Canad. Math. Bull. Vol. 64 (4), 2021, pp. 911–922
http://dx.doi.org/10.4153/S0008439520000946
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An estimate for the composition of rough singular integral operators

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Abstract. Let Ω be homogeneous of degree zero and have mean value zero on the unit sphere S^{d-1}, T_Ω be the convolution singular integral operator with kernel $\frac{\Omega(x)}{|x|^d}$. In this paper, we prove that if $\Omega \in L\log L(S^{d-1})$, and U is an operator which is bounded on $L^2(\mathbb{R}^d)$ and satisfies the weak type endpoint estimate of $L(\log L)^\beta$ type, then the composition operator UT_Ω satisfies a weak type endpoint estimate of $L(\log L)^{\beta+1}$ type.

1 Introduction

We will work on $\mathbb{R}^d$, $d \geq 2$. Let $p \in (1, \infty)$ and T_1, T_2 be two singular integral operators. It is obvious that if T_1 and T_2 are bounded on $L^p(\mathbb{R}^d)$, then the composite operator T_1T_2 is also bounded on $L^p(\mathbb{R}^d)$. However, if T_1 and T_2 are all bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$, we cannot deduce the mapping property of T_1T_2 on $L^1(\mathbb{R}^d)$ immediately.

The composition of singular integral operators arises typically in the algebra of singular integral (see [2, 3, 6]) and the noncoercive boundary-value problems for elliptic equations (see [16, 17]). In the past decades, consideration attention has been paid to the composition of singular integral operators (see [1, 8, 16, 17, 20] and the references therein). Carozza and Passarelli di Napoli [7] considered the weak type endpoint estimates for the composition of maximal operators and proved that for $k \in \mathbb{N}$, M^k, the kth iterated operator of the Hardy–Littlewood maximal operator satisfies

$$|\{x \in \mathbb{R}^d : M^k f(x) > \lambda\}| \lesssim \int_{\mathbb{R}^d} \Psi_{k-1}\left(\frac{|f(x)|}{\lambda}\right)dx,$$

where, and in the following, for $\beta \in [0, \infty)$, $\Psi_\beta(t) = t\log^\beta(e + t)$. Benea and Bernicot [1] and Hu [12] studied the weighted weak type endpoint estimates for the compositions of Calderón–Zygmund operators. It should be pointed out that the results in [1, 12] depend on the regularity of kernel of the Calderón–Zygmund operators.

Received by the editors April 14, 2020; revised October 22, 2020.
Published online on Cambridge Core December 7, 2020.
The research of X.T was supported by the NNSF of China under grant #11771399, and the research of G.H. (corresponding author) was supported by the NNSF of China under grant #11871108.
AMS subject classification: **42B20**, 47B33.
Keywords: Rough singular integral operator, composite operator, weak type endpoint estimate.

https://doi.org/10.4153/S0008439520000946 Published online by Cambridge University Press

Let Ω be homogeneous of degree zero, integrable, and have mean value zero on the unit sphere S^{d-1}. Define the singular integral operator T_Ω by

$$T_\Omega f(x) = \text{p. v.} \int_{\mathbb{R}^d} \frac{\Omega(y')}{|y|^d} f(x-y)dy, \tag{1.2}$$

where, and in the following, $y' = y/|y|$ for $y \in \mathbb{R}^d$. This operator was introduced by Calderón and Zygmund [4] and then studied by many authors in the last 60 years. Calderón and Zygmund [5] proved that if $\Omega \in L\log L(S^{d-1})$, then T_Ω is bounded on $L^p(\mathbb{R}^d)$ for $p \in (1, \infty)$. Ricci and Weiss [18] improved the result of Calderón–Zygmund and showed that $\Omega \in H^1(S^{d-1})$ guarantees the $L^p(\mathbb{R}^n)$ boundedness on $L^p(\mathbb{R}^n)$ for $p \in (1, \infty)$. Seeger [19] showed that $\Omega \in L\log L(S^{d-1})$ is a sufficient condition, such that T_Ω is bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$. For other works about the mapping properties of T_Ω, we refer the papers [9, 10, 18] and the references therein.

