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STOCHASTIC COMPARISONS OF GENERALIZED ORDER STATISTICS

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In this article, we give several results on (multivariate and univariate) stochastic comparisons of generalized order statistics. We give conditions on the underlying distributions and the parameters on which the generalized order statistics are based, to obtain stochastic comparisons in the stochastic, dispersive, hazard rate, and likelihood ratio orders. Our results generalize some recent results for order statistics, record values, and generalized order statistics and provide some new results for other models such as *k*-record values and order statistics under multivariate imperfect repair.

1. INTRODUCTION

The stochastic comparison of probabilistic models is a growing field of research in applied probability. Fields where these results can be applied include reliability, economics, insurance, epidemiology, and so forth (see Müller and Stoyan [26] and Shaked and Shanthikumar [31]).

During the last 10 years, a great number of articles have dealt with stochastic comparisons of order statistics (and the more general model of coherent systems) and the main applications of such results are given in reliability. Some recent articles on the subject are by Belzunce, Franco, Ruiz, and Ruiz [2], Belzunce, Ruiz, and Ruiz [6], Belzunce and Shaked [8], Boland, Hu, Shaked, and Shanthikumar [9], Boland, Shaked, and Shanthikumar [10], Hu and Wei [17], Khaledi and Kochar [21,22], Kochar [23], Lillo, Nanda, and Shaked [25], and Nanda and Shaked [27].

Another model that has received much attention recently is the model of epoch times of nonhomogeneous Poisson and pure birth processes. Nonhomogeneous

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processes are of interest not only in reliability but also in insurance, epidemiology, and so forth. Also, the epoch times of a nonhomogeneous Poisson process correspond with the record values of a proper sequence of independent and identically distributed (i.i.d.) random variables. For these models, Belzunce, Hu, and Khaledi [3], Belzunce, Lillo, Ruiz, and Shaked [4], Belzunce and Ruiz [5], and Belzunce and Shaked [7] provide several results about stochastic comparisons.

Given the similarity between some structural properties of order statistics and record values, Kamps [19,20] proposed the model of generalized order statistics, which includes, as particular cases, random vectors of order statistics and record values. This model covers some other models of interest in applied probability.

There are some partial results about stochastic comparisons of generalized order statistics, as can be seen in Kamps [18,20] and more recently in Franco, Ruiz, and Ruiz [13]. However, most of these results are given in the univariate case and under some restrictive assumptions on the values of the parameters of the generalized order statistics, which limits the applicability of such results.

The purpose of this article is to give conditions on the distributions and the parameters on which the generalized order statistics are based to obtain stochastic comparisons of generalized order statistics. These results are given in most cases in the multivariate case and try to impose less restrictive assumptions. We extend some recent results given for order statistics and record values, and we give applications to some other models. As we will see later, the generalized order statistics are a particular case of the epoch times of a proper nonhomogeneous pure birth process, and from known results about stochastic comparisons of this model, it is possible to derive results for generalized order statistics. However, in some cases, these results can be given under weaker conditions, as we will show later. The article is organized as follows. In Section 2, we introduce several models of random vectors with ordered components, such as order statistics, record values, generalized order statistics, and epoch times of nonhomogeneous pure birth processes. Several distributional properties of interest will also be given. In Section 3, we describe several conditions to obtain stochastic comparisons of generalized order statistics. In Section 4, we consider a similar problem, but in this case for spacings of generalized order statistics. To finish, in Section 5, we give some applications of these results.

In this article, "increasing" and "decreasing" mean "nondecreasing" and "nonincreasing," respectively. Also, given a random variable X with continuous distribution function F and density f, $\bar{F} \equiv 1 - F$ denotes the survival function, $f/\bar{F}$ is the hazard rate, and F^{-1} denotes the inverse of the distribution function, which is understood to be the left-continuous one. In this article, the results are given for continuous distributions, except for the results about the likelihood ratio order, which are given for absolutely continuous distributions.

2. PROBABILISTIC MODELS OF RANDOM VECTORS WITH ORDERED COMPONENTS

Perhaps the most well-known model of random vector with ordered components is the random vector of order statistics. Given a set of n i.i.d. random variables

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$X_1,\dots,X_n$, if we arrange these random variables in ascending order of magnitude, then the random vector of the ordered quantities $(X_{(1)},\dots,X_{(n)})$ is called the random vector of order statistics.

Other models are given from the epoch times of a counting process. For example, we can consider the epoch times of a nonhomogeneous Poisson counting process. A counting process $\{N(t), t \geq 0\}$ is a nonhomogeneous Poisson process with intensity function $r \geq 0$ ($\int_t^\infty r(u)\,du = \infty$ for all $t \geq 0$) if the following hold:

1. $\{N(t), t \geq 0\}$ has the Markov property.
2. $P\{N(t+\Delta t) = n+1 | N(t) = n\} = r(t)\Delta t + o(\Delta t)$, $n \geq 1$.
3. $P\{N(t+\Delta t) > n+1 | N(t) = n\} = o(\Delta t)$, $n \geq 1$.

If we denote by $T_n = \inf\{t : N(t) = n\}$ the epoch times of the nonhomogeneous Poisson process, clearly the components of the random vector of the first n epoch times, $(T_1,\dots,T_n)$, are ordered.

Epoch times of nonhomogeneous Poisson processes can be introduced as record values of a proper sequence of random variables.

Let $\{X_i\}_{i=1}^\infty$ be a sequence of i.i.d. random variables and denote by

$$\begin{aligned} L(1) &= 1, \\ L(n+1) &= \min\{j > L(n) : X_j > X_{L(n)}\} \end{aligned}$$

the record times; then, the record values are defined as

$$\{X_{L(n)}\}_{n=1}^\infty.$$

Now, if we consider a sequence $\{X_i\}_{i=1}^\infty$ of i.i.d. random variables (r.v.'s) with hazard rate r, then

$$T_n =_{\text{st}} X_{L(n)},$$

where the T_n's are the epoch times of a nonhomogeneous Poisson process, with intensity r.

In order to present a unified approach to study order statistics and record values (and a variety of models of ordered random variables), Kamps [19] introduced the concept of generalized order statistics.

Next, we give the definition of generalized order statistics following Kamps [19,20].

DEFINITION 2.1: *Let $n \in \mathbb{N}$, $k \geq 1$, $m_1,\dots,m_{n-1} \in \mathbb{R}$, $M_r = \sum_{j=r}^{n-1} m_j$, $1 \leq r \leq n-1$, be parameters such that $\gamma_r = k + n - r + M_r \geq 1$ for all $r \in 1,\dots,n-1$, and let $\tilde{m} = (m_1,\dots,m_{n-1})$ if $n \geq 2$ ($\tilde{m} \in \mathbb{R}$ arbitrary if $n = 1$). We call uniform generalized order statistics to the random vector $(U_{(1,n,\tilde{m},k)},\dots,U_{(n,n,\tilde{m},k)})$ with joint density function*

$$h(u_1,\dots,u_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{j=1}^{n-1}(1-u_j)^{m_j}\right)(1-u_n)^{k-1}$$

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on the cone $0 \le u_1 \le \cdots \le u_n \le 1$. *Now, given a distribution function F, we call generalized order statistics based on F to the random vector*

$$(X_{(1,n,\tilde{m},k)},\ldots,X_{(n,n,\tilde{m},k)}) \equiv (F^{-1}(U_{(1,n,\tilde{m},k)}),\ldots,F^{-1}(U_{(n,n,\tilde{m},k)})).$$

If F is an absolutely continuous distribution with density f, the joint density function of $(X_{(1,n,\tilde{m},k)},\ldots,X_{(n,n,\tilde{m},k)})$ is given by

$$f(x_1,\ldots,x_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{j=1}^{n-1}\bar{F}(x_j)^{m_j}f(x_j)\right)\bar{F}(x_n)^{k-1}f(x_n) \qquad \textbf{(2.1)}$$

on the cone $F^{-1}(0) \le x_1 \le \cdots \le x_n \le F^{-1}(1)$.

Several models of ordered random variables are included in this model. As we have mentioned previously, order statistics and record values are a particular case of this model. Taking $m_i = 0$ for all $i = 1,\ldots,n-1$ and $k = 1$, we get the order statistics from a distribution F, and taking $m_i = -1$ for all $i = 1,\ldots,n-1$ and $k = 1$, we get the first n record values from a sequence of random variables with distribution F. Some other models contained in the generalized order statistics (GOS) model will be seen in Section 5.

