr{S}$. If one additionally adjoins all finite disjoint unions of crescents, one obtains an *algebra* (a non-empty collection of subsets closed under finite unions, finite intersections, and complements), called the *crescent algebra*.

The *Smiley–Horn–Tarski Theorem* (see König (1997) or Gierz *et al.* (2003), for example) asserts that extending finite valuations from lattices to finitely additive measures is always possible. (Recall that $\mu : \mathscr{A} \to \mathbb{R}_+$ defined on an algebra of sets $\mathscr{A}$ is *finitely additive* if

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$\mu(\varnothing) = 0$ and $\mu(A \cup B) = \mu(A) + \mu(B)$ for disjoint $A, B \in \mathscr{A}$. The other expected properties of a finitely additive measure follow from these.)

Theorem 4.1. Let L be a lattice of subsets of a set X such that $\varnothing, X \in L$, and let $\mathscr{A}$ be the algebra of subsets generated by L. If $\phi : L \to \mathbb{R}_+$ is a valuation, then ϕ extends uniquely to a finitely additive measure on $\mathscr{A}$. In particular, any valuation $\mu : \Omega X \to \mathbb{R}_+$ on the lattice of open sets of a topological space X extends uniquely to a finitely additive measure on the crescent algebra – the smallest algebra containing the open sets.

We thus begin with X a non-empty set, $\mathscr{A}$ an algebra of subsets of X and μ a non-negative finitely additive measure on $\mathscr{A}$, with the particular case of the extension of a valuation on ΩX to the crescent algebra in mind. Note that we are implicitly assuming throughout this section that valuations and finitely additive measures have image contained in $\mathbb{R}_+$, and hence are *bounded*. We will occasionally remind the reader explicitly that we are only considering bounded finitely additive measures.

An *$\mathscr{A}$-partition* of X is a partition by subsets all of which belong to $\mathscr{A}$. Let $f : X \to \mathbb{R}$ be a bounded function. The *Darboux upper sum* and *Darboux lower sum* of f with respect to μ and $\mathscr{P}$, a finite $\mathscr{A}$-partition of X, are defined by

$$S^u(f, \mu, \mathscr{P}) = \sum_{P \in \mathscr{P}} \sup f(P)\mu(P) \text{ and } S^l(f, \mu, \mathscr{P}) = \sum_{P \in \mathscr{P}} \inf f(P)\mu(P),$$

respectively. Since f and μ are bounded, the upper and lower sums, $S^u(f, \mu, \mathscr{P})$ and $S^l(f, \mu, \mathscr{P})$, are well-defined.

The following proposition follows in a straightforward manner from the previous definition.

Proposition 4.2. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two finite $\mathscr{A}$-partitions. If $\mathscr{P}_1$ refines $\mathscr{P}_2$ in the sense that every set in $\mathscr{P}_2$ is a union of sets in $\mathscr{P}_1$, then we have the following inequalities:

$$S^l(f, \mu, \mathscr{P}_2) \leqslant S^l(f, \mu, \mathscr{P}_1) \leqslant S^u(f, \mu, \mathscr{P}_1) \leqslant S^u(f, \mu, \mathscr{P}_2).$$

Since taking all non-empty pairwise intersections of two $\mathscr{A}$-partitions gives rise to an $\mathscr{A}$-partition refining both, the set of $\mathscr{A}$-partitions ordered by refinement forms a directed set. It follows from the previous proposition that assigning to each $\mathscr{A}$-partition its Darboux upper (respectively, lower) sum gives rise to a monotone decreasing (respectively, increasing) net in $\mathbb{R}$ that is bounded below by any Darboux lower sum (respectively, bounded above by any Darboux upper sum). Thus both nets converge. The *upper integral* and *lower integral* of f with respect to μ and $\mathscr{A}$ are defined as the limits of the appropriate net:

$$\overline{\int} f \, d\mu = \inf_{\mathscr{P}} S^u(f, \mu, \mathscr{P}) = \lim_{\mathscr{P}} S^u(f, \mu, \mathscr{P}),$$

$$\underline{\int} f \, d\mu = \sup_{\mathscr{P}} S^l(f, \mu, \mathscr{P}) = \lim_{\mathscr{P}} S^l(f, \mu, \mathscr{P}),$$

where $\mathscr{P}$ is a finite $\mathscr{A}$-partition of X.

Definition 4.3. A bounded function $f : X \to \mathbb{R}$ is said to be *S-integrable* (with respect to an algebra $\mathscr{A}$ and a finitely additive non-negative bounded measure μ on it) if the upper

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integral and the lower integral of f with respect to μ and $\mathscr{A}$ agree. If f is S-integrable, the integral of f is called the *Riemann–Stieltjes* integral, is denoted by $S\int f\,d\mu$, and is defined to be the common value of the lower and upper integral, that is,

$$S\int_X f\,d\mu = \int_- f\,d\mu = \int^- f\,d\mu.$$

Remark 4.4. Suppose that $\mathscr{S}$ is a semialgebra, for example the semialgebra consisting of crescent sets. Then the collection $\mathscr{A}$ consisting of the empty set and all finite disjoint unions of members of $\mathscr{S}$ is an algebra, the smallest algebra of sets containing $\mathscr{S}$, called the algebra generated by $\mathscr{S}$. Given a finitely additive bounded measure μ on $\mathscr{A}$, one can consider only $\mathscr{S}$-partitions and define S-integrals with respect to $\mathscr{S}$ in a manner entirely analogous to the way they are defined for an algebra of sets. Furthermore, since any finite $\mathscr{A}$-partition may be refined to a finite $\mathscr{S}$-partition, the $\mathscr{S}$-partitions are cofinal in the directed set of $\mathscr{A}$-partitions, and thus the upper (respectively, lower) integrals with respect to $\mathscr{S}$ and $\mathscr{A}$ agree. Hence a bounded real-valued function f is S-integrable with respect to $\mathscr{S}$ if and only if it is S-integrable with respect to $\mathscr{A}$, and in this case the integrals agree. Thus the whole theory applies equally well to semialgebras as well as algebras, with the same integral arising for a semialgebra and the algebra it generates, although we present our results in the algebra context.

