Moduli **3**, e11 (2026) 1–53
doi:10.1017/S2949764726100253

How to make log structures

Alessio Corti and Helge Ruddat

Abstract

We introduce the concept of a *viable generically Gorenstein toroidal crossing* (ggtc) space Y. This generalizes the concept of Gorenstein toroidal crossing scheme, which in turn generalizes that of a simple normal crossing scheme. On such a space Y, we define a sheaf $\mathcal{LS}_Y$, intrinsic to Y, by means of an explicit construction. Our main theorem establishes a bijection between the set $\mathrm{LS}_{k^\dagger}(Y)$ of isomorphism classes of log structures on Y over the log point $\operatorname{Spec} k^\dagger$ that are compatible with the ggtc structure and the set $\Gamma(Y, \mathcal{LS}_Y^\times)$ of nowhere-vanishing global sections of $\mathcal{LS}_Y$. The definition of $\mathcal{LS}_Y$ by explicit construction permits the *effective construction* of log structures on Y; it also enables *logarithmic birational geometry*, in particular the construction, in some cases, of resolutions of singular log structures. Our work generalizes [Gross and Siebert, J. Differential Geom. **72** (2006), 169–338, Theorem 3.22], adapting the original proof with techniques from the theory of 2-groups and local line bundle systems.

Contents

1 Introduction **2**
1.1 Informal description of results . . . 3
1.2 Main theorem and its discussion . . . 5
1.3 Summary of previous work . . . 7
1.4 Our work in relation to previous work . . . 7
1.5 Description of contents . . . 8
2 Generically Gorenstein toroidal crossing spaces **8**
2.1 Notation and conventions for monoids and fans . . . 9
2.2 Notation and conventions for stratified spaces . . . 10
2.3 The basic setup and assumptions for toroidal crossing spaces . . . 12
2.4 Generically Gorenstein toroidal crossing space . . . 12
2.5 Viable ggtc spaces . . . 14
2.6 The divisor system . . . 16

Received 26 April 2024 UTC, accepted 17 April 2026 UTC.
2020 Mathematics Subject Classification 14A21, 14D23, 14B07, 14M25, 14E15, 14J33, 14J45
Keywords: log structure, viable generically Gorenstein toroidal crossing, ggtc, logarithmic geometry, toroidal crossing space, slab line bundle, joint condition, ghost sheaf, logarithmic birational geometry, log crepant resolution, Stanley–Reisner ring.
© The Author(s), 2026. Published by Cambridge University Press on behalf of Foundation Compositio Mathematica. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution and reproduction, provided the original article is properly cited.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

3 Construction of the sheaf $\mathcal{LS}_Y$ **18**
3.1 The slab line bundles . . . 18
3.2 The joint condition and the sheaf $\mathcal{LS}_Y$ . . . 22
4 $\mathcal{LS}_Y^\times$ classifies log structures on Y **25**
4.1 Statement of the main result and road-map of its proof . . . 25
4.2 Sheaf of total ring of fractions and invertible fractional ideals . . . 28
4.3 The affine local situation . . . 29
4.4 The sheaf $\mathcal{E}xt^1_c(\mathcal{M},\mathcal{O}_Y^\times)$ . . . 31
4.5 A morphism $\varphi\colon \mathcal{E}xt^1_c(\mathcal{M},\mathcal{O}_Y^\times)\to\mathcal{LS}_Y$ in the affine local situation . . . 33
4.6 Globalizing the construction of φ . . . 34
4.7 The morphism φ is injective . . . 36
4.8 The morphism φ is surjective . . . 37
4.9 Local sections of $\mathcal{E}xt^1_c$ give log structures locally . . . 40
4.10 Proof of Theorem 4.3 . . . 41
5 Examples **42**
5.1 Two components . . . 42
5.2 The easiest singular log structure . . . 45
5.3 A reducible quartic del Pezzo surface . . . 46
5.4 The 3-fold transverse A_1-singularity . . . 48
References **52**

1. Introduction

Assumption 1.1. *Throughout this paper, we work over a perfect field k.*[1] *All varieties, schemes and Deligne–Mumford stacks are of finite type over k.*

Informally speaking, a *generically Gorenstein toroidal crossing (ggtc) space* is a stratified Deligne–Mumford stack Y over k such that, at the generic point of each stratum, the stabilizer is trivial and Y is formally isomorphic to the spectrum of a Stanley–Reisner ring. (See Definition 2.12 for a formal statement.)

Our concept of ggtc space generalizes the concept of Gorenstein toroidal crossing space [SS06]. For simplicity, the reader may think of Y as a scheme with simple normal crossing singularities; however, our main result has a much simpler formulation in this very special situation.[2]

On such a space Y, we define a sheaf $\mathcal{LS}_Y$, intrinsic to Y, by means of an explicit construction. Our main result Theorem 4.3 constructs a bijection from the set $\mathrm{LS}_{k^\dagger}(Y)$ of isomorphism classes of log structures on Y over the standard log point $\operatorname{Spec} k^\dagger$ compatible with the ggtc structure to the set $\Gamma(Y,\mathcal{LS}_Y^\times)$ of nowhere-vanishing sections.

Our construction generalizes that given in [GS06] in the case when Y is a union of toric varieties meeting in boundary strata.[3] Even in that case, our result improves upon [GS06],

[1] We assume that k is perfect in order to use the Cohen structure theorem, see Setup 2.11.

[2] For future applications, we need to include in our consideration a singular space like $(xy=0)$ in $r^{-1}(1,-1,a,-a)$, which is conveniently viewed as a toroidal crossing Deligne–Mumford stack.

[3] More precisely, in the notation of [GS06], if $Y=X_0(B,\mathcal{P},s)$.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

because our construction of the sheaf $\mathcal{LS}_Y$ is more natural and thus it has an independent geometric interpretation: see the discussion in Section 1.2.[4]

To us, the key point of our formulation is that it allows us to construct log structures on Y effectively.

This study is motivated by our program to construct smooth (or mildly singular) Fano and Calabi–Yau varieties. We aim to do so by smoothing a reducible toroidal crossing space equipped with a compatible log structure on a dense open set whose complement Z is of codimension two, while carefully controlling the geometry of Z. In practice, we construct a section of $\mathcal{LS}_Y$ whose zero locus is Z.

Chan, Leung and Ma set up a framework by which one can smoothen a singular log scheme under a list of strong assumptions [CLM23]. The list was then verified in [FFR21] for schemes with Gross–Siebert-type log singularities. The framework was subsequently refined, generalized and placed in the context of curved Gerstenhaber differential graded L_∞ algebras by Felten [Fel22, Fel25]. Our examples of interest in the context of smoothing Fano schemes, e.g., Section 5.4, are not of Gross–Siebert type but we expect them to fall under the notion of *unisingular deformations*, see Section 14.3 in [Fel25], and be therefore amenable to the smoothing framework.

A second motivation is the desire to work with singular log structures and, hence, for a language that allows us to speak of, and construct explicitly, log resolutions of log structures. Our results indeed enable us to do all this; see, for example, (5.5), Section 5.4.3 and [CGR25]. These examples hint at a theory of *log crepant log resolutions* of singular log structures: a subject that we plan to pursue in the near future.

1.1 Informal description of results

We describe our results informally. We begin by stating informally the definition of ggtc space Y over a field k; we proceed to summarize the construction of the sheaf $\mathcal{LS}_Y$; and we conclude with a discussion of our main result, exhibiting a bijection between the set $\mathrm{LS}_{k^\dagger}(Y)$ of isomorphism classes of compatible log structures on Y over $\operatorname{Spec} k^\dagger$ and $\Gamma(Y, \mathcal{LS}_Y^\times)$.

Let $M \cong \mathbb{Z}^r$ be a lattice of rank r, Σ a rational polyhedral fan in M, and K a field. The *Stanley–Reisner ring* $K[\Sigma]$ is the free K-vector space over the monomials z^m, $m \in M \cap |\Sigma|$, where

$$z^m \cdot z^{m'} = \begin{cases} z^{m+m'} & \text{if there is a cone } \sigma \in \Sigma \text{ that contains } m, m', \\ 0 & \text{otherwise.} \end{cases}$$

We consider stratified spaces

$$Y = \coprod_{\eta \in T} Y_\eta^\star$$

over k with locally closed strata indexed by a finite poset T.[5] We denote by Y_η the Zariski closure of $Y_\eta^\star$. We always assume that Y is reduced and equidimensional, and that the irreducible components of Y are normal. We identify a point $\eta \in T$ with the generic point of the

[4] For the technically savvy, a further advantage of our treatment is that the 'gluing data' for the space Y is not explicitly an input of the construction: our space Y comes already glued.

[5] Where $\eta_1 \leqslant \eta_2$ if and only if $Y_{\eta_1}^\star$ is contained in the Zariski closure of $Y_{\eta_2}^\star$. We spell out our terminology on stratified spaces in Section 2.2.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

corresponding stratum and we denote by $T^{[c]} \subset T$ the set of strata of codimension c. It follows that the irreducible components of Y are the closures of the strata of codimension 0 and

$$Y = \bigcup_{\sigma \in T^{[0]}} Y_\sigma.$$

In short, a ggtc space is a stratified space Y that, at the generic point $\eta \in Y$ of every stratum, is formally isomorphic to the spectrum Spec $k(\eta)[\Sigma_\eta]$ of the Stanley–Reisner ring (over the residue field $k(\eta)$) of a fan Σ_η in a lattice M_η of rank

$$\mathrm{rk}(M_\eta) = \mathrm{codim}(\eta).$$

These data are subject to compatibility conditions that are spelled out in Definition 2.12: the most important requirement is that the lattices M_η are the stalks of a sheaf (in the Zariski topology) $\mathcal{M}$ of abelian groups on Y, called the *relative ghost sheaf* of the ggtc space, and that the fans Σ_η in the lattices M_η are the stalks of a sheaf of fans.

In this paper we always assume that Y is *viable*: a technical condition that can be ignored in this informal discussion and that is stated in Definition 2.16.

Given a viable ggtc space Y, we now summarize the construction of the sheaf $\mathcal{LS}_Y$.

Definition 1.2. Let Y be a ggtc space. A *slab* is a codimension one point $\rho \in T^{[1]}$; a *joint* is a codimension two point $\omega \in T^{[2]}$.

We define, for all $\rho \in T^{[1]}$, a line bundle $\mathcal{L}_\rho$ on Y_ρ that we call a *slab bundle*. Given $\rho \in T^{[1]}$, there are exactly two distinct σ, $\sigma' \in T^{[0]}$ such that $Y_\rho \subset Y_\sigma \cap Y_{\sigma'}$. Fix $y \in Y_\rho$ and let η be the generic point of the stratum containing y, so $\eta \leqslant \rho$. We identify σ, σ' with the corresponding maximal cones of the fan Σ_η, and ρ with the submaximal cone in Σ_η that it corresponds to. Next choose $v_y \in M_\eta$ at integral affine distance 1 from ρ.[6] To simplify the discussion, assume that there is a local isomorphism[7] (and not just a formal isomorphism) $f_\eta \colon k(\eta)[\Sigma_\eta]_{\mathfrak{m}} \to \mathcal{O}_{Y,\eta}$ (where $\mathfrak{m} \subset k(\eta)[\Sigma_\eta]$ is the maximal ideal at the origin). Our notion of viability for Y implies that there exists a Zariski open neighbourhood V_y of y in Y such that the divisor germ $\mathrm{div}(f_\eta(z^{v_y}))$ lifts to a Cartier divisor $D_{y,\sigma}$ over $Y_\sigma \cap V_y$ and similarly for z^{-v_y} on $Y_{\sigma'} \cap V_y$. We define the slab bundle $\mathcal{L}_\rho$ on V_y to be

$$(\mathcal{L}_\rho)_{|V_y} := \mathcal{O}_{Y_\sigma \cap V_y}(D_{y,\sigma})_{|Y_\rho \cap V_y} \otimes \mathcal{O}_{Y_{\sigma'} \cap V_y}(D_{y,\sigma'})_{|Y_\rho \cap V_y}.$$

The main result of Section 3.1 is Lemma 3.2 stating that these local definitions glue to a global line bundle $\mathcal{L}_\rho$ on Y_ρ.[8]

The sheaf $\mathcal{LS}_Y$ is defined as the subsheaf of the direct sum of all the slab bundles:

$$\mathcal{LS}_Y \subset \bigoplus_{\rho \in T^{[1]}} \mathcal{L}_\rho \tag{1.1}$$

consisting of sections that, for every joint $\omega \in T^{[2]}$, satisfy the *joint condition* that we describe next.

For every joint $\omega \in T^{[2]}$, we can identify the slabs incident at ω with the rays of the 2-dimensional fan Σ_ω in M_ω. Let $\rho_1, \ldots, \rho_n$ be a cyclic enumeration of the slabs incident at ω, and

[6] We say that v_y is *at integral affine distance* 1 *from* ρ if $\langle d, v_y \rangle = 1$, where $d \in \sigma^\vee \subset \mathrm{Hom}(M_\eta, \mathbb{Z})$ is the primitive vector that pairs to zero with all points in ρ.

[7] It turns out that this assumption holds in many cases of interest.

[8] The tensor product of the divisor of z^{v_y} with the one of z^{-v_y} is reminiscent to the tensor product of the two normal bundles $N_{Y_\sigma} Y_\rho$ and $N_{Y_{\sigma'}} Y_\rho$, but note that in general neither Y_ρ, Y_σ nor $Y_{\sigma'}$ can be assumed smooth.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

let $d_i \in N_\omega = \mathrm{Hom}(M_\omega, \mathbb{Z})$ be the primitive normal to ρ_i such that $d_i > 0$ on ρ_{i+1}. Corollary 3.7 in Section 3.2 states that, for every joint $\omega \in T^{[2]}$, we have a well-defined isomorphism

$$J_\omega \colon \bigotimes_{i=1}^{n} d_i \otimes \mathcal{L}_{\rho_i|Y_\omega} \cong 0 \otimes \mathcal{O}_{Y_\omega} \quad \text{in} \quad N_\omega \otimes \mathrm{Pic}\, Y_\omega.$$

In Section 3.2, Definition 3.9, $\mathcal{LS}_Y \subset \bigoplus_{\rho \in T^{[1]}} \mathcal{L}_\rho$ is defined to be the subsheaf consisting of sections $(f_\rho)_{\rho \in T^{[1]}}$ such that for all joints ω

$$J_\omega(d_i \otimes (f_{\rho_i|Y_\omega})) = 0 \otimes 1$$

at the generic point ω of Y_ω.

We conclude with a statement of our main result. We begin with some preliminaries that are needed before we can talk about our notion of a compatible log structure on a ggtc space. A fuller discussion, including a short summary of basic facts on log structures and a road-map of the proof, can be found in Section 4.1.

Fix a viable ggtc space Y. Recall that a *log structure* on Y is a pair $(\mathfrak{P}, \alpha)$ where $\mathfrak{P}$ is a sheaf (in the Zariski topology) of monoids and $\alpha \colon \mathfrak{P} \to (\mathcal{O}_Y, \times)$ is a homomorphism of sheaves of monoids such that

$$\alpha_{|\alpha^{-1}(\mathcal{O}_Y^\times)} \colon \alpha^{-1}(\mathcal{O}_Y^\times) \to \mathcal{O}_Y^\times$$

is an isomorphism. A log scheme is a scheme equipped with a log structure and we denote by $Y^\dagger$ a log scheme with underlying scheme Y. The *ghost sheaf* is the quotient sheaf of monoids $\overline{\mathfrak{P}} := \mathfrak{P}/\alpha^{-1}(\mathcal{O}_Y^\times)$.

Recall that the *standard log point* is the log scheme $\mathrm{Spec}\, k^\dagger = (\mathrm{Spec}\, k, \mathfrak{P}_k)$, where $\mathfrak{P}_k = k^\times \times \mathbb{N}$ and

$$\alpha(a, n) = \begin{cases} 0 & \text{if } n > 0, \\ a & \text{if } n = 0. \end{cases}$$

A log scheme *over the standard log point*, or simply *over* $k^\dagger$, is a log scheme $Y^\dagger$ equipped with a morphism $Y^\dagger \to \mathrm{Spec}\, k^\dagger$ to the standard log point. We denote a log scheme over $k^\dagger$ by the symbol $Y^\dagger/k^\dagger$. A log scheme $Y^\dagger/k^\dagger$ comes with a global section

$$\mathbf{1}_\mathfrak{P} \in \Gamma(Y, \mathfrak{P}),$$

the image of $1 \in \mathbb{N}$. With $\mathbf{1}_{\overline{\mathfrak{P}}}$ the image of $\mathbf{1}_\mathfrak{P}$ in $\overline{\mathfrak{P}}$, the *relative ghost sheaf* is the quotient sheaf $\overline{\mathcal{M}} = \overline{\mathfrak{P}}/\mathbf{1}_{\overline{\mathfrak{P}}}$. In our context, $\overline{\mathcal{M}}$ is going to be a sheaf of abelian groups.

Definition 4.2 is a precise formulation of our notion of a *compatible log structure* $Y^\dagger/k^\dagger$ *on a ggtc space* Y. The key requirement is the datum of an identification of the relative ghost sheaf of the log structure with the relative ghost sheaf of the ggtc space, $\overline{\mathcal{M}} \xrightarrow{\cong} \mathcal{M}$. We denote by $\mathrm{LS}_{k^\dagger}(Y)$ the set of isomorphism classes of compatible log structures over $k^\dagger$.

1.2 Main theorem and its discussion

The main theorem is about the sheaf $\mathcal{LS}_Y$ in (1.1) constructed in detail in Section 3. This sheaf has a subsheaf $\mathcal{LS}_Y^\times$ of nowhere zero sections.

Theorem 1.3 (Theorem 4.3). *Let Y be a viable ggtc space, and let $\mathcal{LS}_Y \subset \bigoplus_\rho \mathcal{L}_\rho$ be the subsheaf of the direct sum of slab line bundles constructed in Section 3.*

Denote by $\mathrm{LS}_{k^\dagger}(Y)$ the set of isomorphism classes of log structures on Y over $k^\dagger$ compatible with the ggtc structure.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

The set-theoretic function

$$r\colon \mathrm{LS}_{k^\dagger}(Y) \to \Gamma(Y, \mathcal{LS}_Y^\times)$$

constructed in (4.15) is a bijection.

We prove the theorem by considering the *subsheaf of regular extensions*

$$\mathcal{E}xt^1_c(\mathcal{M}, \mathcal{O}_Y^\times) \subset \mathcal{E}xt^1(\mathcal{M}, \mathcal{O}_Y^\times)$$

that already appeared in [GS06, Theorem 3.22].[9] We construct a morphism of sheaves $\varphi\colon \mathcal{LS}_Y^\times \to \mathcal{E}xt^1_c(\mathcal{M}, \mathcal{O}_Y^\times)$ and show that it is bijective. Finally, we prove that the assignment that sends a log structure/$k^\dagger$ to its extension class in $\mathcal{E}xt^1(\mathcal{M}, \mathcal{O}_Y^\times)$ in fact gives a bijection $\mathrm{LS}_{k^\dagger}(Y) \to \Gamma\left(Y, \mathcal{E}xt^1_c(\mathcal{M}, \mathcal{O}_Y^\times)\right)$, and we obtain r as the composition of two bijections.

The content of our main theorem is not that there is *some* sheaf $\mathcal{LS}_Y$ with a natural identification of $\mathrm{LS}_{k^\dagger}(Y)$ with $\Gamma(Y, \mathcal{LS}_Y^\times)$. Indeed, it is a basic general fact that $\mathrm{LS}_{k^\dagger}(-)$ is a sheaf. It is not even that there is *some construction* of a sheaf $\mathcal{LS}_Y^\times$ and a bijection from $\mathrm{LS}_{k^\dagger}(Y)$ to $\Gamma(Y, \mathcal{LS}_Y^\times)$. Indeed, for example, $\mathcal{E}xt^1_c(\mathcal{M}, \mathcal{O}_Y^\times)$ is an example of such a construction.

Our point is that the statement is true with the description of $\mathcal{LS}_Y$ given in Section 3, and that this particular description allows to construct log structures effectively. We illustrate this point here with a very simple example. (More examples can be found in Section 5.) Consider the case, see also Section 5.1, of a scheme Y that is the union of two smooth components Y_1, Y_2 meeting transversally along a smooth irreducible divisor D. We show in Section 5.1 that

$$\mathcal{LS}_Y \cong (N_{Y_1} D) \otimes (N_{Y_2} D), \tag{1.2}$$

where $N_{Y_i} D$ denotes the *normal bundle* of D in Y_i.[10] The description (1.2) is particularly useful when the line bundle $\mathcal{LS}_Y = (N_{Y_1} D) \otimes (N_{Y_2} D)$ is not trivial, and thus it does not have a nowhere-vanishing section. Consider for example the case when $\mathcal{LS}_Y$ is, say, base point free, and let $Z \subset D$ be the vanishing locus of a general section $s \in \Gamma(D, \mathcal{LS}_Y)$ giving an isomorphism $\mathcal{O}_D(Z) \cong \mathcal{LS}_Y$. In language introduced in Definition 5.4(1), we say that such an s gives a log structure/$k^\dagger$ *singular along* Z. The log structure in question is the push forward to Y of the log smooth log structure that we have on $Y \setminus Z$. This push forward log structure is rather badly behaved, for example it is not coherent. However, it has a particularly nice *log resolution*. Indeed, let

$$\widetilde{Y}_1 = \mathrm{Bl}_Z Y_1$$

be the blow up of $Z \subset Y_1$. The strict transform of D in $\widetilde{Y}_1$ is isomorphic to D, so we can glue $\widetilde{Y}_1$ to Y_2 along D to form a scheme $f\colon \widetilde{Y} \to Y$. Denoting by $E = f^{-1}Z \subset \widetilde{Y}$ the exceptional set, we have $\widetilde{Y} \setminus E = Y \setminus Z$. It is clear that

$$N_{\widetilde{Y}_1} D = (N_{Y_1} D)(-Z), \text{ and hence}$$
$$\mathcal{LS}_{\widetilde{Y}} = (N_{\widetilde{Y}_1} D) \otimes (N_{\widetilde{Y}_2} D) = ((N_{Y_1} D) \otimes (N_{Y_2} D))(-Z) = \mathcal{LS}_Y(-Z) = \mathcal{O}_D.$$

[9] Here $\mathcal{M}$ and $\mathcal{O}_Y^\times$ are sheaves of abelian group and $\mathcal{E}xt^1(\mathcal{M}, \mathcal{O}_Y^\times)$ is the sheaf of extensions in the category of sheaves of abelian groups.

[10] The fact that $\mathrm{LS}_{k^\dagger}(Y)$ is in bijective correspondence with $\Gamma(D, ((N_{Y_1} D) \otimes (N_{Y_2} D))^\times)$ is elementary and well-known. As important original references, we recommend taking a look at Definition 1.9 in [Fri83] and Lemma 2.2 in [Kat00]. Further relevant works on the sheaf $(N_{Y_1} D) \otimes (N_{Y_2} D)$ and its cousin in the normal crossing case include [Ste95, KN94, SS06, FFR21]; see also Theorem 1.2 in [Ols03].

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

All of this goes to shows that there exists a unique log structure on $\widetilde{Y}$ smooth over $k^\dagger$ and a log morphism $\widetilde{Y}^\dagger \to Y^\dagger$ over $k^\dagger$ that, when restricted to $\widetilde{Y} \setminus E = Y \setminus Z$, is an isomorphism and so it is the log structure given by the section $s \in \Gamma(Y, \mathcal{LS}_Y)$. We call $\widetilde{Y}^\dagger \to Y^\dagger$ a *log resolution*.[11]

Our point, again, is that it would be awkward to establish these facts directly from the definition of log structure, and impossible to derive it off the shelf from the constructions and the statements in [GS06] (because that paper assumes that all components of Y are toric varieties). It is the independent geometric interpretation of the sheaf $(N_{Y_1} D) \otimes (N_{Y_2} D)$ as the tensor product of normal bundles of D in Y_1, Y_2 that makes the verification straightforward, by tracking the way that normal bundles change under blow ups. Our construction of the sheaf $\mathcal{LS}_Y$, given in Section 3, in the two-component case immediately specializes to $(N_{Y_1} D) \otimes (N_{Y_2} D)$. In the general case of a ggtc space Y, the construction is more involved, but it retains the geometric interpretation, making it possible, in many cases of interest, to construct log structures and log resolutions effectively.

It is well known that, when Y is simple normal crossings, the sheaf $\mathcal{LS}_Y$ is naturally isomorphic to $\mathcal{T}^1_Y = \mathcal{E}xt^1_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y)$, see Theorem 5.5 and Remark 5.1 in [FFR21] for further references, and so $\mathcal{LS}_Y$ is in fact a coherent sheaf in this case. However, this rather special situation is somewhat misleading because the joint condition for gluing $\mathcal{LS}_Y$ from line bundles happens to be linear when Y is normal crossing (or a product of normal crossing spaces) while in general it is a polynomial condition that results in a non-coherent sheaf, see for example (5.3). The precise form of this polynomiality was already shown in [GS06, Theorem 3.22] and in fact everything we do reduces to this explicit local description when choosing a log smooth chart of a compatible log structure.

1.3 Summary of previous work

We already indicated several prior works in the semistable situation [Fri83, Kat00, Ste95, KN94, SS06, Ols03], so we now focus on singular log structures and more general spaces. The paper [GS06] is concerned with toroidal crossing spaces Y that are a union of toric varieties meeting along boundary strata. Among many other things, for such a Y, that paper defines a sheaf $\mathcal{LS}_Y$ and proves a natural identification $\mathrm{LS}_{k^\dagger}(Y) = \Gamma(Y, \mathcal{LS}^\times_Y)$. Essentially, $\mathcal{LS}_Y$ is defined to be $\mathcal{E}xt^1_c(\mathcal{M}, \mathcal{O}^\times_Y) \subset \mathcal{E}xt^1(\mathcal{M}, \mathcal{O}^\times_Y)$, see Sections 1.2 and 4.1, but the paper also gives an explicit local description [GS06, Theorem 3.22] in terms of local functions that satisfy the joint condition, and then shows [GS06, Theorem 3.28] that these local functions are sections of explicit line bundles $\mathcal{N}_\rho$ (corresponding to our $\mathcal{L}_\rho$). The description of the sheaf $\mathcal{LS}_Y$ in [GS06] is sufficiently concrete to enable the effective construction of elements in $\mathrm{LS}_{k^\dagger}(Y)$ when Y is a union of toric varieties meeting along boundary strata.