A property that will be used later is that GOS from a continuous distribution form a Markov chain with transition probabilities

$$P[X_{(r,n,\tilde{m},k)} > t | X_{(r-1,n,\tilde{m},k)} = s] = \left(\frac{\bar{F}(t)}{\bar{F}(s)}\right)^{\gamma_r} \quad \text{for } t \ge s \text{ and } r = 2,\ldots,n. \qquad \textbf{(2.2)}$$

Another random vector with ordered components that has recently received great interest is the model of the first n epoch times of a nonhomogeneous pure birth (NHPB) process. Such processes are called "relevation counting processes" in Pellerey, Shaked, and Zinn [28]. Epoch times of NHPB processes correspond also to Pfeifer's record values. Pfeifer [29] established a model of record values based on a double sequence of nonidentically distributed random variables.

Applications of NHPB processes in insurance, reliability theory, epidemiology, and load-sharing models can be found in Belzunce et al. [4], Hu and Pan [16], Pellerey et al. [28], and Shaked, Spizzichino, and Suter [33].

A NHPB process is a Markovian point process in which the jump intensity at any time t depends not only on t but also on the number n of jumps before time t. Formally, we consider a point process $N = \{N(t),\ t \ge 0\}$ such that the following hold:

1. N has the Markov property.
2. $P\{N(t+\Delta t) = n+1 | N(t) = n\} = r_n(t)\Delta t + o(\Delta t),\ n \ge 1.$
3. $P\{N(t+\Delta t) > n+1 | N(t) = n\} = o(\Delta t),\ n \ge 1,$

The r_n's are nonnegative functions that satisfy

$$\int_t^\infty r_n(x)\,dx = \infty \quad \text{for all } t \ge 0.$$

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In a distributional theoretical sense GOS are contained in the model of epoch times of a NHPB process. Consider GOS based on F with failure rate r and parameters k, n, and M_r, $r = 1,\ldots,n-1$; then,

$$(X_{(1,n,\tilde{m},k)},\ldots,X_{(n,n,\tilde{m},k)}) =_{\text{st}} (T_1,\ldots,T_n),$$

where the T_i's are the epoch times of a NHPB process with intensities $r_i = (k + n - i + M_i)r$, for $i = 1,\ldots,n$. This relationship will be used later to obtain stochastic comparisons of GOS from some previous known results about NHPB processes.

3. STOCHASTIC COMPARISONS OF GENERALIZED ORDER STATISTICS

In this section, we give several results about stochastic comparisons of GOS. As we will see, some of them are given for the whole random vector of generalized order statistics, from which we can obtain stochastic comparisons of the coordinates. These results extend some recent results given by other authors.

We start by considering the usual multivariate stochastic order. Recall that given two n-dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$, then $\mathbf{X}$ is said to be less than $\mathbf{Y}$ in the multivariate stochastic order (denoted by $\mathbf{X} \leq_{\text{st}} \mathbf{Y}$) if $E[\phi(\mathbf{X})] \leq_{\text{st}} E[\phi(\mathbf{Y})]$ for all increasing functions $\phi : \mathbb{R}^n \mapsto \mathbb{R}$, provided the previous expectations exist. In the univariate case given two r.v.'s X and Y with distribution functions F and G, respectively, $X \leq_{\text{st}} Y$ if and only if $\bar{F} \leq \bar{G}$.

As we have mentioned in the previous section, GOS can be considered as a particular case of epoch times of NHPB. For this last model, Belzunce et al. [4] gave conditions in Theorem 3.11 under which two random vectors of epoch times of NHPB processes are ordered in the multivariate stochastic order. Such conditions can be translated into conditions for the multivariate stochastic order of two random vectors of GOS based on distributions F and G. In fact, under the hazard rate order of F and G (see the definition of the hazard rate order later), we get the multivariate stochastic order of the corresponding random vectors of generalized order statistics. However, as we see next, this result can be given under weaker conditions.

THEOREM 3.1: *Let X and Y be r.v.'s with distribution functions F and G, respectively, and let $\mathbf{X} = (X_{(1,n,\tilde{m},k)},\ldots,X_{(n,n,\tilde{m},k)})$ and $\mathbf{Y} = (Y_{(1,n,\tilde{m},k)},\ldots,Y_{(n,n,\tilde{m},k)})$ be random vectors of generalized order statistics based on F and G, respectively. If $X \leq_{\text{st}} Y$, then*

$$\mathbf{X} \leq_{\text{st}} \mathbf{Y}.$$

PROOF: We prove the result by constructing random vectors $\hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}$ on the same probability space with $\hat{\mathbf{X}} =_{\text{st}} \mathbf{X}$ and $\hat{\mathbf{Y}} =_{\text{st}} \mathbf{Y}$, such that $P(\hat{\mathbf{X}} \leq \hat{\mathbf{Y}}) = 1$, and the result follows from Theorem 4.B.1 in Shaked and Shanthikumar [31]. Let $\hat{\mathbf{X}} = (\hat{X}_1,\ldots,\hat{X}_n)$ and $\hat{\mathbf{Y}} = (\hat{Y}_1,\ldots,\hat{Y}_n)$ be random vectors, where

$$\hat{X}_r \equiv F^{-1}(U_{(r,n,\tilde{m},k)})$$

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and

$$\hat{Y}_r \equiv G^{-1}(U_{(r,n,\tilde{m},k)})$$

and $(U_{(1,n,\tilde{m},k)},\ldots,U_{(n,n,\tilde{m},k)})$ be the random vector of uniform generalized order statistics. Clearly, $\hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}$ are defined on the same probability space, and by Definition 2.1, $\hat{\mathbf{X}} =_{\text{st}} \mathbf{X}$ and $\hat{\mathbf{Y}} =_{\text{st}} \mathbf{Y}$. On the other hand,

$$P[\hat{\mathbf{X}} \leq \hat{\mathbf{Y}}] = P\left[\bigcap_{r=1}^{n} (\hat{X}_r \equiv F^{-1}(U_{(r,n,\tilde{m},k)}) \leq G^{-1}(U_{(r,n,\tilde{m},k)}) \equiv \hat{Y}_r)\right] = 1,$$

where the last equality follows from the assumption $X \leq_{\text{st}} Y$. The result follows from Theorem 4.B.1 in Shaked and Shanthikumar [31]. ■

The multivariate stochastic order is preserved under marginalization. Therefore, if $X \leq_{\text{st}} Y$, any subset of coordinates of the random vector of GOS based on F is smaller in the stochastic order than the corresponding subset of coordinates based on G. In particular, we can obtain the following result.

COROLLARY 3.2: *Let X and Y be r.v.'s with distribution functions F and G, respectively, and let $\mathbf{X} = (X_{(1,n,\tilde{m},k)},\ldots,X_{(n,n,\tilde{m},k)})$ and $\mathbf{Y} = (Y_{(1,n,\tilde{m},k)},\ldots,Y_{(n,n,\tilde{m},k)})$ be random vectors of generalized order statistics based on F and G, respectively. If $X \leq_{\text{st}} Y$, then*

$$X_{(r,n,\tilde{m},k)} \leq_{\text{st}} Y_{(r,n,\tilde{m},k)} \quad \text{for all } r = 1,\ldots,n.$$

This corollary extends Theorem 3.2 of Franco et al. [13] to the more general case in which the m_i's are not equal.

Given two nonnegative random variables X and Y with hazard rates r and s, respectively, then X is said to be less than Y in the hazard rate order (denoted by $X \leq_{\text{hr}} Y$) if $r(t) \geq s(t)$ for all $t \geq 0$. This order can be characterized in terms of the survival functions as follows: $X \leq_{\text{hr}} Y$ if $\bar{F}/\bar{G}$ is decreasing, where $\bar{F}$ and $\bar{G}$ are the survival functions of X and Y, respectively.

This partial order is stronger than the stochastic order (i.e., $\leq_{\text{hr}} \Rightarrow \leq_{\text{st}}$), so the previous theorem is given under weaker conditions than those that follow from Theorem 3.11 in Belzunce et al. [4]. Under the hazard rate order assumption, it is possible to obtain the stronger result of the multivariate hazard rate order of generalized order statistics (we refer the reader to [31] for the definition of the multivariate hazard rate order). The following result follows from Theorem 3.12 by Belzunce et al. [4].