The following straightforward generalisations from classical Riemann integration theory are collected from Rao and Rao (1983).

Theorem 4.5. Let $f : X \to \mathbb{R}$ be a bounded function. Then the following are equivalent:

(1) f is S-integrable.

(2) There exists a real number I with the following property: for every $\epsilon > 0$ there exists a finite $\mathscr{A}$-partition $\mathscr{P} = \{A_1, \ldots, A_n\}$ of X such that for every $x_i \in A_i$, $i = 1, \ldots, n$,

$$\left| \sum_1^n f(x_i)\mu(A_i) - I \right| < \epsilon.$$

(3) For every $\epsilon > 0$ there exists a finite $\mathscr{A}$-partition $\mathscr{P} = \{A_1, \ldots, A_n\}$ such that

$$\sum_1^n \sup_{x_i, y_i \in A_i} |f(x_i) - f(y_i)|\mu(A_i) < \epsilon.$$

In case (2), the real number I is unique and is equal to $S\int_X f\,d\mu$. In cases (2) and (3), the inequality continues to hold for any partition that refines $\mathscr{P}$.

Theorem 4.6. If f and g are S-integrable and $a \in \mathbb{R}$, then af and $f + g$ are S-integrable and

$$S\int_X af\,d\mu = a\left(S\int_X f\,d\mu\right), \quad S\int_X (f+g)\,d\mu = S\int_X f\,d\mu + S\int_X g\,d\mu.$$

A *simple* function is one that takes on only finitely many values and an *$\mathscr{A}$-measurable* function $f : X \to \mathbb{R}$ is one that satisfies $f^{-1}(I) \in \mathscr{A}$ for each finite or infinite interval $I \subseteq \mathbb{R}$.

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Lemma 4.7. Let $h = \sum_{i=1}^{n} a_i \chi_{A_i}$, where $\mathscr{P} = \{A_1, \ldots, A_n\}$ is an $\mathscr{A}$-partition, be a simple function. Then h is $\mathscr{A}$-measurable and integrable with integral $\sum_{i=1}^{n} a_i \mu(A_i)$.

Proof. Since the inverse image of every interval is a union of finitely many members of $\mathscr{P}$, h is measurable. For the partition $\mathscr{P}$ and the function h, condition (2) of Theorem 4.5 is satisfied for every $\epsilon > 0$ if I is chosen to be $\sum_{i=1}^{n} a_i \chi_{A_i}$, and hence the latter is the S-integral of h. □

Theorem 4.8. A bounded function $f : X \to \mathbb{R}$ is S-integrable if and only if for each $\epsilon > 0$ there exist simple $\mathscr{A}$-measurable functions g, h such that $h \leqslant f \leqslant g$ and $S \int_X (g - h)\, d\mu < \epsilon$.

Proof. Let $\epsilon > 0$ and pick (by the definition of S-integrability) $\mathscr{A}$-partitions $\mathscr{P}_1$ and $\mathscr{P}_2$ such that

$$S \int_X f\, d\mu - \sum_{P \in \mathscr{P}_1} \inf f(P)\mu(P) < \frac{\epsilon}{2} \text{ and } \sum_{P \in \mathscr{P}_2} \sup f(P)\mu(P) - S \int_X f\, d\mu < \frac{\epsilon}{2}.$$

By Proposition 4.2, the same inequalities hold for an $\mathscr{A}$-refinement $\mathscr{P}$ of $\mathscr{P}_1$ and $\mathscr{P}_2$. For each $A \in \mathscr{P}$, define g on A to be $\sup f(A)$ and h on A to be $\inf f(A)$. Then $h \leqslant f \leqslant g$, and by Lemma 4.7, g and h are $\mathscr{A}$-measurable simple functions. The lemma also implies $\sum_{P \in \mathscr{P}} \inf f(P)\mu(P)$ is the S-integral of h; a similar result holds for g. That $S \int_X (g - h)\, d\mu < \epsilon$ now follows from the choice of $\mathscr{P}$.

Since we are only interested in the preceding implication, we leave the converse as a straightforward exercise for the reader. □

Theorem 4.9. Suppose that $f : X \to \mathbb{R}$ is bounded and upper measurable, that is, $f^{-1}((a, \infty)) \in \mathscr{A}$ for all $a \in \mathbb{R}$, or lower measurable, that is, $f^{-1}((-\infty, a)) \in \mathscr{A}$ for all $a \in \mathbb{R}$ with respect to $\mathscr{A}$. Then f is S-integrable. In particular, for the case of a finitely additive measure on the crescent algebra of a topological space, a lower semicontinuous bounded function is integrable.

Proof. Partition an interval containing the range into small ($< \epsilon/\mu(X)$) appropriately half-open, half-closed intervals and apply condition (3) of Theorem 4.5. □

Corollary 4.10. Let $f : X \to \mathbb{R}$ be a bounded function such that for every $\epsilon > 0$, there exists $A \in \mathscr{A}$ such that $\mu(A) < \epsilon$ and f is upper (respectively, lower) measurable on the complement A^c of A. Then f is S-integrable.

