An alternative formula for the cubic equation

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1. *Introduction*

Consider the general cubic equation:

$$x^3 + ax^2 + bx + c = 0.$$

A change of variable: $x = z + t$ leads to:

$$z^3 + (3t + a)z^2 + \left(3t^2 + 2at + b\right)z + \left(t^3 + at^2 + bt + c\right) = 0. \quad (1)$$

Cardano [1] reduces the cubic by choosing $3t + a = 0$. In this Article, we reduce the cubic in a different way by removing the linear term and retaining the quadratic term, arriving to a new reduced form

$$z^3 + rz^2 + s = 0.$$

Two transformations can be used to achieve this goal:

Transformation 1:

We choose t in (1) such as $3t^2 + 2at + b = 0$. This is always possible in the complex plane. For example we can write:

$$t = \frac{-a + \sqrt{a^2 - 3b}}{3}.$$

This leads to:

$$r = 3t + a \qquad \text{and} \qquad s = t^3 + at^2 + bt + c.$$

If the original coefficients a, b and c are real and $a^2 - 3b \geqslant 0$ then t and the resulting r and s are real.

Transformation 2:

Cardano's change of variable $x = y - \frac{a}{3}$ leads to the traditional reduced form,

$$y^3 + py + q = 0,$$

where

$$p = \frac{3b - a^2}{3} \qquad \text{and} \qquad q = \frac{2a^3 - 9ab + 27c}{27}.$$

Outside of the trivial case where $q = 0$, let us choose

$$z = \frac{1}{y} = \frac{1}{x + \frac{a}{3}}.$$

Then

$$\frac{1}{z^3} + p\frac{1}{z} + q = 0,$$

or

$$z^3 + rz^2 + s = 0 \qquad \text{with} \qquad r = \frac{p}{q} \qquad \text{and} \qquad s = \frac{1}{q}.$$

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2. *Derivation*

Let us start with the new reduced form:

$$z^3 + rz^2 + s = 0 \tag{2}$$

with r and s complex numbers.

When $s = 0$, $z = 0$ or $z = -r$. When $r = 0$, the roots are $-\sqrt[3]{s}$, $-\sqrt[3]{s}\,j$ and $-\sqrt[3]{s}\,j^2$ where $j = e^{2i\pi/3}$. Otherwise, $r \neq 0$ and $s \neq 0$, which implies that $z \neq 0$.

We look for solutions in the following form:

$$z = \frac{v}{1-u} \qquad \text{with} \qquad u \neq 1.$$

The intuition behind this choice is the following: When $z = \frac{v}{1-u}$ $z = uz + v$ and $z^3 = u^3z^3 + 3uvz(uz + v) + v^3 = u^3z^3 + 3uvz^2 + v^3$. Notice that there is now no linear term.

Equation (2) becomes

$$\frac{v^3}{(1-u)^3} + r\frac{v^2}{(1-u)^2} + s = 0,$$

or

$$v^3 + (1-u)\left(rv^2 - 3su\right) + s\left(1-u^3\right) = 0.$$

Let us further impose the condition

$$rv^2 - 3su = 0, \qquad \text{or} \qquad u = \frac{r}{3s}v^2. \tag{3}$$

This is possible because there always exists a complex number v such that

$$z = \frac{v}{1 - \frac{r}{3s}v^2}. \tag{4}$$

Then, since $z \neq 0$, we can choose v as one of the roots of the quadratic equation

$$\frac{r}{3s}v^2 + \frac{1}{z}v - 1 = 0.$$

This additional imposed condition (3) leads to

$$v^3 + s\left(1-u^3\right) = 0. \tag{5}$$

It follows that

$$v^3 + s\left(1 - \left(\frac{r}{3s}v^2\right)^3\right) = 0, \tag{6}$$

Defining $V = v^3$ leads to a quadratic equation:

$$V^2 - s^2\left(\frac{3}{r}\right)^3 V - s^3\left(\frac{3}{r}\right)^3 = 0. \tag{7}$$

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Note that since the constant term of this quadratic is not zero, the solution $V \neq 0$, which means that $v \neq 0$. Using (5), $u \neq 1$, which satisfies our earlier assumption. Also, notice that from (3) $u \neq 0$, which will be used at a later stage.