THEOREM 3.3: *Let X and Y be r.v.'s with distribution functions F and G, respectively, and let $\mathbf{X} = (X_{(1,n,\tilde{m},k)},\ldots,X_{(n,n,\tilde{m},k)})$ and $\mathbf{Y} = (Y_{(1,n,\tilde{m},k)},\ldots,Y_{(n,n,\tilde{m},k)})$ be random vectors of generalized order statistics based on F and G, respectively. If $X \leq_{\text{hr}} Y$, then*

$$\mathbf{X} \leq_{\text{hr}} \mathbf{Y}.$$

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Remark 3.4. The hazard rate order is not known to be preserved under marginalization; therefore, we cannot obtain from the previous result the hazard rate order of the corresponding coordinates. A partial result in this direction is given in Theorem 3.3 by Franco et al. [13] when $m_i = m$, for all $i = 1, \ldots, n-1$.

Next, we provide a result for the general case (i.e., when the $m_i's$ do not need to be equal). To provide the result, we recall, first, the following equality given recently by Cramer and Kamps [11].

The hazard rate, λ_r, of the rth generalized order statistic $X_{(r,n,\tilde{m},k)}$ based on an absolutely continuous distribution function F, with hazard rate $r(t)$, is given by

$$\lambda_r(t) = r(t)h_r(\bar{F}(t)), \tag{3.1}$$

where h_r is defined by

$$h_r(z) = \frac{zG_{r,r}^{r,0}\left[z \left| \begin{matrix} \gamma_1, \ldots, \gamma_r \\ \gamma_1 - 1, \ldots, \gamma_r - 1 \end{matrix} \right.\right]}{\int_0^r G_{r,r}^{r,0}\left[s \left| \begin{matrix} \gamma_1, \ldots, \gamma_r \\ \gamma_1 - 1, \ldots, \gamma_r - 1 \end{matrix} \right.\right] ds}, \qquad z \in (0,1),$$

and G is a particular Meijer's G-function. Cramer and Kamps [11] show also (see Remark 4.7) that $h_r(z)$ is decreasing in z.

Next, we give the following result for the hazard rate order.

THEOREM 3.5: *Let X and Y be absolutely continuous r.v.'s with distribution functions F and G and with hazard rates r and s, respectively. Let $X_{(r,n,\tilde{m},k)}$ and $Y_{(r,n,\tilde{m},k)}$ be the rth generalized order statistics based on F and G, respectively. If $X \leq_{\text{hr}} Y$, then $X_{(r,n,\tilde{m},k)} \leq_{\text{hr}} Y_{(r,n,\tilde{m},k)}$.*

PROOF: From (3.1), the hazard rates of $X_{(r,n,\tilde{m},k)}$ and $Y_{(r,n,\tilde{m},k)}$ are given by

$$r(t)h_r(\bar{F}(t)) \quad \text{and} \quad s(t)h_r(\bar{G}(t)),$$

respectively.

From the condition $X \leq_{\text{hr}} Y$, we get that $r(t) \geq s(t)$ and $\bar{F}(t) \leq \bar{G}(t)$. Now, given that $h_r(z)$ is decreasing in z, we get the following inequality:

$$r(t)h_r(\bar{F}(t)) \geq s(t)h_r(\bar{G}(t))$$

and, therefore, the result. ■

Next, we consider the multivariate likelihood ratio order. Let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional random vectors with density functions $f_{\mathbf{X}}$ and $f_{\mathbf{Y}}$, respectively. We say that $\mathbf{X}$ is less than $\mathbf{Y}$ in the multivariate likelihood ratio order, denoted by $\mathbf{X} \leq_{\text{lr}} \mathbf{Y}$, if ($\wedge$ and $\vee$ denote, respectively, the minimum and the maximum operations)

$$\begin{aligned} &f_{\mathbf{X}}(x_1, x_2, \ldots, x_n) f_{\mathbf{Y}}(y_1, y_2, \ldots, y_n) \\ &\quad \leq f_{\mathbf{X}}(x_1 \wedge y_1, x_2 \wedge y_2, \ldots, x_n \wedge y_n) f_{\mathbf{Y}}(x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n) \end{aligned}$$

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for all $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ (see Shaked and Shanthikumar [31, Chap. 4, and references therein]).

A property that will be used in the proof of the following theorem is that the likelihood ratio order is stronger than the hazard rate order (i.e., $\leq_{\mathrm{lr}} \Rightarrow \leq_{\mathrm{hr}}$).

THEOREM 3.6: *Let X and Y be absolutely continuous r.v.'s with distribution functions F and G and with hazard rates r and s, respectively. Let $\mathbf{X} = (X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)})$ and $\mathbf{Y} = (Y_{(1,n,\tilde{m},k)}, \ldots, Y_{(n,n,\tilde{m},k)})$ be random vectors of generalized order statistics based on F and G, respectively. If (a) $m_i \geq 0$ for all i and $X \leq_{\mathrm{lr}} Y$ or (b) $m_i \geq -1$ for all i, $X \leq_{\mathrm{hr}} Y$, and s/r is increasing, then*

$$\mathbf{X} \leq_{\mathrm{lr}} \mathbf{Y}.$$

PROOF: By (2.1), we have that the joint density function of $\mathbf{X} = (X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)})$ is given by

$$f(\mathbf{x}) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{j=1}^{n-1}(\bar{F}(x_j))^{m_j} f(x_j)\right)(\bar{F}(x_n))^{k-1} f(x_n) \quad \text{for all } 0 < x_1 < \cdots < x_n$$

and the joint density function of $\mathbf{Y} = (Y_{(1,n,\tilde{m},k)}, \ldots, Y_{(n,n,\tilde{m},k)})$ is given by

$$g(\mathbf{y}) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{j=1}^{n-1}(\bar{G}(y_j))^{m_j} g(y_j)\right)(\bar{G}(y_n))^{k-1} g(y_n),$$

$$\text{for all } 0 < y_1 < \cdots < y_n.$$

Therefore, we have to prove that

$$\begin{aligned}
&\left(\prod_{j=1}^{n-1}(\bar{F}(x_j \wedge y_j))^{m_j} f(x_j \wedge y_j)\right)(\bar{F}(x_n \wedge y_n))^{k-1} f(x_n \wedge y_n) \\
&\qquad \times \left(\prod_{j=1}^{n-1}(\bar{G}(x_j \vee y_j))^{m_j} g(x_j \vee y_j)\right)(\bar{G}(x_n \vee y_n))^{k-1} g(x_n \vee y_n) \\
&\geq \left(\prod_{j=1}^{n-1}(\bar{F}(x_j))^{m_j} f(x_j)\right)(\bar{F}(x_n))^{k-1} f(x_n) \\
&\qquad \times \left(\prod_{j=1}^{n-1}(\bar{G}(y_j))^{m_j} g(y_j)\right)(\bar{G}(y_n))^{k-1} g(y_n).
\end{aligned}$$

If we consider the set $E = \{i \leq n-1 : x_i \geq y_i\}$, then the previous inequality is equivalent to

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$$\left(\prod_{j\in E}(\bar F(x_j\wedge y_j))^{m_j}f(x_j\wedge y_j)\right)(\bar F(x_n\wedge y_n))^{k-1}f(x_n\wedge y_n)$$
$$\times\left(\prod_{j\in E}(\bar G(x_j\vee y_j))^{m_j}g(x_j\vee y_j)\right)(\bar G(x_n\vee y_n))^{k-1}g(x_n\vee y_n)$$
$$\geq\left(\prod_{j\in E}(\bar F(x_j))^{m_j}f(x_j)\right)(\bar F(x_n))^{k-1}f(x_n)$$
$$\times\left(\prod_{j\in E}(\bar G(y_j))^{m_j}g(y_j)\right)(\bar G(y_n))^{k-1}g(y_n).$$

From the assumption $X\leq_{\mathrm{lr}}Y$ ($\Rightarrow X\leq_{\mathrm{hr}}Y$) in parts (a) and (b), we have

$$(\bar F(x_n\wedge y_n))^{k-1}f(x_n\wedge y_n)(\bar G(x_n\vee y_n))^{k-1}g(x_n\vee y_n)$$
$$\geq(\bar F(x_n))^{k-1}f(x_n)(\bar G(y_n))^{k-1}g(y_n).$$

Now, under the assumptions in part (a) we have that, for all $j\in E$,

$$(\bar F(y_j))^{m_j}f(y_j)(\bar G(x_j))^{m_j}g(x_j)$$
$$\geq(\bar F(x_j))^{m_j}f(x_j)(\bar G(y_j))^{m_j}g(y_j).$$

Therefore, under the conditions in part (a), we get the result.
On the other hand, under the assumptions in part (b), we get that for all $j\in E$,

$$(\bar F(y_j))^{m_j}f(y_j)(\bar G(x_j))^{m_j}g(x_j)$$
$$=(\bar F(y_j))^{m_j+1}r(y_j)(\bar G(x_j))^{m_j+1}s(x_j)$$
$$\geq(\bar F(x_j))^{m_j}f(x_j)(\bar G(y_j))^{m_j}g(y_j)$$
$$=(\bar F(x_j))^{m_j+1}r(x_j)(\bar G(y_j))^{m_j+1}s(y_j).$$

Again, under the assumptions in part (b) we get the result. ■

The likelihood ratio order is preserved under marginalization; therefore, we have the following result.

