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Topological conjugacy of circle diffeomorphisms

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Abstract. The classical criterion for a circle diffeomorphism to be topologically conjugate to an irrational rigid rotation was given by Denjoy [**1**]. In [**5**] one of us gave a new criterion. There is an example satisfying Denjoy's bounded variation condition rather than the Zygmund condition of [**5**], and vice versa. This paper will give the third criterion which is implied by either of the above criteria.

1. *Introduction*

Given a circle orientation preserving homeomorphism $f : S^1 \to S^1$, the rotation number

$$\rho(f) = \lim_{n\to\infty} \frac{F^n(x) - x}{n} \pmod 1$$

is independent of x and the lift F of f, where $F : R^1 \to R^1$ is a lift of f and $x \in R^1$, and it is invariant under topological conjugations. The rotation number $\rho(f)$ is a rational number if and only if f has a periodic orbit. From the theory of Poincaré, for an orientation preserving homeomorphism $f : S^1 \to S^1$, if f has a periodic orbit then its dynamics turn out trivial: any two periodic orbits have the same period and any orbit tends to a periodic orbit; if f doesn't have any periodic orbit then it is semi-conjugate to an irrational rigid rotation. A natural question to ask is whether or not the semi-conjugation could be improved to be a topological conjugation. In the following context when we say a rigid rotation we always mean an irrational rigid rotation. Denjoy proved the following.

THEOREM A. *Given an orientation preserving homeomorphism f of the circle S^1 with an irrational rotation number, f is topologically conjugate to a rigid rotation provided f is C^1 and the logarithm of the derivative of f is of bounded variation.*

There is an example (Denjoy counterexample) to show that C^1 smoothness is not enough [**6**]. It is shown in [**15**] that even C^∞ smoothness is not enough. Actually, an orientation preserving circle homeomorphism with an irrational rotation number is topologically conjugate to an irrational rigid rotation if and only if it has no wandering interval. Denjoy achieved this by controlling the variation of the derivative. Recently one of us proved the non-existence of a wandering interval by assuming the logarithm of the derivative satisfies the Zygmund condition.

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Definition. A continuous map $f : R^1 \to R^1$ satisfies the Zygmund condition if there exists $B > 0$ such that

$$\sup_{x,t} \left| \frac{f(x+t) + f(x-t) - 2f(x)}{t} \right| \leq B.$$

THEOREM B. *Given an orientation preserving homeomorphism f of the circle S^1 with an irrational rotation number, f is topologically conjugate to a rigid rotation if f is C^1 and the logarithm of the derivative satisfies the Zygmund condition.*

However, there is an example satisfying Denjoy's bounded variation condition and not Zygmund's condition and vice versa (see §5). This paper gives a third criterion which is implied by either of the above two and which implies f is topologically conjugate to a rigid rotation.

Definition. Let I be a closed interval of R^1. A continuous map $f : I \to R^1$ is of bounded Zygmund variation if there exists $B > 0$ such that

$$\sup_{\{x_0,x_1,\ldots,x_n\}} \sum_{i=0}^{n-1} \left| f(x_i) + f(x_{i+1}) - 2f\left(\frac{x_i + x_{i+1}}{2}\right) \right| \leq B,$$

where $\{x_0, x_1, \ldots, x_n\}$ is a partition of the interval I. The supremum is called a Zygmund variation of f over I. It is denoted by $ZV(f|_I)$.

Definition. Let I be a closed interval of R^1. A continuous map $f : I \to R^1$ is of bounded quadratic variation if there exists $B > 0$ such that

$$\sup_{\{x_0,x_1,\ldots,x_n\}} \sum_{i=0}^{n-1} (f(x_{i+1}) - f(x_i))^2 \leq B,$$

where $\{x_0, x_1, \ldots, x_n\}$ is a partition of the interval I. The supremum is called a quadratic variation of f over I. It is denoted by $QV(f|_I)$.

THEOREM C. *Given an orientation preserving homeomorphism f of the circle S^1 with an irrational rotation number, f is topologically conjugate to a rigid rotation if f is C^1 and the logarithm of the derivative has bounded Zygmund variation and bounded quadratic variation.*

2. *Cross ratio distortion*

In this section we control cross ratio distortion for standard 4-tuples in terms of Zygmund variation and quadratic variation (compare [**5**, §1]). Let $a, b, c, d \in R^1$ and $a < b < c < d$.

One cross ratio $[a, b, c, d] = (d - b)(c - a)/[(c - b)(d - a)]$ can be computed by

$$\log[a, b, c, d] = \iint_S \frac{dx\, dy}{(x - y)^2},$$

where S is $\{(x, y) : a \leq x \leq b, c \leq y \leq d\}$.

Another cross ratio (a, b, c, d) is $(c - b)(d - a)/[(b - a)(d - c)]$ and, obviously,

$$[a, b, c, d] = 1 + \frac{1}{(a, b, c, d)}.$$

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Given a homeomorphism h, the distortion of the second cross ratio under h is

$$\frac{(ha, hb, hc, hd)}{(a, b, c, d)}.$$

In this paper, by the cross ratio distortion we mean the distortion of the second cross ratio.