The paper [SS06] introduces the notion of *Gorenstein toroidal crossing space*, and goes on to study log structures on these.

1.4 Our work in relation to previous work

Our definition of ggtc space is a generalization of the Gorenstein toroidal crossing spaces of [SS06]. Our work is closely related to [SS06], but there are two important differences. The first key difference is that we work with log structures over $k^\dagger$, where [SS06] works with absolute log structures: this change of perspective is essential to the applications that we have in mind and it results in surprising simplifications. The second key difference is that we require the Gorenstein

[11] A nice property of this resolution is that $K_{Y^\dagger/k^\dagger}$ is f-trivial: we call a resolution with this property *log crepant*.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

toroidal crossing condition to hold only at the generic point of every stratum, as opposed to everywhere.

Our paper generalizes the corresponding part of [GS06], from toroidal crossing spaces that are union of toric varieties meeting along boundary strata, to the case of viable ggtc spaces. Here the key point of our study is a more natural and more general construction of the sheaf $\mathcal{LS}_Y$.

In outline, our proof follows the proof of [GS06, Theorem 3.22], with changes necessary to work with our construction of the sheaf $\mathcal{LS}_Y$. Indeed, our main innovation is the construction of the sheaf $\mathcal{LS}_Y$, where we use the Picard stack to show that the local descriptions of the slab bundles $\mathcal{L}_\rho$, when formulated not in terms of functions but of divisors, *glue automatically* to give the slab bundles globally, and that the joint conditions automatically make sense globally. Unlike [GS06], we never work with local charts for log structures and our approach is closer to the Deligne–Faltings view of log structures as systems of line bundles with sections.

More detail about where exactly and how specifically we depart from [GS06] can be found in the outline of the proof in Section 4.1. In particular, as was pointed out by Bernd Siebert, our point of view in the proof of Proposition 4.32, where we construct a log structure from a section of $\mathcal{LS}_Y$, is closely related to that of [BV12]. We learned that the approach via line bundle systems is also useful for recasting logarithmic data in symplectic-geometric terms, [FTS25].

1.5 Description of contents

In Section 2 we introduce ggtc spaces. The definition is local in nature, and we take the time to describe two global objects that are naturally attached to them, the *cone sheaf* and the *divisor system*. We also introduce a property, which we call *viability*, that allows us to do log geometry on a ggtc space. We work in the Zariski topology for simplicity and because it is sufficient for many applications.[12]

In Section 3 we explicitly construct a sheaf $\mathcal{LS}_Y$ that naturally exists on every viable ggtc space Y. Later in Section 4 we prove that this sheaf $\mathcal{LS}_Y$ classifies log structures on Y over the standard log point and compatible with the ggtc structure. In this paper, we aim to address a reader whose goal is to make a log structure on Y explicitly. The most efficient way to do this is to construct a nowhere-vanishing global section of $\mathcal{LS}_Y$, and for this she only needs to know how $\mathcal{LS}_Y$ is constructed; she does not need to know the proof that $\mathcal{LS}_Y$ classifies log structures. Our presentation aims to facilitate explicit constructions.

Finally, in Section 5 we give some examples.

2. Generically Gorenstein toroidal crossing spaces

In Section 2.4 we give the formal definition of ggtc space. Basically, a ggtc space is a stratified space Y such that if $\eta \in Y$ is the generic point of a stratum, there is a fan Σ_η in a lattice M_η and an isomorphism from the formal completion $\widehat{k(\eta)[\Sigma_\eta]}$ at the origin to the formal completion $\widehat{\mathcal{O}_{Y,\eta}}$ at the maximal ideal. These isomorphisms need to satisfy certain coherence conditions that are best kept track of by a Kato fan in the sense of [ACM+16, Section 4].

[12] In short, all the examples we are currently interested in are unions $Y = \cup Y_\sigma$ of irreducible components Y_σ each of which admits a *toric model*, i.e., a birational morphism to a toric variety. For certain applications, it will be necessary eventually to work with monodromy in the étale topology, and a future theory will have to implement these features. However, these features become a hindrance when working with examples that do not need them, which gives us another reason for not discussing these matters here.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

In Sections 2.1–2.3 we set out carefully our notation and conventions on monoids, fans and stratified spaces: this material is elementary but tedious.

In Section 2.5 we define a property that we call viability, which allows us to do logarithmic geometry.

The definition of a ggtc space is local in nature. In Section 2.6 we define two global objects that exist on the normalization of a ggtc space, the cone sheaf and the divisor system, that enter crucially the construction of the slab bundles and the precise formulation of the joint condition.

2.1 Notation and conventions for monoids and fans

Our terminology on monoids mostly follows [Ogu18, Chapter I]. The following summary of terminology and notation is intended for reference, not as a complete dictionary on monoids.

Convention 2.1.

(1) A *lattice* is a free abelian group $M \cong \mathbb{Z}^r$ of finite rank r. We denote $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$.

(2) A *monoid* is a commutative semigroup with neutral element. Our default position is to denote the operation and unit of a monoid additively by $+$, 0. (If R is a ring, for example, we may want to consider the monoid $(R, \times, 1)$.)

(3) If M is a lattice and $S \subset M$ a subset, we denote by $\langle S \rangle$ the saturation in M of the subgroup generated by S and by $\langle S \rangle_+$ the saturation in M of the submonoid generated by S. Similarly, if $\tau \subset M_{\mathbb{R}}$, we denote $\langle \tau \rangle := \langle \tau \cap M \rangle$ and $\langle \tau \rangle_+ := \langle \tau \cap M \rangle_+$.

(4) The group of units of a monoid P is denoted by $P^\times$. A monoid P is *sharp* if $P^\times = (0)$.

(5) A monoid P is *toric* if there exist a lattice M of finite rank and a closed convex rational polyhedral cone $\sigma \subset M_{\mathbb{R}}$ such that $P = \sigma \cap M$. In this paper we often work with toric monoids and we almost always assume them to be sharp.

(6) Let P be a monoid. A submonoid $F \subset P$ is a *face* if the following condition is satisfied. For all $u, v \in P$, if $u + v \in F$, then $u, v \in F$. We write $F \leqslant P$ to mean that F is a face of P. The notation $F < P$ means that F is a proper face of P, that is, F is a face and $F \neq P$.

(7) When $F \leqslant P$ we denote by $F^{-1}P$ the *localization* of P at F. We call the monoid homomorphism $P \to F^{-1}P$ a *face localization.* When $F \leqslant P$, the *quotient* P/F is the monoid $F^{-1}P/F$. We call the monoid homomorphism $P \to P/F$ a *face quotient.*

(8) If P is a monoid and $\mathbf{1} \in P$ an element, we use $P/\mathbf{1}$ denote the quotient of P by the submonoid generated by $\mathbf{1}$. For example $\mathbb{N}^2/(1,1) \cong \mathbb{Z}$.

(9) If P is a monoid, we denote by P^{gp} the universal (Grothendieck) group of P.

(10) If R is a ring and P a monoid, we denote by $R[P]$ the monoid ring.

Definition 2.2. Let $M \cong \mathbb{Z}^r$ be a lattice.

(1) A *fan in* M is a finite set Σ of closed convex rational polyhedral cones in $M_{\mathbb{R}}$ such that:
   (i) for all $\tau \in \Sigma$, if $\mu \leqslant \tau$ is a face, then $\mu \in \Sigma$;
   (ii) for all $\tau, \mu \in \Sigma$, $\tau \cap \mu$ is a face of both τ and μ.

   The *support* of the fan, denoted by $|\Sigma| \subset M_{\mathbb{R}}$, is the union of the cones of Σ. The fan is said to be *complete* if $|\Sigma| = M_{\mathbb{R}}$.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

(2) Let Σ be a fan and $\rho \in \Sigma$ a cone.
   (a) The *localization* $\rho^{-1}\Sigma$ of Σ in ρ is the fan in M that consists of the convex cones $\sigma - \rho = \{x - y \in M_{\mathbb{R}} | x \in \sigma, y \in \rho\}$ where σ ranges over all cones in Σ that contain ρ.
   (b) The *quotient* Σ/ρ of Σ by ρ is the fan in $M/\langle\rho\rangle$ obtained from the localization $\rho^{-1}\Sigma$ by projecting each of its cones under the linear map $M \to M/\langle\rho\rangle$.

(3) If Σ is a fan in M and R is a ring, the *Stanley–Reisner ring* $R[\Sigma]$ is the free R-module over the symbols z^m for those $m \in M$ which are also contained in some cone of Σ, with multiplication defined by

$$z^m \cdot z^{m'} = \begin{cases} z^{m+m'} & \text{if there is a cone } \sigma \in \Sigma \text{ that contains } m, m', \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $\mathfrak{m}$ the ideal generated by all the symbols z^m with $m \neq 0$. If R is a field, $\mathfrak{m}$ is a maximal ideal.

Lemma 2.3. *There is an equivalence between the following two categories.*

(1) *The category with objects pairs $(P, \mathbf{1})$ of a sharp toric monoid P and an element $\mathbf{1} \in P$ such that $P \setminus (\mathbf{1} + P)$ is the union of the proper faces of P, and morphisms face quotients.*

(2) *The category whose objects are pairs (Σ, φ) of a rational polyhedral fan Σ, not necessarily complete but with convex support, and a polarization, that is, a strictly convex piecewise linear function $\varphi\colon |\Sigma| \cap M \to \mathbb{Z}$ up to the addition of an integral linear function $M \to \mathbb{Z}$, and morphisms fan quotients.*

Under this equivalence,

$$P \setminus (\mathbf{1} + P) = \cup_{F<P} F$$

if and only if the fan Σ is complete.

In addition, under this equivalence, $\mathbb{Z}[P]/(z^{\mathbf{1}}) = \mathbb{Z}[\Sigma]$.

Sketch of Proof. Starting from (Σ, φ), let P be the supergraph of φ in $M \oplus \mathbb{Z}$ and $\mathbf{1} = (0, 1)$.

Vice versa, starting from $(P, \mathbf{1})$, let M be the universal group of the quotient monoid $P/\mathbf{1}$ of P by the congruence relation generated by the submonoid $\mathbf{1}\mathbb{N}$, and let Σ be the fan in M whose cones are the projections under the obvious homomorphism $P \to M$ of the proper faces of P that do not contain $\mathbf{1}$. □

2.2 Notation and conventions for stratified spaces

The next two definitions correspond to the notion of finite partition of [Sta26], Tag 09XZ and finite good stratification of [Sta26], Tag 09Y0.

Definition 2.4. Let X be a topological space. A *partition* of X is a decomposition

$$X = \coprod_{\eta \in T} X^\star_\eta$$

into locally closed subsets X_η indexed by a finite set T. The $X^\star_\eta$ are called the *parts* of the partition.

We denote by $X_\eta = \overline{X^\star_\eta}$ the closure of $X^\star_\eta$.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

Definition 2.5. Let X be a topological space. A *good stratification* of X is a partition $X = \coprod_{\eta\in T} X^\star_\eta$ such that for all $\mu, \eta \in T$ we have

$$X^\star_\mu \cap X_\eta \neq \emptyset \quad \Rightarrow \quad X^\star_\mu \subset X_\eta.$$

The $X^\star_\eta$ are called the *strata* of the stratification.

Given a good stratification $X = \coprod_{\eta\in T} X^\star_\eta$, we obtain a partial ordering on the index set T by setting $\mu \leqslant \eta$ if and only if $X^\star_\mu \subset X_\eta$. It then follows that

$$X_\eta = \coprod_{\mu\leqslant\eta} X^\star_\mu.$$

Definition 2.6.

(1) A *space* is a scheme or Deligne–Mumford stack of finite type over k.

(2) A *stratified space* is a space Y endowed with a good stratification

$$Y = \coprod_{\eta\in T} Y^\star_\eta$$

such that all strata are irreducible. Under this assumption, T is identified with the set of generic points of the strata, and the partial ordering on T is induced by specialization: for all $\eta_1, \eta_2 \in T$, $\eta_1 \leqslant \eta_2$ if and only if η_1 is a specialization of η_2.
When Y is a Deligne–Mumford stack, we assume in addition that the generic points of strata have trivial stabilizers.

(3) The *codimension* of $\eta \in T$ is the codimension in Y of the corresponding stratum. We denote by $T^{[i]} \subset T$ the set of points of codimension i and we write

$$Y^{[i]} = \coprod_{\eta\in T^{[i]}} Y_\eta, \qquad Y^{(i)} = \bigcup_{\eta\in T^{[i]}} Y_\eta.$$

Lemma 2.7. *Let $Y = \coprod_{\eta\in T} Y^\star_\eta$ be a stratified space.*

The subset topology of $T \subset Y$ is the order topology: $W \subset T$ is open if and only if for all $\tau_1 \in W$, if $\tau_2 \geqslant \tau_1$ then $\tau_2 \in W$. We have the following.

(i) *For all $\eta \in T$,*

$$T_{\geqslant\eta} = \{\mu \in T \mid \mu \geqslant \eta\}$$

is the smallest open subset of T that contains η;

(ii) *The inclusion*

$$a\colon T \hookrightarrow Y \quad \textit{has a continuous retraction} \quad b\colon Y \to T$$

defined such that $b(y) = \eta$ if $y \in Y^\star_\eta$;

(iii) *The map b is open and for all Zariski open subset $U \subset Y$, $b(U) = a^{-1}(U) = U \cap T$.*

Sketch of Proof. The map b is continuous: indeed for all $\eta \in T$ we have that

$$b^{-1}(T_{\geqslant\eta}) = \{y \in Y \mid b(y) \geqslant \eta\} = \coprod_{\mu\geqslant\eta} Y^\star_\mu$$

is the union of all locally closed strata that have η in their Zariski closure and hence it is Zariski open in Y.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

Consider a Zariski open subset $U \subset Y$. If $\eta \in b(U)$, then that means that $Y^\star_\eta \cap U \neq \emptyset$, or, equivalently, that $\eta \in U$. This shows that $b(U) = a^{-1}(U)$ and in particular it is open. □

Definition 2.8. Let $Y = \coprod_{\eta \in T} Y^\star_\eta$ be a stratified space. For all $\eta \in T$, the *open star* of η is the Zariski open subset

$$U_\eta = \{y \in Y \mid b(y) \geqslant \eta\} = b^{-1}(T_{\geqslant \eta}) \subset Y,$$

that is, the union of all strata of Y that have η in their Zariski closure.

Corollary 2.9. *In the situation of Lemma 2.7, if $\mathcal{F}$ is a sheaf on T, then the sheaves $a_\star\mathcal{F}$ and $b^{-1}\mathcal{F}$ on Y are isomorphic.*

Remark 2.10. In the situation of Lemma 2.7, we often use Corollary 2.9 to identify a sheaf $\mathcal{F}$ on T with a sheaf on Y, and we think of it as $a_\star\mathcal{F}$ or $b^{-1}\mathcal{F}$ depending of which point of view is more convenient.

2.3 The basic setup and assumptions for toroidal crossing spaces

In what follows $Y = \coprod_{\eta \in T} Y^\star_\eta$ is a stratified space satisfying the following assumptions:

(1) Y is reduced, equidimensional, and the irreducible components of Y are normal;

(2) Y is normal crossing in codimension 1; denoting by

$$\varepsilon \colon Y^{[0]} = \coprod_{\sigma \in T^{[0]}} Y_\sigma \longrightarrow Y$$

the normalization, the restriction of ε to to $\varepsilon^{-1}(Y^{(1)} \setminus Y^{(2)})$ is a degree-two disconnected finite étale cover over each component of the target;

(3) Y is the push-out of the diagram of spaces $Y^{[1]} \rightrightarrows Y^{[0]}$ where the two maps are obtained from the inclusions $Y_\rho \to Y_\sigma$, $Y_\rho \to Y_{\sigma'}$.

2.4 Generically Gorenstein toroidal crossing space

Setup 2.11. We introduce objects and notation that are used in Definition 2.12.

Fix a finite poset T equipped with the order topology.

(1) To give a sheaf of monoids $\mathcal{P}$ on T is equivalent to give the following data subject to obvious compatibilities:

(i) for all $\eta \in T$ a monoid P_η; and

(ii) for all $\eta_1 \leq \eta_2$, a *generization* homomorphism

$$P_{\eta_1} \to P_{\eta_2}.$$

(2) A *Kato fan* [ACM+16, Section 4] is a sharp monoidal space that is locally isomorphic to $\operatorname{Spec} P$, P a sharp toric monoid. Let $\mathcal{P}$ be a sheaf of monoids on T making T a Kato fan. In particular, all the P_η are sharp toric monoids, and all the generization homomorphisms are face quotients. It follows from the definition that, for all $\eta \in T$, there is a bijective identification of the poset of faces of P_η with $\{\tau \in T \mid \tau \geqslant \eta\}$.

(3) Given a Kato fan $(T, \mathcal{P})$, consider a global section $\mathbf{1} \in \Gamma(T, \mathcal{P})$ corresponding to the datum, for all $\eta \in T$, of sections $\mathbf{1}_\eta \in P_\eta$ such that $P_\eta \setminus (\mathbf{1}_\eta + P_\eta)$ is the union of all the proper faces of P_η. In this situation, by Lemma 2.3, the pair $(\mathcal{P}, \mathbf{1})$ gives rise to a sheaf of complete

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

fans $\mathbf{\Sigma}$ in the sheaf of lattices $\mathcal{M} = \mathcal{P}/\mathbf{1}$. This unpacks in the data, for all $\eta \in T$, of a fan Σ_η in $M_\eta = P_\eta/\mathbf{1}_\eta$, and for all $\eta_1 \leqslant \eta_2$ generization maps (viewing $\eta_2 \in \Sigma_{\eta_1}$) quotient homomorphisms $M_{\eta_1} \to M_{\eta_2} = M_{\eta_1}/\langle \eta_2 \rangle$ identifying Σ_{η_2} with the quotient fan Σ_{η_1}/η_2. For all $\eta \in T$, the poset of faces of P_η is identified with the fan Σ_η, and this induces an identification:

$$\Sigma_\eta = \{\tau \in T \mid \tau \geqslant \eta\}.$$

Now fix a stratified space $Y = \coprod_{\eta \in T} Y_\eta^\star$ satisfying the assumptions of Section 2.3. Assume that T is endowed with a pair $(\mathcal{P}, \mathbf{1})$ of a sheaf of monoids $\mathcal{P}$ making T a Kato fan and global section $\mathbf{1} \in \Gamma(Y, \mathcal{P})$ as in part (3).

(4) Fix $\eta \in T$. The space $\operatorname{Spec} k(\eta)[\Sigma_\eta]$ has a natural stratification indexed by the cones of Σ_η ordered by inclusion. For a cone $\tau \in \Sigma_\eta$ we denote by $O_\tau \subset \operatorname{Spec} k(\eta)[\Sigma_\eta]$ the corresponding stratum. For all $\tau_1, \tau_2 \in \Sigma_\eta$, $\tau_1 \leqslant \tau_2$ if and only if $O_{\tau_1} \subset \overline{O}_{\tau_2}$. We denote by $\widehat{k(\eta)[\Sigma_\eta]}$ the formal completion at the origin, and, for all $\tau \geqslant \eta$, we denote by $\widehat{O_\tau} \subset \operatorname{Spec} \widehat{k(\eta)[\Sigma_\eta]}$ the induced subscheme.

(5) Fix $\eta \in T$. The local ring $(\mathcal{O}_{Y,\eta}, \mathfrak{m}_\eta)$ of Y is a local Noetherian k-algebra (k is the base field) with residue field $k(\eta)$. We denote by $\widehat{\mathcal{O}_{Y,\eta}}$ the formal completion at $\mathfrak{m}_\eta$. By the Cohen structure theorem [Sta26], Tag 032A $\widehat{\mathcal{O}_{Y,\eta}}$ contains a field isomorphic to $k(\eta)$. For all $\tau \geqslant \eta$, we denote by $\widehat{Y_\tau} \subset \operatorname{Spec} \widehat{\mathcal{O}_{Y,\eta}}$ the induced subscheme.

Definition 2.12. Importing Setup 2.11, a ggtc space is a tuple

$$\left(Y = \coprod_{\eta \in T} Y_\eta^\star, (\mathcal{P}, \mathbf{1}), \{\widehat{f}_\eta \mid \eta \in T\}\right)$$

of a stratified space $Y = \coprod_{\eta \in T} Y_\eta^\star$ satisfying the assumptions of Section 2.3, and we have the following.

(a) A pair $(\mathcal{P}, \mathbf{1})$ of a Zariski sheaf $\mathcal{P}$ of monoids on T making T a Kato fan, and a global section $\mathbf{1} \in \Gamma(T, \mathcal{P})$ given by elements $\mathbf{1}_\eta \in P_\eta$ such that $P_\eta \setminus (\mathbf{1}_\eta + P_\eta)$ is the union of all the proper faces of P_η. The section $\mathbf{1}$ induces a sheaf of complete fans $\mathbf{\Sigma}$ in the sheaf of lattices $\mathcal{M} = \mathcal{P}/\mathbf{1}$.

(b) For all $\eta \in T$, a ring isomorphism:

$$\widehat{f}_\eta \colon \widehat{k(\eta)[\Sigma_\eta]} \xrightarrow{\cong} \widehat{\mathcal{O}_{Y,\eta}},$$

subject to the following condition: for all $\tau \in \Sigma_\eta$, the isomorphism $\widehat{f}_\eta$ identifies $\widehat{Y_\tau^\star}$ with $\widehat{O_\tau}$.

The sheaf $\mathcal{P}$ is called the *ghost sheaf* of the ggtc space; $\mathcal{M}$ the *relative ghost sheaf*; $\widehat{f}_\eta$ the *local formal frames*.

Convention 2.13. Sometimes for simplicity we write 'let Y be a ggtc space'. When we do this, it is understood that Y is a stratified space, and that it has a ghost sheaf, a relative ghost sheaf, etc., and we take it for granted that they will be denoted by $\mathcal{P}$, $\mathcal{M}$, etc.

Remark 2.14.

(1) The reader should not be overly concerned about the formal local frames. We could replace part (b) with the following (possibly more familiar) requirement: for all $\eta \in T$, there is

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

a Zariski closed subset $Z_\eta \subsetneq Y_\eta^\star$ such that, for all $y \in Y_\eta \setminus Z_\eta$, there is a neighbourhood $y \in W \subset Y_\eta \setminus Z_\eta$ and a smooth morphism

$$W \to \operatorname{Spec} k[\Sigma_\eta].$$

These morphisms would have to satisfy a more-or-less obvious coherence condition.

(2) Let Y be a ggtc space. For all $\eta \in T$, $\operatorname{Spec} \mathbb{Z}[P_\eta]$ is a Gorenstein toric variety with reduced boundary $B = \operatorname{div} z^{\mathbf{1}}$.

(3) In [SS06] 'gtc' is an acronym of 'Gorenstein toroidal crossing' in the étale topology. On the one hand, in this paper, we work in the Zariski topology. On the other hand, we only require a Gorenstein toroidal crossing structure at the generic points of strata.

Example 2.15. In many applications of interest,[13] the formal frame isomorphisms $\widehat{f}_\eta \colon \widehat{k(\eta)[\Sigma_\eta]} \xrightarrow{\cong} \widehat{\mathcal{O}_{Y,\eta}}$ arise from bona fide frame isomorphisms

$$f_\eta \colon k(\eta)[\Sigma_\eta]_{\mathfrak{m}} \xrightarrow{\cong} \mathcal{O}_{Y,\eta}.$$

In general, however, this is not possible. Consider, for example, a situation where $Y = Y_1 \cup Y_2$ is a union of two smooth elliptic curves that meet in a point P with residue field $k = k(P)$. The discrete valuation ring of P in either component is not isomorphic to $k[x]_{(x)}$.

2.5 Viable ggtc spaces

In this section we define the key notion of viability of a ggtc space.

Definition 2.16. Fix a ggtc space $(Y = \coprod_{\eta \in T} Y_\eta^\star, (\mathcal{P}, \mathbf{1}), \{\widehat{f}_\eta \mid \eta \in T\})$.

For all $\eta \in T$, $\operatorname{Spec} k(\eta)[\Sigma_\eta]$ has a stratification with strata $\{O_\tau^\star \mid \tau \in \Sigma_\eta\}$, $O_\tau = \overline{O_\tau^\star}$. For $\tau \in \Sigma_\eta$, denote by

$$\Omega_\tau = \coprod_{\tau \leqslant \tau'} O_{\tau'}^\star$$

the open star of τ, and denote by $\overline{\Omega}_\tau$ its Zariski closure. In addition, denote by

$$\operatorname{Div}_\flat^+ \overline{\Omega}_\tau$$

the monoid of effective Cartier divisors on $\overline{\Omega}_\tau$ supported on $\bigcup_{\rho \in T^{[1]}} O_\rho$.

For $\tau \in \Sigma_\eta$ and $m \in \tau \cap M_\eta$, denote by $z^m \in k(\eta)[\Sigma_\eta]$ the corresponding monomial. For all $\tau \in \Sigma_\eta$ we have a monoid homomorphism

$$\operatorname{div}_{\eta,\tau} \colon \langle \tau \rangle_+ \ni m \mapsto \operatorname{div} z^m \in \operatorname{Div}_\flat^+ \overline{\Omega}_\tau.$$

For $m \in \langle \tau \rangle_+$, write $\operatorname{div}_{\eta,\tau}(m) = \sum_{\rho \in T^{[1]}} m_\rho O_\rho$ and consider the effective divisor:

$$D_{\eta,\tau}(m) = \sum_{\rho \in T^{[1]}} m_\rho Y_\rho \quad \text{on } \overline{U}_\tau \cap U_\eta.$$

We say that the ggtc space is viable if, for all $\eta \in T$, for all $\tau \in \Sigma_\eta$ and $m \in \tau \cap M_\eta$ as above, the divisor $D_{\eta,\tau}(m)$ is a Cartier divisor.