COROLLARY 3.7: *Let X and Y be r.v.'s with distribution functions F and G, respectively, and let $\mathbf{X}=(X_{(1,n,\tilde m,k)},\ldots,X_{(n,n,\tilde m,k)})$ and $\mathbf{Y}=(Y_{(1,n,\tilde m,k)},\ldots,Y_{(n,n,\tilde m,k)})$ be random vectors of generalized order statistics based on F and G, respectively. If (a) $m_i\geq 0$ for all i and $X\leq_{\mathrm{lr}}Y$ or (b) $m_i\geq -1$ for all i, $X\leq_{\mathrm{hr}}Y$ and s/r is increasing, then*

$$X_{(r,n,\tilde m,k)}\leq_{\mathrm{lr}}Y_{(r,n,\tilde m,k)}\quad\text{for all } r=1,\ldots,n.$$

Part (a) of this corollary extends Theorem 3.4 by Franco et al. [13] to the more general case in which the m_i's are not equal.

Now, we consider a different problem. We consider stochastic comparisons of two random vectors of GOS with different parameters but based on the same dis-

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tribution F. We start by giving a simple result for random vectors of uniform generalized order statistics.

THEOREM 3.8: *Let* $\mathbf{U} = (U_{(1,n,\tilde{m},k)},\ldots,U_{(n,n,\tilde{m},k)})$ *and* $\mathbf{V} = (V_{(1,n,\tilde{m}',k')},\ldots,V_{(n,n,\tilde{m}',k')})$ *be random vectors of uniform generalized order statistics with parameters* k *and* m_i, $i = 1,\ldots,n-1$, *and* k' *and* m_i', $i = 1,\ldots,n-1$, *respectively. Then,* $\mathbf{U} \leq_{\mathrm{lr}} \mathbf{V}$ *if and only if* $k \geq k'$ *and* $m_i \geq m_i'$, *for all* $i = 1,\ldots,n-1$.

PROOF: From Definition 2.1 we need to prove that

$$\frac{(1-u_n)^{k-1}(1-v_n)^{k'-1}\prod_{j=1}^{n-1}(1-u_j)^{m_j}(1-v_j)^{m_j'}}{(1-(u_n\wedge v_n))^{k-1}(1-(u_n\vee v_n))^{k'-1}\prod_{j=1}^{n-1}(1-(u_j\wedge v_j))^{m_j}(1-(u_j\vee v_j))^{m_j'}} \leq 1,$$

for all $0 < u_1 \leq \cdots \leq u_n < 1$ and $0 < v_1 \leq \cdots \leq v_n < 1$.

The previous inequality holds if and only if

$$\frac{(1-u_j)^{m_j}(1-v_j)^{m_j'}}{(1-(u_j\wedge v_j))^{m_j}(1-(u_j\vee v_j))^{m_j'}} \leq 1, \quad \text{for all } j = 1,\ldots,n-1$$

and

$$\frac{(1-u_n)^{k-1}(1-v_n)^{k'-1}}{(1-(u_n\wedge v_n))^{k-1}(1-(u_n\vee v_n))^{k'-1}} \leq 1$$

for all $0 < u_1 \leq \cdots \leq u_n < 1$ and $0 < v_1 \leq \cdots \leq v_n < 1$.

Now, it is easy to see that the previous inequalities are equivalent to $k \geq k'$ and $m_j \geq m_j'$, for all $j = 1,\ldots,n-1$. ■

Now, from the preservation of the multivariate likelihood ratio order under increasing transformations of each individual coordinate (see [31, Thm. 4.E.2]), we can obtain the following result.

THEOREM 3.9: *Let* $\mathbf{X} = (X_{(1,n,\tilde{m},k)},\ldots,X_{(n,n,\tilde{m},k)})$ *and* $\mathbf{X}' = (X_{(1,n,\tilde{m}',k')},\ldots,X_{(1,n,\tilde{m}',k')})$ *be random vectors of generalized order statistics based on the same distribution* F *and with parameters* k *and* m_i, $i = 1,\ldots,n-1$, *and* k' *and* m_i', $i = 1,\ldots,n-1$, *respectively. Then,* $\mathbf{X} \leq_{\mathrm{lr}} \mathbf{X}'$ *if and only if* $k \geq k'$ *and* $m_i \geq m_i'$ *for all* $i = 1,\ldots,n-1$.

Now, a combination of Theorem 3.9 and Theorems 3.1, 3.3, and 3.6 leads to the following result.

THEOREM 3.10: *Let* X *and* Y *be r.v.'s with distribution functions* F *and* G, *respectively, and let* $\mathbf{X} = (X_{(1,n,\tilde{m},k)},\ldots,X_{(n,n,\tilde{m},k)})$ *and* $\mathbf{Y} = (Y_{(1,n,\tilde{m}',k')},\ldots,Y_{(n,n,\tilde{m}',k')})$ *be random vectors of generalized order statistics based on* F *and* G *and with parameters* k *and* m_i, $i = 1,\ldots,n-1$, *and* k' *and* m_i', $i = 1,\ldots,n-1$, *respectively. Let us*

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suppose that $k \geq k'$ *and* $m_i \geq m_i'$ *for all* $i = 1, \ldots, n-1$. *Then, we have the following results:*

(i) *If* $X \leq_{\text{st}} Y$, *then* $\mathbf{X} \leq_{\text{st}} \mathbf{Y}$.
(ii) *If* $X \leq_{\text{hr}} Y$, *then* $\mathbf{X} \leq_{\text{hr}} \mathbf{Y}$.
(iii) *If (a)* $m_i \geq 0$ *or* $m_i' \geq 0$ *for all* $i = 1, \ldots, n-1$ *and* $X \leq_{\text{lr}} Y$ *or (b)* $m_i \geq -1$ *or* $m_i' \geq -1$ *for all* $i = 1, \ldots, n-1$, $X \leq_{\text{hr}} Y$, *and s/r is increasing, then* $\mathbf{X} \leq_{\text{lr}} \mathbf{Y}$.

To finish this section, we consider two results under the dispersive order assumption.

Given two random variables X and Y, with distribution functions F and G, respectively, we say that X is less than Y in the dispersive order (denoted by $X \leq_{\text{disp}} Y$) if (see [24]):

$$G^{-1}(p) - F^{-1}(p) \text{ is increasing in } p \in (0,1) \tag{3.2}$$

or, equivalently, in the continuous case, if

$$F^{-1}(G(y)) - F^{-1}(G(x)) \leq x - y \quad \text{for all real } x < y.$$

Shaked and Shanthikumar [32, p. 16] consider the following multivariate generalization of (3.2).

Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random vector and consider the transformation $\Psi_{\mathbf{X}}^* : (0,1)^n \to \mathbf{R}^n$ defined by

$$\Psi_{\mathbf{X}}^*(u_1, \ldots, u_n) = (x_1, \ldots, x_n),$$

where the x_i's are defined recursively by

$$x_1 = F_1^{-1}(u_1)$$

and

$$x_i = F_{i|1,\ldots,i-1}^{-1}(u_i | x_1, \ldots, x_{i-1}) \quad \text{for all } i = 2, \ldots, n,$$

where F_1^{-1} is the inverse of the marginal distribution of X_1 and $F_{i|1,\ldots,i-1}^{-1}(u_i | x_1, \ldots, x_{i-1})$ is the inverse of the conditional distribution of X_i given that $X_1 = x_1, \ldots, X_{i-1} = x_{i-1}$.

Now, given $\Psi_{\mathbf{X}}^*$ and $\Psi_{\mathbf{Y}}^*$ for two random vectors, $\mathbf{X}$ and $\mathbf{Y}$, of dimension n, a multivariate generalization of (3.2) is the condition

$$\Psi_{\mathbf{Y}}^*(\mathbf{u}) - \Psi_{\mathbf{X}}^*(\mathbf{u}) \text{ is increasing in } \mathbf{u} \in (0,1)^n. \tag{3.3}$$

Next, we show that under the dispersive order of F and G, then (3.3) holds for the random vectors of GOS based on F and G.