We call a 4-tuple $a < b < c < d$ standard if $b - a = c - b = d - c$. The cross ratio distortion under h of a standard 4-tuple is bounded away from zero and from above if and only if (ha, hb, hc, hd) is also. If h is C^1 diffeomorphism, then

$$\log\left[1 + \frac{1}{(ha, hb, hc, hd)}\right] = \log[ha, hb, hc, hd] = \iint_S (h \times h)^* \mu,$$

where μ is the measure $dx\,dy/(x - y)^2$.

Clearly the cross ratio distortion under f of a standard 4-tuple is bounded away from zero and from above if and only if $\log[ha, hb, hc, hd]$ is also. Calculating the integrand, we get

$$\frac{h'xh'y}{(hx - hy)^2} = \frac{1}{(x - y)^2} \frac{h'xh'y}{[h']^2_{xy}},$$

where $[h']_{xy}$ is the average of h' over the interval $[x, y]$.

Since $b - a = c - b = d - c$, $\iint_S dx\,dy/(x - y)^2 = \log([a, b, c, d]) = \log \frac{4}{3}$. Thus a bound on $h'xh'y/[h']^2_{xy}$ yields a bound on the cross ratio distortions for standard 4-tuples.

We say that h satisfies the *bounded Koebe condition* if one of the following equivalent conditions hold:

$$(1) \quad \frac{1}{M} \le \frac{h'xh'y}{[h']^2_{xy}} \le M \quad \text{for some } M > 0;$$

$$(2) \quad \left|\log \frac{h'xh'y}{[h']^2_{xy}}\right| \le M' \quad \text{for some } M' > 0.$$

The following proposition is trivial.

PROPOSITION 1. *If h satisfies the bounded Koebe condition, then the cross ratio distortion under h of a standard 4-tuple is bounded away from zero and from above.*

In order to estimate the log in (2), i.e.

$$\log h'x + \log h'y - 2\log[h']_{xy},$$

let us consider the following two terms:

$$(a) \quad \log h'x + \log h'y - 2[\log h']_{xy}$$

$$(b) \quad \log[h']_{xy} - [\log h']_{xy}.$$

Remark. If both (a) and (b) are bounded, then (1) and (2) hold.

Expression (a) can be controlled by the Zygmund variation of $\log h'$ on the interval $[x, y]$ because of the following proposition.

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PROPOSITION 2. *Let ϕ be a continuous function from R^1 to R^1. Then,*

$$|\phi(x) + \phi(y) - 2[\phi]_{xy}|$$

is no more than the Zygmund variation $ZV(\phi|_{[x,y]})$ of ϕ over $[x, y]$.

Remark. As we define the Zygmund variation of ϕ on the interval $[a, b]$ in the introduction, we can also define the average Zygmund variation of ϕ on $[a, b]$ by replacing the value $\phi((x_i + x_{i+1})/2)$ of ϕ at the middle point by the average $1/(|x_{i+1} - x_i|)\int_{[x_i, x_{i+1}]}\phi$ of ϕ over $[x_i, x_{1+1}]$. Proposition 2 tells us that the average Zygmund variation of ϕ over $[a, b]$ is no more than the Zygmund variation of ϕ over $[a, b]$. Conversely, one can show that the Zygmund variation of ϕ over $[a, b]$ is no more than twice the average Zygmund variation of ϕ over $[a, b]$. Hence, these two conditions are actually equivalent.

Proof. Without loss of generality, assume $[x, y] = [0, 1]$. Then

$$[\phi]_{01} = \int_0^1 \phi\, dx.$$

Suppose that we define successive approximations to the average of ϕ over $[a, b]$ by

$$A_0[a, b] = (\phi(a) + \phi(b))/2$$

and

$$A_{n+1}[a, b] = (A_n[a, m] + A_n[m, b])/2,$$

where $m = (a + b)/2$. Similarly, measure the non-linearity by expressions

$$N_0[a, b] = A_0[a, b] - A_1[a, b] = (\phi(a) - 2\phi(m) + \phi(b))/4$$

and

$$N_{n+1}[a, b] = (N_n[a, m] + N_n[m, b])/2,$$

or equivalently

$$N_n[a, b] = A_n[a, b] - A_{n+1}[a, b].$$

Then

$$A_0[0, 1] - A_n[0, 1] = N_0[0, 1] + \cdots + N_{n-1}[0, 1]$$

with

$$|N_k[0, 1]| \le ZV(\phi|_{[0,1]})/2^{k+2}.$$

Hence

$$2|A_0[0, 1] - \lim_{n\to\infty} A_n[0, 1]| \le ZV(\phi|_{[0,1]}),$$

i.e.

$$|\phi(0) + \phi(1) - 2[\phi]_{01}| \le ZV(\phi|_{[0,1]}).$$

Next we estimate expression (b) in terms of the quadratic variation.

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LEMMA 1. *If* $\epsilon \geq \delta > -1$, *assume*

$$\log(1+\epsilon) = \epsilon - \frac{\epsilon^2}{2}\Delta(\epsilon).$$

Then there exists $B(\delta) > 0$ *depending on* δ *such that* $|\Delta(\epsilon)| \leq B(\delta)$.