[13] For instance if the irreducible components of Y are toric varieties, or more generally if they have toric models.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

Notation 2.17. Let Y be a viable ggtc space. For all $\eta \in T$ and $\tau \in \Sigma_\eta$, we denote by

$$D_{\eta,\tau} \colon \langle \tau \rangle_+ \to \mathrm{Div}_\flat^+(\overline{U}_\tau \cap U_\eta)$$

the monoid homomorphism provided by Definition 2.16.

The monoid homomorphism $D_{\eta,\tau}$ extends to a group homomorphism that, abusing notation, we still denote by $D_{\eta,\tau} \colon \langle \tau \rangle \to \mathrm{Div}_\flat(\overline{U}_\tau \cap U_\eta)$.

Remark 2.18. In the situation of Definition 2.16, for all ggtc spaces, not necessarily viable, it can be shown that for all $\eta \in T$, $\tau \in \Sigma_\eta$ and $m \in \tau \cap M_\eta$, there is a Zariski neighbourhood

$$\eta \subset W \subset \overline{U}_\tau \cap U_\eta$$

such that $D_{\eta,\tau}(m)$ is Cartier on W.

Example 2.19. Consider $r, a > 0$ with $\gcd(r, a) = 1$ and $\mathrm{char}(k) \nmid r$. The natural ggtc structure on the 3-dimensional Deligne–Mumford stack

$$Y = (uv = 0) \subset [\mathbb{A}^4/\mu_r]\,,$$

where u, v, w, z are coordinates on $\mathbb{A}^4$ and μ_r acts with weights $(1, -1, a, -a)$ is viable, as we now explain. By assumption, there is an integer s so that $sa = r - 1$ modulo r. This is one of those cases where the formal frames arise from bona fide frame isomomorphisms: an example of a frame at $\eta = (u, v)$ is the isomorphism

$$f_\eta \colon \ (k(\eta)[x, y]/(xy))_{(x,y)} \to \mathcal{O}_{Y,\eta}, \quad x \mapsto uw^s, \ y \mapsto vz^s.$$

There are other possibilities for frame maps. We could post-compose this frame with any equivariant change of coordinates at η that preserves strata, e.g., $u \mapsto uw^i z^j$, $v \mapsto vw^k z^l$ with $(i - j)$ and $(k - l)$ divisible by r, so there is a variety of choices.

Most importantly for the ggtc property, the divisor $\mathrm{div}(f_\eta(x))$ equals $\mathrm{div}(u)$ in $\mathrm{Spec}\, \mathcal{O}_{Y,\eta}$ and therefore extends as a Cartier divisor in the component $(v = 0) \subset Y$, supported on $(u = v = 0)$, even at $y = (0, 0, 0, 0) \in (v = 0)$. Indeed, because we are working with the Deligne–Mumford stack Y, rather than its coarse moduli space, the divisors $\mathrm{div}(u), \mathrm{div}(v)$ are well-defined Cartier divisors, also at the point $y = (0, 0, 0, 0) \in Y$: even though the functions u, v are not well-defined, each gives rise to a descent datum for a Cartier divisor from $\mathbb{A}^4$ to Y. These divisors are not Cartier on the coarse moduli on the other hand.

This example turns up very frequently in applications to smoothing toric Fano varieties, and it is precisely to allow for this example that we need to be working with log structures on ggtc Deligne–Mumford stacks, as opposed to just ggtc schemes.

The coarse moduli scheme of Y is a ggtc scheme that is not viable if $r > 1$.

Example 2.20. It is easy to construct examples of ggtc spaces that are not toroidal crossing spaces. For example, assume f, g, h are mutually coprime and glue

$$X_1 = (yg(x_1, \ldots, x_n) + f(x_1, \ldots x_n) = 0) \subset \mathbb{A}^{n+1},$$

$$X_2 = (zh(x_1, \ldots, x_n) + f(x_1, \ldots x_n) = 0) \subset \mathbb{A}^{n+1}$$

along the common subvariety $(f(x_1, \ldots, x_n) = 0) \subset \mathbb{A}^n$. If $f(x_1, \ldots, x_n)$ is reduced, then the total space is ggtc. For it to be toroidal crossing, X_1, X_2 and their respective subset given by $f = 0$ must be smooth.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

2.6 The divisor system

Construction 2.21. Fix a ggtc space $(Y=\coprod_{\eta\in T} Y_\eta^\star, (\mathcal{P},\mathbf{1}), \{\widehat{f}_\eta \mid \eta\in T\})$. As usual we denote by $\mathcal{M}$ the relative ghost sheaf. Strictly speaking $\mathcal{M}$ is a sheaf on T, but we identify it with a sheaf on Y as in Remark 2.10.

Denote by

$$\varepsilon\colon X=Y^{[0]}\to Y$$

the normalization. We construct a sheaf of monoids on X, which we call the cone sheaf.

The space X is naturally stratified by strata indexed by the finite poset

$$S=\{(\eta,\sigma)\mid \eta\in T,\ \sigma\in T^{[0]},\ \eta\leqslant\sigma\}$$

where $(\eta_1,\sigma_1)\leqslant(\eta_2,\sigma_2)$ if $\sigma_1=\sigma_2$ and $\eta_1\leqslant\eta_2$. The stratum corresponding to $\xi=(\eta,\sigma)\in S$ is

$$X_\xi^\star\cong Y_\eta^\star.$$

We construct a sheaf of monoids $\mathcal{C}$ on S and we identify it with a sheaf of monoids on X as in Remark 2.10.

For all $\xi\in S$, we need to assign a monoid C_ξ and for all $\xi_1\leqslant\xi_2$ we need to assign generization homomorphisms $C_{\xi_1}\to C_{\xi_2}$.

For $\xi=(\eta,\sigma)\in S$, identify σ with a maximal cone of the fan Σ_η and set

$$C_\xi=\sigma\cap M_\eta.$$

If $\xi_1=(\eta_1,\sigma_1)\leqslant\xi_2=(\eta_2,\sigma_2)$, then $\eta_1\leqslant\eta_2$ so $\Sigma_{\eta_2}=\Sigma_{\eta_1}/\eta_2$ and the generization morphism

$$M_{\eta_1}\to M_{\eta_2}=M_{\eta_1}/\langle\eta_2\rangle$$

maps C_{ξ_1} to C_{ξ_2}.

It is clear that these assignments define a sheaf of monoids $\mathcal{C}$ on S.

Definition 2.22. Let Y be a viable ggtc space. The *cone sheaf* is the subsheaf of monoids

$$\mathcal{C}\subset\varepsilon^\star\mathcal{M}$$

of Construction 2.21.

Construction 2.23. Let Y be a ggtc space, $\varepsilon\colon X\to Y$ the normalization. The space $X=\coprod_{\xi\in S} X_\xi^\star$ is stratified as explained in Construction 2.21. The boundary of X is the union of the strata of codimension $\geqslant 1$, and we denote by $\mathcal{D}iv_\flat^+$ the sheaf in the Zariski topology of effective Cartier divisors on X supported on the boundary.

Let $\mathcal{C}$ be the cone sheaf on X. Assuming that Y is viable, we construct a homomorphism of sheaves of monoids on X:

$$\widetilde{D}\colon\mathcal{C}\to\mathcal{D}iv_\flat^+,$$

which we call the divisor system.

For $\sigma\in T^{[0]}$, denote by $\varepsilon_\sigma\colon Y_\sigma\hookrightarrow Y$ the natural closed immersion, so that $\varepsilon=\bigsqcup_{\sigma\in T^{[0]}}\varepsilon_\sigma\colon X\to Y$.

We use the notation set out in Construction 2.21.

Denote by $\widetilde{a}\colon S\to X$ the natural inclusion and by $\widetilde{b}\colon X\to S$ the retraction of Lemma 2.7: we have that $a(\eta,\sigma)=a(\eta)\in Y_\sigma$, and for $x\in Y_\sigma$, $\widetilde{b}(x)=(b(x),\sigma)$.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

More precisely, $\mathcal{C}$ is the sheaf on S of Construction 2.21 and we construct a sheaf homomorphism $\widetilde{b}^{-1}\mathcal{C} \to \mathcal{D}iv_\flat^+$, which is the same as a homomorphism $\mathcal{C} \to \widetilde{b}_\star \mathcal{D}iv_\flat^+$ of sheaves on S.

For all $\xi = (\eta, \sigma) \in S$, let

$$\widetilde{U}_\xi = \widetilde{b}^{-1}(S_{\geqslant \xi}) = \{x \in Y_\sigma \mid b(x) \geqslant \eta\}.$$

To construct our sheaf homomorphism, for all $\xi \in S$ we need to assign a monoid homomorphism:

$$\widetilde{D}_\xi \colon C_\xi \to \mathrm{Div}_\flat^+(\widetilde{U}_\xi)$$

and these monoid homomorphisms need to be compatible with generization in the obvious way.

In Definition 2.16, for $m \in C_\xi = \sigma \cap M_\eta$ we defined a Cartier divisor

$$D_{\eta,\sigma}(m) \in \mathrm{Div}_\flat^+(\overline{U_\sigma} \cap U_\eta)$$

Noting that $\widetilde{U}_\xi = \varepsilon_\sigma^{-1}(\overline{U_\sigma} \cap U_\eta)$, we define $\widetilde{D}_\xi(m) = \varepsilon_\sigma^\star D_{\eta,\sigma}(m)$.

Definition 2.24. Let Y be a viable ggtc space. The *divisor system* is the homomorphism of Zariski sheaves of monoids on $X = Y^{[0]}$

$$\widetilde{D} \colon \mathcal{C} \to \mathcal{D}iv_\flat^+ \quad \text{on } X$$

of Construction 2.23.

Remark 2.25. One can show that $\widetilde{D}$ is an isomorphism.

Notation 2.26. Let $Y = \coprod_{\eta \in T} Y^\star$ be a ggtc space, $a \colon T \to Y$ the inclusion and $b \colon Y \to T$ the retraction of Lemma 2.7. For all $y \in Y$, the open star of y is

$$U_y = U_\eta, \quad \text{where } y \in Y_\eta^\star$$

and $U_\eta = b^{-1}T_{\geqslant \eta}$ is the open star of η, cf. Definition 2.8.

Similarly let $\varepsilon \colon X = \coprod_{\xi \in S} X_\xi^\star \to Y$ be the normalization, stratified as in Construction 2.21, $\widetilde{a} \colon S \to X$ the inclusion and $\widetilde{b} \colon X \to S$ the retraction of Lemma 2.7. For all $x \in X$, the open star of x is

$$\widetilde{U}_x = \widetilde{U}_\xi, \quad \text{where } x \in X_\xi^\star$$

and $\widetilde{U}_\xi = \widetilde{b}^{-1}(S_{\geqslant \xi})$ is the open star of ξ, cf. Definition 2.8.

Lemma 2.27. *Let Y be a viable ggtc space, $\varepsilon \colon X \to Y$ the normalization stratified as in Construction 2.21. For all $y \in Y$, consider the fan $\Sigma_y \subset M_y$, and let $\sigma_1, \sigma_2 \in \Sigma_y$ be two maximal cones adjacent along a submaximal cone $\rho = \sigma_1 \cap \sigma_2$. Then $y \in Y_\rho$ and let $x_1 \in Y_{\sigma_1} \subset X$, $x_2 \in Y_{\sigma_2} \subset X$ be the two lifts, so $C_{x_1} = \sigma_1$, $C_{x_2} = \sigma_2$ and in this sense $C_{x_1} \cap C_{x_2} = \rho$. There are obvious inclusions*

$$\iota_1 \colon Y_\rho \hookrightarrow Y_{\sigma_1} \quad \textit{and} \quad \iota_2 \colon Y_\rho \hookrightarrow Y_{\sigma_2}$$

and we write (following Notation 2.26)

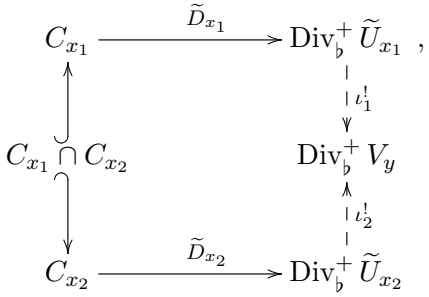
$$V_y = U_y \cap Y_\rho.$$

Note that $V_y = \iota_1^{-1}(\widetilde{U}_{x_1}) = \iota_2^{-1}(\widetilde{U}_{x_2})$.

In this situation, denote by $\iota_1^! \colon \mathrm{Div}_\flat^+ \widetilde{U}_{x_1} \dashrightarrow \mathrm{Div}_\flat^+ V_y$ the partially defined restriction homomorphism that is defined for each divisor that intersects V_y properly, and similarly $\iota_2^!$.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

The following diagram is commutative.



Here we note that the restriction $\iota_1^!$ is well-defined on $\widetilde{D}_{x_1}(C_{x_1} \cap C_{x_2})$ and, similarly, $\iota_2^!$ is well-defined on $\widetilde{D}_{x_2}(C_{x_1} \cap C_{x_2})$).

The diagram is compatible with generization in the obvious way.

Proof. Straightforward. □

3. Construction of the sheaf $\mathcal{LS}_Y$

In this section, given a viable ggtc space Y, we construct a sheaf of sets $\mathcal{LS}_Y$ on Y, intrinsic to Y. In Section 4, we prove that $\mathcal{LS}_Y$ classifies compatible log structures on Y over $k^\dagger$. We refer to Section 1.1 for an informal summary of the construction of $\mathcal{LS}_Y$.

In Section 3.1, for every slab $\rho \in T^{[1]}$, we give the construction of the slab line bundle $\mathcal{L}_\rho$ on Y_ρ. In Corollary 3.7 of Section 3.2, we show that, for every joint $\omega \in T^{[2]}$, the restrictions $\mathcal{L}_{\rho|Y_\omega}$ for all $\omega < \rho$ satisfy the joint condition. This is seen as an easy consequence of an abstract joint condition that is stated in Lemma 3.6. In Definition 3.9, the joint conditions are used to define the sheaf $\mathcal{LS}_Y$.

The construction of the slab line bundles $\mathcal{L}_\rho$ and, in particular, the formulation of the joint condition, are delicate. In order to glue the slab bundles from local data, and in order to formulate the joint condition, we need carefully to keep track of isomorphisms between line bundles and not just the line bundles themselves. For this reason it is necessary that we work with the Picard 2-group $\underline{\mathrm{Pic}}\, Y$ (and, implicitly, the Picard stack $\underline{\mathcal{P}ic}_Y$), for which we refer the reader to [Del73, Section 1.4].

3.1 The slab line bundles

In this subsection we fix throughout a viable ggtc space Y. The goal is to construct, for all $\rho \in T^{[1]}$, a line bundle $\mathcal{L}_\rho \in \underline{\mathrm{Pic}}\, Y_\rho$ that we call a slab bundle.

Before describing the construction, we recall the following.

Definition 3.1. Let M be an abelian group and $(\underline{A}, \otimes, \mathbf{1})$ a strictly commutative 2-group, where:

(a) for objects A_1, A_2 of $\underline{A}$, we denote by

$$t\colon A_1 \otimes A_2 \xrightarrow{\cong} A_2 \otimes A_1$$

the isomorphism provided by the 2-group structure;

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

(b) for objects A_1, A_2, A_3 of $\underline{A}$, we denote by

$$s\colon A_1 \otimes (A_2 \otimes A_3) \xrightarrow{\cong} (A_1 \otimes A_2) \otimes A_3$$

the isomorphism provided by the 2-group structure.

A 2-*homomorphism* $\mu\colon M \to \underline{A}$ is an assignment $\mu\colon M \to \mathrm{Ob}\,\underline{A}$ together with the datum, for all $m_1, m_2 \in M$, of an isomorphism

$$\mu(m_1 + m_2) \xrightarrow{\cong} \mu(m_1) \otimes \mu(m_2)$$

such that:

(i) $\mu(0) = \mathbf{1}$;

(ii) for all $m_1, m_2 \in M$ the following diagrams are commutative

$$\begin{array}{ccc} \mu(m_1 + m_2) & \longrightarrow & \mu(m_1) \otimes \mu(m_2) \\ \| & & \downarrow t \\ \mu(m_2 + m_1) & \longrightarrow & \mu(m_2) \otimes \mu(m_1) \end{array}$$

(iii) for all $m_1, m_2, m_3 \in M$ the following diagrams are commutative

$$\begin{array}{ccccc} \mu\left(m_1 + (m_2 + m_3)\right) & \longrightarrow & \mu(m_1) \otimes \mu(m_2 + m_3) & \longrightarrow & \mu(m_1) \otimes (\mu(m_2) \otimes \mu(m_3)) \\ \| & & & & \downarrow s \\ \mu\left((m_1 + m_2) + m_3\right) & \longrightarrow & \mu(m_1 + m_2) \otimes \mu(m_3) & \longrightarrow & (\mu(m_1) \otimes \mu(m_2)) \otimes \mu(m_3) \end{array}$$

Lemma 3.2. *Let Y be a viable ggtc space and $\rho \in T^{[1]}$ a slab. Fix $y \in Y_\rho$, and use the notation of Lemma 2.27: in particular, denote by σ_1, σ_2 the maximal cones of $\Sigma_y \subset M_y$ that meet along ρ. Denote by*

$$\mu_i\colon M_y \to \underline{\mathrm{Pic}}(V_y)$$

the unique 2-homomorphism such that, for $m \in C_{x_i} \subset M_y$, $\mu_i(m) = \iota_i^\star(\mathcal{O}_{\widetilde{U}_{x_i}}(\widetilde{D}_{x_i}(m)))$. Note that $\mu_1 = \mu_2$ on the subspace $\langle C_{x_1} \cap C_{x_2} \rangle = \langle \rho \rangle$ and denote this restricted 2-homomorphism simply by μ.

Let $d \in \sigma_2^\vee \subset \mathrm{Hom}(M_y, \mathbb{Z})$ be the unique primitive vector that pairs to zero with all points in ρ. Next choose $v \in M_y$ such that $\langle d, v \rangle = 1$. We define a line bundle $\lambda(v)$ on V_y as

$$\lambda(v) = \mu_1(-v) \otimes \mu_2(v).$$

Then we have the following.

(1) *The line bundle $\lambda(v)$ is independent on the choice of v in the following sense: for all $r \in \langle \rho \rangle$, we construct an isomorphism*

$$\psi_r\colon \lambda(v) \xrightarrow{\cong} \lambda(r + v).$$

(2) *The set of these isomorphisms has the cocycle property:*

(i) *for all $v \in \sigma_2$ such that $\langle d, v \rangle = 1$, $\psi_0 = \mathrm{id}_{\lambda(v)}$;*

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

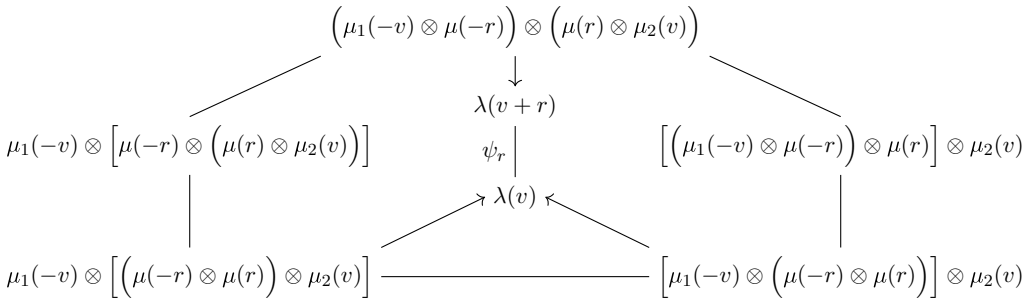
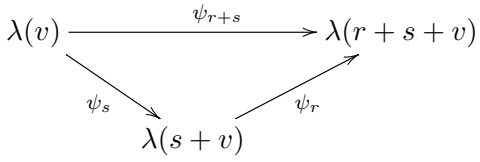


FIGURE 1. Construction of $\psi_r : \lambda(v) \to \lambda(r+v)$.

(ii) *for all* $v \in \sigma_2$ *such that* $\langle d, v\rangle = 1$, *and for all* $r, s \in \langle\rho\rangle$, *the following diagram is commutative.*



(3) *Similarly,* $\lambda(v)$ *does not depend on the choice of numbering of* σ_1, σ_2 *in the sense that if we swap numbering, then* d *is changed into* $-d$, v *into* $-v$, *and the line bundle into* $\mu_2(v)\otimes\mu_1(-v)$.

Proof. In the diagram in Figure 1, the outer pentagon is commutative by the pentagon axiom and the internal part of the diagram gives the construction of $\psi_r\colon \lambda(v) \to \lambda(r+v)$. (Note that, since $r \in \langle\rho\rangle$, $\mu_1(r) = \mu_2(r)$, and we write simply $\mu(r)$ to signify either of these two equal line bundles.)

The statement that for all $r, s \in \langle\rho\rangle$, $\psi_{r+s} = \psi_r \circ \psi_s$ follows from contemplating the diagram in Figure 2, where the outer circle is commutative by MacLane coherence theorem for monoidal categories. For reasons of space we are suppressing the $\otimes$ symbols. □

Construction 3.3. For every slab $\rho \in T^{[1]}$, we construct a line bundle $\mathcal{L}_\rho$ on Y_ρ, which we will call a slab bundle.

Fix throughout $\rho \in T^{[1]}$.

First, we construct a covering of Y_ρ by Zariski open subsets. Note that

$$Y_\rho = \coprod_{\{\tau\in T \mid \tau\leqslant\rho\}} Y_\tau^\star$$

is itself a stratified space. We denote by $T_\rho = \{\tau \in T \mid \tau \leqslant \rho\}$ the index poset for the strata of Y_ρ. For all $\tau \in T_\rho$, we denote by $V_\tau \subset Y_\rho$ the open star of τ in Y_ρ. It is clear that $\{V_\tau \mid \tau \in T_\rho\}$ is a Zariski open cover of Y_ρ.

Now choose a *global numbering* for the maximal cones incident at ρ: in other words, for all $\tau \in T_\rho$, a numbering $\sigma_1(\tau)$, $\sigma_2(\tau)$ of the two maximal cones in Σ_τ incident at ρ,[14] satisfying

[14] For all $\tau \in T_\rho$, we can identify ρ with a submaximal cone in Σ_τ. To be precise, we ought to denote this cone by ρ_τ but we abuse notation and denote all these cones by ρ trusting that this will not cause confusion.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

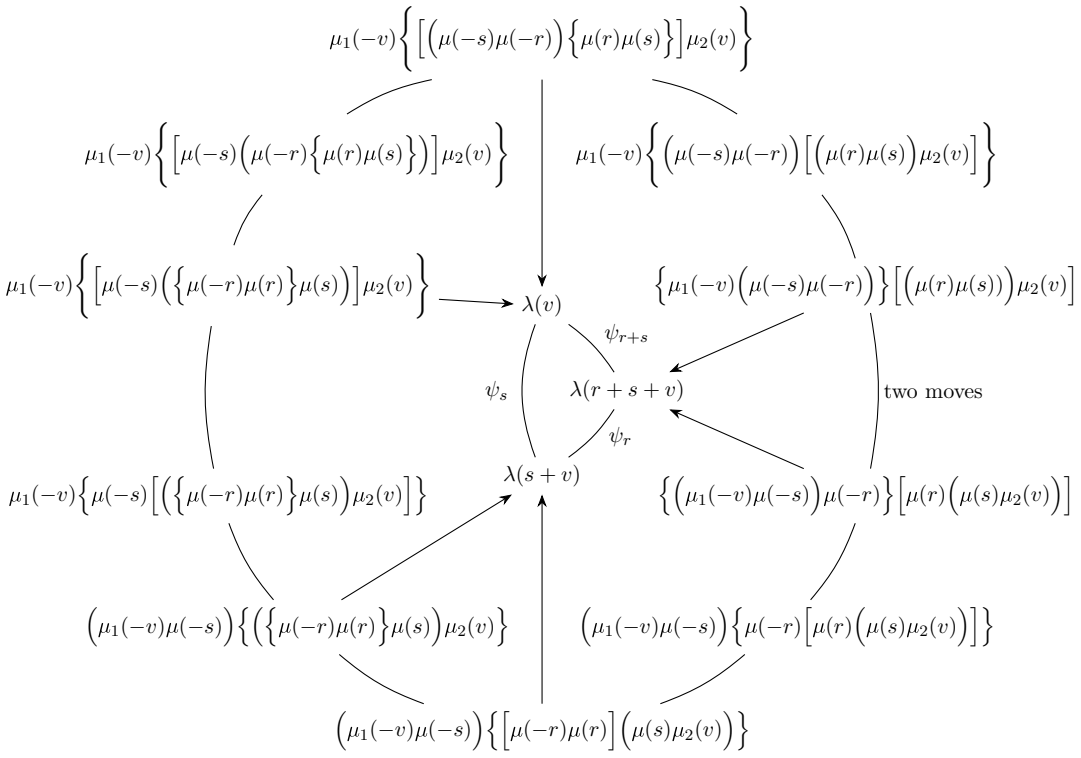


FIGURE 2. Proof that $\psi_{r+s} = \psi_r \circ \psi_s$.

the consistency condition. If $\tau_1 \leqslant \tau_2$, then, for $i = 1, 2$, $\sigma_i(\tau_1)$ maps to $\sigma_i(\tau_2)$ under the natural projection $M_{\tau_1} \to M_{\tau_2} = M_{\tau_1}/\langle \tau_2 \rangle$.[15] Note that there are precisely two global numberings.

For all τ, Let $d_\tau \in \sigma_2(\tau)^\vee \subset \mathrm{Hom}(M_\tau, \mathbb{Z})$ be the unique primitive vector that pairs to zero with all points in ρ. Next choose $v_\tau \in M_\tau$ such that $\langle d_\tau, v_\tau \rangle = 1$. Following Lemma 3.2 with $y = \tau$, we define a line bundle $\lambda(v_\tau)$ on V_τ as

$$\lambda(v_\tau) = \mu_1(\tau)(-v_\tau) \otimes \mu_2(\tau)(v_\tau)$$

where we denote by $\mu_i(\tau)\colon M_\tau \to \underline{\mathrm{Pic}}(V_\tau)$ the 2-homomorphisms of Lemma 3.2.