THEOREM 3.11: *Let* X *and* Y *be r.v.'s with distribution functions* F *and* G, *respectively, and let* $\mathbf{X} = (X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)})$ *and* $\mathbf{Y} = (Y_{(1,n,\tilde{m},k)}, \ldots, Y_{(n,n,\tilde{m},k)})$ *be*

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random vectors of generalized order statistics based on F and G, respectively. If $X \leq_{\text{disp}} Y$, *then*

$$\mathbf{X} \leq_{\text{disp}} \mathbf{Y}.$$

PROOF: We observe first that

$$F^{-1}_{X_{(1,n,\tilde{m},k)}}(u_1) = F^{-1}(1-(1-u_1)^{1/\gamma_1})$$

and

$$F^{-1}_{(X_{(r,n,\tilde{m},k)}|X_{(1,n,\tilde{m},k)}=x_1\ldots X_{(r-1,n,\tilde{m},k)}=x_{r-1})}(u_r) = F^{-1}(1-\bar{F}(x_{r-1})(1-u_r)^{1/\gamma_r})$$

and the same holds for $\mathbf{Y}$; that is,

$$G^{-1}_{Y_{(1,n,\tilde{m},k)}}(u_1) = G^{-1}(1-(1-u_1)^{1/\gamma_1})$$

and

$$G^{-1}_{(Y_{(r,n,\tilde{m},k)}|Y_{(1,n,\tilde{m},k)}=y_1\ldots Y_{(r-1,n,\tilde{m},k)}=y_{r-1})}(u_r) = G^{-1}(1-\bar{G}(y_{r-1})(1-u_r)^{1/\gamma_r}).$$

Therefore, we just need to prove that

$$G^{-1}(1-(1-u_1)^{1/n}) - F^{-1}(1-(1-u_1)^{1/n}) \quad \text{is increasing in } u_1 \in (0,1) \quad \textbf{(3.4)}$$

and

$$G^{-1}(1-\bar{G}(y_{r-1})(1-u_r)^{1/\gamma_r}) - F^{-1}(1-\bar{F}(x_{r-1})(1-u_r)^{1/\gamma_r})$$
$$\text{is increasing in } (u_1,\ldots,u_r) \in (0,1)^r \quad \textbf{(3.5)}$$

for all $r = 2,\ldots,n$, where $x_r = F^{-1}(1-\bar{F}(x_{r-1})(1-u_r)^{1/\gamma_r})$ and $y_r = G^{-1}(1-\bar{G}(y_{r-1})(1-u_r)^{1/\gamma_r})$. If we define recursively the values $p_1 = 1-(1-u_1)^{1/n}$ and $p_r = 1-(1-p_{r-1})(1-u_r)^{1/(n-r+1)}$ for $r = 2,\ldots,n$, it is not difficult to see that (3.4) and (3.5) are equivalent to

$$G^{-1}(p_r) - F^{-1}(p_r) \quad \text{is increasing in } p_r \in (0,1) \quad \textbf{(3.6)}$$

for all $r = 1,\ldots,n$ and that p_r is an increasing function of $(u_1,\ldots,u_r)$. Therefore, the result follows from the assumption $X \leq_{\text{disp}} Y$. ■

Again, condition (3.3) does not imply the dispersive order among coordinates. However, it is possible to obtain such result, as we show in Theorem 3.12. The result is easy to obtain observing that the distribution function of $X_{(r,n,\tilde{m},k)}$ based on an arbitrary distribution function F is given by

$$F_{X_{(r,n,\tilde{m},k)}}(x) = \Phi_{r,n}(F(x)),$$

where $\Phi_{r,n}$ is the distribution function of the uniform generalized order statistic $U_{(r,n,\tilde{m},k)}$.

Now, we have the following result:

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THEOREM 3.12: *Let X and Y be r.v.'s with distribution functions F and G, respectively, and let $\mathbf{X} = (X_{(1,n,\tilde{m},k)},\ldots,X_{(n,n,\tilde{m},k)})$ and $\mathbf{Y} = (Y_{(1,n,\tilde{m},k)},\ldots,Y_{(n,n,\tilde{m},k)})$ be random vectors of generalized order statistics based on F and G, respectively. If $X \leq_{\text{disp}} Y$, then*

$$X_{(r,n,\tilde{m},k)} \leq_{\text{disp}} Y_{(r,n,\tilde{m},k)}.$$

PROOF: The proof follows by observing that

$$F^{-1}_{X_{(r,n,\tilde{m},k)}} G_{Y_{(r,n,\tilde{m},k)}}(x) = F^{-1}\Phi^{-1}_{r,n}\Phi_{r,n}G(x) = F^{-1}G(x). \quad \blacksquare$$

Similar results hold for the convex, star-shaped, and superadditive orders.

Given two random variables X and Y, with distribution functions F and G, respectively, we say that X is less than Y in the convex (star-shaped, superadditive) order (denoted by $X \leq_{\text{cx}[^*,\text{su}]} Y$) if

$$G^{-1}(F(x)) - x \text{ is convex (star-shaped, superadditive) in } x \geq 0. \qquad \textbf{(3.7)}$$

Under the same arguments as in Theorem 3.12, we get the following result.

THEOREM 3.13: *Let X and Y be r.v.'s with distribution functions F and G, respectively, and let $\mathbf{X} = (X_{(1,n,\tilde{m},k)},\ldots,X_{(n,n,\tilde{m},k)})$ and $\mathbf{Y} = (Y_{(1,n,\tilde{m},k)},\ldots,Y_{(n,n,\tilde{m},k)})$ be random vectors of generalized order statistics based on F and G, respectively. If $X \leq_{\text{cx}[^*,\text{su}]} Y$, then*

$$X_{(r,n,\tilde{m},k)} \leq_{\text{cx}[^*,\text{su}]} Y_{(r,n,\tilde{m},k)}.$$

4. STOCHASTIC COMPARISONS OF RANDOM VECTORS OF SPACINGS OF GENERALIZED ORDER STATISTICS

In this section, we provide conditions for the stochastic comparisons of spacings from random vectors of generalized order statistics.

Given a random variable X with distribution function F and a random vector of GOS $\mathbf{X} = (X_{(1,n,\tilde{m},k)},\ldots,X_{(n,n,\tilde{m},k)})$ based on F, we will denote the spacings from $\mathbf{X}$ by $S_{X(r,n,\tilde{m},k)} \equiv X_{(r,n,\tilde{m},k)} - X_{(r-1,n,\tilde{m},k)}$ for $r \geq 2$ and $S_{X(1,n,\tilde{m},k)} \equiv X_{(1,n,\tilde{m},k)}$, and the random vector of spacings from $\mathbf{X}$ by

$$\mathbf{S}_{\mathbf{X}} \equiv (S_{X(1,n,\tilde{m},k)},\ldots,S_{X(n,n,\tilde{m},k)}).$$

We start giving conditions for the multivariate stochastic order of random vectors of spacings from GOS. As we will see later, the following result is given under weaker conditions than those that follow from Theorem 4.10 of [4].

THEOREM 4.1: *Let X and Y be r.v.'s with distribution functions F and G, respectively, and let $\mathbf{X} = (X_{(1,n,\tilde{m},k)},\ldots,X_{(n,n,\tilde{m},k)})$ and $\mathbf{Y} = (Y_{(1,n,\tilde{m},k)},\ldots,Y_{(n,n,\tilde{m},k)})$ be random vectors of generalized order statistics based on F and G, respectively. If*

$$X \leq_{\text{disp}} Y, \qquad \textbf{(4.1)}$$

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then

$$\mathbf{S_X} \leq_{\text{st}} \mathbf{S_Y}. \tag{4.2}$$

PROOF: We proceed, as in Theorem 3.1, to construct two random variables $\tilde{\mathbf{S}}_\mathbf{X}$ and $\tilde{\mathbf{S}}_\mathbf{Y}$ on the same probability space, such that $\tilde{\mathbf{S}}_\mathbf{X} =_{\text{st}} \mathbf{S_X}$, $\tilde{\mathbf{S}}_\mathbf{Y} =_{\text{st}} \mathbf{S_Y}$, and $P[\tilde{\mathbf{S}}_\mathbf{X} \leq \tilde{\mathbf{S}}_\mathbf{Y}] = 1$.