The discriminant is

$$4s^2\left(\frac{3}{r}\right)^6\left(\frac{s^2}{4} + s\frac{r^3}{27}\right).$$

The solutions of (7) are

$$V_1 = \left(\frac{3}{r}\right)^3\left(\frac{s^2}{2} + s\sqrt{\frac{s^2}{4} + s\frac{r^3}{27}}\right) \quad \text{and} \quad V_2 = \left(\frac{3}{r}\right)^3\left(\frac{s^2}{2} - s\sqrt{\frac{s^2}{4} + s\frac{r^3}{27}}\right).$$

Using Vieta's formula notice that

$$V_2 = -\left(\frac{3s}{r}\right)^3\frac{1}{V_1}. \tag{8}$$

The solutions of (6) derived from V_1 are

$$v_k = j^k\sqrt[3]{V_1} = \frac{3}{r}j^k\sqrt[3]{\frac{s^2}{2} + s\sqrt{\frac{s^2}{4} + s\frac{r^3}{27}}}, \qquad k = 0,\ 1,\ 2.$$

The other three roots are obtained from (8) and can be defined as follow:

$$v_{k+3} = -\frac{3s}{r}\frac{1}{v_k}, \qquad k = 0,\ 1,\ 2. \tag{9}$$

It appears that the six choices of v may lead to six roots of (2). However, we show that there are only three. Indeed, if z_k and z_{k+3} are the roots derived respectively from v_k and v_{k+3} then using (4) and (9):

$$z_k = \frac{v_k}{1 - \frac{r}{3s}v_k^2} = \frac{v_k}{1 + \frac{v_k}{v_{k+3}}} = \frac{v_kv_{k+3}}{v_k + v_{k+3}},$$

$$z_{k+3} = \frac{v_{k+3}}{1 - \frac{r}{3s}v_{k+3}^2} = \frac{v_{k+3}}{1 + \frac{v_{k+3}}{v_k}} = \frac{v_kv_{k+3}}{v_k + v_{k+3}} = z_k.$$

For $k = 0$,

$$v_0 = \frac{3}{r}\sqrt[3]{\frac{s^2}{2} + s\sqrt{\frac{s^2}{4} + s\frac{r^3}{27}}} \quad \text{and} \quad u_0 = \frac{rv_0^2}{3s} = \frac{3}{r}\sqrt[3]{\frac{s}{2} + \frac{r^3}{27} + \sqrt{\frac{s^2}{4} + s\frac{r^3}{27}}}. \tag{10}$$

Therefore

$$z_0 = \frac{\frac{3}{r}\sqrt[3]{\frac{s^2}{2} + s\sqrt{\frac{s^2}{4} + s\frac{r^3}{27}}}}{1 - \frac{3}{r}\sqrt[3]{\frac{s}{2} + \frac{r^3}{27} + \sqrt{\frac{s^2}{4} + s\frac{r^3}{27}}}}.$$

Notice that this formula does not resemble Cardano's. In the following section, we will prove that they are equal.

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The other roots can be obtained from v_1 and v_2. The general formula for $k = 0, 1, 2$ is

$$z_k = \frac{\frac{3}{r}j^k\sqrt[3]{\frac{s^2}{2} + s\sqrt{\frac{s^2}{4} + s\frac{r^3}{27}}}}{1 - \frac{3}{r}j^{2k}\sqrt[3]{\frac{s}{2} + \frac{r^3}{27} + \sqrt{\frac{s^2}{4} + s\frac{r^3}{27}}}}. \tag{11}$$

3. *Link to Cardano's formula*

The identity $1 - u_0^3 = (1 - u_0)(1 + u_0 + u_0^2)$ leads to

$$z_0 = \frac{v_0}{1 - u_0} = \frac{v_0(1 + u_0 + u_0^2)}{1 - u_0^3}.$$

Recall that $u_0 \neq 0$ therefore;

$$z_0 = \frac{u_0 v_0}{1 - u_0^3}\left(1 + \frac{1}{u_0} + u_0\right).$$

Using (3) and (5):

$$z_0 = \frac{r}{3s}\frac{v_0^3}{1 - u_0^3}\left(1 + \frac{1}{u_0} + u_0\right) = -\frac{r}{3}\left(1 + \frac{1}{u_0} + u_0\right). \tag{12}$$

Recall from (10):

$$u_0 = \frac{3}{r}\sqrt[3]{A + \sqrt{B}} \quad \text{with } A = \frac{s}{2} + \frac{r^3}{27} \quad \text{and } B = \frac{s^2}{4} + s\frac{r^3}{27}.$$