Proof. Since

$$\Delta(\epsilon) = \frac{\epsilon - \log(1+\epsilon)}{\epsilon^2/2},$$

the proof is an elementary calculation. □

Definition. A quantity C_1 (depending on parameters) is a big O of another quantity C_2 (depending on the parameters) if there exists a constant B (independent of the parameters) such that

$$|C_1| \leq B|C_2|.$$

PROPOSITION 3. *Suppose that the derivative* h' *satisfies* $1/C \leq h' \leq C$ *for some* $C > 0$. *Then expression (b) is equal to a big* O *of the quadratic variation of* $\log h'$ *over the interval* $[x, y]$.

Proof. Let $h'(x) = a$. Expression (b) is unchanged if we multiply $h'x$ by $1/a$. Write $(1/a)h'$ on $J = [x, y]$ as $1+\epsilon$, where ϵ is a function of $(t-x)$, $t \in J$. Expand the two terms of (b):

$$\begin{aligned}
&\log \frac{1}{|J|}\int_J (1+\epsilon) - \frac{1}{|J|}\int_J \log(1+\epsilon) \\
&= \log\left(1 + \frac{1}{|J|}\int_J \epsilon\right) - \frac{1}{|J|}\int_J \left[\epsilon - \frac{\epsilon^2}{2}\Delta(\epsilon)\right] \\
&= \left[\frac{1}{|J|}\int_J \epsilon - \frac{1}{2}\left(\frac{1}{|J|}\int_J \epsilon\right)^2 \Delta\left(\frac{1}{|J|}\int_J \epsilon\right)\right] - \left[\frac{1}{|J|}\int_J \epsilon - \frac{1}{|J|}\int_J \frac{\epsilon^2}{2}\Delta(\epsilon)\right] \\
&= -\frac{1}{2}\left(\frac{1}{|J|}\int_J \epsilon\right)^2 \Delta\left(\frac{1}{|J|}\int_J \epsilon\right) + \frac{1}{|J|}\int_J \frac{\epsilon^2}{2}\Delta(\epsilon).
\end{aligned}$$

By the Cauchy inequality,

$$\left(\frac{1}{|J|}\int_J \epsilon\right)^2 \leq \frac{1}{|J|}\int_J \epsilon^2.$$

Since $1/C \leq h' \leq C$ for some $C > 0$, there exists $\delta(C) > -1$ such that $\epsilon = (h't/h'x) - 1 \geq \delta$ for any $t \in J$, $J = [x, y]$. Hence $(1/|J|)\int_J \epsilon \geq \delta$. By Lemma 1, there exists $B(\delta) > 0$ such that $|\Delta(\epsilon)| \leq B(\delta)$. Hence $|\Delta((1/|J|)\int_J \epsilon)| \leq B(\delta)$. Furthermore, we can get that $\epsilon = (h't/h'x) - 1$ is a big O of $\log(h't/h'x) = \log h't - \log h'x$. So expression (b) is a big O of the quadratic variation of $\log h'$ over J. □

The following proposition will be applied in §4 to the iterates of a circle diffeomorphism f.

PROPOSITION 4. *Suppose* $h : I \to R^1$ *is a* C^1 *diffeomorphism with* $h' > 0$, *and* $\log h'$ *has bounded Zygmund variation and bounded quadratic variation over* I. *Assume* $J_0 \subset I$

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and J_0 $J_1 = h(J_0), \ldots, J_n = h^n(J_0)$ are pairwise disjoint. Then the cross ratio distortion under h^n of a standard 4-tuple in the interval J_0 is a big O of the sum of the Zygmund variation and the quadratic variation of $\log h'$ *on* $\cup_{i=0}^{n-1} J_i$.

Proof. From expression (2) above Proposition 1, we want to estimate

$$\log \frac{(h^n)'(x)(h^n)'(y)}{[(h^n)']^2_{xy}}.$$

By the chain rule of calculating the derivative of h^n,

$$\log \frac{(h^n)'(x)(h^n)'(y)}{[(h^n)']^2_{xy}} = \sum_{i=0}^{n-1} \log \frac{h'(h^i(x))h'(h^i(y))}{[h']^2_{h^i(x)h^i(y)}}.$$

Each summand can be decomposed into a sum of expressions (a) and (b). By Proposition 2 and Proposition 3, each summand is a big O of the sum of the Zygmund variation and the quadratic variation of $\log h'$ over the interval $[h^i(x), h^i(y)]$, where $i = 0, 1, 2, \ldots, n-1$. So the cross ratio distortion under h^n of a standard 4-tuple in J_0 is a big O of the sum of the Zygmund variation and the quadratic variation of $\log h'$ on $\cup_{i=0}^{n-1} J_i$. □

3. *Non-wandering set and ergodicity*

In this section we review some basic techniques due to Denjoy [**1**]. Suppose $f : S^1 \to S^1$ is an orientation preserving homeomorphism with an irrational rotation number.