Proof. Suppose that $|f(x)| \leqslant B$ for all $x \in X$. Pick A such that $\mu(A) < \epsilon/4B$ and f is upper measurable on A^c. By the preceding theorem and (3) of Theorem 4.5, there exists an $\mathscr{A}$-partition $\{P_1, \ldots, P_n\}$ of A^c such that $\sum_{i=1}^{n} |f(x_i) - f(y_i)|\mu(P_i) < \epsilon/2$ for every choice $x_i, y_i \in P_i$ for $i = 1, \ldots n$. Then adding A to the partition gives a partition of X that satisfies condition (3) of Theorem 4.5. □

A set $G \subseteq X$ is called a *null* set if for every $\epsilon > 0$ there exists $A \in \mathscr{A}$ such that $G \subseteq A$ and $\mu(A) < \epsilon$. Two functions f and g are *equal almost everywhere* (a.e.) if they differ on a null set.

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Corollary 4.11. If bounded real-valued functions f and g are equal a.e. and f is S-integrable, then g is S-integrable and the two integrals agree.

Proof. Since $g - f$ is equal to 0 a.e., it follows easily from the preceding corollary that $g - f$ is S-integrable, and then it follows directly from the definition of the integral that it must be 0. Then

$$S\int_X g\,d\mu = S\int_X f\,d\mu + S\int_X (g-f)\,d\mu = S\int_X f\,d\mu. \qquad \square$$

Let X be a topological space equipped with a finitely additive, non-negative, bounded measure. A function from X into another topological space is *continuous a.e.* if the set of points on which it is discontinuous is a null set.

Theorem 4.12. Let X be a topological space equipped with a finitely additive, non-negative bounded measure on the crescent algebra. If a bounded function $f : X \to (\mathbb{R}, \text{Scott})$ is continuous a.e., then f is integrable with respect to the crescent algebra, and hence also with respect to the semialgebra of crescent sets.

Proof. For each $\epsilon > 0$, we can find $A \in \mathscr{A}$ such that $\mu(A) < \epsilon$ and f is continuous for the Scott topology, and hence upper measurable, on A^c. Thus, by Corollary 4.10, f is integrable. $\square$

Finally, we consider the case that $\mathscr{A}$ is a non-empty σ-algebra on a set X and μ is a countably additive bounded measure (always assumed non-negative) defined on $\mathscr{A}$. Suppose that $f : X \to \mathbb{R}$ is bounded and S-integrable with respect to the σ-algebra. Then, using Theorem 4.8 (and the construction in its proof), we obtain inductively for each n a finite $\mathscr{A}$-partition $\mathscr{P}_n$ refining $\mathscr{P}_{n-1}$, and simple functions g_n and h_n with level sets the members of $\mathscr{P}_n$ such that $h_{n-1} \leqslant h_n \leqslant f \leqslant g_n \leqslant g_{n-1}$ and $S\int_X (g_n - h_n)\,d\mu < 1/n$. Set $g = \inf_n g_n$ and $h = \sup_n h_n$. Then g and h are Lebesgue integrable from the Lebesgue Dominated Convergence Theorem, $h \leqslant f \leqslant g$, and $\int_X h\,d\mu = \sup_n \int_X h_n\,d\mu$ and $\int_X g\,d\mu = \inf_n \int_X g_n\,d\mu$. It follows that $S\int_X f\,d\mu = \int_X h\,d\mu = \int_X g\,d\mu$. Thus $g - h \geqslant 0$ and has integral 0, and therefore must be non-zero on a null set. Since it dominates $f - h$, it follows that f is equal to h (and hence g) a.e. We summarise this in the following theorem.

Theorem 4.13. Let $\mathscr{A}$ be a σ-algebra on a set X and μ be a countably additive bounded measure (always assumed non-negative) defined on $\mathscr{A}$. An S-integrable function f is *almost measurable*, that is, equal to a measurable function a.e. (indeed to one that is integrable), and its S-integral is equal to the Lebesgue integral of any such function. In particular, if f is measurable (which will be the case if the measure is complete) and S-integrable, then it is Lebesgue integrable and the two integrals agree.

We close this section by connecting the S-integral with the horizontal integral. Let μ denote both a finite valuation on a lattice ΩX of subsets of X, $\varnothing, X \in \Omega X$, and its extension to a finitely additive measure on the algebra of sets $\mathscr{A}$ generated by ΩX. Let $f : X \to \mathbb{R}_+$ be a bounded function that is upper measurable with respect to ΩX. Let $[0, n]$ be an interval containing the image of f. Partition X by defining $A_i = f^{-1}(](i-1)/2^n, i/2^n])$ for integers i, $1 \leqslant i \leqslant n2^n$. Note that many A_i may be empty, so they are simply 'ghost'

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summands in the following computations. The upper and lower Darboux sums for this partition differ by no more than $\mu(X)/2^n$, and hence each converge to the S-integral as $n \to \infty$. For $x_i = i/2^n$, the lower Darboux sum will be bounded below by

$$\begin{aligned}\sum_{i=1}^{n2^n} x_{i-1}\mu(A_i) &= \sum_{i=1}^{n2^n} x_{i-1}(\mu(f^{-1}(]x_{i-1}, \infty)) - \mu(f^{-1}(]x_i, \infty])))\\ &= \sum_{i=1}^{n2^n} x_{i-1}(f^\mu(x_{i-1}) - f^\mu(x_i)).\end{aligned}$$

Grouping together the $f^\mu(x_i)$ terms and observing that $x_0 = 0 = f^\mu(x_{n2^n})$, we obtain

$$\ldots = \sum_{i=1}^{n2^n} f^\mu(x_i)(x_i - x_{i-1}) = \sum_{i=1}^{n2^n} \frac{1}{2^n} f^\mu\left(\frac{i}{2^n}\right).$$

Since the final expression converges to the horizontal integral $⨍ f\, d\mu$ by formula (3.1), we conclude that $⨍_X f\, d\mu \leqslant S\int_X f\, d\mu$. A similar analysis with the upper Darboux sum yields the fact that it is bounded above by $\mu(X)/2^n + \sum_{i=1}^{n2^n}(1/2^n)f^\mu(i/2^n)$; since the latter also converges to $⨍_X f\, d\mu$ by (3.1), we obtain the reverse inequality. We have thus established the following result.