Next we glue the $\lambda(v_\tau)$ to a line bundle on all of Y_ρ. If $\tau_1 \leqslant \tau_2$, then:

(1) $V_{\tau_2} \subset V_{\tau_1}$;
(2) we have an identification $\pi = \pi_{\tau_1,\tau_2}\colon M_{\tau_1} \to M_{\tau_1}/\langle \tau_2 \rangle = M_{\tau_2}$;
(3) under the dual injection $N_{\tau_2} \subset N_{\tau_1}$, d_{τ_2} is identified with d_{τ_1} and $\langle d_{\tau_2}, \pi(v_{\tau_1}) \rangle = 1$ and hence we can form $r_{\tau_1,\tau_2} = \pi(v_{\tau_1}) - v_{\tau_2} \in \langle \rho \rangle \cap M_{\tau_2}$.

At this point, we

$$\text{glue } \lambda(v_{\tau_1})_{|V_{\tau_2}} = \lambda(v_{\pi(v_{\tau_1})}) \text{ to } \lambda(v_{\tau_2}) \text{ with } \psi_{r(\tau_1,\tau_2)}\colon \lambda(v_{\pi(v_{\tau_1})}) \xrightarrow{\cong} \lambda(v_{\tau_2})$$

whose construction is given by Lemma 3.2.

[15] The existence of such a global numbering follows easily from the conditions spelled out in Section 2.3.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

Next, we keep gluing on all the line bundles on all the V_τ formed from both choices of numbering and all system of vectors $(v_\tau)_{\tau\in M_\tau}$ as above by using the ψ isomorphisms as above. The cocycle condition ensures that all these gluings can be done consistently.

We denote by $\mathcal{L}_\rho$ the resulting line bundle on Y_ρ.

Definition 3.4. Let $Y=\coprod_{\tau\in T} Y_\tau^\star$ be a viable ggtc space and $\rho\in T^{[1]}$. We call the line bundle $\mathcal{L}_\rho$ on Y_ρ of Construction 3.3 the *slab bundle*.

Abusing notation, we also denote by $\mathcal{L}_\rho$ the direct image of this line bundle under the inclusion $Y_\rho\hookrightarrow Y$.

3.2 The joint condition and the sheaf $\mathcal{LS}_Y$

We show that for every joint $\omega\in T^{[2]}$ the restrictions of all $\mathcal{L}_\rho$ with $\omega<\rho$ to Y_ω satisfy a relation. We then use this relation to define a subsheaf $\mathcal{LS}_Y\subset\bigoplus_{\rho\in T^{[1]}}\mathcal{L}_\rho$. A similar relation first appeared as a relation of elements in a local computation in Theorem 3.22 in [GS06].

The following definition and theorem are stated in terms of a strictly commutative 2-group $\underline{A}$, because this is what we need. The statement contains as a special case the situation where $\underline{A}$ is the categorification of an abelian group A, where a more concrete formulation of the definition and theorem are possible.

Definition 3.5. Let M be a finitely generated free abelian group and Σ a complete fan in M. Let $\underline{A}$ be a strictly commutative 2-group. A *continuous piecewise linear 2-homomorphism* $\mu\colon M\to\underline{A}$ with respect to Σ is a collection of 2-homomorphisms

$$\mu_\sigma\colon M\to\underline{A}\,,$$

one for each maximal cone $\sigma\in\Sigma$, with the property that whenever two maximal cones σ_1,σ_2 share a submaximal cone $\rho=\sigma_1\cap\sigma_2$ then the restrictions of $\mu_{\sigma_1},\mu_{\sigma_2}$ to $\langle\rho\rangle$ are equal.

Lemma 3.6 (Abstract joint condition lemma). *Let M be a lattice, $N=\mathrm{Hom}(M,\mathbb{Z})$, and assume given a surjective homomorphism $\pi\colon M\to\overline{M}$, where $\overline{M}$ is a rank two lattice endowed with a complete fan $\overline{\Sigma}$. Endow M with the fan*

$$\Sigma=\{\pi^{-1}(\eta)\mid\eta\in\overline{\Sigma}\}$$

We denote the cones of Σ as follows:

- *the codimension-2 subspace (also known as a joint) $\omega=\pi^{-1}\{0\}$;*
- *cyclically ordered codimension-1 cones (also known as slabs) $\rho_1,\dots,\rho_n$ incident at ω;*
- *maximal cones $\sigma_i=\langle\rho_i\cup\rho_{i+1}\rangle_+$ (for $i=1,\dots,n$ with the convention that $\rho_{n+1}=\rho_1$).*

Let $(\underline{A},+,0_{\underline{A}})$ be a strictly commutative 2-group and $\mu\colon M\to\underline{A}$ a continuous piecewise linear 2-homomorphism with respect to Σ. Denote by $\mu_i=\mu_{\sigma_i}$ the 2-homomorphism for σ_i that forms part of the datum μ. Let $d_1,\dots,d_n\in N$ be the primitive normals to the slabs $\rho_1,\dots,\rho_n$, such that $d_i\geq 0$ on ρ_{i+1}.

Choose $v_i\in M$ such that $\langle d_i,v_i\rangle=1$ and define

$$\lambda(v_i)=\mu_{i-1}(-v_i)+\mu_i(v_i).\tag{3.1}$$

(1) *We construct the joint isomorphism*

$$\sum_{i=1}^n d_i\otimes\lambda(v_i)\xrightarrow{\cong}0\otimes 0_{\underline{A}}\ \text{ in }\ N\otimes\underline{A}=\underline{2\text{-Hom}}(M,\underline{A}),\tag{3.2}$$

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

(2) *The isomorphism constructed in part (1) does not depend on the choice of* v_i *in the following sense. For all* $i = 1, \dots, n$ *and* $r_i \in \langle \rho_i \rangle$ *we construct isomorphisms*

$$\psi_{r_i} \colon \lambda(v_i) \xrightarrow{\cong} \lambda(v_i + r_i)$$

such that the following diagram is commutative

$$\begin{array}{ccc} \sum_{i=1}^n d_i \otimes \lambda(v_i) & & \\ \Big\downarrow \{\psi_{r_i}\} & \searrow^{\cong} & \\ & & 0 \otimes 0_{\underline{A}} \\ & \nearrow_{\cong} & \\ \sum_{i=1}^n d_i \otimes \lambda(v_i + r_i) & & \end{array},$$

and, furthermore, the set of these isomorphisms has the cocycle property:

(i) $\psi_0 = \mathrm{id}$;
(ii) *for all* $r_i, s_i \in \langle \rho_i \rangle$, $\psi_{r_i + s_i} = \psi_{r_i} \circ \psi_{s_i}$.

(3) *The joint isomorphism does not depend on the cyclic ordering in the sense that the isomorphism from the opposite ordering relates to the given one by a global multiplication by* (-1).

Proof. For all $w \in M$ we produce an isomorphism $\sum_{i=1}^n d_i(w)\lambda(v_i) \cong 0_{\underline{A}}$ in $\underline{A}$. The key observation is that for all i

$$d_i \left(-d_i(w)v_i + w\right) = -d_i(w) \underbrace{d_i(v_i)}_{=1} + d_i(w) = 0.$$

Hence, $-d_i(w)v_i + w \in \langle \rho_i \rangle$ and therefore $\mu_{i-1}(-d_i(w)v_i + w) = \mu_i(-d_i(w)v_i + w)$. Rewriting and using the natural transformations of functors provided by the monoidal structure of $\underline{A}$ yields

$$d_i(w)\mu_{i-1}(-v_i) \cong d_i(w)\mu_i(-v_i) + \mu_i(w) - \mu_{i-1}(w). \tag{3.3}$$

The sum (3.2) in the assertion can be identified as a telescoping sum when writing

$$\begin{aligned} \sum_{i=1}^n d_i(w)\lambda(v_i) &\overset{(3.1)}{\cong} d_1(w)(\mu_n(-v_1) + \mu_1(v_1)) \\ &\quad + d_2(w)(\mu_1(-v_2) + \mu_2(v_2)) \\ &\quad + d_3(w)(\mu_2(-v_3) + \mu_3(v_3)) \\ &\quad \vdots \\ &\overset{(3.3)}{\cong} d_1(w)(\mu_1(-v_1) + \mu_1(v_1)) + \mu_1(w) - \mu_n(w) \\ &\quad + d_2(w)(\mu_2(-v_2) + \mu_2(v_2)) + \mu_2(w) - \mu_1(w) \\ &\quad + d_3(w)(\mu_3(-v_3) + \mu_3(v_3)) + \mu_3(w) - \mu_2(w) \\ &\quad \vdots \\ &\cong 0_{\underline{A}}. \end{aligned}$$

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

The resulting isomorphism to $0_{\underline{A}}$ does not depend on choices, that is, the order of association and commutation in $\underline{A}$, by the MacLane coherence theorem for 2-groups.

For the independence on the choice of v_i, part (1) is proved in the same way as the corresponding statement in Lemma 3.2. The proof of part (2) is straightforward and we omit the details. □

Corollary 3.7. *Let Y be a viable ggtc space, $\omega \in T^{[2]}$ a joint, and $\rho_1, \dots, \rho_n$ a cyclical ordering of the slabs incident at ω. Let $d_1, \dots, d_n \in N_\omega$ be the primitive normals to the slabs $\rho_1, \dots, \rho_n$, such that $d_i \geq 0$ on ρ_{i+1}.*

We construct a joint isomorphism:

$$J_\omega \colon \bigotimes_{i=1}^{n} d_i \otimes \mathcal{L}_{\rho_i|Y_\omega} \cong 0 \otimes \mathcal{O}_{Y_\omega} \quad \text{in } N_\omega \otimes \underline{\operatorname{Pic}}\, Y_\omega. \tag{3.4}$$

Sketch of proof. Fix throughout a joint $\omega \in T^{[2]}$.

The space $Y_\omega = \coprod_{\{\tau \in T \mid \tau \leqslant \omega\}} Y_\tau^\star$ is stratified. We denote by $T_\omega = \{\tau \in T \mid \tau \leqslant \omega\}$ the index poset for the strata of Y_ω. For all $\tau \in T_\omega$, we denote by $U_\tau \subset Y$ the open star of τ in Y. It is clear that $\{U_\tau \cap Y_\omega \mid \tau \leqslant \omega\}$ is a Zariski open cover of Y_ω.

Now Σ_ω is a fan in a rank two lattice M_ω. Denote the cones of Σ_ω as follows:

- the cone (0), corresponding to the joint ω itself;
- cyclically ordered rays $\overline{\rho}_1, \dots, \overline{\rho}_n$, also know as slabs;
- maximal cones $\overline{\sigma}_i = \langle \overline{\rho}_i, \overline{\rho}_{i+1} \rangle_+$.

Denote by $d_1, \dots, d_n \in N_\omega$ the primitive normals to the slabs $\overline{\rho}_1, \dots, \overline{\rho}_n$ such that $d_i > 0$ on $\overline{\rho}_{i+1}$.

For all i, Construction 3.3 constructs a slab bundle $\mathcal{L}_{\rho_i}$ on Y_{ρ_i}.

The bundles $\mathcal{L}_{\rho_i}$ are obtained from gluing together certain bundles constructed on certain open covers of Y_{ρ_i}. We are only interested in the open subsets $U_\tau \cap Y_{\rho_i}$ for $\tau \in T_\omega$: these open subsets do not cover all of Y_{ρ_i} but they do cover all of Y_ω and, hence, they are sufficient for working with $\mathcal{L}_{\rho_i|Y_\omega}$. Let us fix $\tau \in T_\omega$ and focus on one of these open subsets U_τ.

We are going to apply Lemma 3.6 to the situation $M = M_\tau$, $\overline{M} = M_\omega = M/\langle\omega\rangle$, $\pi \colon M \to \overline{M}$ the projection to the quotient, $\overline{\Sigma} = \Sigma_\omega$, and $\Sigma = \omega^{-1}\Sigma_\tau$ the localized fan. Abusing notation slightly, we denote by $\omega = \pi^{-1}(0)$, $\rho_i = \pi^{-1}(\overline{\rho}_i)$, $\sigma_i = \pi^{-1}(\overline{\sigma}_i)$ the cones of Σ. Furthermore, we take $\underline{A} = \underline{\operatorname{Pic}}\, Y_\omega$, and $\mu_i \colon M \to \underline{A}$ the unique 2-homomorphism such that, for $m \in \sigma_i \cap M$,[16]

$$\mu_i(m) = \iota_i^\star\Big(\mathcal{O}_{\widetilde{U}_{(\tau,\sigma_i)}}(\widetilde{D}_{(\tau,\sigma_i)}(m))\Big).$$

Consider local charts for the bundles $\mathcal{L}_{\rho_i}$ constructed by choosing a global numbering compatible with the cyclic order of the rays ρ_i. The construction of the local charts for $\mathcal{L}_{\rho_i}$ further depend on vectors $v_i \in M$ such that $\langle v_i, d_i \rangle = 1$. The corresponding local chart for $\mathcal{L}_{\rho_i|U_\tau \cap Y_\omega}$ is $\lambda(v_i)$. Lemma 3.6 then gives a joint isomorphism

$$J_\tau(\mathbf{v}) \colon \bigotimes_{i=1}^{n} d_i \otimes \lambda(v_i) \cong 0 \otimes \mathcal{O}_{U_\tau \cap Y_\omega} \quad \text{in } N_\omega \otimes \underline{\operatorname{Pic}}(U_\tau \cap Y_\omega)$$

defined locally on U_τ and depending on $\mathbf{v} = (v_1, \dots v_n)$ and the global numbering.

[16] Recall that $\widetilde{X}$ is the normalization of Y: it is a stratified space where strata are pairs (τ, σ) of $\tau \leqslant \sigma \in T$ and $\sigma \in T^{[0]}$.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

We need to prove that these local joint isomorphism glue to give a global joint isomorphism. For this purpose, for all $\tau \leqslant \tau'$ (the case $\tau = \tau'$ is included), we need to check that

$$J_\tau(\mathbf{v})_{|U_{\tau'} \cap Y_\omega} = J_{\tau'}(\mathbf{v}).$$

For a fixed list of vectors $\mathbf{v} = (v_i)$ this is obvious, but we also need to address the possibility of changing $\mathbf{v} = (v_i)$. The required consistency follows from the way that the local joint condition behaves under change of (v_i) given in Lemma 3.6(2). We also need to check consistency under change of global numbering around each of the ρ_i; this follows from Lemma 3.2(3) and Lemma 3.6(3). □

Remark 3.8. If e_1, e_2 is a lattice basis of M_ω, the map J_ω is equivalent to two isomorphisms of line bundles

$$J_{\omega,e_1} \colon \bigotimes_{i=1}^{n} (\mathcal{L}_{i|Y_\omega})^{\otimes d_i(e_1)} \cong \mathcal{O}_{Y_\omega}; \qquad J_{\omega,e_2} \colon \bigotimes_{i=1}^{n} (\mathcal{L}_{i|Y_\omega})^{\otimes d_i(e_2)} \cong \mathcal{O}_{Y_\omega}.$$

Definition 3.9. We define the sheaf of sets $\mathcal{LS}_Y$ as the subsheaf of the direct sum $\bigoplus_{\rho \in T^{[1]}} \mathcal{L}_\rho$ on Y satisfying the following condition. For every point $y \in Y$, the stalk $\mathcal{LS}_y$ consist of those tuples of sections $(f_\rho)_{\rho \in T^{[1]}}$ whose restrictions to Y_ω for every $\omega \in T^{[2]}$ satisfies the *joint condition*

$$J_\omega(d_i \otimes (f_{i|Y_\omega})) = 0 \otimes 1$$

at the generic point ω of Y_ω.

Notation 3.10. We denote by

$$\mathcal{LS}_Y^\times = \mathcal{LS}_Y \cap (\oplus_{\rho \in T^{[1]}} \mathcal{L}_\rho^\times)$$

the subsheaf of nowhere-vanishing sections.

Remark 3.11.

(1) In the special situation where $T^{[2]} = \emptyset$, we get $\mathcal{LS}_Y = \bigoplus_{\rho \in T^{[1]}} \mathcal{L}_\rho$ and in this situation $\mathcal{LS}_Y$ is a coherent sheaf. It can be seen that, more generally, if Y is a simple normal crossing scheme, then $\mathcal{LS}_Y$ is a line bundle on $Y^{(1)}$.

(2) In Definition 3.9 we require the joint condition to hold at the generic point of Y_ω only. The only reason for not requiring it everywhere is to allow the sections f_ρ to vanish somewhere.

4. $\mathcal{LS}_Y^\times$ classifies log structures on Y

4.1 Statement of the main result and road-map of its proof

In this section, we prove the main result of the paper, Theorem 4.3. Before we can give the precise statement, we need to recall a few notions about log structures and define some key concepts that enter it. The next definition is really just meant to fix our notation.

Definition 4.1. Let X be a space.

(1) A *log structure* on X is a pair $(\mathfrak{P}, \alpha)$ where $\mathfrak{P}$ is a sheaf of monoids[17] and $\alpha \colon \mathfrak{P} \to (\mathcal{O}_X, \times)$ is a homomorphism of sheaves of monoids such that

[17] In this paper we take $\mathfrak{P}$ to be a sheaf in the Zariski topology. When reading this section, it helps to internalize early on that our sheaves are sheaves of monoids or groups and rarely are they coherent sheaves.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

$$\alpha_{|\alpha^{-1}(\mathcal{O}_X^\times)}\colon \alpha^{-1}(\mathcal{O}_X^\times)\to\mathcal{O}_X^\times$$

is an isomorphism.

(2) A *log scheme* is a pair $(X,\mathfrak{P})$ of a scheme X and a log structure $\mathfrak{P}$. The symbol $X^\dagger$ signifies a log scheme with underlying scheme X.

(3) A *morphism* of log schemes $f\colon (X,\mathfrak{P})\to(Y,\mathfrak{Q})$ is an ordinary morphism of schemes, together with a homomorphism of sheaves of monoids $f^{-1}\mathfrak{Q}\to\mathfrak{P}$ that commutes with $f^{-1}\mathcal{O}_Y\to\mathcal{O}_X$ under the respective maps α.

(4) Let k be a field. The *standard log point* is the log scheme $\operatorname{Spec} k^\dagger=(\operatorname{Spec} k,\mathfrak{P}_k)$, where $\mathfrak{P}_k=k^\times\times\mathbb{N}$ and $\alpha\colon\mathfrak{P}_k\to k$ maps (a,n) to 0 if $n>0$ and to a if $n=0$.

(5) Let k be a field, X a scheme over $\operatorname{Spec} k$, and $X^\dagger=(X,\mathfrak{P})$ a log scheme. Note that to give a morphism $X^\dagger\to\operatorname{Spec} k^\dagger$ is equivalent to give a global section $\mathbf{1}_{\mathfrak{P}}\in\Gamma(X,\mathfrak{P})$ with $\alpha(\mathbf{1}_{\mathfrak{P}})=0$.[18]
A log structure on X *over the standard log point,* or simply a log structure on X over $k^\dagger$, written $X^\dagger/k^\dagger$, is a morphism $X^\dagger\to\operatorname{Spec} k^\dagger$ of log schemes; equivalently, it is a pair $(X^\dagger,\mathbf{1}_{\mathfrak{P}})$ of a log scheme $X^\dagger=(X,\mathfrak{P})$ and section $\mathbf{1}_{\mathfrak{P}}\in\Gamma(X,\mathfrak{P})$ as just described.

(6) The *ghost sheaf* of a log structure $\mathfrak{P}$ is the quotient sheaf $\overline{\mathfrak{P}}:=\mathfrak{P}/\alpha^{-1}(\mathcal{O}_X^\times)$. We denote by $\mathbf{1}_{\overline{\mathfrak{P}}}\in\Gamma(X,\overline{\mathfrak{P}})$ the image of $\mathbf{1}_{\mathfrak{P}}\in\Gamma(X,\mathfrak{P})$. The *relative ghost sheaf* of a log scheme $X^\dagger/k^\dagger$ is the quotient sheaf $\overline{\mathcal{M}}:=\overline{\mathfrak{P}}/\mathbf{1}_{\overline{\mathfrak{P}}}$.

Before reading the upcoming definition, the reader is advised to recall the definition of viable ggtc space.

Definition 4.2. Let $(Y=\coprod_{\eta\in T}Y_\eta^\star,(\mathcal{P},\mathbf{1}),\{\widehat{f}_\eta\mid\eta\in T\})$ be a viable ggtc space.

(1) A *log structure compatible with the ggtc structure* on Y, or simply a *compatible log structure*, is a log structure on Y over $k^\dagger$, $((Y,\mathfrak{P}),\mathbf{1}_{\mathfrak{P}})$, together with a homomorphism of sheaves of monoids

$$\psi\colon\mathfrak{P}\to\mathcal{P}$$

such that we have the following.

(a) We have $\psi(\mathbf{1}_{\mathfrak{P}})=\mathbf{1}_{\mathcal{P}}$ and ψ induces an isomorphism

$$\overline{\psi}\colon\mathfrak{P}/\mathcal{O}_Y^\times=\overline{\mathfrak{P}}\xrightarrow{\cong}\mathcal{P}$$

of ghost sheaves. In particular, ψ also induces an isomorphism $\overline{\mathcal{M}}\to\mathcal{M}$ of the relative ghost sheaf of the log structure $Y^\dagger/k^\dagger$ to the relative ghost sheaf of the ggtc space Y.

(b) For all $y\in Y$ and $p\in\mathfrak{P}_y$, denote by $[\psi(p)]\in\mathcal{M}_y$ the image of p, and let $\tau\in\Sigma_y$ be the smallest cone that contains $[\psi(p)]$.[19]
The condition is $\alpha(p)\in\mathcal{O}_{Y,y}$ does not vanish identically on any component of $\overline{U}_\tau\cap U_y$, and

$$\operatorname{div}(\alpha(p))=D_{\eta,\tau}([\psi(p)])\quad\text{in }\operatorname{Div}_\flat^+(\overline{U}_\tau\cap U_y),$$

where $\eta=b(y)$ and $D_{\eta,\tau}([\psi(p)])$ is the Cartier divisor of Definition 2.16 (its existence guaranteed by the viability condition) and Notation 2.17.

[18] Given $X^\dagger\to\operatorname{Spec} k^\dagger$, $\mathbf{1}_{\mathfrak{P}}$ is the image of $1\in\mathbb{N}$.

[19] If $\psi(p)$ does not lie on a proper face of $\mathcal{P}_y$, then of course $[\psi(p)]=(0)$.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

(2) A *morphism* of compatible log structures $(\mathfrak{P}, \psi)$, $(\mathfrak{P}', \psi')$ on Y is a morphism of log structures $\varphi\colon \mathfrak{P} \to \mathfrak{P}'$ such that for all sections $p \in \mathfrak{P}$, $\psi(p) = \psi'(\varphi(p))$.

(3) We denote by $\mathrm{LS}_{k^\dagger}(Y)$ the set of isomorphism classes of compatible log structures on Y.

Theorem 4.3. *Let Y be a viable ggtc space, and let $\mathcal{LS}_Y \subset \bigoplus_\rho \mathcal{L}_\rho$ be the sheaf of Definition 3.9.*

Denote by $\mathrm{LS}_{k^\dagger}(Y)$ the set of isomorphism classes of log structures on Y over $k^\dagger$ compatible with the ggtc structure.

The set-theoretic function

$$r\colon \mathrm{LS}_{k^\dagger}(Y) \to \Gamma(Y, \mathcal{LS}_Y^\times)$$

constructed in (4.15) is a bijection.

Next we give an outline of the proof. Details are carried out in the subsections that follow. In outline, our proof follows closely the proof of [GS06, Theorem 3.22]; however, there are important differences due to the fact that we start out from an independent construction of the sheaf $\mathcal{LS}_Y$.

We conclude with a synopsis of the following subsections.

Outline of the proof of Theorem 4.3. Fix a viable ggtc space Y as above. A compatible log structure $Y^\dagger = (Y, \mathfrak{P})$ sits in an extension sequence

$$0 \to \mathcal{O}_Y^\times \to \mathfrak{P} \xrightarrow{\psi} \mathcal{P} \to 0.$$

Recall that the relative ghost sheaf $\mathcal{M} = \mathcal{P}/\mathbf{1}_{\mathcal{P}}$ is a sheaf of groups. We have that

$$\mathfrak{P} = \mathfrak{M} \times_{\mathcal{M}} \mathcal{P},$$

where $\mathfrak{M} = \mathfrak{P}/\mathbf{1}_{\mathfrak{P}}$ is a sheaf of abelian groups[20] that is an extension in the category of sheaves of abelian groups on Y,

$$0 \to \mathcal{O}_Y^\times \to \mathfrak{M} \to \mathcal{M} \to 0.$$

The relative ghost sheaf $\mathcal{M}$ is supported in codimension one and Y is reduced, so $\mathcal{H}om(\mathcal{M}, \mathcal{O}_Y^\times) = 0$. The local-to-global Ext spectral sequence then gives $\mathrm{Ext}^1(\mathcal{M}, \mathcal{O}_Y^\times) = H^0(Y, \mathcal{E}xt^1(\mathcal{M}, \mathcal{O}_Y^\times))$.

A key point of the proof is to characterize the extensions that give rise to compatible log structures. We introduce a subsheaf

$$\mathcal{E}xt_c^1(\mathcal{M}, \mathcal{O}_Y^\times) \subset \mathcal{E}xt^1(\mathcal{M}, \mathcal{O}_Y^\times)$$

which we call the sheaf of *regular extensions*, defined by a prescribed asymptotic behaviour at the boundary. This subsheaf is the same as the corresponding subsheaf defined in [GS06] by fixing a *ghost type*. Here, we get around choosing local charts for the log structure by defining the intrinsic data of the *local divisor system* (Definition 4.12) and embedding it in the total ring of fractions.

The next step is to construct a sheaf homomorphism

$$\varphi\colon \mathcal{E}xt_c^1(\mathcal{M}, \mathcal{O}_Y^\times) \to \mathcal{LS}_Y^\times.$$

Here we need to depart from [GS06]: we introduce the *local line bundle system* (Definition 4.20) and construct φ as an application of the abstract joint condition Lemma 3.6.