Let us show that

$$\frac{1}{u_0} = u_0' = \frac{3}{r}\sqrt[3]{A - \sqrt{B}}.$$

This is due to

$$u_0 u_0' = \frac{3^2}{r^2}\sqrt[3]{A^2 - B} = \frac{3^2}{r^2}\sqrt[3]{\frac{r^6}{27^2}} = \sqrt[3]{\frac{3^6}{r^6}\frac{r^6}{27^2}} = 1.$$

This simplifies (12) to a form that resembles Cardano's

$$z_0 = -\frac{r}{3} - \sqrt[3]{\frac{s}{2} + \frac{r^3}{27} + \sqrt{\frac{s^2}{4} + s\frac{r^3}{27}}} - \sqrt[3]{\frac{s}{2} + \frac{r^3}{27} - \sqrt{\frac{s^2}{4} + s\frac{r^3}{27}}}.$$

Let us now show that Cardano's approach leads to the same formula. Starting from (2) we define

$$p' = -\frac{r^2}{3} \qquad \text{and} \qquad q' = \frac{2r^3 + 27s}{27}.$$

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Using Cardano's formula

$$z_0' = -\frac{r}{3} - \sqrt[3]{\frac{q'}{2} + \sqrt{\frac{q'^2}{4} + \frac{p'^3}{27}}} - \sqrt[3]{\frac{q'}{2} - \sqrt{\frac{q'^2}{4} + \frac{p'^3}{27}}}.$$

Since

$$\frac{q'}{2} = \frac{s}{2} + \frac{r^3}{27}$$

and

$$\frac{q'^2}{4} + \frac{p'^3}{27} = \left(\frac{s}{2} + \frac{r^3}{27}\right)^2 - \frac{1}{27}\frac{r^6}{27} = \frac{s^2}{4} + s\frac{r^3}{27}.$$

It follows that

$$z_0' = -\frac{r}{3} - \sqrt[3]{\frac{s}{2} + \frac{r^3}{27} + \sqrt{\frac{s^2}{4} + s\frac{r^3}{27}}} - \sqrt[3]{\frac{s}{2} + \frac{r^3}{27} - \sqrt{\frac{s^2}{4} + s\frac{r^3}{27}}} = z_0.$$

4. *Example*

$$x^3 - 2x^2 + x + 3 = 0.$$

Using transformation 1 $a = -2, b = 1$ and $c = 3$.

Removing the linear term in (1) leads to solving

$$3t^2 + 2at + b = 3t^2 - 4t + 1 = (t - 1)(3t - 1) = 0.$$

This leads to $t = 1$ or $t = \frac{1}{3}$.

We choose $t = 1$ and the change of variable $z = x - 1$.

$$r = 3t + a = 1 \qquad s = t^3 + at^2 + bt + c = 3.$$

Using (10)

$$v_0 \approx 6.26552, \qquad u_0 \approx 4.3619.$$

Which leads to a reduced solution:

$$z_0 = \frac{v_0}{1 - u_0} \approx -1.8637 \qquad \text{and} \qquad x_0 = z_0 + t = -0.8637.$$

Similarly, using (11),

$$x_{1,2} \approx 1.4319 \pm 1.1930i.$$

Using transformation 2,

$$p = \frac{3b - a^2}{3} \approx -0.3333$$

and

$$q = \frac{2a^3 - 9ab + 27c}{27} \approx 3.0741.$$

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This leads to

$$r = \frac{p}{q} \approx -0.1084, \qquad s = \frac{1}{q} \approx 0.3253.$$

Using (10),

$$v_0 \approx -13.08558, \qquad u_0 \approx -19.0258.$$

This leads to a reduced solution

$$z_0 = \frac{v_0}{1 - u_0} \approx -0.6534 \qquad \text{and} \qquad x_0 = \frac{1}{z} - \frac{a}{3} \approx -0.8637.$$

Similarly, using (11),

$$x_{1,2} \approx 1.4319 \pm 1.1930i.$$

Using Cardano's approach

$$p \approx -0.3333 \qquad \text{and} \qquad q \approx 3.0741.$$

The roots from Cardano's formula for $k = 0,\ 1,\ 2$ are

$$x_k = -\frac{a}{3} + j^k \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + j^{-k} \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

This leads to

$$x_0 \approx -0.8637$$

and

$$x_{1,2} \approx 1.4319 \pm 1.1930i.$$

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Reference

1. Girolamo Cardano, *Ars Magna* (1545).

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