Proposition 4.14. Given the setting of the previous paragraph, and an upper (lattice) measurable bounded function $f: X \to \mathbb{R}_+$, the horizontal integral and the S-integral both exist and agree. This is true in particular for bounded lower (respectively, upper) semicontinuous functions on a topological space, where the integrals arise from a finite valuation on the lattice of open sets.

Note that if we have a finitely additive bounded measure defined on an algebra of sets, we can take the algebra of sets to be the lattice, and thus have the two integrals agreeing for all non-negative bounded upper measurable (with respect to the algebra) functions.

5. Bilinear integration

Let X be a topological space, $\mathrm{LSC}(X, \overline{\mathbb{R}}_+)$ be the set of lower semicontinuous functions from X into $\overline{\mathbb{R}}_+$ endowed with the pointwise order, and $\mathbf{V}(X)$ be the set of continuous valuations into $\overline{\mathbb{R}}_+$. We also endow $\mathbf{V}(X)$ with the pointwise order: $\nu \leqslant \mu$ if and only if $\nu(U) \leqslant \mu(U)$ for all U open. This order is called the *stochastic order* and $\mathbf{V}(X)$ equipped with this order is called the *extended probabilistic powerdomain*. Since the pointwise directed supremum of continuous valuations is easily seen to be again a continuous valuation, $\mathbf{V}(X)$ is always a dcpo.

Definition 5.1. A *dcpo-cone* is a dcpo C equipped with a distinguished element $0 \in C$, an addition $+ : C \times C \to C$, and a scalar multiplication $\cdot : \mathbb{R}^+ \times C \to C$ such that the usual axioms of a vector space hold, except for the existence of an additive inverse (in this case, one must also postulate that $0 \cdot a = 0$ for all $a \in C$). We assume further that addition and scalar multiplication are Scott-continuous functions from the Scott topology on the product into C.

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Lemma 5.2. For X a topological space, $\mathrm{LSC}(X, \overline{\mathbb{R}}_+)$ and $\mathbf{V}(X)$ are dcpo-cones.

Proof. One can verify directly that sums and scalar products of lower semicontinuous functions and of continuous valuations are again continuous valuations. Since the operations are defined pointwise, all the appropriate algebraic laws of a cone are satisfied.

Let $\{r_\alpha\}$ be a directed net in $\mathbb{R}^+$ with supremum r and $\{\mu_\alpha\}$ and $\{\nu_\alpha\}$ be directed nets in $\mathbf{V}(X)$ with suprema μ and ν, respectively. Then for any open set U in X, $(\mu_\alpha + \nu_\alpha)(U)$ and $r_\alpha \mu_\alpha(U)$ are directed sets with suprema $(\mu + \nu)(U)$ and $r\mu(U)$, respectively. Since the pointwise suprema of a directed set of continuous valuations is again a continuous valuation and the supremum of the collection, we have that addition and scalar multiplication are Scott-continuous. An analogous argument holds for $\mathrm{LSC}(X, \overline{\mathbb{R}}_+)$. □

We want to consider the function $I : \mathrm{LSC}(X, \overline{\mathbb{R}}_+) \times \mathbf{V}(X) \to \overline{\mathbb{R}}_+$ defined by $I(f, \mu) = \int_X f \, d\mu$, the horizontal integral.

Lemma 5.3. The function I is Scott-continuous in each variable, and hence Scott-continuous from the Scott topology on the product $\mathrm{LSC}(X, \overline{\mathbb{R}}_+) \times \mathbf{V}(X)$ into $\overline{\mathbb{R}}_+$.

Proof. Let f be the (pointwise) supremum of a directed family $\{f_\alpha\}$ in $\mathrm{LSC}(X, \overline{\mathbb{R}}_+)$, and let $\mu \in \mathbf{V}(X)$. Then for $t \in \mathbb{R}_+$, by continuity of μ,

$$f^\mu(t) = \mu(f^{-1}(]t, \infty])) = \mu\left(\bigcup_\alpha f_\alpha^{-1}(]t, \infty])\right) = \sup_\alpha \{\mu(f_\alpha^{-1}(]t, \infty]))\} = \sup_\alpha f_\alpha^\mu(t),$$

where the suprema are directed suprema. In particular, note that the directed family f_α^μ converges pointwise to f^μ on $\mathbb{R}_+$.

Pick $r < \int_X f \, d\mu$ and $\epsilon > 0$. By (3.1), we may pick n such that

$$r < \sum_{i=1}^{n2^n} \frac{1}{2^n} f^\mu\left(\frac{i}{2^n}\right) \leqslant \int_X f \, d\mu.$$

By the preceding paragraph, we may pick α large enough to ensure that $f^\mu(i/2^n) - f_\alpha^\mu(i/2^n) < \epsilon/n$ for each $i/2^n$, $1 \leqslant i \leqslant n2^n$. Again by (3.1),

$$r - \int_X f_\alpha \, d\mu < \sum_{i=1}^{n2^n} \frac{1}{2^n} f^\mu\left(\frac{i}{2^n}\right) - \sum_{i=1}^{n2^n} \frac{1}{2^n} f_\alpha^\mu\left(\frac{i}{2^n}\right) < \frac{\epsilon}{n} \sum_{i=1}^{n2^n} \frac{1}{2^n} = \epsilon;$$

thus $r - \epsilon < \int_X f_\alpha \, d\mu$. Since r was arbitrarily below $\int_X f \, d\mu$ and ϵ was arbitrarily positive, we conclude that $\int_X f \, d\mu$ is the supremum of the $\int_X f_\alpha \, d\mu$.