[20] It is important to appreciate that $\mathfrak{M}$ is not in any sensible way a log structure: we lack the monoid homomorphism $\mathfrak{M} \to \mathcal{O}_Y$.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

While the proof of injectivity of φ is very similar to [GS06, Theorem 3.22], the proof of surjectivity departs somewhat from [GS06]. It requires us to introduce 2-homomorphisms from a lattice into the local line bundle system (Proposition 4.31), sections of these and then gluing them using the abstract joint condition once more.

Finally, we construct

$$\psi\colon \mathrm{LS}_{k^\dagger}(Y) \to \mathcal{E}xt^1_c(\mathcal{M}, \mathcal{O}_Y^\times)$$

and prove that it is also bijective. Composing with φ gives a bijection

$$r\colon \mathrm{LS}_{k^\dagger}(Y) \xrightarrow{\cong} \mathcal{LS}_Y^\times.$$

This 2-homomorphism in the surjectivity of φ also plays a central role in the construction of a log structure from a section of $\mathcal{LS}_Y$, see Proposition 4.32. Our point of view in the proof of Proposition 4.32 closely matches that of [BV12] as was pointed out to us by Bernd Siebert. A log structure in [BV12] is defined as a certain symmetric monoidal functor. In fact, our 2-group $\underline{T}(Y)$ of Section 4.5 agrees with the category $\mathrm{Div}(X)$ considered in [BV12, Example 2.5]. □

Synopsis of the following sections.

In Section 4.2 we recall the basics of total rings of fractions and fractional ideals.

In Section 4.3 we work in the affine local situation $y \in Y$ and we construct a canonical resolution of $\mathcal{M}$ (in the category of sheaves of abelian groups on Y).

In Section 4.4 we use this resolution to compute $\mathcal{E}xt^1(\mathcal{M}, \mathcal{O}_Y^\times)$ in the affine local situation, and to define the subsheaf $\mathcal{E}xt^1_c(\mathcal{M}, \mathcal{O}_Y^\times)$. The local properties that define it in fact define it for every Y, not necessarily local.

In Section 4.5 we define a homomorphism $\varphi\colon \mathcal{E}xt^1_c(\mathcal{M}, \mathcal{O}_Y^\times) \to \mathcal{LS}_Y$ in the affine local situation. In Section 4.6 we show that the definition globalizes to all Y, not necessarily local.

In Section 4.7 we show that $\varphi\colon \mathcal{E}xt^1_c(\mathcal{M}, \mathcal{O}_Y^\times) \to \mathcal{LS}_Y$ is injective; in Section 4.8 we show that it is surjective.

In the final two sections we complete the proof of the main theorem: Section 4.9 does it in the affine local situation and Section 4.10 does it in the general case.

4.2 Sheaf of total ring of fractions and invertible fractional ideals

For a scheme X, the sheaf of total rings of fractions $\mathcal{K}_X$ is the sheafification of the presheaf $U \mapsto S(U)^{-1}\Gamma(U, \mathcal{O}_U)$ where $S(U) \subset \Gamma(U, \mathcal{O}_U)$ is the subset of those non-zero-divisors of $\Gamma(U, \mathcal{O}_U)$ that are also non-zero-divisors at every stalk of $\mathcal{O}_U$. If U is affine, then the stalk condition can be shown to be vacuous and $S(U)$ is just the set of non-zero-divisors in $\Gamma(U, \mathcal{O}_U)$, see p. 204 in [Kle79]. The case that concerns us is when X is a reduced scheme whose set of irreducible components is locally finite, case (b) on p. 205 in [Kle79]. In this case,

$$\mathcal{K}_X = j_*(\mathcal{O}_{X|\,\mathrm{Ass}(X)})$$

where $j\colon \mathrm{Ass}(X) \to X$ is the inclusion of the set of points $x \in X$ for which the maximal ideal in $\mathcal{O}_{X,x}$ is associated to zero. The subsheaf of groups consisting of sections that are nowhere zero-divisors is denoted $\mathcal{K}_X^\times \subset \mathcal{K}_X$.

Example 4.4. Consider $X = \mathrm{Spec}\, A$ where $A = k[x,y]/(xy)$, so X has irreducible components $X_1 = (y=0)$ and $X_2 = (x=0)$. For all $a, b \in k[x]$, $c, d \in k[y]$ with $b, d \neq 0$, consider the rational function $f = \frac{ax+cy}{bx+dy} \in K(A)$. Then $f_{|X_1} = \frac{a}{b}$ and $f_{|X_2} = \frac{c}{d}$, so $f = (\frac{a}{b}, \frac{c}{d})$ under the natural isomorphism $K(A) = k(x) \times k(y)$.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

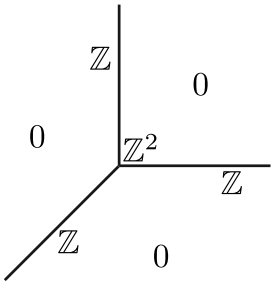


Figure 3. The stalks of the sheaf $\mathcal{M} = \mathcal{P}^{\mathrm{gp}}/\mathbf{1}$ at various points of T for the normal crossing surface $xyz = 0$.

Following [EGA67, Sections 19–21], an *invertible fractional ideal* on a scheme X is a coherent $\mathcal{O}_X$-submodule $\mathcal{I} \subset \mathcal{K}_X$ that is locally principal. In other words, there is an affine cover of X such that for all open subsets $U = \operatorname{Spec} A \subset X$ that belong to this cover,

$$\mathcal{I}(U) = A \cdot \frac{s}{t} \subset \mathcal{K}_X(U) \tag{4.1}$$

where s, t are both non-zero-divisors in A. The product of two fractional ideals inside $\mathcal{K}_X$ is again a fractional ideal giving the set of fractional ideals the structure of an abelian group. The sheaf of groups of fractional ideals on X is denoted by $\mathcal{I}d.inv_X$. The sheaf of Cartier divisors on X is, by definition, the sheaf of groups

$$\mathcal{D}iv_X = \mathcal{K}_X^{\times}/\mathcal{O}_X^{\times}.$$

The natural homomorphism $\mathcal{D}iv_X \to \mathcal{I}d.inv_X$ is bijective [EGA67], Proposition (21.2.6). We denote by $\mathcal{D}iv_X^{+} \subset \mathcal{D}iv_X$ the subsheaf of those divisors whose invertible sheaf has local forms (4.1) with $t = 1$. We set $\operatorname{Div} X = \Gamma(X, \mathcal{D}iv_X)$ and $\operatorname{Div}^{+} X = \Gamma(X, \mathcal{D}iv_X^{+})$.

4.3 The affine local situation

Setup 4.5. In this section we work with the following setup, which we refer to as the *affine local situation*:

(a) $Y = \coprod_{\tau \in T} Y_\tau^\star$ is an affine viable ggtc space;
(b) Y has a unique smallest stratum $Y_\eta^\star = Y_\eta$, which is necessarily closed;
(c) we write $M = M_\eta$, $\Sigma = \Sigma_\eta$, etc.

The purpose of this section is to prove Lemma 4.8, giving a canonical resolution of the quotient sheaf

$$\mathcal{M} = \mathcal{P}^{\mathrm{gp}}/\mathbf{1}.$$

Recall $\mathcal{M}$ is so defined that, for a cone $\tau \in \Sigma$ the stalk M_τ is the quotient $M/\langle\tau\rangle$ (see Figure 3).

Notation 4.6. For all $\tau \in \Sigma$, denote by $\underline{\langle\tau\rangle}_{U_\tau}$ the constant sheaf on U_τ with group $\langle\tau\rangle$. Denoting by $j_\tau \colon U_\tau \hookrightarrow Y$ the inclusion, we write

$$L_!(\tau) = j_{\tau\,!}\underline{\langle\tau\rangle}_{U_\tau},$$

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

where $j_{\tau\,!}$ is the extension by zero. More explicitly, if $U \subset Y$ is a connected open subset, we have

$$L_!(\tau)(U) = \begin{cases} \langle \tau \rangle & \text{if } U \subset U_\tau, \\ 0 & \text{otherwise.} \end{cases}$$

Construction 4.7. Recall that for an open embedding $j\colon U \to U'$, we have $j^* = j^!$ and hence for a sheaf $\mathcal{F}$ on U', we have an adjunction morphism $j_! j^* \mathcal{F} \to \mathcal{F}$. With that in mind, and using that, for $\tau_1 < \tau_2$, the open embedding $j_{\tau_2}\colon U_{\tau_2} \to Y$ factors through $j_{\tau_1}\colon U_{\tau_1} \to Y$, we have a natural map

$$d_{\tau_1\tau_2}\colon j_{\tau_1\,!}\underline{\langle \tau_1 \rangle}_{U_{\tau_1}} \to j_{\tau_2\,!}\underline{\langle \tau_2 \rangle}_{U_{\tau_2}}$$

which at a stalk of a point $p \in Y$ is either the natural inclusion $\langle \tau_1 \rangle \subset \langle \tau_2 \rangle$ or the zero map, depending on whether $p \in U_{\tau_2}$ or not. We define a homomorphism

$$\delta_i\colon \bigoplus_{\tau_0 \lneq \ldots \lneq \tau_i} L_!(\tau_i) \to \bigoplus_{\tau_0 \lneq \ldots \lneq \tau_{i-1}} L_!(\tau_{i-1})$$

via $\delta_i(\sum_{\tau_0 \lneq \ldots \lneq \tau_i} a_{\tau_0 \lneq \ldots \lneq \tau_i}) = \sum_{\tau_0 \lneq \ldots \lneq \tau_i} \delta_i(a_{\tau_0 \lneq \ldots \lneq \tau_i})$ and a component of $\delta_i(a_{\tau_0 \lneq \ldots \lneq \tau_i})$ is trivial except for

$$\delta_i(a_{\tau_0 \lneq \ldots \lneq \tau_i})_{\tau_0 \lneq \ldots \widehat{\tau}_j \ldots \lneq \tau_i} = (-1)^j a_{\tau_0 \lneq \ldots \lneq \tau_i},$$

where $\tau_0 \lneq \ldots \widehat{\tau}_j \ldots \lneq \tau_i$ refers to the result of removing τ_j from the sequence $\tau_0 \lneq \ldots \lneq \tau_i$ for $0 \le j < i$; and for

$$\delta_i(a_{\tau_0 \lneq \ldots \lneq \tau_i})_{\tau_0 \lneq \ldots \lneq \tau_{i-1}} = (-1)^i d_{\tau_{i-1}\tau_i}(a_{\tau_0 \lneq \ldots \lneq \tau_i}).$$

In particular,

$$\delta_1 \left(\sum_{\tau_0 \lneq \tau_1} a_{\tau_0 \lneq \tau_1} \right)_\tau = \sum_{\tau_0 \lneq \tau} a_{\tau_0 \lneq \tau} - \sum_{\tau \lneq \tau_1} d_{\tau\tau_1}(a_{\tau \lneq \tau_1}). \tag{4.2}$$

Lemma 4.8. *Let $\underline{M}$ denote the constant sheaf on Y with group M. Construction 4.7 gives a resolution of $\mathcal{M}$ by sheaves on Y:*

$$\cdots \to \bigoplus_{\tau_0 \leqslant \tau_1} L_!(\tau_1) \xrightarrow{\delta_1} \bigoplus_{\tau \in \Sigma} L_!(\tau) \xrightarrow{\delta_0} \underline{M} \to \mathcal{M} \to 0.$$

Proof. Exactness is to be checked at the stalk level, so let us fix a point $p \in Y$. The complex of stalks only depends on the stratum $Y_\tau^\star$ that contains p. We have $p \in U_{\tau'}$ if an only if $\tau' < \tau$. In particular, $L_!(\tau')_p = 0$ unless $\tau' < \tau$. The complex has an increasing filtration F_k by subcomplexes given by requiring the maximum for the dimension of the cones in the the chain $\tau_0 \lneq \ldots \lneq \tau_i$ to be at most k, that is, $\dim \tau_i \le k$. To show the exactness of the original sequence, it suffices to show the exactness of the graded quotients with respect to the filtration. The graded quotient decomposes as $F_k/F_{k-1} = \bigoplus_{\dim \tau = k} C_\tau$ and C_τ is the complex that result from applying $L_!(\tau) \otimes \cdot$ to the complex

$$\cdots \to \bigoplus_{\tau_0 \lneq \ldots \lneq \tau_i = \tau} \mathbb{Z} \to \cdots \to \bigoplus_{\tau_0 \lneq \tau_1 = \tau} \to \mathbb{Z}$$

with differential similar to δ_i as before. This complex is exact, as it can be identified with the augmented chain complex for the homology of a contractible space. □

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

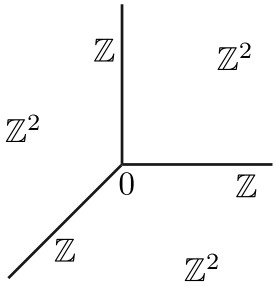


Figure 4. The stalks of the relation sheaf $\mathcal{R}$ at various points of T for the normal crossing surface $xyz = 0$.

Definition 4.9. The *relation sheaf* is the sheaf $\mathcal{R}$ on Y defined by the exact sequence:

$$0 \to \mathcal{R} \to \underline{M} \to \mathcal{M} \to 0. \tag{4.3}$$

See Figure 4 for an illustration of the relation sheaf.

4.4 The sheaf $\mathcal{E}xt^1_c(\mathcal{M}, \mathcal{O}_Y^\times)$

Setup 4.10. We continue with the *affine local situation* of Setup 4.5.

In this section we define a subsheaf $\mathcal{E}xt^1_c(\mathcal{M}, \mathcal{O}_Y^\times) \subset \mathcal{E}xt^1(\mathcal{M}, \mathcal{O}_Y^\times)$, which we call the sheaf of regular extensions, consisting of sections with a prescribed asymptotic behaviour towards the Zariski closure $\overline{U}_\tau$ of U_τ in Y. Before we can state this definition, we need to discuss some preliminaries.

Lemma 4.11. *For all cones $\tau' \leqslant \tau$ in Σ, denote by $\rho_\tau^{\tau'} \colon \mathcal{O}_Y^\times(U_{\tau'}) \to \mathcal{O}_Y^\times(U_\tau)$ the restriction homomorphism.*

We have the following concrete description of the relation sheaf: for every open $U \subset Y$

$$\begin{aligned}\Gamma(U, \mathcal{H}om(\mathcal{R}, \mathcal{O}_Y^\times)) &= \left\{ h \in \mathcal{H}om\left(\bigoplus_\tau L_!(\tau), \mathcal{O}_Y^\times\right)(U) \;\middle|\; h_{|\operatorname{im}\delta_1} = 0 \right\} \\ &= \left\{ (h_\tau)_{\tau\in\Sigma} \;\middle|\; \begin{array}{l} h_\tau \colon \langle\tau\rangle \to \mathcal{O}_Y(U \cap U_\tau)^\times \text{ is a group homomorphism} \\ \text{and for every } \tau' \leqslant \tau \text{ we have } \rho_\tau^{\tau'} \circ h_{\tau'} = h_{\tau|\langle\tau'\rangle} \end{array} \right\}. \end{aligned} \tag{4.4}$$

Proof. Straightforward from Lemma 4.8 and the adjoint properties of the functor $j_!$. □

Definition 4.12. The *local divisor system* is the system of group homomorphisms $\{D_{\eta,\tau} | \tau \in \Sigma\}$ where, for all $\tau \in \Sigma$,

$$D_{\eta,\tau} \colon \langle\tau\rangle \to \operatorname{Div} \overline{U}_\tau$$

is the group homomorphism of Definition 2.16 and Notation 2.17.

Notation 4.13. Because in this section $\eta \in T$ is the unique smallest stratum, in this section we suppress the subscript η from the notation and simply write D_τ instead of $D_{\eta,\tau}$.

Lemma 4.14. *The local divisor system satisfies the conditions (A), (B) and (C) given below.*

Note that if $\tau' \leqslant \tau$, then $U_{\tau'} \supset U_\tau$. In the statement of the conditions we denote by $\overline{\rho}_\tau^{\tau'} \colon \operatorname{Div} \overline{U}_{\tau'} \to \operatorname{Div} \overline{U}_\tau$ the restriction of Cartier divisors: this restriction is defined because $\overline{U}_\tau$ is a union of irreducible components of $\overline{U}_{\tau'}$. Figure 5 gives an illustration.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

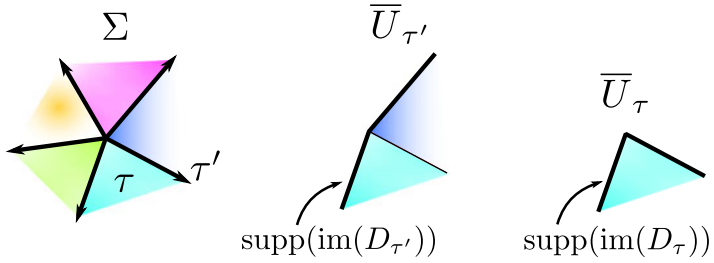


Figure 5. $\overline{U}_\tau \subset \overline{U}_{\tau'}$ for $\tau' \leqslant \tau$.

The conditions are as follows.

(A) Fan property. *If* $\tau' \leqslant \tau$, *then*

$$\overline{\rho}_\tau^{\tau'} \circ D_{\tau'} = D_{\tau|\langle\tau'\rangle}.$$

(B) Positivity. *We have* $D_\tau(\langle\tau\rangle_+) \subset \operatorname{Div}^+ \overline{U}_\tau$.

(C) Support property. *The composition of* D_τ *with the restriction* $\operatorname{Div} \overline{U}_\tau \to \operatorname{Div} U_\tau$ *gives the trivial map. In particular, none of the divisors in* $D_\tau(\langle\tau\rangle)$ *contain the stratum* Y_τ *in their support; hence all of these divisors are restrictable to* Y_τ.

Proof. The statement is a straightforward consequence of the definition. By construction of D, see Definition 2.16, we may assume that $Y = \operatorname{Spec} k(\eta)[\Sigma_y]$ with $D_\tau(m) = \operatorname{div} z^m$, where the result is basically obvious.

Part (A) follows from observing that for $m \in \tau'$, by definition, $D_{\tau'}(m) = \operatorname{div} z^m \in \operatorname{Div} \overline{\Omega}_{\tau'}$ and so $\overline{\rho}_\tau^{\tau'} \circ D_{\tau'}(m)$ is its restriction to $\overline{\Omega}_\tau$ which of course agrees with $D_\tau(m) = \operatorname{div} z^m \in \operatorname{Div} \overline{\Omega}_\tau$.

Part (B) is clear because $D_\tau(m) = \operatorname{div} z^m$ is a principal divisor of a regular function.

Moreover, since z^m is invertible on Ω_τ for $m \in \tau$, its support is contained in $\overline{\Omega}_\tau \setminus \Omega_\tau$, so we deduce part (C). □

We are now ready to define the subsheaf $\mathcal{E}xt^1_c(\mathcal{M}, \mathcal{O}_Y^\times)$. As in Section 4.2, we denote by $\mathcal{K}_Y$ be the sheaf of total rings of fractions on Y and a Cartier divisor D on Y gives the subsheaf $\mathcal{O}_Y(D) \subset \mathcal{K}_Y$.

Definition 4.15.

(1) Let $U \subset Y$ be an open subset, and consider a section $(h_\tau)_{\tau\in\Sigma} \in \Gamma(U, \mathcal{H}om(\mathcal{R}, \mathcal{O}_Y^\times))$. Denote by $\widetilde{h}_\tau \colon \langle\tau\rangle \to \mathcal{K}_Y(\overline{U}_\tau \cap U)$ the composition of $h_\tau \colon \langle\tau\rangle \to \mathcal{O}_Y^\times(U_\tau \cap U)$ with the inclusion $\mathcal{O}_Y^\times(U_\tau \cap U) \hookrightarrow \mathcal{K}_Y(\overline{U}_\tau \cap U)$.
We say that h is *regular* if for all $\tau \in \Sigma$ and $m \in \tau$, $\widetilde{h}_\tau(m)$ is a generator of $\mathcal{O}_{\overline{U}_\tau \cap U}(-D_\tau(m)) \subset \mathcal{K}_{\overline{U}_\tau \cap U}$ as a $\mathcal{O}_{\overline{U}_\tau \cap U}$-module. We denote by

$$\mathcal{H}om_c(\mathcal{R}, \mathcal{O}_Y^\times) \subset \mathcal{H}om(\mathcal{R}, \mathcal{O}_Y^\times)$$

the subsheaf of regular sections.

(2) Consider the exact sequence:

$$\mathcal{H}om(\underline{M}, \mathcal{O}_Y^\times) \to \mathcal{H}om(\mathcal{R}, \mathcal{O}_Y^\times) \to \mathcal{E}xt^1(\mathcal{M}, \mathcal{O}_Y^\times) \to 0. \tag{4.5}$$

The *sheaf of regular extensions*, denoted by $\mathcal{E}xt^1_c(\mathcal{M}, \mathcal{O}_Y^\times)$, is the image of $\mathcal{H}om_c(\mathcal{R}, \mathcal{O}_Y^\times)$ in $\mathcal{E}xt^1(\mathcal{M}, \mathcal{O}_Y^\times)$.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

4.5 A morphism $\varphi\colon \mathcal{E}xt^1_c(\mathcal{M},\mathcal{O}_Y^\times)\to\mathcal{LS}_Y$ in the affine local situation

Setup 4.16. We continue with the *affine local situation* of Setup 4.5, and we assume in addition the following.

(d) For all $\tau\in\Sigma$, and all $m\in\tau$, $\mathcal{O}(-D_\tau(m))$ is a trivial line bundle on $\overline{U}_\tau$.

Construction 4.17. Let $U\subset Y$ be open and $h\in\Gamma(U,\mathcal{E}xt^1_c(\mathcal{M},\mathcal{O}_Y^\times))$. Because of assumption (d) of Setup 4.16, h lifts to

$$(h_\tau)_{\tau\in\Sigma}\in\mathcal{H}om_c(\mathcal{R},\mathcal{O}_Y^\times)(U).$$

Consider two maximal cones $\sigma_1,\sigma_2\in\Sigma$ meeting along a common facet ρ. We choose $e_2\in(\langle\rho\rangle+\sigma_2)\cap M$ at integral distance 1 from ρ and set[21]

$$\varphi_\rho(h)=\widetilde{h}_{\sigma_1}(e_2)_{|Y_\rho}\otimes\widetilde{h}_{\sigma_2}(-e_2)_{|Y_\rho} \tag{4.6}$$

and the regularity of h implies that $\varphi_\rho(h)$ is a generator of the chart

$$\lambda_\rho(e_2)=\mathcal{O}_{U_1}(D_{\sigma_1}(-e_2))_{|Y_\rho}\otimes\mathcal{O}_{U_2}(D_{\sigma_2}(e_2))_{|Y_\rho} \tag{4.7}$$

of the line bundle $\mathcal{L}_{\rho|U}$.

Lemma-Definition 4.18. *Let $U\subset Y$ be open and $h\in\Gamma(U,\mathcal{E}xt^1_c(\mathcal{M},\mathcal{O}_Y^\times))$.*

(1) *The section $\varphi_\rho(h)\in\Gamma(Y_\rho\cap U,\mathcal{L}_\rho)$ of Construction 4.17 does not depend on the choice of the lift $(h_\tau)_{\tau\in\Sigma}\in\mathcal{H}om_c(\mathcal{R},\mathcal{O}_Y^\times)(U)$, nor on the choice of e_2. Thus, Construction 4.17 provides for all $\rho\in T^{[1]}$ a morphism of sheaves*

$$\varphi_\rho\colon\mathcal{E}xt^1_c(\mathcal{M},\mathcal{O}_Y^\times)\to\mathcal{L}_\rho.$$

(2) *Assembling the morphisms of part (1) gives a morphism $\bigoplus_{\rho\in T^{[1]}}\varphi_\rho\colon\mathcal{E}xt^1_c(\mathcal{M},\mathcal{O}_Y^\times)\to\bigoplus_{\rho\in T^{[1]}}\mathcal{L}_\rho$. The image of this morphism lies in $\mathcal{LS}_Y^\times$.*

The two parts show that Construction 4.17 provides a morphism $\varphi\colon\mathcal{E}xt^1_c(\mathcal{M},\mathcal{O}_Y^\times)\to\mathcal{LS}_Y^\times$.

Beginning of the proof of Lemma-Definition 4.18. We show that $\varphi(h)_\rho$ does not depend on the choice of lift $(h_\tau)_{\tau\in\Sigma}\in\mathcal{H}om_c(\mathcal{R},\mathcal{O}_Y^\times)(U)$ and also that it does not depend on the choice of e_2. Another choice of lift differs from the chosen one by an element

$$u\in\mathcal{H}om(\underline{M},\mathcal{O}_Y^\times)(U)=\mathrm{Hom}_{\mathrm{groups}}(M,\mathcal{O}_Y(U)^\times),$$

so it can be written as $(uh_\tau)_{\tau\in\Sigma}$ and then

$$\varphi(uh)_\rho=(u(e_2)\widetilde{h}_{\sigma_1}(e_2))_{|Y_\rho}\otimes(u(e_2)^{-1}\widetilde{h}_{\sigma_2}(-e_2))_{|Y_\rho}$$

agrees with $\varphi(h)_\rho$.

A different choice e_2' of e_2 differs by $r=e_2'-e_2\in\langle\rho\rangle$ and leads to a different chart $\lambda_\rho(e_2')$ of $\mathcal{L}_\rho$ which is isomorphic to the one in (4.7) via the isomorphism ψ_r from Lemma 3.2. It is straightforward to see that the assignment of the section $\varphi(h)_\rho$ to the tuple $(h_\tau)_{\tau\in\Sigma}$ is compatible with ψ_r.