Let us consider a random vector $\mathbf{V} = (V_1, V_2, \ldots, V_n)$ such that $\mathbf{V} =_{\text{st}} \mathbf{X}$. Then, we define

$$\tilde{\mathbf{S}}_\mathbf{X} \equiv (V_1, V_2 - V_1, \ldots, V_n - V_{n-1})$$

and

$$\tilde{\mathbf{S}}_\mathbf{Y} \equiv (G^{-1}F(V_1), G^{-1}F(V_2) - G^{-1}F(V_1), \ldots, G^{-1}F(V_n) - G^{-1}F(V_{n-1})).$$

Clearly, $\tilde{\mathbf{S}}_\mathbf{X} =_{\text{st}} \mathbf{S_X}$ and

$$\begin{aligned} &P(G^{-1}F(V_1) \leq y_1, G^{-1}F(V_2) \leq y_2, \ldots, G^{-1}F(V_n) \leq y_n) \\ &\quad = P(F(V_1) \leq G(y_1), F(V_2) \leq G(y_2), \ldots, F(V_n) \leq G(y_n)) \\ &\quad = P(U_{(1,n,\tilde{m},k)} \leq G(y_1), U_{(2,n,\tilde{m},k)} \leq G(y_2), \ldots, U_{(n,n,\tilde{m},k)} \leq G(y_n)) \\ &\quad = P(Y_{(1,n,\tilde{m},k)} \leq y_1, Y_{(2,n,\tilde{m},k)} \leq y_2, \ldots, Y_{(n,n,\tilde{m},k)} \leq y_n); \end{aligned}$$

therefore, $\tilde{\mathbf{S}}_\mathbf{Y} =_{\text{st}} \mathbf{S_Y}$.

Now, since $F \leq_{\text{disp}} G$ and $V_{j-1} \leq V_j$,

$$G^{-1}F(V_j) - G^{-1}F(V_{j-1}) \geq V_j - V_{j-1}$$

and, therefore, $P[\tilde{\mathbf{S}}_\mathbf{X} \leq \tilde{\mathbf{S}}_\mathbf{Y}] = 1$. ■

If we denote by r and s the failure rates of X and Y, respectively, then the conditions given in Theorem 4.10 of [4], translated to this context are $r(t) \geq s(t + x)$, $t, x \geq 0$. Now, from Proposition 4.2 of [4], this condition is stronger than the dispersive order of X and Y. By the preservation of the stochastic order under marginalization, we obtain Theorem 2.7 by Kamps [20, p. 183].

Next, we proceed to give a result for the stochastic comparison of univariate spacings with gaps and from random vectors of GOS with different dimensions.

In order to state the result, we need first the following two technical results. Let X be a random variable with continuous distribution function F and let $\mathbf{X} = (X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)})$ be a random vector of generalized order statistics based on F. Let us denote the spacing between pth and qth coordinates from $\mathbf{X}$ by $S_{q,p:n} = X_{(q,n,\tilde{m},k)} - X_{(p,n,\tilde{m},k)}$ for $p < q$. Then, we have the following result.

LEMMA 4.2: *Let X be a random variable with continuous distribution function F, let $\mathbf{X} = (X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)})$ be a random vector of generalized order statistics based on F, and let $m_r = m \geq -1$ for all $r = 1, \ldots, n-1$. Then, we have the following:*

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(i) If X has a decreasing failure rate (DFR), then $[S_{q,p:n}|X_{(p,n,\tilde{m},k)} = y]$ is stochastically increasing in $y \in \mathbb{R}$.

(ii) $[S_{q,p:n}|X_{(p,n,\tilde{m},k)} = y] \geq_{\mathrm{st}} [S_{j,i:n}|X_{(i,n,\tilde{m},k)} = y]$ for all y and for all $p \geq i$ and $q - j \geq p - i \geq m - n$.

PROOF: (i) We consider first the case $m \neq -1$. After some computations, we have that the survival function of $[S_{q,p:n}|X_{(p,n,\tilde{m},k)} = y]$ at a point x is given by

$$\overline{M}_{p,q}(x|y) = H_{p,q}\left(\frac{\bar{F}(x+y)}{\bar{F}(y)}\right),$$

where

$$H_{p,q}(t) = K_{p,q} \sum_{j=0}^{q-p-1} \binom{q-p-1}{j} \frac{(-1)^j}{(m+1)j + \gamma_q} t^{(m+1)j+\gamma_q}$$

with

$$K_{p,q} = \frac{\prod_{i=p+1}^{q} \gamma_i}{(q-p-1)!(m+1)^{q-p-1}}.$$

Clearly, $H_{p,q}(t)$ is an increasing function and $\bar{F}(x+y)/\bar{F}(y)$ is increasing in y if X is DFR, and, therefore, $\overline{M}_{p,q}(x|y)$ is increasing in y.

For $m = -1$, the proof follows taking limits when $m \to -1$. It is not hard to prove that $[S_{q,p:n}|X_{(p,n,\tilde{m},k)} = y]$ converges in distribution to $[S_{q,p:n}|X_{(p,n,\tilde{-1},k)} = y]$ when $m \to -1$. By the preservation of the stochastic order by the convergence in distribution, we get part (i) when $m = -1$.

(ii) Let us consider $p \geq i$ and $q - j \geq p - i \geq m - n$. In this case, it can be easily checked that $[U_{(q,n,\tilde{m},k)}|U_{(p,n,\tilde{m},k)} = x] \geq_{\mathrm{st}} [U_{(j,n,\tilde{m},k)}|U_{(i,n,\tilde{m},k)} = x]$, where $U_{(r,n,\tilde{m},k)}$ denotes the uniform generalized order statistics. The proof is easy and can be obtained by showing, instead of the stochastic order, the likelihood ratio order. Now, by the preservation of the stochastic order by increasing transformations, we get $[S_{q,p:n}|X_{(p,n,\tilde{m},k)} = y] \geq_{\mathrm{st}} [S_{j,i:n}|X_{(i,n,\tilde{m},k)} = y]$ for all y and for all $p \geq i$ and $q - j \geq p - i \geq m - n$. ■

Now, we recall the following result.

PROPOSITION 4.3: *Let $\{X_1(\theta), \theta \in \mathcal{X}\}$ and $\{X_2(\theta), \theta \in \mathcal{X}\}$ be two families of random variables. Let Θ_1 and Θ_2 be two random variables with supports in $\mathcal{X}$. Let Y_1 and Y_2 be two random variables such that $Y_i =_{\mathrm{st}} X_i(\Theta_i)$, $i = 1,2$. Sufficient conditions for the implication*

$$\Theta_1 \leq_{\mathrm{st}} \Theta_2 \quad \Rightarrow \quad Y_1 \leq_{\mathrm{st}} Y_2$$

to be true for every such Θ_1 and Θ_2 are

$$X_1(\theta) \leq_{\mathrm{st}} X_2(\theta) \quad \text{for all } \theta \in \mathcal{X}$$

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and $E[\phi(X_1(\theta))]$ or $E[\phi(X_2(\theta))]$ or both are increasing in θ for every increasing function ϕ.

Belzunce and Shaked [7] prove similar results for the (increasing) convex and concave order and mention that these results can be generalized to the (increasing) convex *s*-orders described in Denuit, Lefèvre, and Shaked [12]. In particular, when $s = 1$, we get the stochastic order and, therefore, the previous result.

Let X and Y be r.v.'s with continuous distribution functions F and G, respectively, and let $\mathbf{X} = (X_{(1,n,\tilde{m},k)},\ldots,X_{(n,n,\tilde{m},k)})$ and $\mathbf{Y} = (Y_{(1,l,\tilde{m},k)},\ldots,Y_{(l,l,\tilde{m},k)})$ be random vectors of generalized order statistics based on F and G, respectively. Let us denote $S_{j,i:n} = X_{(j,n,\tilde{m},k)} - X_{(i,n,\tilde{m},k)}$ for $0 \le i < j \le n$ and $V_{j,i:l} = Y_{(j,l,\tilde{m},k)} - Y_{(i,l,\tilde{m},k)}$ for $0 \le i < j \le l$. Then, we have the following result.

THEOREM 4.4: *Under the above notation, if either X or Y or both are DFR and $X \le_{\mathrm{hr}} Y$, then*

$$S_{j,i:n} \le_{\mathrm{st}} V_{q,p:l} \text{ whenever } p \ge i \text{ and } q - j \ge p - i \ge l - n. \tag{4.3}$$

PROOF: We give the proof for the case when F is DFR; in the other case, the proof is similar.