For $x \in S^1$, let

$$\omega(x) = \cap_{n\in N} \mathrm{Cl}(\{f^k(x) \mid k \geq n\}),$$
$$\alpha(x) = \cap_{n\in N} \mathrm{Cl}(\{f^{-k}(x) \mid k \geq n\}),$$

where $\mathrm{Cl}(A)$ means the closure of the set A. They are called the ω limit set of the orbit of x and the α limit set of the orbit of x respectively.

$x \in S^1$ is called a wandering point of f if there exists a neighborhood U of x such that

$$f^k(U) \cap U = \emptyset, \quad \forall k \in Z \setminus \{0\}.$$

A point is called a non-wandering point if it is not a wandering point. $\Omega(f)$ denotes the set of all non-wandering points, which is called the non-wandering set. Clearly it is a closed subset.

A subset A is invariant under f if

$$f(A) \subset A, \quad f^{-1}(A) \subset A.$$

A non-empty subset A is minimal for f if it is closed, invariant under f and there is no non-empty proper closed subset of A which is invariant under f.

PROPOSITION 5. *Suppose f has no periodic point. Then:*

(1) $\Omega(f) = \omega(x) = \alpha(x), \ \forall x \in S^1$;

(2) $\Omega(f)$ *is a minimal set of* f;

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(3) *either $\Omega(f)$ is a nowhere dense perfect subset of S^1 or $\Omega(f) = S^1$.*

Proof. (1) $\omega(x)$ is a non-empty closed invariant subset of S^1. Let (γ, δ) be a component of $S^1 \setminus \omega(x)$, then $f^j((\gamma, \delta))$ is also a component of $S^1 \setminus \omega(x)$ for any $j \in Z$. Since f has no periodic point, $\{f^j([\gamma, \delta]) \mid j \in Z\}$ must be pairwise disjoint and hence (γ, δ) is a wandering interval of f. So $S^1 \setminus \omega(x) \subset S^1 \setminus \Omega(f)$. Hence, $\Omega(f) \subset \omega(x)$. Clearly, $\omega(x) \subset \Omega(f)$. So $\Omega(f) = \omega(x)$. Similarly, $\Omega(f) = \alpha(x)$.

(2) Clearly from (1).

(3) Let $\partial\Omega$ denote the boundary of Ω; $\partial\Omega$ is closed. Since

$$\partial\Omega \subset \Omega, \quad f(\partial\Omega) = \partial f(\Omega) = \partial\Omega,$$

either $\partial\Omega = \emptyset$ hence $\Omega = S^1$, or $\partial\Omega = \Omega$ hence Ω is nowhere dense. For the second case, Ω is perfect since $\Omega = \omega(y), \ \forall y \in \Omega$. □

Definition 2. Suppose f has no periodic point. We say f is ergodic if $\Omega(f) = S^1$, otherwise we say f is not ergodic.

The following result is well known and its proof can be found in several references (e.g. [**1**–**4**]).

PROPOSITION 6. *Suppose an orientation preserving homeomorphism $f : S^1 \to S^1$ has no periodic point and is ergodic, $\alpha = \rho(f)$. Then f is topologically conjugate to an irrational rigid rotation $\tau_\alpha : S^1 \to S^1$ given by*

$$\tau_\alpha(\xi) = e^{2\pi i\alpha}\xi.$$

4. *Proofs of results*

A circle homeomorphism with an irrational rotation number is topologically conjugate to a rigid rotation if and only if it is ergodic, in other words if and only if it has no wandering interval. Denjoy's $C^{1+b.v}$-condition and the C^{1+Z}-condition in [**5**] both guarantee the non-existence of a wandering interval. In this section we prove that the C^1-plus bounded Zygmund variation and bounded quadratic variation guarantee the non-existence of a wandering interval. Before we get into the proofs of these results, we need the following technical lemmas.

PROPOSITION 7. (Contraction principle [**10**, **6**]) *Suppose $f : S^1 \to S^1$ is a circle homeomorphism that has no periodic orbits and that I is a subinterval of S^1. If $\inf_{n\geq 0}\{|f^n(I)|\} = 0$, then I is a wandering interval of f.*

Proof. Let $I_n = f^n(\text{int}\, I)$ and let $\Sigma = \cup_{n\geq 0} I_n$, where $\text{int}(I)$ denotes the interior of I.

Case 1. If $\Sigma = S^1$, then S^1 is covered by finite I_{n_i}, $i = 1, 2, \ldots, k$. Since $\inf_{n\geq 0} |I_n| = 0$, the Lebesgue lemma implies that there is I_l, $l \in N$, contained in one of I_{n_i}, $i = 1, 2, \ldots, k$; name it I_j. Without loss of generality we assume $l > j$. Note $I_l = f^{l-j}(I_j)$. So f^{l-j} will have a periodic point in the closure of I_j. This is a contradiction.