We have thus constructed a morphisms of sheaves of sets $\oplus_\rho\varphi_\rho\colon\mathcal{E}xt^1_c(\mathcal{M},\mathcal{O}_Y^\times)\to\bigoplus_\rho\mathcal{L}_\rho$. We next explain why its image is contained in $\mathcal{LS}_Y^\times$, i.e., why its elements are nowhere-vanishing

[21] It is intentional that $\lambda(v_i)=\mu_{i-1}(-v_i)+\mu_i(v_i)$ from (3.1) has a different sign in the arguments when compared with $\varphi_\rho(h)=\widetilde{h}_{\sigma_1}(e_2)_{|Y_\rho}\otimes\widetilde{h}_{\sigma_2}(-e_2)_{|Y_\rho}$.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

and satisfy the joint condition. That they are nowhere-vanishing follows immediately from the regularity property.

Before showing that the joint condition holds we need to discuss some preliminaries. □

Definition 4.19. Let X be a space. The 2-*group* $\underline{T}(X)$ *of trivialized line bundles* on X is the strictly commutative 2-group where:

- objects of $\underline{T}(X)$ are trivialized line bundles on X, that is, pairs $(\mathcal{L}, s)$ where $\mathcal{L}$ is a (trivial) line bundle on X, and $s\colon \mathcal{O}_X \to \mathcal{L}$ an isomorphism;
- a morphism from $(\mathcal{L}_1, s_1)$ to $(\mathcal{L}_2, s_2)$ is an isomorphisms $s\colon \mathcal{L}_1 \to \mathcal{L}_2$ such that $s_2 = s \circ s_1$.

The monoidal structure in $\underline{T}(X)$ is given by tensor product

$$(\mathcal{L}_1, s_1)(\mathcal{L}_2, s_2) = (\mathcal{L}_1 \otimes \mathcal{L}_2, s_1 \otimes s_2)$$

and the distinguished neutral element is the trivial line bundle $\mathcal{O}_X$ with the unit trivializing section 1. For $(\mathcal{L}, s)$ we have the inverse $(\mathcal{L}^\star, (s^\star)^{-1})$ where $\mathcal{L}^\star = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$ denotes the dual and $s^\star$ the dual map. For any pair $(\mathcal{L}_1, s_1), (\mathcal{L}_2, s_2) \in \underline{T}(X)$, there is a unique morphism $(\mathcal{L}_1, s_1) \to (\mathcal{L}_2, s_2)$ in $\underline{T}(X)$, so, in particular, we have unique morphisms

$$(\mathcal{L}_1 \otimes \mathcal{L}_2, s_1 \otimes s_2) \to (\mathcal{L}_2 \otimes \mathcal{L}_1, s_2 \otimes s_1),$$

$$(\mathcal{L}, s)(\mathcal{L}^\star, s^{-1}) \to (\mathcal{O}_X, 1)$$

and $\underline{T}(X)$ is thus strictly commutative.

Definition 4.20. Fix a codimension-two cone $\omega \in \Sigma$ and denote by $\omega^{-1}\Sigma$ the localized fan. The *local line bundle system* associated to a regular element $(h_\tau)_{\tau\in\Sigma} \in \mathcal{H}om_c(\mathcal{R}, \mathcal{O}_Y^\times)(U)$ is the continuous piecewise linear 2-homomorphism (recall Definition 3.5)

$$h^\omega\colon M \to \underline{T}(Y_\omega \cap U)$$

such that if $\sigma \in \omega^{-1}\Sigma$ is a maximal cone and $m \in \sigma$, $h^\omega(m) = (\mathcal{O}(-D_\sigma(m)), \widetilde{h}_\sigma)_{|Y_\omega\cap U}$. Regularity of $(h_\tau)_{\tau\in\Sigma}$ makes h^ω well-defined.

End of the proof of Lemma-Definition 4.18. To see that φ is well-defined as a morphism to $\mathcal{LS}_Y$, we need to argue that elements in the image of φ satisfy the joint condition. We have defined the local line bundle system associated to a regular element $(h_\tau)_{\tau\in\Sigma} \in \mathcal{H}om_c(\mathcal{R}, \mathcal{O}_Y^\times)(U)$ on an open $U \subset Y$. The abstract joint condition Lemma 3.6 applied to h^ω for every codimension-two cell ω implies that the image of φ is indeed contained in $\mathcal{LS}_Y(U)$. □

4.6 Globalizing the construction of φ

Lemma 4.21. *Let Y be a ggtc space. Then Y has a cover by affines that satisfy properties (a)–(d) of Setups 4.5 and 4.16.*

Proof. The proof is straightforward. To satisfy property (d), we need to show that every $y \in Y$ has an affine neighbourhood $y \in U$ such that property (d) holds on U. If $y \in Y_\eta^\star$, we may restrict to $Y = U_\eta$. We want a neighbourhood $y \in U$ such that for all $\tau \in \Sigma_\eta$ and all $m \in \tau \cap M_\eta$, the line bundles $\mathcal{O}(-D_{\eta,\tau}(m))$ are trivial on U. This is easy to achieve based on the facts that: T is finite, and all monoids $\tau \cap M_\eta$ are finitely generated. □

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

The goal of this section is to show that for all ggtc spaces Y the morphisms of sheaves defined in Lemma-Definition 4.18 in the local situation automatically glue to define a morphism of sheaves $\varphi\colon \mathcal{E}xt^1_c(\mathcal{M},\mathcal{O}_Y^\times)\to\mathcal{LS}_Y^\times$. The key result that makes everything work is the following.

Lemma 4.22. *Let Y be an affine ggtc space satisfying conditions (a)–(d) of Setup Setup 4.5 and Setup 4.16 and $\eta\in T$ unique smallest stratum. Write $M=M_\eta$, $\Sigma=\Sigma_\eta$, etc. We identify the poset of strata with the poset of cones of Σ; note that, under this identification, η corresponds to the cone $\{0\}\in\Sigma$.*

Consider now an affine open $Y'\subset Y$, also satisfying conditions (a)–(d). In particular, Y' has a smallest stratum $\eta'\in\Sigma$ (and we allow the possibility $\eta'=\{0\}$). We write

$$M'=M_{\eta'}=M/\langle\eta'\rangle,\ \pi\colon M\to M' \text{ the projection},\quad \Sigma'=\Sigma_{\eta'}=\Sigma/\eta',\quad \text{etc.}$$

Note that the cones of Σ' (also known as the strata of Y') are the projections of the cones $\tau\in\Sigma$ such that $\eta'\subset\tau$.

In the notation of Definition 4.12, we have the following.[22]

(D) *Sheaf property For all $\eta'\subset\tau$ and all $m\in\tau\cap M$,*

$$D_{\eta,\tau}(m)_{|Y'\cap\overline{U}_\tau}=D_{\eta',\pi(\tau)}(\pi(m)).$$

Proof. As for the proof of Lemma 4.14, the statement is a straightforward consequence of the definition. By construction of D, see Definition 2.16, we may assume that $Y=\operatorname{Spec}k(\eta)[\Sigma_y]$ with $D_\tau(m)=\operatorname{div}z^m$, where the result it is basically obvious. □

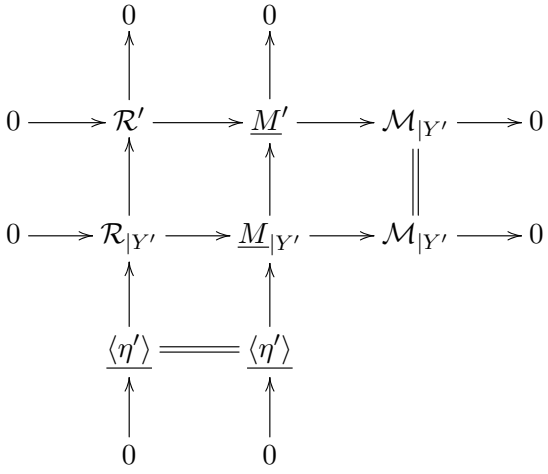
Lemma 4.23. *For all ggtc spaces Y the morphisms of sheaves defined in Lemma-Definition 4.18 glue to define a morphism of sheaves*

$$\varphi\colon \mathcal{E}xt^1_c(\mathcal{M},\mathcal{O}_Y^\times)\to\mathcal{LS}_Y^\times.$$

Proof. By Lemma 4.21, it is enough to consider the situation $Y'\subset Y$ of Lemma 4.22.

We want to check that the respective definitions of φ resulting from using either Y or Y' agree on the smaller open set Y'. This is, basically, an entirely straightforward exercise on unpacking the definitions, but we spell it out in some detail.

The issue is that we are working with two relation sheaves, defined and related by the following commutative diagram of sheaves on Y' with exact rows and columns, where the notation is self-explanatory.



[22] In the display, U_τ is the star of τ in Y, and $\overline{U}_\tau$ is the Zariski closure in Y.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

The two relation sheaves lead to two computations of the extension sheaf, summarized in the following commutative diagram of sheaves on Y' with exact rows and columns, where the notation is self-explanatory.

$$\begin{array}{ccccccc}
0 & & 0 & & & & \\
\downarrow & & \downarrow & & & & \\
\mathcal{H}om(\underline{M}', \mathcal{O}_{Y'}^\times) & \longrightarrow & \mathcal{H}om(\mathcal{R}', \mathcal{O}_{Y'}^\times) & \longrightarrow & \mathcal{E}xt^1(\mathcal{M}_{|Y'}, \mathcal{O}_{Y'}^\times) & \longrightarrow & 0 \\
\downarrow & & \downarrow \chi & & \| & & \\
\mathcal{H}om(\underline{M}_{|Y'}, \mathcal{O}_{Y'}^\times) & \longrightarrow & \mathcal{H}om(\mathcal{R}_{|Y'}, \mathcal{O}_{Y'}^\times) & \longrightarrow & \mathcal{E}xt^1(\mathcal{M}_{|Y'}, \mathcal{O}_{Y'}^\times) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
\mathcal{H}om(\underline{\langle \eta' \rangle}, \mathcal{O}_{Y'}^\times) & = & \mathcal{H}om(\underline{\langle \eta' \rangle}, \mathcal{O}_{Y'}^\times) & & & &
\end{array}$$

Let $\rho \in \Sigma'$ be a submaximal cone at which maximal cones σ_1', σ_2' are incident. Denote by $\varphi_\rho' \colon \mathcal{E}xt_c^1(\mathcal{M}_{|Y'}, \mathcal{O}_{Y'}^\times) \to \mathcal{L}_\rho$ the morphism of sheaves obtained by construction 4.17 on the space Y', and by $\varphi_\rho \colon \mathcal{E}xt_c^1(\mathcal{M}_{|Y'}, \mathcal{O}_{Y'}^\times) \to \mathcal{L}_\rho$ the morphism of sheaves obtained by Construction 4.17 on the space Y. We want to show that $\varphi' = \varphi$.

Consider a regular extension $h \in \mathcal{E}xt_c^1(\mathcal{M}_{|Y'}, \mathcal{O}_{Y'}^\times)$. In order to compute $\varphi'(h)$, we need to choose a lift in the first row

$$(h_{\tau'})_{\tau' \in \Sigma'} \in \mathcal{H}om_c(\mathcal{R}', \mathcal{O}_{Y'}^\times)$$

and follow Construction 4.17 on Y'. Recall that we denote by $\pi \colon M \to M'$ the natural projection: the cones $\tau' \in \Sigma'$ are the projections of the cones of Σ that contain η'. The result follows from the observation that

$$(h_{\tau'} \circ \pi)_{\eta' \subset \tau' \in \Sigma} = \chi((h_{\tau'})_{\tau' \in \Sigma'})$$

is a lift of h in the second row; that because of Lemma 4.22 it lies in $\mathcal{H}om_c(\mathcal{R}_{|Y'}, \mathcal{O}_{Y'}^\times)$; and that therefore it is good to feed to Construction 4.17 on Y. Unpacking the results leads to $\varphi(h) = \varphi'(h)$. □

4.7 The morphism φ is injective

Lemma 4.24. *Let Y be a ggtc space. The morphism φ of Lemma 4.23 is injective.*

Proof. In order to show the injectivity of φ, we may work on Y affine satisfying properties (a)–(d) of Setups 4.5 and 4.16.

Assume that we are given two regular tuples $(h_\sigma)_{\sigma \in \Sigma}, (h'_\sigma)_{\sigma \in \Sigma} \in \mathcal{H}om(\mathcal{R}, \mathcal{O}_Y^\times)$ that map to the same section of $\mathcal{LS}_Y^\times$ under φ.

For $\sigma \in \Sigma$ a maximal cone, the quotient $g = (\widetilde{h}_\sigma / \widetilde{h}'_\sigma)_{\sigma \in \Sigma}$ is a homomorphism $g_\sigma \colon M \to \mathcal{O}(\overline{U}_\sigma)^\times$. To prove the injectivity of Y, we want to glue these maps to a homomorphism $g \colon M \to \mathcal{O}(Y)^\times$, that is, $g \in \Gamma(Y, \mathcal{H}om(\underline{M}, \mathcal{O}_Y^\times))$.

For gluing along a slab ρ with σ_1, σ_2 the maximal cones containing ρ, since $g_{\sigma_1}, g_{\sigma_2}$ already agree on $\langle \rho \rangle$, we only need to consider $e_2 \in \sigma_2$ at integral distance one from ρ and we need

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

to have $g_{\sigma_1}(e_2)_{|Y_\rho} = g_{\sigma_2}(e_2)_{|Y_\rho}$. Assuming $\varphi(h) = \varphi(h')$ gives that $\varphi((h_\sigma)_{\sigma\in\Sigma})_\rho$ and $\varphi((h'_\sigma)_{\sigma\in\Sigma})_\rho$ define the same section of $\mathcal{L}_\rho$. In view of the definition (4.6), we then have

$$\widetilde{h}_{\sigma_1}(e_2)_{|Y_\rho} \otimes \widetilde{h}_{\sigma_2}(-e_2)_{|Y_\rho} = \widetilde{h}'_{\sigma_1}(e_2)_{|Y_\rho} \otimes \widetilde{h}'_{\sigma_2}(-e_2)_{|Y_\rho}.$$

and rearranging factors yields

$$g(e_2)_{|Y_\rho} = g(-e_2)^{-1}_{|Y_\rho} \tag{4.8}$$

as desired. We can use (4.8) to glue the homomorphisms $g_{\sigma_1}\colon M \to \Gamma(\overline{U}_{\sigma_1}\mathcal{O}_Y^\times)$ and $g_{\sigma_2}\colon M \to \Gamma(\overline{U}_{\sigma_2}, \mathcal{O}_Y^\times)$ to a homomorphism $g\colon M \to \Gamma(\overline{U}_{\sigma_1} \cup \overline{U}_{\sigma_2}, \mathcal{O}_Y^\times)$. Attaching similar gluings along other slabs eventually yields a homomorphism $g\colon M \to \Gamma(Y, \mathcal{O}_Y^\times)$ with the property that on each $\overline{U}_\tau$ it equals $\widetilde{h}_\tau/\widetilde{h}'_\tau$. It follows from Equation (4.5) that the tuples (h_σ) and (h'_σ) project to the same section in $\mathcal{E}xt^1_c(\mathcal{M}, \mathcal{O}_Y^\times)$ and hence φ is injective. □

4.8 The morphism φ is surjective

Theorem 4.25. *For every viable ggtc space Y, the morphism $\varphi\colon \mathcal{E}xt^1_c(\mathcal{M}, \mathcal{O}_Y^\times) \to \mathcal{LS}_Y^\times$ is bijective.*

The statement is local, hence we may work in the following.

Setup 4.26. In the rest of this section we assume that Y satisfies properties (a)–(d) of Setup 4.5 and Setup 4.16 and use freely the notation of Setup 4.5.

We begin with some preparations; the proof of the theorem is at the end of the section.

Notation 4.27. For all maximal cones $\sigma \in \Sigma$ we denote by

$$E_\sigma \in N \otimes_{\mathbb{Z}} \underline{\mathrm{Pic}}(Y_\sigma) = \underline{2\text{-Hom}}(M, \underline{\mathrm{Pic}}\, Y_\sigma)$$

the 2-homomorphism such that for all $m \in M$, $E_\sigma(m) = \mathcal{O}_{Y_\sigma}(-D_\sigma(m))$.

Construction 4.28. Consider two maximal cones $\sigma_1, \sigma_2 \in \Sigma$ meeting along a slab ρ. Denote by $d_{\sigma_1\sigma_2} \in N$ the primitive normal to ρ which evaluates non-negatively on σ_2.

We construct an isomorphism:

$$\psi_{\sigma_1\sigma_2}\colon [E_{\sigma_1|Y_\rho} - d_{\sigma_1\sigma_2} \otimes \mathcal{L}_\rho] \xrightarrow{\cong} E_{\sigma_2|Y_\rho} \quad \text{in } N \otimes_{\mathbb{Z}} \underline{\mathrm{Pic}}\, Y_\rho. \tag{4.9}$$

Note that this thing unpacks into a collection of isomorphisms, one for each $m \in M$:

$$\psi_{\sigma_1\sigma_2}(m)\colon E_{\sigma_1}(m)_{|Y_\rho} \otimes \mathcal{L}_\rho^{-d_{\sigma_1\sigma_2}(m)} \xrightarrow{\cong} E_{\sigma_2}(m)_{|Y_\rho}$$

and such that $\psi_{\sigma_1\sigma_2}(m)$ depends 'linearly' on m.

For $m \in \langle\rho\rangle \cap M$ we have $d_{\sigma_1\sigma_2}(m) = 0$: in this case, we define $\psi_{\sigma_1\sigma_2}(m)$ to be the obvious isomorphism obtained by restricting to Y_ρ the restrictable divisors $D_{\sigma_i}(m)$.

Now fix $e_2 \in M$ with $d_{\sigma_1\sigma_2}(e_2) = 1$: this gives us the chart

$$\lambda(e_2) = \mathcal{O}_{Y_{\sigma_1}}(D_{\sigma_1}(-e_2))_{|Y_\rho} \otimes \mathcal{O}_{Y_{\sigma_2}}(D_{\sigma_2}(e_2))_{|Y_\rho}$$

for $\mathcal{L}_\rho$ and we carry out the construction by working in this chart.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

For $m = e_2$, we define

$$\begin{aligned}\psi_{\sigma_1\sigma_2}(e_2)\colon & (E_{\sigma_1|Y_\rho} - d_{\sigma_1\sigma_2} \otimes \lambda(e_2))(e_2)\\ &= E_{\sigma_1}(e_e)_{|Y_\rho} \otimes \lambda(e_2)^{-1}\\ &= \mathcal{O}_{Y_{\sigma_1}}(-D_{\sigma_1}(e_2))_{|Y_\rho} \otimes (\mathcal{O}_{Y_{\sigma_1}}(D_{\sigma_1}(-e_2))_{|Y_\rho} \otimes \mathcal{O}_{Y_{\sigma_2}}(D_{\sigma_2}(e_2))_{|Y_\rho})^{-1}\\ &\xrightarrow{\cong} \mathcal{O}_{Y_{\sigma_2}}(-D_{\sigma_2}(e_2)) = E_{\sigma_2|Y_\rho}(e_2).\end{aligned}$$

For general $m \in M$, there is a unique way to write $m = m' + ke_2$ with $m' \in \langle\rho\rangle$ and $k \in \mathbb{Z}$, and we define $\psi_{\sigma_1\sigma_2}(m) = \psi_{\sigma_1\sigma_2}(m') \otimes \psi_{\sigma_1\sigma_2}(e_2)^{\otimes k}$.

We leave it to the reader to check that all the isomorphisms that one constructs similarly by working in different charts for $\mathcal{L}_\rho$ glue (see the proof of Corollary 3.7 for a model discussion) and that the construction is linear in $m \in M$.

Notation 4.29. Assume Setup 4.26 and Notation 4.27.

Fix a nowhere-vanishing section $f_\rho \in \Gamma(Y_\rho, \mathcal{L}_\rho)$.

Construction 4.28 provides an isomorphism

$$\psi_{\sigma_1\sigma_2}(f_\rho)\colon E_{\sigma_1|Y_\rho} \xrightarrow{\cong} E_{\sigma_2|Y_\rho} \quad \textit{in} \quad N \otimes \underline{\mathrm{Pic}}(Y_\rho) \tag{4.10}$$

given, for all $m \in M$, as the composition

$$E_{\sigma_1}(m)_{|Y_\rho} \xrightarrow{(\ )\times f_\rho^{-d_{\sigma_1\sigma_2}(m)}} E_{\sigma_1}(m)_{|Y_\rho} \otimes \mathcal{L}_\rho^{-d_{\sigma_1\sigma_2}(m)} \xrightarrow{\psi_{\sigma_1\sigma_2}(m)} E_{\sigma_2}(m)_{|Y_\rho}.$$

Definition 4.30. Let Y be a space, M a lattice, $N = \mathrm{Hom}(M, \mathbb{Z})$ and E an object of the category $N \otimes_{\mathbb{Z}} \underline{\mathrm{Pic}}(Y)$.

A *section upgrade* of E is an object $\widehat{E}$ of $N \otimes_{\mathbb{Z}} \underline{T}(Y)$ that maps to E under the forgetful functor; in other words, $\widehat{E} = (E, e)$ where $e\colon N \otimes \mathcal{O}_Y \to E$ is an isomorphism.

Proposition 4.31. *Let Y be an affine viable ggtc space as in Setup 4.26, $y \in Y$ a (closed) point, and let E be as in Notation 4.27.*

For all $(f_\rho)_{\rho \in T^{[1]}} \in \Gamma(Y, \mathcal{LS}_Y^\times)$, possibly after shrinking Y to a smaller affine neighbourhood of $y \in Y$, there exists $\widehat{H} \in N \otimes \underline{T}(Y)$ such that:

(1) *for every maximal cone σ, $\widehat{H}_{|Y_\sigma} = (E_\sigma, e_\sigma)$ is a section upgrade of E_σ;*

(2) *for all pairs of maximal cones σ_1, σ_2 that meet in a slab ρ, denoting by $\psi_{\sigma_1\sigma_2}(f_\rho)\colon E_{\sigma_1|Y_\rho} \xrightarrow{\cong} E_{\sigma_2|Y_\rho}$ the isomorphism of Notation 4.29, we have*

$$\psi_{\sigma_1\sigma_2}(f_\rho)(e_{\sigma_1|Y_\rho}) = e_{\sigma_2|Y_\rho}.$$

Proof. The boundary complex of a polytope or polyhedral cone is shellable. Recall that this means that there is a shelling, that is an enumeration

$$\sigma_1, \sigma_2, \ldots$$

of its maximal cones such that for every $k \geqslant 1$ we have that $B_k = \left(\bigcup_{i=1}^k \sigma_i\right) \cap \sigma_{k+1}$ is a pure polyhedral complex of dimension $\dim M - 1$ homeomorphic to either the cone over a ball or the cone over a sphere. Pick a shelling of Σ and fix it for the rest of the proof.

Choose a section upgrade $\widehat{H_1} = (E_{\sigma_1}, e_{\sigma_1})$. The cones σ_1 and σ_2 share exactly one submaximal face ρ and Notation 4.29 implies that

$$(E_{\sigma_2|Y_\rho}, \psi_{\sigma_1\sigma_2}(f_\rho)(e_{\sigma_1|Y_\rho}))$$

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

is a section upgrade of $E_{\sigma_2|Y_\rho}$. We extend it to a section upgrade $(E_{\sigma_2}, e_{\sigma_2})$ of E_{σ_2}. (This extension might require us to shrink Y to a smaller affine neighbourhood of y.) After this step, we have produced section upgrades $\widehat{E_{\sigma_1}}, \widehat{E_{\sigma_2}}$ of $E_{\sigma_1}, E_{\sigma_2}$ that glue along Y_ρ to give

$$\widehat{H_2} \in N \otimes \underline{T}(Y_{\sigma_1} \cup Y_{\sigma_2}).$$

Writing $Y_{k-1} = Y_{\sigma_1} \cup \cdots \cup Y_{\sigma_{k-1}}$, assume by induction that we have constructed:

$$\widehat{H_{k-1}} \in N \otimes \underline{T}(Y_{k-1})$$

such that parts (1) and (2) hold for all cones σ_i and for all pairs of cones σ_i, σ_j with $i, j \leqslant k-1$.

We construct $\widehat{H_k}$ as follows. In the following, we write $\widehat{H_{k-1}} = (H_{k-1}, u_{k-1})$ where $H_{k-1} \in N \otimes \underline{\mathrm{Pic}}(Y_{k-1})$ and $\widehat{H_{k-1}}$ is a section upgrade. Let $i_1 < \cdots < i_r \leqslant k-1$ be the indices such that $\sigma_{i_m} \cap \sigma_k = \rho_m \in T^{[1]}$. We have that

$$Y_{k-1} \cap Y_{\sigma_k} = Y_{\rho_1} \cup \cdots \cup Y_{\rho_r}.$$

The first step is to construct $H_k \in N \otimes \underline{\mathrm{Pic}}(Y_k)$ by gluing the H_{k-1} to E_{σ_k} by using the isomorphisms

$$\psi_{\sigma_{i_m}\sigma_k}(f_{\rho_m})\colon H_{k-1|Y_{\rho_m}} \xrightarrow{\cong} E_{\sigma_k|Y_{\rho_m}}.$$

Claim. These isomorphisms agree on the joints Y_ω where two of the Y_ρ meet.