By Theorem 4.1 and the preservation under convolution of the stochastic order, we get that $S_{q,p:l} \le_{\mathrm{st}} V_{q,p:l}$ and, therefore, we just need to prove

$$S_{j,i:n} \le_{\mathrm{st}} S_{q,p:l} \text{ whenever } p \ge i \text{ and } q - j \ge p - i \ge l - n. \tag{4.4}$$

Now, consider $X_1(\theta) = [S_{j,i,n} | X_{(i,n,\tilde{m},k)} = \theta]$, $\Theta_1 = X_{(i,n,\tilde{m},k)}$, $X_2(\theta) = [S_{p,q,l} | X_{(p,l,\tilde{m},k)} = \theta]$, and $\Theta_2 = X_{(p,l,\tilde{m},k)}$. Then $S_{j,i:n} =_{\mathrm{st}} X_1(\Theta_1)$ and $S_{q,p:l} =_{\mathrm{st}} X_2(\Theta_2)$. Now from our Lemma 4.2 and Theorem 2.6 in Kamps [20, p. 182], these random variables satisfy the conditions of Proposition 4.3 and, therefore, $S_{j,i:n} \le_{\mathrm{st}} S_{q,p:l}$ whenever $p \ge i$ and $q - j \ge p - i \ge l - n$. ■

Now, we consider the multivariate likelihood ratio order of random vectors of spacings from GOS.

THEOREM 4.5: *Let X and Y be r.v.'s with distribution functions F and G, density functions f and g, and failure rates r and s, respectively, and let $\mathbf{X} = (X_{(1,n,\tilde{m},k)},\ldots,X_{(n,n,\tilde{m},k)})$ and $\mathbf{Y} = (Y_{(1,n,\tilde{m},k)},\ldots,Y_{(n,n,\tilde{m},k)})$ be random vectors of generalized order statistics based on F and G, respectively. If (a) $X \le_{\mathrm{lr}} Y$, $m_j \ge 0$ for all $j = 1,\ldots,n-1$ and if f or g or both are logconvex, or (b) $X \le_{\mathrm{hr}} Y$, $m_j \ge -1$ for all $j = 1,\ldots,n-1$, r and f or s and g are logconvex, and s/r is increasing, then*

$$(S_{X(1,n,\tilde{m},k)},\ldots,S_{X(n,n,\tilde{m},k)}) \le_{\mathrm{lr}} (S_{Y(1,n,\tilde{m},k)},\ldots,S_{Y(n,n,\tilde{m},k)}) \quad \text{for all } n \ge 1. \tag{4.5}$$

PROOF: (a) First, we note that the multivariate density functions of $\mathbf{S}_X$ and $\mathbf{S}_Y$ are given respectively by

$$l_{\mathbf{S}_X}(x_1,\ldots,x_n) = k\left(\prod_{j=1}^{n-1}\gamma_j f\left(\sum_{i=1}^{j}(x_i)\right)\bar{F}\left(\sum_{i=1}^{j}(x_i)\right)^{m_j}\right) f\left(\sum_{i=1}^{n}(x_i)\right)\bar{F}\left(\sum_{i=1}^{n}(x_i)\right)^{k-1}$$

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and

$$l_{\mathbf{S}_Y}(x_1,\dots,x_n)=k\left(\prod_{j=1}^{n-1}\gamma_j g\left(\sum_{i=1}^{j}(x_i)\right)\bar{G}\left(\sum_{i=1}^{j}(x_i)\right)^{m_j}\right)g\left(\sum_{i=1}^{n}(x_i)\right)\bar{G}\left(\sum_{i=1}^{n}(x_i)\right)^{k-1}$$

for $x_i \geq 0$, for all $i=1,\dots,n$.

Let us suppose that f is logconvex (and, in fact, $\bar{F}$ is also logconvex); the proof in the other case is similar. Under this assumption, we have the following inequality:

$$\begin{aligned}
&l_{\mathbf{S}_X}(x_1\wedge y_1,\dots,x_n\wedge y_n)\,l_{\mathbf{S}_Y}(x_1\vee y_1,\dots,x_n\vee y_n)\\
&\quad\geq k\,\frac{f\left(\sum_{i=1}^{n}x_i\right)f\left(\sum_{i=1}^{n}y_i\right)}{f\left(\sum_{i=1}^{n}x_i\vee y_i\right)}\,\frac{\bar{F}\left(\sum_{i=1}^{n}x_i\right)^{k-1}\bar{F}\left(\sum_{i=1}^{n}y_i\right)^{k-1}}{\bar{F}\left(\sum_{i=1}^{n}x_i\vee y_i\right)^{k-1}}\\
&\qquad\times\prod_{j=1}^{n-1}\gamma_j\,\frac{f\left(\sum_{i=1}^{j}x_i\right)f\left(\sum_{i=1}^{j}y_i\right)}{f\left(\sum_{i=j}^{n}x_i\vee y_i\right)}\,\frac{\bar{F}\left(\sum_{i=1}^{j}x_i\right)^{m_j}\bar{F}\left(\sum_{i=1}^{j}y_i\right)^{m_j}}{\bar{F}\left(\sum_{i=1}^{j}x_i\vee y_i\right)^{m_j}}\\
&\qquad\times kg\left(\sum_{i=1}^{n}x_i\vee y_i\right)\bar{G}\left(\sum_{i=1}^{n}x_i\vee y_i\right)^{k-1}\prod_{j=1}^{n-1}\gamma_j g\left(\sum_{i=1}^{j}x_i\vee y_i\right)\bar{G}\left(\sum_{i=1}^{j}x_i\vee y_i\right)^{m_j}\\
&\quad=kf\left(\sum_{i=1}^{n}x_i\right)f\left(\sum_{i=1}^{n}y_i\right)\bar{F}\left(\sum_{i=1}^{n}x_i\right)^{k-1}\bar{F}\left(\sum_{i=1}^{n}y_i\right)^{k-1}\\
&\qquad\times\prod_{j=1}^{n-1}\gamma_j f\left(\sum_{i=1}^{j}x_i\right)f\left(\sum_{i=1}^{j}y_i\right)\bar{F}\left(\sum_{i=1}^{j}x_i\right)^{m_j}\bar{F}\left(\sum_{i=1}^{j}y_i\right)^{m_j}\\
&\qquad\times k\,\frac{g\left(\sum_{i=1}^{n}x_i\vee y_i\right)}{f\left(\sum_{i=1}^{n}x_i\vee y_i\right)}\,\frac{\bar{G}\left(\sum_{i=1}^{n}x_i\vee y_i\right)^{k-1}}{\bar{F}\left(\sum_{i=1}^{n}x_i\vee y_i\right)^{k-1}}\\
&\qquad\times\prod_{j=1}^{n-1}\gamma_j\,\frac{g\left(\sum_{i=1}^{j}x_i\vee y_i\right)}{f\left(\sum_{i=1}^{j}x_i\vee y_i\right)}\,\frac{\bar{G}\left(\sum_{i=1}^{j}x_i\vee y_i\right)^{m_j}}{\bar{F}\left(\sum_{i=1}^{j}x_i\vee y_i\right)^{m_j}}.
\end{aligned}$$

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Now, given that g/f and $\bar{G}/\bar{F}$ are increasing functions and $m_j \geq 0$ and from the previous inequality, we get the following inequality;

$$\begin{aligned}
&l_{\mathbf{S}_X}(x_1 \wedge y_1, \ldots, x_n \wedge y_n) l_{\mathbf{S}_Y}(x_1 \vee y_1, \ldots, x_n \vee y_n) \\
&\quad \geq k f\left(\sum_{i=1}^{n} x_i\right) f\left(\sum_{i=1}^{n} y_i\right) \bar{F}\left(\sum_{i=1}^{n} x_i\right)^{k-1} \bar{F}\left(\sum_{i=1}^{n} y_i\right)^{k-1} \\
&\qquad \times \prod_{j=1}^{n-1} \gamma_j f\left(\sum_{i=1}^{j} x_i\right) f\left(\sum_{i=1}^{j} y_i\right) \bar{F}\left(\sum_{i=1}^{j} x_i\right)^{m_j} \bar{F}\left(\sum_{i=1}^{j} y_i\right)^{m_j} \\
&\qquad \times k \frac{g\left(\sum_{i=1}^{n} y_i\right)}{f\left(\sum_{i=1}^{n} y_i\right)} \frac{\bar{G}\left(\sum_{i=1}^{n} y_i\right)^{k-1}}{\bar{F}\left(\sum_{i=1}^{n} y_i\right)^{k-1}} \cdot \prod_{j=1}^{n-1} \gamma_j \frac{g\left(\sum_{i=1}^{j} y_i\right)}{f\left(\sum_{i=1}^{j} y_i\right)} \frac{\bar{G}\left(\sum_{i=1}^{j} y_i\right)^{m_j}}{\bar{F}\left(\sum_{i=1}^{j} y_i\right)^{m_j}} \\
&\quad = l_{\mathbf{S}_X}(x_1, \ldots, x_n) l_{\mathbf{S}_Y}(y_1, \ldots, y_n)
\end{aligned}$$

and, therefore, the result.