Now fix a lower semicontinuous $f : X \to \overline{\mathbb{R}}_+$ and let μ_α be a directed family of continuous valuations with pointwise supremum μ. We have for $t \in \mathbb{R}_+$

$$f^\mu(t) = \mu(f^{-1}(]t, \infty])) = \sup_\alpha \{\mu_\alpha(f^{-1}(]t, \infty]))\} = \sup_\alpha f^{\mu_\alpha}(t),$$

where the suprema are directed. As in the previous paragraph, we argue that $\int_X f \, d\mu = \int_0^\infty f^\mu(t) dt$ is the directed supremum of the $\int_X f \, d\mu_\alpha = \int_0^\infty f^{\mu_\alpha}(t) dt$. □

Lemma 5.4. Every continuous valuation on the lattice of open sets ΩX is the directed supremum of bounded continuous valuations.

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Proof. Let μ be a continuous valuation. Let $\Phi(\mu) = \{O \in \Omega X : \mu(O) < \infty\}$; note that $\Phi(\mu)$ is directed by modularity. Set $W := \bigcup\{O \in \Phi(\mu)\}$, and let C be the complement of W. For each $O \in \Phi(\mu)$, the function $\mu_O : \Omega X \to \mathbb{R}$ defined by $\mu_O(U) = \mu(U \cap O)$ is a bounded continuous valuation with $\mu_O \leqslant \mu$. For each $O \in \Phi(\mu)$, each finite $F \subseteq C$, and each $n \in \mathbb{N}_0$, set $\mu_{O,F,n} = \mu_O + \sum_{x \in F} n\eta_x$, where η_x is the point valuation defined by $\eta_x(U) = 1$ if $x \in U$ and 0 otherwise. Let $D = \{\mu_{O,F,n} : O \in \Phi(\mu),\ F \subseteq C,\ |F| < \infty, n \in \mathbb{N}_0\}$. This set is non-empty, since it contains $\mu_{\varnothing,\varnothing,0}$, and directed, since μ_{O_1,F_1,n_1} and μ_{O_2,F_2,n_2} are bounded by $\mu_{O_1 \cup O_2, F_1 \cup F_2, n_1 \vee n_2}$. Let $\nu = \sup D$.

If $U \cap C \neq \varnothing$ for some open U, let $x \in U \cap C$. Then $\nu(U) \geqslant (n\mu_x)(U) = n$ for all $n \in \mathbb{N}_0$, and hence $\nu(U) = \infty$. On the other hand, if $\mu(U)$ is finite, then $U \subseteq W = X \backslash C$, in contradition to $U \cap C \neq \varnothing$. Thus μ and ν agree on sets meeting C.

If $U \subseteq W$, then $U = \bigcup\{U \cap O : O \in \Phi(\mu)\}$. Since $\Phi(\mu)$ is directed and μ is continuous,

$$\begin{aligned}\mu(U) &= \sup\{\mu(U \cap O) : O \in \Phi(\mu)\} = \sup\{\mu_O(U) : O \in \Phi(\mu)\} \\ &= \sup\{\mu_{O,F,n}(U) : \mu_{O,F,n} \in D\} = \nu(U).\end{aligned}$$

Note that all $\mu_{O,F,n}$ are finite. □

Proposition 5.5. For a space X, we have that the map $I : \mathrm{LSC}(X, \overline{\mathbb{R}}_+) \times \mathbf{V}(X) \to \overline{\mathbb{R}}_+$ defined by $I(f, \mu) = \int_X f \, d\mu$ is bilinear and Scott-continuous from the Scott topology on $\mathrm{LSC}(X, \overline{\mathbb{R}}_+) \times \mathbf{V}(X)$.

Proof. The Scott-continuity follows from Lemma 5.3. Suppose that $f \in \mathrm{LSC}(X, \overline{\mathbb{R}}_+)$ and $\mu, \nu \in \mathbf{V}(X)$. Then

$$f^{\mu+\nu}(t) = (\mu + \nu)(f^{-1}(]t, \infty])) = \mu(f^{-1}(]t, \infty])) + \nu(f^{-1}(]t, \infty])) = f^{\mu}(t) + f^{\nu}(t),$$

and thus, by additivity of the improper Riemann integral for non-negative functions,

$$\int_X f \, d(\mu + \nu) = \int_0^\infty f^{\mu+\nu}(t)\, dt = \int_0^\infty f^{\mu}(t)\, dt + \int_0^\infty f^{\nu}(t)\, dt = \int_X f \, d\mu + \int_X f \, d\nu.$$

Thus I is additive in the $\mathbf{V}(X)$-coordinate, and a similar argument establishes homogeneity.