Let $\eta \in (0, 1)$ and $\mathcal{S} = \{Q_j\}$ be a family of cubes. We say that $\mathcal{S}$ is η-sparse, if for each fixed $Q \in \mathcal{S}$, there exists a measurable subset $E_Q \subset Q$, such that $|E_Q| \geq \eta|Q|$ and E_Qs are pairwise disjoint. Let T be a sublinear operator acting on $\cup_{p\geq 1}L^p(\mathbb{R}^d)$, β, $q \in (0, \infty)$. We say that T enjoys a $(L(\log L)^\beta, L^q)$-bilinear sparse domination, if for each bounded function f with compact support, there exists a sparse family $\mathcal{S}$ of cubes, such that for all $h \in L^q_{\text{loc}}(\mathbb{R}^d)$,

$$\Big|\int_{\mathbb{R}^d} h(x)Tf(x)dx\Big| \lesssim \sum_{Q\in\mathcal{S}} \|f\|_{L(\log L)^\beta, Q}\langle|h|\rangle_{Q,q}|Q|,$$

where

$$\|f\|_{L(\log L)^\beta, Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \Psi_\beta\left(\frac{|f(y)|}{\lambda}\right)dy \leq 1\right\},$$

$\langle|h|\rangle_Q$ denotes the mean value of $|h|$ on Q, and $\langle|h|\rangle_{Q,r} = (\langle|h|^r\rangle_Q)^{\frac{1}{r}}$. Recently, Hu et al. [13] considered the composition of operators in abstract settings and obtained the following result.

Theorem 1.1 *Let $q \in (1, 3/2]$, U_1 be a sublinear operator and U_2 be a linear operator on $\cup_{p\geq 1}L^p(\mathbb{R}^d)$. Suppose that*

(i) *U_1 is bounded on $L^2(\mathbb{R}^d)$ with bound 1;*

(ii) *there exists a positive constant β_1, such that for any $\lambda > 0$,*

$$|\{x \in \mathbb{R}^d : |U_1 f(x)| > \lambda\}| \leq \int_{\mathbb{R}^d} \Psi_{\beta_1}\left(\frac{|f(x)|}{\lambda}\right)dx; \tag{1.3}$$

(iii) *for some $q \in (1, \frac{3}{2}]$ and $\beta_2 \in [0, \infty)$, U_2 enjoys a bilinear $(L(\log L)^{\beta_2}, L^q)$-sparse domination with bound 1.*

Then, for any $\lambda > 0$,

$$|\{x \in \mathbb{R}^d : |U_1U_2 f(x)| > \lambda\}| \lesssim \int_{\mathbb{R}^d} \Psi_{1+\beta_1+\beta_2}\left(\frac{|f(x)|}{\lambda}\right)dx. \tag{1.4}$$

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Recall that when $\Omega \in L^\infty(S^{d-1})$, T_Ω satisfies the (L^1, L^r)-bilinear sparse domination for any $r \in (1, \infty)$ (see [13, 14]), we deduce from Theorem 1.1 that

Corollary 1.2 *Let Ω be homogeneous of degree zero, have mean value zero on S^{d-1}, and $\Omega \in L^\infty(S^{d-1})$, T_Ω be the operator defined by equation (1.2). Let U be a sublinear operator which is bounded on $L^2(\mathbb{R}^d)$ and satisfies equation (1.3) for some $\beta_1 \in [1, \infty)$. Then for any $\lambda > 0$,*

$$|\{x \in \mathbb{R}^d : |U(T_\Omega f)(x)| > \lambda\}| \lesssim \int_{\mathbb{R}^d} \Psi_{\beta_1+1}\left(\frac{|f(x)|}{\lambda}\right) dx. \tag{1.5}$$

For the case of $\Omega \in L\log L(S^{d-1})$, we know that T_Ω does not satisfy a bilinear sparse domination of $(L(\log L)^\beta, L^q)$ type for some $\beta \in [1, \infty)$ and $q \in (1, \infty)$. In fact, if T_Ω satisfies a bilinear sparse domination of $(L(\log L)^\beta, L^q)$ type, then for any bounded functions f and h, a standard duality argument tells us that there exists a sparse family $\mathcal{S}$, such that

$$\left| \int_{\mathbb{R}^d} h(x) T_\Omega f(x) dx \right| \lesssim \sum_{Q \in \mathcal{S}} \langle |f| \rangle_{Q,q} \|h\|_{L(\log L)^\beta, Q} |Q|. \tag{1.6}$$