(b) In this case, we write the multivariate density functions of $\mathbf{S}_X$ and $\mathbf{S}_Y$ respectively by

$$\begin{aligned}
&l_{\mathbf{S}_X}(x_1, \ldots, x_n) \\
&\quad = k\left(\prod_{j=1}^{n-1} \gamma_j\right)\left(\prod_{j=1}^{n-1} r\left(\sum_{i=1}^{j}(x_i)\right) \bar{F}\left(\sum_{i=1}^{j}(x_i)\right)^{m_j+1}\right) f\left(\sum_{i=1}^{n}(x_i)\right) \bar{F}\left(\sum_{i=1}^{n}(x_i)\right)^{k-1}
\end{aligned}$$

and

$$\begin{aligned}
&l_{\mathbf{S}_Y}(x_1, \ldots, x_n) \\
&\quad = k\left(\prod_{j=1}^{n-1} \gamma_j\right)\left(\prod_{j=1}^{n-1} s\left(\sum_{i=1}^{j}(x_i)\right) \bar{G}\left(\sum_{i=1}^{j}(x_i)\right)^{m_j+1}\right) g\left(\sum_{i=1}^{n}(x_i)\right) \bar{G}\left(\sum_{i=1}^{n}(x_i)\right)^{k-1}
\end{aligned}$$

for $x_i \geq 0$, for all $i = 1, \ldots, n$. Now, the proof follows as in the previous case. ■

To finish this section, we consider again the situation of generalized order statistics based on the same distribution but with different parameters. First, we give a result for the multivariate stochastic order, which follows from Theorem 4.10 in [4].

THEOREM 4.6: *Let* $\mathbf{X} = (X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)})$ *and* $\mathbf{X}' = (X_{(1,n,\tilde{m}',k')}, \ldots, X_{(1,n,\tilde{m}',k')})$ *be random vectors of generalized order statistics based on the same distribution* F *and with parameters* k *and* m_i, $i = 1, \ldots, n-1$, *and* k' *and* m_i', $i = 1, \ldots, n-1$, *respectively. If* F *has a decreasing failure rate and* $k \geq k'$ *and* $m_i \geq m_i'$ *for all* $i = 1, \ldots, n-1$, *then* $\mathbf{X} \leq_{\text{st}} \mathbf{X}'$.

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For the multivariate likelihood ratio order, it is possible to get a result under stronger conditions, which improves the conditions that follow from Theorem 4.11 in [4].

THEOREM 4.7: *Let* $\mathbf{X} = (X_{(1,n,\tilde{m},k)},\ldots,X_{(n,n,\tilde{m},k)})$ *and* $\mathbf{X}' = (X_{(1,n,\tilde{m}',k')},\ldots,X_{(1,n,\tilde{m}',k')})$ *be random vectors of generalized order statistics based on the same distribution* F *and with parameters* k *and* m_i, $i = 1,\ldots,n-1$, *and* k' *and* m_i', $i = 1,\ldots,n-1$, *respectively. If* X *has a decreasing and logconvex failure rate and* $k \geq k'$ *and* $m_i \geq m_i'$ *for all* $i = 1,\ldots,n-1$, *then* $\mathbf{X} \leq_{\mathrm{lr}} \mathbf{X}'$.

PROOF: The proof follows the steps similar to the proof of Theorem 4.5. ■

Again, as in Section 3, we can combine Theorems 4.1 and 4.5 with Theorems 4.6 and 4.7 to get stochastic comparisons of random vectors of spacings of GOS based on different distributions and different parameters.

5. APPLICATIONS

In this section, we apply the previous results to several models of random vectors with ordered components. As a particular case, we obtain several results given by other authors and we also give some new results. As we have mentioned in Section 1, most of the results given by other authors for generalized order statistics are given under the assumption that $m_i = m$ for all $i = 1,\ldots,n$, which is not valid for some models, as we will see in this section.

5.1. Order Statistics

Order statistics from a sample of i.i.d. random variables are a particular case of GOS when $k = 1$ and $m_i = 0$ for all $i = 1,\ldots,n-1$. Therefore, most of the previous results can be applied to order statistics, and in fact these results extend some known and recent results about order statistics. In particular, from Theorems 3.5, 3.6(a), 3.11, 3.12, and 3.13, we obtain Theorems 2.1, 2.2, and 2.3 in [6], Lemma 3(a) in [1], and Theorem 3.C.6 in [31]. Also, from Theorems 4.1, 4.4, and 4.5(a), we obtain Theorem 6.1 in [9] (see also Lemmas 3(b) and 3(c) in [1]), Theorem 3.1 in [17], and Theorem 2.2 in [23], respectively.

5.2. Record Values

When $k = 1$ and $m_i = -1$ for all $i = 1,\ldots,n-1$, we get the random vector of the first n epoch times of a nonhomogeneous Poisson process. In fact, the results given in Sections 2 and 3 extend recent and known results. In particular, from Theorems 3.1, 3.5, 3.6(b), 3.11, 3.12, and 3.13, we obtain Proposition 3.9 in [34], Theorems 3.4 and 3.6 in [4], Theorem 3.1 in [5], Theorem 3.10 in [4], and Theorem 2 in [14]. From Theorems 4.1 and 4.5(b), we obtain Proposition 3.10 in [34] and Theorem 4.5 in [4].

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5.3. *k*-Record Values

A generalization of the previous model is the case in which $k \in \mathbb{N}$, obtaining what is called *k*-records (see [20, Sect. 1.5] for references).

Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of i.i.d r.v.'s with distribution function F and let $k \in \mathbb{N}$. Denoting the sequence of kth largest order statistics by

$$(X_{j-k+1,j})_{j=k,k+1,\ldots},$$

the r.v.'s given by

$$\begin{aligned} L^k(1) &= 1, \\ L^k(n+1) &= \min\{j > L(n) : X_{j,j+k-1} > X_{L^k(n),L^k(n)+k-1}\} \end{aligned}$$

are called the record times and the quantities

$$X_{L^k(n)} \equiv X_{L^k(n),L^k(n)+k-1}$$

are called kth records or *k*-records.

For this model, there are no results about stochastic comparisons, at least as far as we know; thus, from the results given in the previous sections, we get a great variety of results in this context, which generalize the results given for the usual record values ($k = 1$).

5.4. Order Statistics Under Multivariate Imperfect Repair

An interesting model contained in the model of generalized order statistics is the model of order statistics under multivariate imperfect repair. We consider the following minimal repair policy introduced by Shaked and Shanthikumar [30]. n items start to function at the same time 0. Upon failure, an item undergoes a repair. If i items ($i = 0,1,\ldots,n-1$) have already been scrapped, then, with probability p_{i+1}, the repair is unsuccessful and the item is scrapped, and with probability $1 - p_{i+1}$, the repair is successful and minimal.

Let us now consider n items with i.i.d. random lifetimes $T_1,\ldots,T_n$, with the same distribution F and density function f. Let $(T_{(1)},\ldots,T_{(n)})$ be the ordered random lifetimes resulting from $T_1,\ldots,T_n$ under a minimal repair policy as above; the joint multivariate density function of $(T_{(1)},\ldots,T_{(n)})$ is given by

$$f(t_1,\ldots,t_n) = n!\prod_{j=1}^{n} p_j f(t_j)(\bar{F}(t_j))^{(n-j+1)p_j-(n-j)p_{j+1}-1} \quad \text{for } 0 \le t_1 \le \cdots \le t_n.$$

This density function is a particular case of the density of generalized order statistics based on F with parameters $k = p_n$ and $m_j = (n-j+1)p_j - (n-j)p_{j+1} - 1$. Therefore, we can apply previous results to get stochastic comparisons for this model. For example, from Theorem 3.8, we get as a particular case Theorem 1 given by Hu [15]. Theorems 4.6 and 4.7 can be applied to provide stochastic comparisons of spacings for this model.

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If we choose $\bar{F}(x) = 1 - e^{-x}$, $x > 0$, then the joint density of the corresponding generalized order statistics is the density of Weinman's multivariate exponential distribution. For this model, it is not required that p_i be a probability; it can be any positive real number. Again, we can apply the previous results for this model.

Recall also that Theorem 3.8 and Theorems 4.6 and 4.7 and their consequences can be applied to compare different models (e.g., order statistics from i.i.d. r.v.'s and order statistics under multivariate imperfect repair).

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