On the other hand, let $f, g \in \mathrm{LSC}(X, \overline{\mathbb{R}}_+)$ and $\mu \in \mathbf{V}(X)$. If f, g and μ are all bounded, the additivity and homogeneity in the first coordinate follow from Theorem 4.6 and Proposition 4.14. If μ is bounded, but f and g are not assumed so, then the additivity can be deduced by considering the canonical sequences of simple (hence bounded) functions converging to f and g, respectively (see Section 3), and observing

$$\begin{aligned}I(f + g, \mu) &= \sup\{I(f_n + g_n, \mu) : n \in \mathbb{N}\} \\ &= \sup\{I(f_n, \mu) : n \in \mathbb{N}\} + \sup\{I(g_n, \mu) : n \in N\} \\ &= I(f, \nu) + I(g, \mu),\end{aligned}$$

where the first and last equalities follow from the dcpo-cone property and Lemma 5.3, and the middle one follows from additivity on bounded functions and directedness of the sequences $I(f_n, \mu)$ and $I(g_n, \mu)$. For μ unbounded, pick a directed sequence of bounded

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valuations μ_n with supremum μ by the preceding lemma; then

$$\begin{aligned} I(f+g,\mu) &= \sup\{I(f+g,\mu_n) : n \in \mathbb{N}\} \\ &= \sup\{I(f,\mu_n) : n \in \mathbb{N}\} + \sup\{I(g,\mu_n) : n \in \mathbb{N}\} \\ &= I(f,\mu) + I(g,\mu), \end{aligned}$$

by the previous step and Lemma 5.3. Homogeneity in the first variable follows along the same lines. □

Theorem 5.6. For a continuous domain D, the map $I : \mathrm{LSC}(D, \overline{\mathbb{R}}_+) \times \mathbf{V}(D) \to \overline{\mathbb{R}}_+$ given by $I(f,\mu) = \int_D f\, d\mu$ is the unique map that is bilinear, Scott-continuous and satisfies $I(\chi_U, \eta_x) = \chi_U(x)$ for all characteristic functions χ_U of open sets U and all point valuations η_x.

Proof. In the light of the previous proposition, it only remains to establish the uniqueness. Let J be another such map. One verifies directly from the definition of the horizontal integral that $I(\chi_U, \eta_x) = \chi_U(x)$, and so by linearity and equation (3.2), $I(f,\mu) = J(f,\mu)$ for any function f belonging to a canonical sequence and simple valuation μ, that is, any valuation consisting of a finite linear combination of point valuations. Since any continuous valuation on a continuous domain D is a directed supremum of simple valuations (see Kirch (1993) or Gierz *et al.* (2003, Chapter IV-9)), continuity arguments along the lines of those in the proof of the last proposition establish that I and J agree everywhere. □

For a much more extensive development of the bilinear pairing by integration for the case when X is a domain equipped with the Scott topology, see Kirch (1993). In addition, many of the developments of this section have been treated in more generality by R. Heckmann in Heckmann (1996).

6. Integrals via semicontinuity

Let X be a topological space, and let $f : X \to \mathbb{R}$ be a bounded function. Then there exists a largest lower semicontinuous function $f_* \leqslant f$ and a smallest upper semicontinuous function $f^* \geqslant f$ defined by

$$f_*(x) := \sup\{\inf f(U) : x \in U \text{ open}\}, \quad f^*(x) := \inf\{\sup f(U) : x \in U \text{ open}\};$$

see, for example, Gierz *et al.* (2003, Chapter II-3), particularly 3.19. Furthermore, it follows directly from the definitions that f_* (respectively, f^*) agrees with f at any point x where f is lower (respectively, upper) semicontinuous. The function f is said to be μ-continuous a.e. if it is discontinuous on a μ-null set (see Section 4). We observe that f is μ-continuous a.e. if and only if $f_* = f = f^*$ a.e. The next proposition now follows from Theorem 4.9 and Corollary 4.11.

Proposition 6.1. Let μ be a finite valuation on ΩX, the lattice of open sets of a topological space. If $f : X \to \mathbb{R}$ is a bounded function that is continuous a.e., then f_* is upper measurable, f^* is lower measurable, f, f_*, and f^* are all S-integrable, and

$$S\int_X f_* d\mu = S\int_X f\, d\mu = S\int_X f^* d\mu.$$

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We next consider simple valuations.

Lemma 6.2. Let $v = \sum_{b\in B} r_b\eta_b$ be a simple valuation on a topological space X and let $f : X \to \mathbb{R}$ be a bounded function that is S-integrable with respect to v and some algebra $\mathscr{A}$ of sets. Then $S\int_X f\,d\mu = \sum_{b\in B} r_b f(b)$.

Proof. Suppose that there exists some $\mathscr{A}$-partition that separates all the points of B. Then all the finer ones do so also, and we can always choose a Riemann sum with the value $\sum_{b\in B} r_b f(b)$ for all these partitions. It then follows from Theorem 4.5 that $S\int_X f\,dv = \sum_{b\in B} r_b f(b)$.

Suppose that $b_1 \neq b_2$, $b_1, b_2 \in B$, lie in the same partition member for every $\mathscr{A}$-partition. If $f(b_1) \neq f(b_2)$, then for any $\mathscr{A}$-partition we have two Riemann sums that differ by at least $(r_{b_1} + r_{b_2})|f(b_1) - f(b_2)|$, which contradicts (3) of Theorem 4.5. Thus $f(b_1) = f(b_2)$ whenever b_1 and b_2 cannot be $\mathscr{A}$-separated. It then follows that $\sum_{b\in B} r_b f(b)$ is one of the Riemann sums for every $\mathscr{A}$-partition and hence by Theorem 4.5 is equal to $S\int_X f\,dv$. □

Proposition 6.3. Let $f : X \to \mathbb{R}$ be a bounded function, and let μ be a finite valuation on ΩX that is the directed supremum of a monotone net v_α of simple valuations. Extend μ to the crescent algebra.

(i) If f is lower semicontinuous and non-negative, then $S\int_X f\,d\mu$ is the directed supremum of the $S\int_X f\,dv_\alpha$.
(ii) If f is lower or upper semicontinuous, then $S\int_X f\,dv_\alpha$ converges to $S\int_X f\,d\mu$.

Proof. (i) Part (i) follows from Proposition 4.14 and Lemma 5.3.