Case 2. Suppose $\Sigma \neq S^1$. If there is no component U of Σ such that some iterate of U intersects with U, then U and hence I are wandering intervals. If there is a component

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U of Σ such that $f^n(U) \cap U \neq \emptyset$ for some $n \geq 0$, then $f^n(U) \subseteq U$ and hence f^n has a periodic point in the closure of U. This is again a contradiction. □

LEMMA 2. (Real Koebe principle) *If $h : I \to R^1$ does not increase the cross ratio distortions for standard 4-tuples too much, then the quasisymmetric distortions for standard interior triples are controlled. More precisely, if $x, y \in I$ satisfy $|x - y|$ is as small as the distance to the boundary ∂I of I and $z = (x + y)/2$, then*

$$\frac{1}{C} \leq |h(x) - h(z)|/|h(z) - h(y)| \leq C,$$

where C only depends on the bound of the cross ratio distortions for standard 4-tuples.

Proof. See [**5**, §2]. The idea to prove this lemma is to use the four interval argument. Let J, L, M, R be four contiguous equal length intervals. Suppose the length of $h(L)$ is much smaller than $h(M)$. Since the cross ratio distortion $[(|h(M)||h(T)|)/(|h(L)||h(R)|)]/3$ on L, M, R is greater than the ratio distortion $|h(M)|/|h(L)|$, no bound of ratio distortions implies no bound of cross ratio distortions. □

It is easy to use the real Koebe principle to get the following macroscopic Koebe distortion principle.

Definition 3. Let M and T be two intervals with $M \subset T$, and let L and R be components of $T \setminus M$. If $\epsilon > 0$ we say T is an ϵ-scaled neighborhood of M if

$$\frac{|L|}{|M|} \geq \epsilon \quad \text{and} \quad \frac{|R|}{|M|} \geq \epsilon.$$

PROPOSITION 8. (Macroscopic Koebe distortion principle) *Given any $B > 0$, $\epsilon > 0$, there exists $\delta > 0$ only depending on B and ϵ such that, for any homeomorphism f of the circle, any subintervals $M \subset T$ and any $n \geq 0$, if the cross ratio distortion under f^{-n} of any standard 4-tuple in $f^n(T)$ is bounded by B and $f^n(T)$ contains an ϵ-scaled neighborhood of $f^n(M)$, then T contains a δ-scaled neighborhood of M.*

Proof. Let $T \setminus M = L \cup R$. Let $\hat{T} = f^n(T)$, $\hat{M} = f^n(M)$ and $\hat{T} \setminus \hat{M} = \hat{L} \cup \hat{R}$. Clearly $|\hat{L}|/|\hat{M}| \geq \epsilon$ and $|\hat{R}|/|\hat{M}| \geq \epsilon$. We want to show that there exists $\delta > 0$, only depending on B and ϵ, such that $|L|/|M| \geq \delta$ and $|R|/|M| \geq \delta$.

Let $\hat{L}_0$ be the interval connected to the left of $\hat{M}$ with length $\epsilon|\hat{M}|/2$. Let $\hat{L}_i$ be the interval connected to the right of $\hat{L}_{i-1}$ with length $\epsilon|M|/2$, where $i \in N$. Let k be the integer part of $1/\epsilon$. Since $|\hat{L}|/|\hat{M}| \geq \epsilon$, $\cup_{i=1}^{k} \hat{L}_i \subset \hat{M} \subset \cup_{i=1}^{k+1} \hat{L}_i$. By applying the real Koebe principle to any two adjacent $\hat{L}$'s, one has

$$|f^{-n}(\hat{L}_0)| \sum_{i=1}^{k} \frac{1}{C^i} \leq |f^{-n}(\hat{M})| \leq |f^{-n}(\hat{L}_0)| \sum_{i=1}^{k+1} C^i,$$

where C is the constant in the real Koebe principle, which is only depending on B.

Now we have

$$|f^{-n}(\hat{L})| > |f^{-n}(\hat{L}_0)| \geq \frac{|f^{-n}(\hat{M})|}{\sum_{i=1}^{k+1} C^i},$$

which means that $|L|/|M| \geq 1/\sum_1^{k+1} C^i$. Let $\delta = 1/\sum_1^{i=1} C^i$. Then $|L|/|M| \geq \delta$, and clearly δ only depends on B and ϵ. Similarly we can show that $|R|/|M| \geq \delta$. □

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Definition 4. The intersection multiplicity of a collection of sets $X_{\alpha\in\Lambda}$ is the maximal cardinality of subcollections with non-empty intersection.

Using the contraction principle (Proposition 7), it is easy to get the following proposition.

PROPOSITION 9. *Suppose $f : S^1 \to S^1$ is an orientation preserving homeomorphism without periodic orbits. Let I be a wandering interval, not contained in any larger wandering interval. If I is a proper subset of an interval J, then the intersection multiplicity of the pullbacks $\{f^{-i}(J) : i = 0, 1, 2, \ldots\}$ is infinity.*

Proof. Suppose the intersection multiplicity of the pullbacks $\{f^{-i}(J) : i = 0, 1, 2, \ldots\}$ is finite. Then $|f^{-n}(J)| \to 0$ as $n \to \infty$. Now apply the contraction principle to J and the map f^{-1}. It says that J is a wandering interval. But this is false because I was a maximal wandering interval. □

Definition 5. Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism. The variation of the logarithm of cross ratio distortion under f is defined as

$$\sup_{\{x_0,x_1,\ldots,x_n\}} \sum_{i=0}^{n-1} \sup_{b_i,c_i\in(x_i,x_{i+1})} \log \frac{(f(x_i), f(b_i), f(c_i), f(x_{i+1}))}{(x_i, b_i, c_i, x_{i+1})},$$

where b_i and c_i belong to the open interval (x_i, x_{i+1}) from x_i to x_{i+1} counter clockwise, and $\{x_0, x_1, \ldots, x_n\}$ is a partition of S^1.

Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism with an irrational rotational number. Let I be a wandering interval for f. The following combinatorial machinery on wandering intervals $I_n = f^n(I) : n = 0, 1, 2, \ldots,$ was developed in [**10**] and can be found in [**6**].

Definition 6. If $n \in N$, we say I_k is a left (or right) predecessor of I_n if there is no I_l, $0 \le l < n$, in the gap (I_k, I_n) (or (I_n, I_k)), where (I_k, I_n) denotes the counter-clockwise gap from I_k to I_n. We denote them by $I_{L(n)}$ and $I_{R(n)}$.

I_n has a successor I_{n+a} if:

(1) I_{n-a} is a left (or right) predecessor (with $0 < a \le n$);

(2) $f^a|_{[I_{n-a},I_{n+a}]}$ (or $f^a|_{[I_{n+a},I_{n-a}]}$) contains no predecessor of I_n;

(3) if I_n is to the left (or right) of I_{n+a}, then there is no I_k, $0 \le k < n + a$ in the gap (I_n, I_{n+a}) (or (I_{n+a}, I_n)).

Furthermore, we define the natural neighborhood T_n of I_n to be the biggest closed interval containing I_n which contains no I_i, $0 \le i < n$, except its nearest predecessor or successor.

Remark. Of course, I_n can have at most one predecessor on each side. Moreover, I_n has at most one successor, denote it by $I_{S(n)}$. Therefore $T_n = [I_{L(n)}, I_{R(n)}]$ if I_n has two predecessors and no successor and $T_n = [I_{L(n)}, I_{S(n)}]$ (or $T_n = [I_{S(n)}, I_{R(n)}]$) if I_n has a successor.

One can prove the following lemmas ([**6**, p. 309]).

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LEMMA 3. *For every $n \in N$, I_n can have at most one successor.*

LEMMA 4. *Assume the interval I_n has two predecessors $I_{L(n)}$, $I_{R(n)}$, and a successor $I_{S(n)}$. If this successor is to the right of I_n then the predecessors of $I_{S(n)}$ are I_n and $I_{R(n)}$, and if $I_{S(n)}$ has a successor then this successor must be, again, to the right of $I_{S(n)}$.*

Remark. This lemma implies that if I_n has a successor $I_{S(n)}$, and $I_{S(n)}$ also has a successor $I_{S(S(n))}$, then $S(n)-n = S(S(n))-S(n)$, and $I_{S(n)}$ is between I_n and $I_{S(S(n))}$. Continuing this if there exists a maximal integer k such that $I_{S^{i+1}(n)}$ is a successor of $I_{S^i(n)}$ for $0 \le i \le k-1$, then the intervals $I_{S^i(n)}$, $0 \le i \le k-1$, are ordered and f^a acts as a translation on these intervals, where $a = S(n) - n$.

THEOREM 1. ([**6**, p. 310]) *Let $n \in N$ and assume that I_n has two predecessors $I_{L(n)}$ and $I_{R(n)}$. Let $M_n \supset I_n$ be an interval contained either in $[I_{L(n)}, I_n]$ or in $[I_n, I_{R(n)}]$. Assume that $\{M_{t_0}, M_{t_0+1}, \ldots, M_n\}$ are pullbacks of M_n. If the intersection multiplicity of this collection is at least $2m$ and $m \ge 2$, then there exists $t \in \{t_0, \ldots, n\}$ such that:*
(1) *$I_{S(t)}, I_{S^2(t)}, \ldots, I_{S^{2m-2}(t)}$ are defined;*
(2) *$n = S^m(t)$ and $I_{S^j(t)}$ is contained in M_n for $j = m, \ldots, 2m-2$.*

COROLLARY 1. *Assume an interval $T \supset I_n$ and T is contained in the natural neighborhood T_n of I_n. Then the intersection multiplicity of the pullbacks of T is at most* 15.

Proof. Consider the pullbacks of $T \cap [I_{L(n)}, I_n]$ and $T \cap [I_n, I_{R(n)}]$ separately. Suppose the intersection multiplicity of the pullbacks of T is at least 16. Then either the pullbacks of $T \cap [I_{L(n)}, I_n]$ or the pullbacks of $T \cap [I_n, I_{R(n)}]$ has intersection multiplicity ≥ 8. Take $m = 4$. The previous theorem implies that $I_{S^2(n)}$ is contained in $T \cap [I_n, I_{R(n)}]$. This is impossible because of $T \subset T_n$. □

Now we can prove the following theorem.