To prove the claim, let us choose such a ω where two of the ρ meet and see what is going on on Y_ω. Since

$$B_{k-1} = \rho_1 \cup \cdots \cup \rho_r,$$

shellability implies that exactly two ρ meet at ω, say ρ_a and ρ_b. We need to show that

$$\psi_{\sigma_{i_a}\sigma_k}(f_{\rho_a})_{|Y_\omega}\colon H_{k-1|Y_\omega} \to E_{\sigma_k|Y_\omega} \quad \text{equals} \quad \psi_{\sigma_{i_b}\sigma_k}(f_{\rho_b})_{|Y_\omega}\colon H_{k-1|Y_\omega} \to E_{\sigma_k|Y_\omega}.$$

Shellability implies that there are two increasing sequences of indices:

$$a_1 < \cdots < a_s \leqslant k-1 \quad \text{and} \quad b_1 < \cdots < b_t \leqslant k-1 \quad \text{where } a_1 = b_1,\ a_s = i_a,\ b_t = i_b$$

such that

$$\sigma_{a_1}, \sigma_{a_2}, \ldots, \sigma_{a_s}, \sigma_k, \sigma_{b_t}, \ldots, \sigma_{b_2}$$

is a cyclic enumeration of all the maximal cones of Σ incident at ω. In what follows, we denote by f_{a_i} the slab section on $\sigma_{a_i} \cap \sigma_{a_{i+1}}$ and by f_{b_i} the slab section on $\sigma_{b_i} \cap \sigma_{b_{i+1}}$ The claim follows from the identity:

$$\psi_{\sigma_{a_s}\sigma_k}(f_{a_s}) \cdots \psi_{\sigma_{a_1}\sigma_{a_2}}(f_{a_1}) = \psi_{\sigma_{b_t}\sigma_k}(f_{b_t}) \cdots \psi_{\sigma_{b_1}\sigma_{b_2}}(f_{b_1}) \tag{4.11}$$

on Y_ω. Evaluating at $m \in M$ and using the description in Notation 4.29, identity (4.11) is the joint condition:

$$f_{a_s}^{d_{\sigma_{a_s}\sigma_k}(m)} \cdots f_{a_1}^{d_{\sigma_{a_1}\sigma_{a_2}}(m)} = f_{b_t}^{d_{\sigma_{b_t}\sigma_k}(m)} \cdots f_{b_1}^{d_{\sigma_{b_1}\sigma_{b_2}}(m)}.$$

This proves the claim, and the claim readily implies the result.

Proof of Theorem 4.25. We have shown injectivity in Section 4.7. It suffices to show surjectivity of φ at a stalk $y \in Y$. Given $s = (f_\rho)_\rho \in \Gamma(Y, \mathcal{LS}_Y^\times)$, let $\widehat{H} \in N \otimes \underline{T}(Y)$ denote the object given

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

by Proposition 4.31. By the construction of $\widehat{H}$, for every cone $\tau \in \Sigma_y$, the restriction of $\widehat{H}\colon M \to \underline{T}(Y)$ gives a group homomorphism

$$h_\tau\colon \langle\tau\rangle \to \mathcal{O}(U_\tau)^\times.$$

Moreover, the regularity condition from Definition 4.15 is satisfied by the construction of h because $m \in \langle\tau\rangle$ maps to a generator of $\mathcal{O}_{\overline{U}_\tau}(-D_\tau(m))$. The collection of maps $h = (h_\tau)_{\tau\in\Sigma}$ is compatible in the sense that $\rho_\tau^{\tau'} \circ h_{\tau'} = h_{\tau|\langle\tau'\rangle}$ whenever $\tau' \le \tau$ and so, by (4.4), the collection h of h_τ constitutes a section of $\mathcal{H}om_c(\mathcal{R}, \mathcal{O}_Y^\times)$ and thus it gives a class in $\Gamma(Y, \mathcal{E}xt_c^1(\mathcal{M}, \mathcal{O}_Y^\times))$. By the the key property of $\widehat{H}$ stated in Proposition 4.31(2), we have $\varphi(h) = s$. □

4.9 Local sections of $\mathcal{E}xt_c^1$ give log structures locally

Proposition 4.32. *Let Y be an affine viable ggtc space as in Setup 4.26. Consider a regular extension of sheaves of groups*

$$0 \to \mathcal{O}_Y^\times \to \mathfrak{M} \to \mathcal{M} \to 0.$$

Then the fiber product $\mathfrak{P} := \mathfrak{M} \times_{\mathcal{M}} \mathcal{P}$ comes equipped with a monoid homomorphism α to $(\mathcal{O}_Y, \times)$ that yields a compatible log structure on $Y/k^\dagger$.

Proof. Consider the relation sheaf sequence (4.3) (this is the same as the first line in (4.12)). By definition $\mathfrak{M}$ is in $\mathcal{E}xt_c^1(\mathcal{M}, \mathcal{O}_Y^\times)$ if $\mathfrak{M} = \partial(h)$ for a regular $h \in \mathcal{H}om_c(\mathcal{R}, \mathcal{O}_Y^\times)$, where ∂ denotes the boundary map

$$\partial\colon \mathcal{H}om(\mathcal{R}, \mathcal{O}_Y^\times) \to \mathcal{E}xt^1(\mathcal{M}, \mathcal{O}_Y^\times).$$

In concrete terms, given $h\colon \mathcal{R} \to \mathcal{O}_Y^\times$, the extension ∂h is constructed by push out:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{R} & \longrightarrow & \underline{M} & \longrightarrow & \mathcal{M} & \longrightarrow & 0 \\ & & {\scriptstyle h}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{O}_Y^\times & \longrightarrow & \mathfrak{M} & \longrightarrow & \mathcal{M} & \longrightarrow & 0 \end{array} \tag{4.12}$$

with exact rows and cocartesian squares, where

$$\mathfrak{M} = \mathcal{O}_Y^\times \oplus_{\mathcal{R}} \underline{M} = (\mathcal{O}_Y^\times \oplus \underline{M})/\{(h(m), -m) \mid m \in \mathcal{R}\}. \tag{4.13}$$

Our $\mathfrak{M}$ arises in this way from a regular $h \in \mathcal{H}om_c(\mathcal{R}, \mathcal{O}_Y^\times)$. From the concrete description of Lemma 4.11, $h = (h_\tau)_{\tau\in\Sigma}$ where $h_\tau\colon \langle\tau\rangle \to \mathcal{O}(U_\tau)^\times$ is a group homomorphism and, for every $\tau' \leqslant \tau$, $\rho_\tau^{\tau'} \circ h_{\tau'} = h_\tau$. Definition 4.15 states that h is regular if, denoting by

$$\iota_\tau\colon \mathcal{O}_Y^\times(U_\tau) \hookrightarrow \mathcal{K}(\overline{U}_\tau)$$

the natural inclusion, we have that for all $\tau \in \Sigma$ and all $m \in \langle\tau\rangle$

$$\widetilde{h}_\tau(m) = \iota_\tau \circ h_\tau(m) \quad \text{is a generator of } \mathcal{O}_{\overline{U}_\tau}(-D_\tau(m)).$$

We are now ready to define the log structure $\alpha\colon \mathfrak{P} := \mathfrak{M} \times_{\mathcal{M}} \mathcal{P} \to \mathcal{O}_Y$. For all $\tau \in \Sigma$, we describe

$$\alpha_\tau\colon \mathfrak{P}(U_\tau) = \mathfrak{M}(U_\tau) \times_{M_\tau} P_\tau \to \mathcal{O}(U_\tau).$$

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

For all $\tau \in \Sigma$, denote by $\pi_\tau \colon M \to M_\tau = M/\langle\tau\rangle$ and by $\psi_\tau \colon P_\tau \to M_\tau = P_\tau/\mathbf{1}_\tau$ the natural projections. An element of $\mathfrak{P}(U_\tau)$ is a pair $((\xi, m), p)$ where $\xi \in \mathcal{O}(U_\tau)^\times$, $m \in M$, $p \in P_\tau$, and $\pi_\tau(m) = \psi_\tau(p)$.

If $\psi_\tau(p) = 0$, that is, $p \in \mathbf{1}_\tau + P_\tau$, then we set $\alpha_\tau((\xi, m), p) = 0$.

Otherwise, there is a smallest cone $\tau \leqslant \tau'$ such that $m \in \tau'$ and $\pi_\tau(m) = \psi_\tau(p)$. Note that, in this case, $\overline{U}_{\tau'} \cap U_\tau \subset U_\tau$ is a union of irreducible components. In this case, we set

$$\alpha_\tau((\xi, m), p) = \begin{cases} \widetilde{h}_{\tau'}(m)\xi_{|\overline{U_{\tau'}} \cap U_\tau} & \text{on } \overline{U}_{\tau'} \cap U_\tau; \\ 0 & \text{on } U_\tau \setminus \overline{U}_{\tau'}. \end{cases}$$

The key observation here is that, when $m \in \tau'$, the divisor $D_{\tau'}(m) \in \operatorname{Div}^+ \overline{U}_{\tau'}$ is effective; and hence the rational function $\widetilde{h}_{\tau'}(m)\xi_{|\overline{U}_{\tau'} \cap U_\tau}$ is in fact regular and it vanishes on the boundary, and hence it can be extended by zero to a regular function on all of $\overline{U}_\tau$.

It is straightforward to check that α is well-defined; that it is a log structure; and that the global section $((1, 0), \mathbf{1}_\mathcal{P})$ defines a morphism of log structures $Y^\dagger \to (\operatorname{Spec} k)^\dagger$. □

4.10 Proof of Theorem 4.3

We want to generalize the constructions in the previous section from the affine situation to the general situation. We begin by rephrasing Proposition 3.11 in [GS06] as the following lemma.

Lemma 4.33. *Let X be a reduced Deligne–Mumford stack with two log structures $\alpha, \alpha' \colon \mathfrak{P} \to \mathcal{O}_X$ with identical monoid sheaf. We denote $\beta = \alpha'_{|(\alpha')^{-1}(\mathcal{O}_X^\times)}$, so β is an isomorphism onto $\mathcal{O}_X^\times$. If $\alpha \circ \beta^{-1} \colon \mathcal{O}_X^\times \to \mathcal{O}_X^\times$ is the identity, then $\alpha = \alpha'$.*

Proof. Consider α, α' at the generic point η of a component of X, $\alpha_\eta, \alpha'_\eta \colon \mathfrak{P}_\eta \to \mathcal{O}_{X,\eta}$. Since $\mathcal{O}_{X,\eta}$ is a field, given $m \in \mathfrak{P}_\eta$ either $\alpha_\eta(m) = 0$ or $\alpha_\eta(m)$ is invertible, similarly for $\alpha'_\eta(m)$. It follows from the assumptions that $\alpha_\eta = \alpha'_\eta$. Since X is reduced, for any open set U, $\mathcal{O}_X(U)$ injects into $\bigoplus_{\eta \in U} \mathcal{O}_{X,\eta}$ where η runs over the generic points of the components of X. The homomorphism α is thus determined by the homomorphisms α_η and thus $\alpha = \alpha'$. □

Proof of Theorem 4.3. Let Y be a viable ggtc space.

Step 1. We construct a morphism of Zariski sheaves of sets on Y:

$$\psi \colon \mathrm{LS}_{k^\dagger} \to \mathcal{E}xt^1_c(\mathcal{M}, \mathcal{O}_Y^\times). \tag{4.14}$$

We construct a set-theoretic function $\mathrm{LS}_{k^\dagger}(Y) \to \Gamma(Y, \mathcal{E}xt^1_c(\mathcal{M}, \mathcal{O}_Y^\times))$ that assembles to a sheaf morphism in the input Y. Let $\alpha \colon \mathfrak{P} \to \mathcal{O}_Y$ be a compatible log structure. Using $\psi \colon \mathfrak{P} \to \mathcal{P}$ from part (1) of Definition 4.2, we can write $\mathfrak{P} = \mathfrak{M} \times_\mathcal{M} \mathcal{P}$ where $\mathfrak{M} = \mathfrak{P}/\mathbf{1}_\mathfrak{P}$. All relevant maps being bijective, we identify the projection of $\alpha^{-1}(\mathcal{O}_Y^\times)$ in $\mathfrak{M}$ with $\mathcal{O}_Y^\times$ and obtain an exact sequence as shown as the bottom row in (4.12). We need to show that its extension class lies in $\mathcal{E}xt^1_c(\mathcal{M}, \mathcal{O}_Y^\times)$. This is a local question, so we may assume that $[\mathfrak{M}] = \partial(h)$ for some $h \in \Gamma(Y, \mathcal{H}om(\mathcal{R}, \mathcal{O}_Y^\times))$ so we have a diagram like (4.12). We may also assume that Y is affine as in Setup 4.26. We want to show that $h = (h_\tau)_{\tau \in \Sigma}$ is regular. We need to 'extract' $h_\tau(m)$ from the datum of the log structure. Fix $m \in M$ and let $\tau \in \Sigma$ be the smallest cone that contains it. There is a unique $\widehat{m} \in P_\tau \setminus (\mathbf{1}_\tau + P_\tau)$ that maps to m under the projection $P_\tau \to M_\tau = P_\tau/\mathbf{1}_\tau$, and it is tautologically the case that, interpreting the pair $((1, m), \widehat{m})$ as an element of $\mathfrak{P}(U_\tau) = \mathfrak{M}(U_\tau) \times_{M_\tau} P_\tau$,

$$h_\tau(m) = \alpha(\widehat{m}) \in \mathcal{O}(U_\tau).$$

By part (b) of Definition 4.2, $\operatorname{div}(h_\tau(m)) = D_\tau(m)$. This holds for all $m \in M$, hence h is regular.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

Step 2. The morphism ψ is injective.

We need to show that if two compatible log structures give rise to the same extensions, then they are isomorphic as compatible log structures. If the extensions are the same then the two log structures must have isomorphic monoid sheaves compatibly with the respective embeddings of $\mathcal{O}_Y^\times$ and the result follows from Lemma 4.33.

Step 3. The morphism ψ is surjective.

Start from a class in $\Gamma(Y, \mathcal{E}xt^1_c(\mathcal{M}, \mathcal{O}_Y^\times)) \subset \Gamma(Y, \mathcal{E}xt^1(\mathcal{M}, \mathcal{O}_Y^\times))$. Since $\mathcal{M}$ is supported on the union of slabs, which is of codimension one, we have that $\mathcal{H}om(\mathcal{M}, \mathcal{O}_Y^\times) = 0$. The local-to-global Ext spectral sequence then implies that $\Gamma(Y, \mathcal{E}xt^1(\mathcal{M}, \mathcal{O}_Y^\times)) = \mathrm{Ext}^1(\mathcal{M}, \mathcal{O}_Y^\times)$, so we obtain a sheaf $\mathfrak{M}$ that sits in the middle of an exact sequence as in Proposition 4.32. We then produce the monoid sheaf $\mathfrak{P} := \mathfrak{M} \times_{\mathcal{M}} \mathcal{P}$ which locally comes with a homomorphism to $(\mathcal{O}_Y, \cdot)$ by Proposition 4.32. These local maps glue to a global one by the uniqueness of these maps according to Lemma 4.33.

Step 4. End of proof of Theorem 4.3.

We define the composition

$$r = \varphi \circ \psi \colon \mathrm{LS}_{k^\dagger} \to \mathcal{LS}_Y^\times \tag{4.15}$$

of ψ from (4.14) with the bijection φ from Theorem 4.25. □

5. Examples

5.1 Two components

In this section, Y consists of two smooth components Y_1, Y_2 meeting along a smooth irreducible divisor $D \subset Y_i$ $(i = 1, 2)$ all defined over k. First, we describe a sheaf of monoids $\mathcal{P}$ on Y that allows us to equip Y with the structure of a ggtc space; then, we study the log structures $\mathfrak{P}$ on Y that have ghost sheaf $\mathcal{P}$, and finally we see how to keep track of a morphism to the standard log point $\operatorname{Spec} k^\dagger$. This is the most elementary example of the theory but it is nevertheless quite rich and it is well worth it to invest the time necessary to understand it completely.

The simplest source of examples of log structures on Y are embeddings $i \colon Y \hookrightarrow X$ into a smooth scheme X. More generally, we can look at an embedding where locally analytically (or étale locally) $Y = (t = 0) \subset X$ where $X = (xy + t^r = 0) \subset \mathbb{A}^3_{x,y,t} \times \mathbb{A}^m$ for $m = \dim D$.

Given such an embedding one can form the *divisorial log structure* $\mathfrak{P}_{X,Y} = \mathcal{O}_X \cap \mathcal{O}^\times_{X\setminus Y}$ on X, and the log structure $\mathfrak{P}_Y = i^\star \mathfrak{P}_{X,Y}$ on Y. In the following we make log structures on Y of this étale local type without reference to an embedding. Fix an integer $r > 0$, consider the sublattice

$$M = \{(n_1, n_2) \mid n_1 \equiv n_2 (\mathrm{mod}\ r)\} \subset \mathbb{Z}^2,$$

illustrated in Figure 6, and let $P = \langle e_1, e_2 \rangle_+ \cap M$ be the monoid of lattice points in the positive quadrant. This monoid is generated by $(r, 0)$, $(0, r)$ and $(1, 1)$ with the obvious relation. We use P to define a sheaf of monoids on Y as follows. For a connected Zariski open subset $U \subset Y$, we set

$$\mathcal{P}(U) = \begin{cases} P & \text{if } U \cap D \neq \emptyset, \\ P/\langle re_2 \rangle \cong \mathbb{N} & \text{if } U \subset Y \setminus Y_2, \\ P/\langle re_1 \rangle \cong \mathbb{N} & \text{if } U \subset Y \setminus Y_1. \end{cases}$$

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

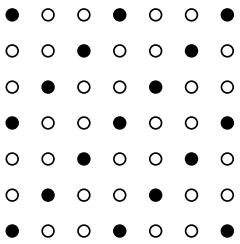


Figure 6. The sublattice $M \subset \mathbb{Z}^2$ for $r = 3$.

Now Y is a ggtc space with ghost sheaf $\mathcal{P}$ in a natural way: if locally at the generic point $\eta \in D$ we have $Y = (xy = 0)$ where say $Y_1 = (y = 0)$ and $Y_2 = (x = 0)$, then we have $k[P] = k[x, y, t]/(xy + t^r)$. In what follows, it is crucial to understand that $x \in k[P]$ vanishes with multiplicity r along Y_2.

Proposition 5.1. *Let $r > 0$ and $Y = Y_1 + Y_2$ a toroidal crossing space as just described. To give a log structure $\mathfrak{P}$ on Y is equivalent to giving line bundles $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}$ on Y, homomorphisms $\alpha_1 \colon \mathcal{L}_1 \to \mathcal{O}_Y$, $\alpha_2 \colon \mathcal{L}_2 \to \mathcal{O}_Y$ such that*

$$\alpha_{1|Y_2} \colon \mathcal{L}_{1|Y_2} \xrightarrow{\cong} \mathcal{O}_{Y_2}(-D) \subset \mathcal{O}_{Y_2} \quad \text{and} \quad \alpha_{1|Y_1} = 0, \tag{5.1}$$

$$\alpha_{2|Y_1} \colon \mathcal{L}_{2|Y_1} \xrightarrow{\cong} \mathcal{O}_{Y_1}(-D) \subset \mathcal{O}_{Y_1} \quad \text{and} \quad \alpha_{2|Y_2} = 0; \tag{5.2}$$

and an identification $s \colon \mathcal{L}^{\otimes r} \xrightarrow{\cong} \mathcal{L}_1 \otimes \mathcal{L}_2$.

Sketch of proof. We construct a log structure from the data spelled out in the proposition. First of all, for $p = n_1(r, 0) + q(1, 1) \in P$, that is, $q \geqslant 0, n_1 \in \mathbb{Z}$ satisfying $n_1 + q/r \geqslant 0$, write

$$\mathcal{L}^p = \mathcal{L}_1^{\otimes n_1} \otimes \mathcal{L}^{\otimes q}.$$

Note how this choice singles out a 2-homomorphism from the monoid P, viewed as a category, to the 2-group $\underline{\mathrm{Pic}}\, Y$. The log structure is the sheaf of monoids:

$$\mathfrak{P}(U) = \begin{cases} \coprod_{p \in P} \mathcal{L}^p(U)^\times & \text{if } U \cap D \neq \emptyset, \\ \coprod_{q \in \mathbb{N}} \mathcal{L}^{\otimes q}(U)^\times & \text{if } U \subset Y \setminus Y_2, \\ \coprod_{q \in \mathbb{N}} \mathcal{L}^{\otimes q}(U)^\times & \text{if } U \subset Y \setminus Y_1, \end{cases}$$

together with the obvious homomorphism $\alpha \colon \mathfrak{P} \to \mathcal{O}_Y$. □

We want to understand isomorphism classes of log structures on Y over $\operatorname{Spec} k^\dagger$, where we map $\mathbb{N} \to P$ by taking 1 to $\mathbf{1} = (1, 1)$: it is surprising to see that this set has a simpler description as follows.

Proposition 5.2. *Let $Y = Y_1 + Y_2$ be a toroidal crossing space as above. The set of isomorphism classes of log structures on Y over $\operatorname{Spec} k^\dagger$ is the set of nowhere-vanishing sections of the sheaf*

$$\mathcal{L}_D = (N_{Y_1} D) \otimes (N_{Y_2} D).$$

Sketch of proof. Suppose given a log structure $\mathfrak{P}$ on Y, making it into a log scheme $Y^\dagger$. In the notation of the previous proposition, a morphism $Y^\dagger \to \operatorname{Spec} k^\dagger$ is precisely the datum of a

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

nowhere-vanishing global section $\sigma_0 \in H^0(Y, \mathcal{L})$. This also gives a section $\sigma = \sigma_0^r \in H^0(Y, \mathcal{L}^{\otimes r})$ that we pass through the isomorphism s to trivialize the line bundle $\mathcal{L}_1 \otimes \mathcal{L}_2$. Thus, for example, we have

$$\mathcal{L}_{1|Y_2} = \mathcal{O}_{Y_2}(-D),$$
$$\mathcal{L}_{1|Y_1} \overset{s(\sigma)}{=} \mathcal{L}_{2|Y_1}^{\star} = \mathcal{O}_{Y_1}(D).$$

Thus, $\mathcal{L}_1$ is glued together from an isomorphism

$$f\colon \mathcal{O}_{Y_2}(-D)_{|D} = N_{Y_2}^{\star} D \xrightarrow{\cong} N_{Y_1} D = \mathcal{O}_{Y_1}(D)_{|D}$$

or, equivalently, a nowhere-vanishing section of

$$\mathcal{LS}_Y = (N_{Y_1} D) \otimes (N_{Y_2} D).$$

□

In particular. in order for the log structure over $\operatorname{Spec} k^{\dagger}$ to even exist, one must have that $\mathcal{LS}_Y$ is a trivial line bundle.

Remark 5.3.

(1) Perhaps surprisingly, the sheaf $\mathcal{LS}_Y = \mathcal{L}_D$ in the example, and thus the fact of the existence of a compatible log structure over the standard log point, does not depend on r.

(2) It is a matter of convention whether $\mathcal{LS}_Y = N_{Y_1} D \otimes N_{Y_2} D$ or its dual. Our convention is motivated by the fact that we want $\mathcal{LS}_Y$ to have lots of sections when $N_{Y_1} D \otimes N_{Y_2} D$ has lots of sections. These sections of $\mathcal{LS}_Y$ are useful even when they vanish somewhere and we think of them as giving 'singular' log structures, as justified in Definition 5.4. Furthermore, sections with vanishing loci in $\mathcal{LS}_Y$ may arise from restrictions of divisorial log structures of a pair (X, Y) to the divisor Y. The convention also fits with the existence of an isomorphism $\mathcal{LS}_Y \cong \mathcal{T}_Y^1$ in the normal crossing case [FFR21].

Definition 5.4. We adopt the following language from the Gross–Siebert program initiated in [GS06].

(i) We say that a section of $\mathcal{LS}_Y$ whose vanishing locus is a divisor $Z \subset D$ is a log structure on Y *singular along* Z. A priori, the log structure is only defined on $Y \setminus Z$. A log structure on all of Y can be produced by taking the direct image of the log structure on $Y \setminus Z$. This direct image log structure fails to be coherent along Z. Failure of coherence prevents the log scheme from satisfying the formal log smoothness lifting criterion as we explain in the simplest example below, so calling the log structure *singular* along Z is justified. With regards to part (2) of the previous remark we view a log structure from a section of $\mathcal{LS}_Y$ on an open set that extends with *poles* in its complement as pathological.

(ii) In the above example, the sheaf $\mathcal{LS}_Y$ was a line bundle on D, identical with the unique slab bundle. More generally, as in Definition 3.9, $\mathcal{LS}_Y$ is defined as a subsheaf of the direct sum of slab bundles cut out by the joint condition. We call a section f_ρ of a slab bundle $\mathcal{L}_\rho$ a *slab section*, typically studied in a local trivialization of the bundle where we may also call it a *slab function.*

(iii) Each slab bundle comes with a positive integer like the integer r above which appears in its frame from the relation $xy = z^r$. We refer to this integer as the *kink* of the slab.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

5.2 The easiest singular log structure

It pays off to study the easiest singular log structure that there is and understand it completely.

Consider $\mathbb{A}^3$ with coordinates x, y, u; we take X to be the surface

$$X = (xy = 0) \subset \mathbb{A}^3.$$

Note that X consists of two components

$$X_1 = \mathbb{A}^2_{x,u} = (y = 0), \quad X_2 = \mathbb{A}^2_{y,u} = (x = 0).$$

Let us write $S = X_1 \cap X_2 = \mathbb{A}^1_u$. The slab bundle for the slab S is trivial. We endow X with the log structure over $k^\dagger$ given by the slab function

$$f_\rho(u) = u$$

on S. Denoting the origin by $\mathbf{0}$, we set $Z = \{\mathbf{0}\} \subset S$, write $U = X \setminus Z$, $S^\star = S \setminus Z$ and denote by $j\colon U \to X$ the natural inclusion. Since f_ρ vanishes outside U, the log structure only exists on U and we denote its sheaf of monoids by $\mathfrak{P}_U$. To obtain a log structure on X we use $\mathfrak{M}_X = j_\star \mathfrak{P}_U$ and arrive at a log morphism $f\colon X^\dagger \to \operatorname{Spec} k^\dagger$ which is not formally log smooth at Z. This can be seen as follows. One first verifies that the stalk $\alpha_{\mathbf{0}}\colon \mathfrak{M}_{X,\mathbf{0}} \to \mathcal{O}_{X,\mathbf{0}}$ agrees with the stalk of log structure obtained as pull-back from $\operatorname{Spec} k^\dagger$. That is, the log morphism f is *strict* at $\mathbf{0}$. Since the underlying morphism is not formally smooth, there is a test diagram with a square zero extension as in the definition of formal smoothness that has no diagonal map. Enrich this diagram with pull-back log structures from $\operatorname{Spec} k^\dagger$ and it also will not have a diagonal map, so formal log smoothness fails at $\mathbf{0}$. On the other hand, formal log smoothness works over U which we leave as an exercise.