(ii) Assume that f is lower semicontinuous and that $-a$ is a lower bound of $f(X)$, where $a \geqslant 0$. Then $g(x) := f(x)+a$ is non-negative and lower semicontinuous, hence S-integrable, and, by part (i), $S\int_X g\,d\mu$ is the directed supremum of the $S\int_X g\,dv_\alpha$. We thus have

$$\begin{aligned} S\int_X f\,d\mu + S\int_X a\,d\mu &= S\int_X g\,d\mu = \lim_\alpha S\int_X g\,dv_\alpha \\ &= \lim_\alpha \left(S\int_X f\,dv_\alpha + S\int_X a\,dv_\alpha \right) = \lim_\alpha S\int_X f\,dv_\alpha + a\mu(X). \end{aligned}$$

Cancelling $S\int_X a\,d\mu = a\mu(X)$, we conclude that

$$S\int_X f\,d\mu = \lim_\alpha S\int_X f\,dv_\alpha.$$

If f is upper semicontinuous, then $-f$ is lower semicontinuous, and

$$S\int_X f\,d\mu = (-1)S\int_X -f\,d\mu = (-1)\lim_\alpha S\int_X -f\,dv_\alpha = \lim_\alpha S\int_X f\,dv_\alpha$$

by the preceding paragraph. □

The preceding results provide computational schemes for the S-integral provided a valuation μ is a directed supremum of known simple valuations v_α.

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Corollary 6.4. Suppose that $f : X \to \mathbb{R}$ is bounded and μ-continuous almost everywhere, where μ is a finite valuation that is a directed supremum of simple valuations v_α. Then $S\lim_X f\, d\mu = \lim_\alpha v_\alpha(f)$, where $v_\alpha(f) = \sum_{b\in B} r_b f(b)$ if $v_\alpha = \sum_{b\in B} r_b \eta_b$.

Proof. By Proposition 6.1, f is equal almost everywhere to f_* and f^*, and they all have the same S-integral. By Proposition 6.3, the net $S\int_X f_*\, dv_\alpha$ converges to $S\int_X f_* d\mu$, which by Lemma 6.2 means $v_\alpha(f_*)$ converges to $S\int_X f_* d\mu = S\int_X f\, d\mu$. Similarly, $v_\alpha(f^*)$ converges to $S\int_X f\, d\mu$. By the sandwich principle, $v_\alpha(f)$ converges to $S\int_X f\, d\mu$. □

7. The Riemann–Edalat integral

In this section we outline the theory of the Riemann–Edalat integral, which was first introduced by A. Edalat for compact spaces (Edalat 1995b) for the upper domain, extended by Edalat and Negri to locally compact spaces (Edalat and Negri 1998), and extended to general domains by J. Howroyd (Howroyd 2000). Our approach to connecting it with the preceding developments is closely tied to the developments in Lu (1999) and Lawson and Lu (2003).

Let D be a continuous domain (a dcpo in which every element is a supremum of a directed set of elements that approximate it) equipped with the Scott topology ΩD, and let X be a Scott-dense subset of D that is Hausdorff in the relative Scott topology. A prototypical example is the case for which X is the set of maximal points of D (if a Hausdorff space X can be topologically embedded as the set of maximal points of a domain D equipped with the relative Scott topology, then the embedding is sometimes called a *domain environment*, see, for example, Lawson (1997), Lawson (1998) and Gierz *et al.* (2003)).

Let $f : X \to \mathbb{R}$ be a bounded function. For $b \in D$, we set $[b] := {\uparrow}b \cap X$ and define $f_\star, f^\star : D \to \mathbb{R}$ by

$$f_\star(b) = \inf f[b], \qquad f^\star(b) = \sup f[b],$$

where $\inf f[b] = \sup f(X)$ (respectively, $\sup f[b] = \inf f(X)$) if ${\uparrow}b \cap X = \varnothing$. We also slightly modify the definitions of f_* and f^* given in the last section as follows:

$$f_*(b) := \sup\{\inf f(U \cap X) : b \in U \text{ open}\}, \quad f^*(b) := \inf\{\sup f(U \cap X) : b \in U \text{ open}\}.$$

One sees directly that $f_* \leqslant f_\star$ and $f^\star \leqslant f^*$.

For a simple valuation $v = \sum_{b\in B} r_b \eta_b$ on ΩD, we define the *Darboux lower sum* of f with respect to v by

$$S^\ell(f, v) = \int f_\star\, dv = \sum_{b\in B} r_b f_\star(b) = \sum_{b\in B} r_b \inf f[b],$$

and the *Darboux upper sum* by

$$S^u(f, v) = \int f^\star dv = \sum_{b\in B} r_b f^\star(b) = \sum_{b\in B} r_b \sup f[b].$$

We assume now that the domain D is pointed (that is, has a smallest element $\perp$) and consider the *normalised probabilistic powerdomain* P^1D consisting of all continuous

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valuations μ on ΩD; P^1D is ordered with the relative stochastic order. The powerdomain P^1D is again a continuous domain (see Edalat (1995b)), but the approximation relation $\ll^1$ is slightly different from $\mathbf{V}(D)$: if $\mu \ll \nu$ in $\mathbf{V}(D)$, $\nu(X) = 1$, and $\mu(X) < 1$, then $\mu_1 \ll^1 \nu$ in P^1D for $\mu_1 := \mu + (1 - \mu(X))\eta_\perp$.