THEOREM 2. *Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism with an irrational rotation number. If the logarithm of the cross ratio distortion under f has bounded variation B, then f has no wandering interval, hence it is topologically conjugate to a rigid rotation.*

Proof. Suppose I is a maximal wandering interval for f and $I_n = f^n(I)$, $n \ge 0$, $n \in Z$. There exists arbitrarily large $n \in N$ and $l, r < n$ such that $I_n \subset (I_l, I_r)$, $I_k \cap (I_l, I_r) = \emptyset$, $0 \le k < n$, and $|I_n| \le \min\{|I_l|, |I_r|\}$. This property is proved as follows. Pick up I_l and I_r such that the gap (I_l, I_r) contains no I_k for $0 \le k \le \max\{l, r\}$. By the density of any orbit under an irrational rotation, there exists an I_n that first gets into the gap (I_l, I_r). If $|I_n| \le \min\{|I_l|, |I_r|\}$, then it is done, otherwise replace I_r by I_n and go on. Since the sum of the lengths of I_k is bounded, eventually we will get $|I_n| \le |I_r|$, furthermore we get $|I_n| \le |I_l|$. We have seen I_l and I_r are two predecessors of I_n.

Let T_n be the natural neighborhood of I_n. If I_n has no successor, then $T_n = [I_l, I_r]$. By Corollary 1, the intersection multiplicity of the pullbacks of T_n is bounded by 15. As n varies to the positive infinite, the length of T_n goes to zero. Using the macroscopic

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Koebe distortion principle, we get a bigger wandering interval J strictly containing I. This contradicts with the maximality of I. Hence I_n has a successor $I_{s(n)}$. Use the same method as above, we get $|I_{s(n)}| < |I_n|$. Inductively, we get infinitely many successors $I_{s^i(n)}, i = 1, 2, \ldots$, and by Theorem 1 all successors are contained in $[I_l, I_r]$ and are ordered. Moreover, $s^i(n) - s^{i-1}(n)$ is a constant $a = s(n) - n$. It follows that $I_{s^i(n)}$ converges to a fixed point of f^a as $i \to \infty$. This contradicts f having no periodic points. □

Proof of Theorem C. Let I be a maximal wandering interval for f and $I_n = f^n(I)$, $n \geq 0$, $n \in Z$. Let T_n be the natural neighborhood of I_n. The intersection multiplicity of the pullbacks of T_n is bounded by 15. By Proposition 4 the cross ratio distortion of f^n on T_0 is uniformly bounded by a constant B. The rest of the proof follows the proof of Theorem 2. □

The remainder of this section explains why the conditions of Theorem C is weaker than Denjoy's condition and the condition of [**5**]. It is almost trivial that Denjoy's condition implies the conditions of the Theorem C.

PROPOSITION 10. *Let $h : I \to R^1$ be a C^1 smooth function and let $\log h'$ be of bounded variation. Then $\log h'$ is of bounded Zygmund variation and bounded quadratic variation.*

Proof. By the triangle inequality, the Zygmund variation of $\log h'$ is no more than the variation of $\log h'$ on the interval I. Let M be the maximal value of $|\log h'|$ on the interval I, then the quadratic variation of $\log h'$ is no more than $2M$ multiplied by the variation of $\log h'$ on the interval I.

Clearly the Zygmund condition implies bounded Zygmund variation. Furthermore, the Zygmund condition implies α-Hölder continuous for $0 < \alpha < 1$. The $(1/2)$-Hölder continuity implies the bounded quadratic variation. □

LEMMA 5. *If $\phi : I \to R^1$ satisfies the Zygmund condition, there exists $B > 0$ such that*

$$\sup_{x,t} \left| \frac{\phi(x+t) + \phi(x-t) - 2\phi(x)}{t} \right| \leq B.$$

Then ϕ is α-Hölder continuous for any $0 < \alpha < 1$.

Proof. Denote

$$D(x,t) = \frac{\phi(x+t) - \phi(x)}{t}.$$

Then

$$\begin{aligned}
&D(x,t/2) + D(x+t/2,t/2) = 2D(x,t), \quad |D(x,t/2) - D(x+t/2,t/2)| \leq B,\\
&D(x,t/4) + D(x+t/4,t/4) = 2D(x,t/2), \quad |D(x,t/4) - D(x+t/4,t/4)| \leq B,\\
&\qquad\qquad \vdots\\
&D\left(x, \frac{t}{2^n}\right) + D\left(x + \frac{t}{2^n}, \frac{t}{2^n}\right) = 2D\left(x, \frac{t}{2^{n-1}}\right),\\
&\left| D\left(x, \frac{t}{2^n}\right) - D\left(x + \frac{t}{2^n}, \frac{t}{2^n}\right) \right| \leq B.
\end{aligned}$$

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These give us

$$|D(x, t/2^n)| \leq |D(x, t)| + nB,$$

i.e.

$$\left|\frac{\phi(x + t/2^n) - \phi(x)}{t/2^n}\right| \leq |D(x, t)| + nB.$$

Then

$$\left|\frac{\phi(x + t/2^n) - \phi(x)}{(t/2^n)^\alpha}\right| \leq (|D(x, t)|/n + B)n(|t|/2^n)^{1-\alpha},$$

which tells us that ϕ is α-Hölder continuous for any $0 < \alpha < 1$. □

PROPOSITION 11. *If $\phi : I \to R^1$ satisfies the Zygmund condition, then ϕ is of bounded Zygmund variation and bounded quadratic variation over the interval I.*

5. *Three examples*

In the introduction it is mentioned that there exists an example satisfying Denjoy's bounded variation condition rather than [**5**]'s Zygmund condition and vice versa. In this section, we will give these two examples and we will also give an example to show that there is an example being of bounded quadratic variation but not being of bounded Zygmund variation.