As explained in Section 5.1, the log structure on U is determined by line bundles

$$\mathcal{L}_{1\,U}, \quad \mathcal{L}_{2\,U}, \quad \mathcal{L}_U$$

on U, and homomorphisms $\alpha_i\colon \mathcal{L}_{i\,U} \to \mathcal{O}$ satisfying the conditions in Proposition 5.1, and $s\colon \mathcal{L}_U \xrightarrow{\cong} \mathcal{L}_{1\,U} \otimes \mathcal{L}_{2\,U}$. Following the recipe in the proof of Proposition 5.2, the line bundle $\mathcal{L}_{2U}$, for example, is obtained by assembling the line bundles $\mathcal{O}_{U_1}$ on U_1 and $\mathcal{O}_{U_2}$ on U_2 via the isomorphism

$$\text{multiplication by } u : \mathcal{O}_{U_1|S^\star} = \mathcal{O}_{S^\star} \xrightarrow{\cong} \mathcal{O}_{S^\star} = \mathcal{O}_{U_2|S^\star}.$$

It follows from this that $\mathcal{L}_2 = j_\star \mathcal{L}_{2\,U}$ is the reflexive sheaf on X associated to the $k[x,y,u]/_{(xy)}$-module

$$M_1 = \Gamma\left(\mathcal{L}_{1\,U}, U\right) = \langle e_1 = (1,u),\ e_2 = (0,y) \mid ye_1 = ue_2, xe_2 \rangle.$$

Note for example that $(x,0) = xe_1$, $(1, u+y) = e_1 + e_2$, etc. Although $\mathcal{L}_2$ is not a line bundle on X, the restrictions $\mathcal{L}_{2|X_1}$ and $\mathcal{L}_{2|X_2}$ are line bundles.

The homomorphism $\alpha\colon M \to k[x,y]/_{(xy)}$ is defined as

$$\alpha(1,u) = x, \quad \alpha(0,y) = 0,$$

that is,

$$\alpha\colon \mathcal{L}_{2|X_1} \xrightarrow{\cong} \mathcal{O}_{X_1}(-S), \quad \alpha\colon \mathcal{L}_{2|X_2} = 0$$

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

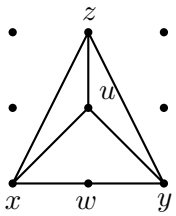


Figure 7. The moment polyhedral complex of the surface X.

Naturally $\mathcal{L}_1 = \mathcal{L}_2^\vee$ and $\mathcal{L} = \mathcal{O}_X = \mathcal{L}_1[\otimes]\mathcal{L}_2$ where $[\otimes]$ refers to the reflexive tensor product.[23]

Take $Y_1 = X_1$, let $Y_2 \to X_2$ be the blow up of X_2 at the origin, assemble Y_1 and Y_2 to obtain Y and denote by $f\colon Y \to X$ the proper birational map; the exceptional set is $E \cong \mathbb{P}^1 \subset Y$ and $f_{|Y\setminus E}\colon Y \setminus E \to X \setminus Z$ is an isomorphism. Then

$$\mathcal{LS}_Y = f^\star\left(\mathcal{LS}_X(-Z)\right).$$

There is a unique log structure on Y over $k^\dagger$ that agrees on

$$Y \setminus E \overset{f}{=} X \setminus Z$$

with the given one and is log smooth over $k^\dagger$ because the section $f_\rho = u \in \Gamma(S^\star, \mathcal{LS}_Y)$ extends as a nowhere-vanishing section of $\mathcal{LS}_Y$ on S.

5.3 A reducible quartic del Pezzo surface

5.3.1 *Description of the surface and log structure.* We start from X the toroidal crossing surface given as the union of three toric surfaces as in the moment polygonal complex pictured in Figure 7. From the picture we see that X is naturally embedded in $\mathbb{P}^4$ and given by equations:

$$X = \begin{cases} xy - w^2 = 0 \\ zw = 0 \end{cases}$$

in homogeneous coordinates $x : y : z : u : w$. The three components of X are

$$X_1 = (z = xy - w^2 = 0); \quad X_2 = (w = x = 0); \quad \text{and} \quad X_3 = (w = y = 0).$$

The obvious and, essentially, unique polarization (see Lemma 2.3) has kink $\kappa = 1$ along the x and y axes and $\kappa = 2$ along the z-axis.[24] The datum of a log structure on X over $k^\dagger$ consists of slab sections:

$$\begin{aligned} f_{\rho_x} &= a_0 + a_1 x + a_2 x^2 \\ f_{\rho_y} &= b_0 + b_1 y + b_2 y^2 \\ f_{\rho_z} &= c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 \end{aligned}$$

[23] It is an interesting (and surprisingly nontrivial) exercise to verify that the reflexive sheaf of relative log differentials $\mathscr{W}^1_{X^\dagger/k^\dagger} := j_\star \Omega^1_{U^\dagger/k^\dagger}$ is isomorphic to $\mathcal{L}_1 \oplus \mathcal{L}_2$. The associated $k[x,y]/_{(x,y)}$-module is

$$\langle D_x, D_y, D_x^u, D_y^u \mid xD_y + yD_x = 0, xD_x^u = uD_x, yD_y^u = uD_y \rangle$$

where, morally, $D_x = dx$, $D_y = dy$, $D_x^u = u(dx/x)$, $D_y^u = u(dy/y)$. Note for example that $D_x^u + D_y^u = du$.

[24] In fact, this is the simplest real life example where we see a necessity for allowing a kink > 1.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

(in the affine patch $u=1$) and the joint condition at the origin states that

$$a_0 = b_0 \quad \text{and} \quad c_0 = a_0^2. \tag{5.3}$$

From now on we fix general slab sections and denote by $X^\dagger/k^\dagger$ the surface X equipped with the log structure and structure morphism $X^\dagger \to \operatorname{Spec} k^\dagger$ given by these functions.

The morphism is not log smooth: it is singular along the union of points $Z \subset X$ where the slab sections vanish. For generic coefficients, we find 8 such points. We next show how these log structures arise naturally when we deform the surface X.

5.3.2 *Log deformations of $X^\dagger/k^\dagger$.* Consider the family of surfaces over $\mathbb{A}^1$ with coordinate t given by the equations

$$\mathfrak{X} = \begin{cases} xy - w^2 = ts_1wu + t^2(c_2u^2 + c_3uz + c_4z^2) \\ zw = t(s_0u^2 + a_1xu + a_2x^2 + b_1yu + b_2y^2) \end{cases} \tag{5.4}$$

in $\mathbb{P}^4 \times \mathbb{A}^1$. Note the following.

(1) Together with the projection $\pi\colon \mathfrak{X} \to \mathbb{A}^1$ the equations represent a flat deformation, in fact, a smoothing, of the surface $X = \mathfrak{X}_0$. The general fiber is a del Pezzo surface of degree 4: a smooth intersection of two quadrics in $\mathbb{P}^4$.

(2) Denote by $i\colon X \hookrightarrow \mathfrak{X}$ the natural inclusion. The total space $\mathfrak{X}$ is endowed with a natural (singular) divisorial log structure $\mathfrak{M}_{\mathfrak{X},X}$; let us denote by $\mathfrak{X}^\dagger$ the corresponding log scheme. Similarly, let $(\mathbb{A}^1)^\dagger$ be $\mathbb{A}^1$ together with the divisorial log structure from $t=0$. There is an obvious morphism of log schemes $\pi^\dagger\colon \mathfrak{X}^\dagger \to (\mathbb{A}^1)^\dagger$ which when pulled back to X gives a log structure $X^\dagger/k^\dagger$. The conflict of notation will be resolved momentarily when we see that this log structure is the same one that we defined above in Section 5.3.1. The slab sections for this log structure on X can be easily read from the equations; they are

$$\begin{aligned} f_{\rho_x} &= s_0 + a_1x + a_2x^2, \\ f_{\rho_y} &= s_0 + b_1y + b_2y^2, \\ f_{\rho_z} &= s_0^2 + s_0s_1z + c_2z^2 + c_3z^3 + c_4z^4. \end{aligned}$$

For example, we compute f_{ρ_z} in the affine open set $u=1$ by localizing at $z=0$, using the second equation to solve for w,

$$w = \frac{ts_0}{z} \mod t(x,y)$$

and plugging into the first equation which gives

$$(xz)(yz) = t^2(s_0^2 + s_0s_1z + c_2z^2 + c_3z^3 + c_4z^4) \mod t^2(x,y).$$

From this expression, we also read off from the exponent of t that gives the kink $k=2$.[25]

5.3.3 *Log crepant log resolutions.* The following facts are elementary and easy to verify. We construct a surface Y and proper birational morphism $f\colon Y \to X$ as follows. First let $Y_2 \to X_2$ be the blow up of the two points $(f_{\rho_y}=0)$ on the y-axis. Next let $Y_3 \to X_3$ be the blow up of:

[25] Staring at the formula for the slab section f_{ρ_z} together with Equation (5.4) we see the slightly curious fact that 'half' of f_{ρ_z} contributes to the first-order deformation of X, and half to the second-order deformation.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

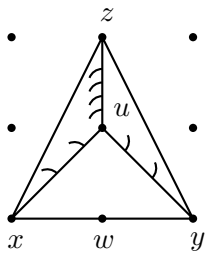


Figure 8. A picture of the resolved surface Y with exceptional curves indicated as little arcs in the triangles that correspond to the components of Y that contain them.

- the two points $(f_{\rho_x}=0)$ on the x-axis; and
- the four points $(f_{\rho_z}=0)$ on the z-axis.

Denote by Y be the surface obtained by re-assembling the three components $Y_1=X_1$, Y_2, Y_3 in the obvious way, sketched in Figure 8; and by $f\colon Y\to X$ the obvious proper birational morphism.

Our construction of $\mathcal{LS}$ gives a way to compare $\mathcal{LS}_Y$ with $\mathcal{LS}_X$. Denote by $S=\operatorname{Sing} X$ and $T=\operatorname{Sing} Y$ the singular sets with the reduced scheme structure; and note that $f_{|T}\colon T\to S$ is an isomorphism. We use $f_{|T}$ to view Z as a subset of T. The resolution was constructed precisely so that the intersection of the union of all exceptional curves with T equals Z. Moreover, we precisely understand how the slab bundles change under this blow-up. A concise way to write this is as follows. The sheaf $\mathcal{LS}_Y$ is supported on T and $\mathcal{LS}_X$ on S and we have

$$\mathcal{LS}_Y=f^\star\mathcal{LS}_X(-Z). \tag{5.5}$$

The slab sections f_{ρ_x}, f_{ρ_y} and f_{ρ_z} give a section s of $\mathcal{LS}_X$ in the affine neighbourhood $u=1$ of the unique joint and this section extends uniquely to all of X without receiving extra zeros outside the affine chart. The vanishing set of s is Z. Therefore, the section s is the image of a section $\widetilde{s}$ under the natural inclusion $\mathcal{LS}_X(-Z)\subset\mathcal{LS}_X$ and, moreover, $\widetilde{s}$ is nowhere-vanishing. Hence, $\widetilde{s}$ gives a compatible log structure on Y that is smooth over $k^\dagger$, in notation $Y^\dagger/k^\dagger$. Furthermore, since $\widetilde{s}$ and s agree over the set

$$Y\setminus f^{-1}(Z)\overset{f}{=}X\setminus Z.$$

and $\widetilde{s}$ maps to s under the inclusion $\mathcal{LS}_X(-Z)\subset\mathcal{LS}_X$, we also obtain that the log structure given by $\widetilde{s}$ on $Y\setminus X$ is that given by s on $X\setminus Z$. The morphism of schemes $f\colon Y\to X$ is *log crepant* in the sense that

$$\Omega^2_{Y^\dagger/k^\dagger}$$

is f-trivial. We say that $Y^\dagger/k^\dagger$ together with $f\colon Y\to X$ is a log crepant log resolution of $X^\dagger$ over $k^\dagger$.

5.4 The 3-fold transverse A_1-singularity

5.4.1 *Description of the space and its log structure.* We take X to be the affine cone over the projective toroidal crossing surface of Section 5.3.1: the toroidal crossing 3-fold union of

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

three toric affine pieces given in $\mathbb{A}^5$ by the equations

$$X = \begin{cases} xy - w^2 = 0 \\ zw = 0 \end{cases}$$

in affine coordinates x, y, z, u, w. Note that u does not appear in the equations: X is the product of a toroidal crossing surface with $\mathbb{A}^1_u$.

The three components of X are

$$X_1 = \big(z = xy - w^2 = 0\big), \quad X_2 = (w = x = 0) \quad \text{and} \quad X_3 = (w = y = 0).$$

The obvious and, essentially, unique polarization has kink $k = 1$ along the coordinate surfaces $\mathbb{A}^2_{x,u}$ and $\mathbb{A}^2_{y,u}$, and $k = 2$ along $\mathbb{A}^2_{z,u}$.

We choose the very special log structure on X over $k^\dagger$ given by the slab sections:

$$\begin{aligned} f_{\rho_x} &= u, \\ f_{\rho_y} &= u, \\ f_{\rho_z} &= u^2 - z^2. \end{aligned} \tag{5.6}$$

The joint condition along the u-axis is satisfied. We denote by $X^\dagger/k^\dagger$ the 3-fold X equipped with the log structure and structure morphism $X^\dagger \to \operatorname{Spec} k^\dagger$ given by these slab sections.

The singular locus of X has three irreducible components as follows:

$$\begin{aligned} S_1 &= \mathbb{A}^2_{z,u} = X_2 \cap X_3, \\ S_2 &= \mathbb{A}^2_{x,u} = X_1 \cap X_3, \\ S_3 &= \mathbb{A}^2_{y,u} = X_1 \cap X_2. \end{aligned}$$

The structure morphism $\pi\colon X^\dagger \to \operatorname{Spec} k^\dagger$ is singular along the locus $Z \subset X$ where the slab sections vanish. This singular locus (see Figure 9) has three irreducible components as follows:

$$Z_1 = (u^2 - z^2 = 0) \subset S_1,$$

$$Z_2 = (u = 0) \subset S_2,$$

$$Z_3 = (u = 0) \subset S_3.$$

5.4.2 *Log deformations of $X^\dagger/k^\dagger$.* Consider the family of 3-folds over $\mathbb{A}^1$ with coordinate t given by the equations

$$\mathfrak{X} = \begin{cases} xy - w^2 = -t^2 \\ zw = tu \end{cases} \tag{5.7}$$

in $\mathbb{A}^5 \times \mathbb{A}^1$, together with the projection $\pi\colon \mathfrak{X} \to \mathbb{A}^1$. The equations describe a smoothing of X; and the restriction to X of the divisorial log structure $\mathfrak{M}_{\mathfrak{X},X}$ is that given in (5.6).

5.4.3 *Log crepant log resolutions.* The following facts are elementary and easy to verify. We construct a 3-fold Y and proper birational morphism $f\colon Y \to X$ as follows. First let $Y_2 \to X_2$ be the blow up of the singular curve $Z_3 \subset X_2$.

Next let $Y_3 \to X_3$ be as follows:

– first, let $f'\colon Y'_3 \to X_3$ be the blow up of the curve $Z_2 \subset X_3$, and

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

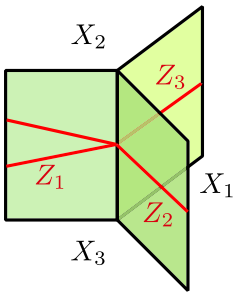


Figure 9. The log singular locus of $X^\dagger$.

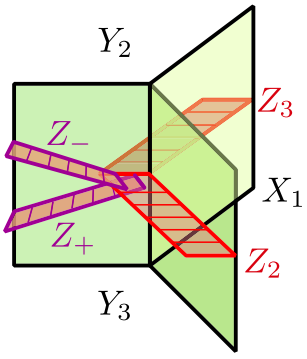


Figure 10. Schematic of the resolution.

– second, let $Y_3 \to Y_3'$ be the blow up of the strict transform $Z_1' \subset Y_3'$ of the curve $Z_1 \subset X_3$ given by $(u^2 - z^2 = 0) \subset \mathbb{A}^2_{z,u}$; note that Z_1' is nonsingular and consists of two components that we call Z_+, Z_- corresponding to the factors $u+z$, $u-z$.

Denote by Y be the 3-fold obtained by re-assembling the three components $Y_1 = X_1$, Y_2, Y_3 in the obvious way (see Figure 10); and by $f\colon Y \to X$ the obvious proper birational morphism.

Our construction of $\mathcal{LS}$ gives a way to compare $\mathcal{LS}_Y$ with $f^\star(\mathcal{LS}_X)$, which we next explain. Denote by $T = \operatorname{Sing} Y$ the singular set with the reduced scheme structure; T consists of three irreducible components

$$T_1 = Y_2 \cap Y_3, \quad T_2 = Y_3 \cap Y_1, \quad T_3 = Y_1 \cap Y_3.$$

We have that $f_{|T_1}\colon T_1 \to S_1$ is the blow up of the origin, $f_{|T_2}\colon T_2 \to S_2$ is an isomorphism, and so is $f_{|T_3}\colon T_3 \to S_3$.

Next we identify the slab line bundles $\mathcal{L}_{Y,1}$, $\mathcal{L}_{Y,2}$, $\mathcal{L}_{Y,3}$ on T_1, T_2, T_3.

We need a notation for the exceptional divisors of the morphisms $Y_i \to X_i$. The first of these morphisms, $f_{|Y_1}\colon Y_1 \to X_1$, is an isomorphism, so there is nothing to be done here. Denote by $E_2 \subset Y_2$ the exceptional divisor of $f_{|Y_2}\colon Y_2 \to X_2$, so E_2 is a $\mathbb{P}^1$-bundle over Z_3. The exceptional divisor of $f_{|Y_3}\colon Y_3 \to X_3$ consists of three components: the strict transform E_3 of the exceptional divisor of $f'\colon Y_3' \to X_3$, and the divisors F_+, F_- that dominate the curves $u+z=0$, $u-z=0$ in S_1.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

We denote by $\Xi = E_2 \cap T_1 = E_3 \cap T_1 \subset T_1$ the exceptional divisor of $T_1 \to S_1$, and we write $Z_+ = F_+ \cap T_1$, $Z_- = F_- \cap T_1$. Note that

$$(f_{|T_1})^\star(Z_1) = F_+ + F_- + 2\Xi.$$

With this notation in hand we are ready to compute the slab bundles. The Stanley–Reisner ring along the joint u is given by the fan at the monomial u in Figure 7. We see that the monomials xz and yz correspond to opposite lattice vectors, so we can use them to compute the slab bundle on T_1. We also see from the figure that the monomial xz defines the boundary divisor ∂Y_3 on Y_3 and the monomial yz defines the boundary divisor ∂Y_2 on Y_2, so we arrive at

$$\begin{aligned}\mathcal{L}_{Y,1} &= \mathcal{O}_{Y_2}(\partial Y_2)_{|T_1} \otimes \mathcal{O}_{Y_3}(\partial Y_3)_{|T_1} = (N_{Y_2}T_1) \otimes (N_{Y_3}T_1) \otimes \mathcal{O}_{T_1}(2Y_1) \\ &= (f^\star N_{X_2}S_1) \otimes (f^\star N_{X_3}S_1)(-F_+ - F_-) \otimes \mathcal{O}_{S_1}(2X_1)(-2\Xi) \\ &= (f^\star \mathcal{L}_{X_1})(-F_+ - F_- - 2\Xi) = f^\star(\mathcal{L}_{X,1}(-Z_1)).\end{aligned}$$

The formulas for $\mathcal{L}_{Y,2}$ and $\mathcal{L}_{Y,3}$ are much easier to understand:

$$\mathcal{L}_{Y,2} = \mathcal{L}_{X,2}(-Z_2), \quad \mathcal{L}_{Y,3} = \mathcal{L}_{X,3}(-Z_3).$$

We summarize these calculations with the formula

$$\mathcal{LS}_Y = f^\star(\mathcal{LS}_X(-Z)). \tag{5.8}$$

Pulling back the section of $\mathcal{LS}_X$ defined by the slab sections (5.6) results in a nowhere-vanishing section of $\mathcal{LS}_Y$. We see from this the following.

(1) There is a unique log structure on Y over $k^\dagger$ that on

$$Y \setminus f^{-1}(Z) \overset{f}{=} X \setminus Z$$

is the given one on $X \setminus Z$;

(2) The resulting log scheme $Y^\dagger/k^\dagger$ is log smooth over $\operatorname{Spec} k^\dagger$.

One can check that the resulting morphism $Y \to X$ is log crepant, so it gives a log crepant log resolution of $X^\dagger$ over $k^\dagger$.

This example shows how the sheaf $\mathcal{LS}_X$ can be used in constructing log smooth resolutions.

Acknowledgements

We are grateful to Simon Felten and Andrea Petracci for many useful discussions on the subject of this paper. We thank Kevin Buzzard, Simon Felten, Martin Olsson, Bernd Siebert, Mattia Talpo, Anna-Maria Raukh and an anonymous referee for comments on earlier versions of this article and helpful suggestions at various stages of the long revision process.

Kevin had a strong influence on shaping our view of the subject; in particular, he explained to us that the words 'canonical', 'natural', distinguished', etc. are meaningless, see [Buz25]. As a result of our conversations with him, we eliminated all occurrences of these words in the text, and instead we made the effort of stating all the properties that our constructions need to satisfy. Kevin also explained to us some key points about coherence theorems in monoidal categories.

The authors were hosted by Mathematisches Forschungsinstitut Oberwolfach as Oberwolfach Research Fellows in 2021 and 2022. Substantial parts of the work on this project were produced during these retreats. Gratitude for hospitality also goes to the Department of Mathematics at

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

Imperial College London, the University of Hamburg and the Department of Mathematics and Physics at the University of Stavanger.

Financial support

A.C. received support from an EPSRC Programme Grant EP/N03189X/1. H.R. received support from the grants DFG RU 1629/4-1 and NFR Fripro Shape2030.

Conflicts of interest

None.

Journal information

Moduli is published as a joint venture of the Foundation Compositio Mathematica and the London Mathematical Society. As not-for-profit organisations, the Foundation and Society reinvest 100% of any surplus generated from their publications back into mathematics through their charitable activities.

References

[ACM+16] D. Abramovich, Q. Chen, S. Marcus, M. Ulirsch and J. Wise, *Skeletons and fans of logarithmic structures*, in Nonarchimedean and tropical geometry, Simons Symposia (Springer, Cham, 2016), pp. 287–336.

[BV12] N. Borne and A. Vistoli, *Parabolic sheaves on logarithmic schemes*, Adv. Math. **231** (2012), 1327–1363.

[Buz25] K. Buzzard, *Grothendieck's use of equality*, in The mathematical and philosophical legacy of Alexander Grothendieck, Chapman Mathematical Notes (Birkhäuser/Springer, Cham, 2025), 337–354.

[CLM23] K. Chan, N. C. Leung and Z. N. Ma, *Geometry of the Maurer-Cartan equation near degenerate Calabi-Yau varieties*, J. Differential Geom. **125** (2023), 1–84.

[CGR25] A. Corti, T. Graefnitz and H. Ruddat, *Singular log structures and log crepant log resolutions i, March 2025. 15 pages, 6 figures* (2025).

[Del73] P. Deligne, *Exposés XVIII (la formule de dualité globale)*, in Artin, M. Grothendieck, A. Verdier, J. L. and Avec la collaboration de P. Deligne et B. Saint-Donat (eds.) Séminaire de Géométrie Algébrique du Bois-Marie 1963-1964 (SGA 4), Lecture Notes in Mathematics, vol. 305 (Springer-Verlag, Berlin-New York, 1973), vi+640.

[EGA67] *Étude locale des schémas et des morphismes de schémas (quatrième partie)*, Inst. Hautes Études Sci. Publ. Math. **32** (1967), 5–361.

[FTS25] M. Farajzadeh-Tehrani and M. Swaminathan, *What is a stable log map?* (2025), 86 p.

[Fel22] S. Felten, *Log smooth deformation theory via Gerstenhaber algebras*, Manuscripta Math. **167** (2022), 1–35.

[Fel25] S. Felten, Global logarithmic deformation theory, Lecture Notes in Mathematics, vol. 2373 (Springer, Cham, 2025).

[FFR21] S. Felten, M. Filip and H. Ruddat, *Smoothing toroidal crossing spaces*, Forum Math. Pi **9** (2021), Paper No. e7, 36.

[Fri83] R. Friedman, *Global smoothings of varieties with normal crossings*, Ann. of Math. **118** (1983), 75–114.

[GS06] M. Gross and B. Siebert, *Mirror symmetry via logarithmic degeneration data. I*, J. Differential Geom. **72** (2006), 169–338.

[Kat00] F. Kato, *Log smooth deformation and moduli of log smooth curves*, Internat. J. Math. **11** (2000), 215–232.

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press

[KN94] Y. Kawamata and Y. Namikawa, *Logarithmic deformations of normal crossing varieties and smoothing of degenerate Calabi-Yau varieties*, Invent. Math. **118** (1994), 395–409.
[Kle79] S. L. Kleiman, *Misconceptions about K_X*, Enseign. Math. **25** (1980), 203–206, 1979.
[Ogu18] A. Ogus, Lectures on logarithmic algebraic geometry, Cambridge Studies in Advanced Mathematics, vol. 178 (Cambridge University Press, Cambridge, 2018).
[Ols03] M. C. Olsson, *Universal log structures on semi-stable varieties*, Tohoku Math. J. **55** (2003), 397–438.
[SS06] S. Schröer and B. Siebert, *Toroidal crossings and logarithmic structures*, Adv. Math. **202** (2006), 189–231.
[Ste95] J. H. M. Steenbrink, *Logarithmic embeddings of varieties with normal crossings and mixed Hodge structures*, Math. Ann. **301** (1995), 105–118.
[Sta26] The Stacks Project Authors, *The Stacks Project Authors* (2026), https://stacks.math.columbia.edu.

Alessio Corti a.corti@imperial.ac.uk
Department of Mathematics, Imperial College London, South Kensington Campus, UK

Helge Ruddat helge.ruddat@uis.no
Department of Mathematics and Physics, University of Stavanger, Norway

https://doi.org/10.1017/S2949764726100253 Published online by Cambridge University Press