Definition 7.1. Let μ be a regular Borel probability measure on X. As we have seen, there arises a continuous valuation on ΩD defined by $\hat{\mu}(U) = \mu(U \cap X)$. For a bounded function $f : X \to \mathbb{R}$, the *lower integral* is defined by

$$\underline{\mathbf{E}} \int_X f \, d\mu = \sup\{S^\ell(f, \nu) : \nu \ll^1 \hat{\mu}, \nu \text{ simple}\},$$

and the *upper integral* by

$$\overline{\mathbf{E}} \int_X f \, d\mu = \inf\{S^u(f, \nu) : \nu \ll^1 \hat{\mu}, \ \nu \text{ simple}\}.$$

We say that f is *E-integrable* if its upper and lower integrals coincide; the common value is called the *Riemann–Edalat integral.*

We remark that a Riemann-like theory emerges in the preceding framework. As one moves up the directed set $\{\nu : \nu \ll^1 \hat{\mu}, \ \nu \text{ simple}\}$, one obtains a directed family of Darboux lower sums and a filtered family of Darboux upper sums, with all upper sums bounding all lower sums. Standard integral properties such as linearity hold for the Riemann–Edalat integral. See Edalat (1995b) and Howroyd (2000) for a direct development of these topics.

We will now consider the lower integral in more detail. We first recall a basic fact about the approximation relation in the normalised probabilistic power domain (see Edalat (1995b)).

Lemma 7.2. For two simple probability valuations, we have

$$\zeta = \sum_{b \in B} r_b \eta_b \ll^1 \xi = \sum_{c \in C} s_c \eta_c$$

iff $\perp \in B$ with $r_\perp > 0$ and there exist $t_{b,c} \geqslant 0$ such that

$$\sum_{c \in C} t_{b,c} = r_b, \qquad \sum_{b \in B} t_{b,c} = s_c$$

and $t_{b,c} \neq 0$ implies $b \ll c$.

Lemma 7.3. Let X be a dense subspace of a continuous domain D, and let $f : X \to \mathbb{R}$ be bounded. Let μ be a regular Borel probability measure on X and let $\hat{\mu}$ be the corresponding continuous valuation on ΩD and its extension to the crescent algebra. Then the lower integral $\underline{\mathbf{E}} \int_X f \, d\mu$ is equal to $S \int_D f_* d\hat{\mu}$.

Proof. For $\nu = \sum_{b \in B} r_b \eta_b$ and $g : D \to \mathbb{R}$, we write $\nu(g)$ for $\sum_{b \in B} r_b g(b)$, which is the S-integral of g with respect to ν by Lemma 6.2. Since for each $b \in D$ we have $f_*(b) \leqslant f^\star(b)$, it follows that $\nu(f_*) \leqslant \nu(f^\star)$ for all simple $\nu \ll^1 \hat{\mu}$. Since, by Proposition 6.3, the directed set $\{\nu(f_*)\}$ for ν simple, $\nu \ll^1 \mu$, converges to $S \int_D f_* d\hat{\mu}$ and the lower integral is the directed sup of the $\nu(f^\star)$, we conclude that the S-integral is less than or equal to the lower integral.

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Conversely, consider some $\zeta(f_\star)$ for simple $\zeta \ll^1 \mu$. Pick a simple ξ such that

$$\zeta = \sum_{b\in B} r_b \eta_b \ll^1 \xi = \sum_{c\in C} s_c \eta_c \ll^1 \mu.$$

We then have, using the previous lemma,

$$\begin{aligned}\zeta(f_\star) &= \sum_{b\in B} r_b f_\star(b) = \sum_{b\in B}\sum_{c\in C} t_{b,c} f_\star(b)\\ &\leqslant \sum_b \sum_c t_{b,c} f_*(c) = \sum_c \sum_b t_{b,c} f_*(c) = \sum_c s_c f_*(c) = \xi(f_*),\end{aligned}$$

where the inequality at the beginning of the second line follows from the fact that for any non-zero term $t_{b,c}$, $b \ll c$, hence $\uparrow b$ is a neighbourhood of c, and thus $f_\star(b) = \inf(f(\uparrow b \cap X)) \leqslant f_*(c)$. Therefore, for any simple $\zeta \ll^1 \mu$, we have $\zeta(f_\star)$ is an eventual lower bound for the $v(f_*)$, v simple, $v \ll^1 \mu$, and thus their supremum, the lower integral must be less than or equal to the limit point of the $v(f_*)$. □

Lemma 7.4. Under the conditions of the preceding lemma, we have

$$S\int_D f_* d\hat{\mu} = S\int_X f_* d\mu.$$

Proof. For each open set U of D, $\hat{\mu}(U) = \mu(U \cap X)$. Since the collection of subsets for which this equality holds forms a subalgebra, we conclude that the equality holds for all members of the crescent algebra of D. By picking Riemann sums for crescent partitions of D such that the selected point always comes from X whenever the partition element meets X, and noting that these Riemann sums must converge to both integrals as the partitions become finer, one obtains the conclusion of the lemma. □

Theorem 7.5. Let X be a dense subspace of a continuous domain D, Hausdorff in the relative topology, and let $f : X \to \mathbb{R}$ be bounded. Let μ be a regular Borel probability measure on X and let $\hat{\mu}$ be the corresponding continuous valuation on ΩD and its extension to the crescent algebra. Then the following conditions are equivalent:

(1) f is Riemann–Edalat integrable;
(2) $f : X \to \mathbb{R}$ is continuous a.e.;
(3) $S\int_X f_* d\mu = S\int_X f\, d\mu = S\int_X f^* d\mu$.

When these conditions hold, the Riemann–Edalat integral, $S\int_X f\, d\mu$ and the Lebesgue integral are all equal.

Proof. For the implication (1) implies (2), see Howroyd (2000). That (2) implies (3) follows from Proposition 6.1. That (3) implies (1) follows from the preceding two lemmas and their dual versions for f^*. These also imply the equality of the Riemann–Edalat integral and the S-integral. The equality with the Lebesgue integral follows from Theorem 4.13. □

For a survey of applications of the Riemann–Edalat integral, see Edalat (1995a) and Edalat (1995c).

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