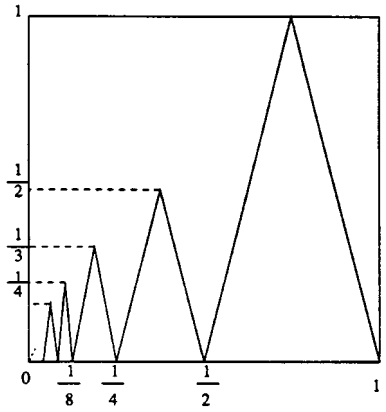


FIGURE 1.

Example 1. Let $\phi : [-1, 1] \to [-1, 1]$ be the following function:

$$\phi(x) = x, \quad x \in [-1, 0],$$
$$\phi(x) = \sqrt{x}, \quad x \in (0, 1].$$

Clearly, ϕ is monotone hence it is of bounded variation, but the Zygmund condition fails since the right derivative of ϕ at the point 0 is infinite but the left derivative is 1.

Example 2. Let $\phi_0(x) = 2x$ for $x \in [0, \frac{1}{2}]$ and $\phi_0(x) = 2 - 2x$ for $x \in [\frac{1}{2}, 1]$, and let

$$\phi_n = \frac{\phi(2^n x - i)}{2^n} \quad \text{for } x \in \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right],$$

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where $i = 0, 1, \ldots, 2^n - 1$.

Let $\phi(x) = \sum_{n=0}^{\infty} \phi_n(x)$. ϕ is differentiable at a set of measure 0 only. It cannot be of bounded variation, otherwise it is differentiable almost everywhere which is a contradiction. [**7**] and [**8**] study the general theory about the differentiability of a function satisfying the Zygmund condition.

Example 3. Let $\phi : [0, 1] \to [0, 1]$ be defined by Figure 1. It is easy to obtain that the quadratic variation of ϕ on $[0, 1]$ is equal to $\sum_{n=1}^{\infty} 2/n^2$ which is finite. But the difference between the left derivative and the right derivative of ϕ at $1/2^n$ is equal to

$$\frac{2^{n+2}}{n+1} - \frac{2^{n+1}}{n} = \frac{2^{n+1}}{n}\frac{n-1}{n+1}$$

which tends to ∞ as $n \to \infty$. Therefore it has bounded quadratic variation but has no bounded Zygmund variation.

Actually, the remaining question is to study whether or not the bounded Zygmund variation property implies the bounded quadratic variation property.

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Appendix

The non-existence of wandering domains for any rational map of the complex sphere was proved by Sullivan in 1985 [**9**]. The analogue of this theorem for one-dimensional dynamical systems was done for certain smooth multimodal maps by Martens *et al* in 1992 [**10**]. In the latest publication [**6**], the smooth condition used by de Melo and van Strien is that a multimodal map piecewise satisfies $C^{1+b.v}$ (or C^{1+Z}) and the map can be written as a power map ($x \mapsto |x|^{\alpha}$, $\alpha > 1$) composed by a $C^{1+b.v}$ (or C^{1+Z}) diffeomorphism around every turning point. Combining the analysis work of getting a bound of cross ratio distortions in this paper and the combinatorial machinery on wandering intervals in [**10**] or [**6**, p. 308–312], we can get a weak version of the theorem of Martens *et al* of no wandering intervals for multimodal maps. Before we state the theorem, let us give the definition of a wandering interval for a multimodal map of an interval.

Definition 7. Let $f : I \to I$ be a continuous map of an interval I. An open interval $J \subset I$ is called a wandering interval of f if:

(1) $f^n(J) \cap f^m(J) = \emptyset$ for any $n \neq m$, $n, m \in N$;

(2) $f^n(J)$ does not converge to a periodic orbit.

THEOREM 3. ([**11**]) *Let $f : I \to I$ be a C^1 smooth map satisfying the following.*

(1) *f is $C^{1+b.Z.v+b.q.v}$ away from critical points.*

(2) *Let K_f be the set of critical points of f. For each $x_0 \in K_f$, there exist $\alpha > 1$, a neighborhood $U(x_0)$ of x_0 and a $C^{1+b.Z.v+b.q.v}$ diffeomorphism $\phi : U(X_0) \to (-1, 1)$*

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such that $\phi(x_0) = 0$ *and*

$$f(x) = f(x_0) \pm |\phi(x)|^{\alpha}, \quad \forall x \in U(x_0).$$

Then f *has no wandering intervals.*

Norton, Sullivan and Velling [**12**–**14**] have begun the work of generalizing the setting of Denjoy's theorem to two-dimensional dynamical systems by considering diffeomorphisms of the torus. The quasiconformal theory has found a place there.

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