For $p \in [1, \infty)$, let $A_p(\mathbb{R}^d)$ be the weight function class of Muckenhoupt (see [11, Chapter 9] for definitions and properties of $A_p(\mathbb{R}^d)$), $M_q f(x) = \left[M(|f|^q)(x)\right]^{1/q}$, and

$$M_{L(\log L)^\beta} h(x) = \sup_{Q \ni x} \|h\|_{L(\log L)^\beta, Q}.$$

By inequality (1.6) and sparseness of $\mathcal{S}$, we know that for $p \in (q, \infty)$ and $w \in A_{p/q}(\mathbb{R}^d)$,

$$\begin{aligned} \left| \int_{\mathbb{R}^d} h(x) T_\Omega f(x) dx \right| &\lesssim \int_{\mathbb{R}^d} M_q f(x) M_{L(\log L)^\beta} h(x) dx \\ &\lesssim \|M_q f\|_{L^p(\mathbb{R}^d, w)} \|M_{L(\log L)^\beta} h\|_{L^{p'}(\mathbb{R}^d, w^{-1/(p-1)})} \\ &\lesssim \|f\|_{L^p(\mathbb{R}^n, w)} \|h\|_{L^{p'}(\mathbb{R}^d, w^{-1/(p-1)})}. \end{aligned}$$

But [15, Theorem 1] tells us that the last inequality is false when $\Omega \in L\log L(S^{d-1})$. Thus, Theorem 1.1 does not apply to T_Ω in this case. Our main purpose in this paper is to prove that inequality (1.5) still holds when $\Omega \in L\log L(S^{d-1})$, namely we will prove the following result.

Theorem 1.3 *Let $\beta \in [0, \infty)$, U be a sublinear operator which is bounded on $L^2(\mathbb{R}^d)$ and satisfies the weak type endpoint estimate of $L(\log L)^\beta$ type, that is, for any $\lambda > 0$,*

$$|\{x \in \mathbb{R}^d : |Uf(x)| > \lambda\}| \lesssim \int_{\mathbb{R}^d} \Psi_\beta\left(\frac{|f(x)|}{\lambda}\right) dx. \tag{1.7}$$

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Let Ω be homogeneous of degree zero, have mean value zero on S^{d-1} and $\Omega \in L\log L(S^{d-1})$, T_Ω be the operator defined by equation (1.2). Then for $\lambda > 0$ and bounded function f with compact support,

$$\left|\{x \in \mathbb{R}^d : |UT_\Omega f(x)| > \lambda\}\right| \lesssim \int_{\mathbb{R}^d} \Psi_{\beta+1}\left(\frac{|f(x)|}{\lambda}\right)dx. \tag{1.8}$$

As an application of Theorem 1.3, we obtain the following result.

Corollary 1.4 *Let $\Omega_1, \ldots, \Omega_k$ be homogeneous of degree zero, have mean value zero on S^{d-1}, and $\Omega_1, \ldots, \Omega_k \in L\log L(S^{d-1})$. Then for $\lambda > 0$,*

$$\left|\{x \in \mathbb{R}^d : |T_{\Omega_1}(T_{\Omega_2} \ldots T_{\Omega_k} f)(x)| > \lambda\}\right| \lesssim \int_{\mathbb{R}^d} \Psi_{k-1}\left(\frac{|f(x)|}{\lambda}\right)dx.$$

Remark 1.5 Let M_Ω be the operator defined by

$$M_\Omega f(x) = \sup_{r>0} r^{-d} \int_{|y-x|<r} |\Omega(x-y)||f(y)|dy. \tag{1.9}$$

Using the estimates relating to M_Ω in [9], and the argument used in the proof of Theorem 1.3, we can verify the following Theorem 1.6.

Theorem 1.6 *Let $\beta \in [0, \infty)$, U be a sublinear operator which is bounded on $L^2(\mathbb{R}^d)$ and satisfies the weak type endpoint estimate of $L(\log L)^\beta$ type. Let Ω be homogeneous of degree zero and $\Omega \in L\log L(S^{d-1})$. Then for $\lambda > 0$ and bounded function f with compact support,*

$$\left|\{x \in \mathbb{R}^d : |UM_\Omega f(x)| > \lambda\}\right| \lesssim \int_{\mathbb{R}^d} \Psi_{\beta+1}\left(\frac{|f(x)|}{\lambda}\right)dx.$$

It was proved in [9] that if $\Omega \in L\log L(S^{d-1})$, then M_Ω is bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$. By Theorem 1.6, we know that if $\Omega_1, \ldots, \Omega_k \in L\log L(S^{d-1})$, then for any $\lambda > 0$,

$$\left|\{x \in \mathbb{R}^d : M_{\Omega_1}(M_{\Omega_2} \ldots M_{\Omega_k} f)(x) > \lambda\}\right| \lesssim \int_{\mathbb{R}^d} \Psi_{k-1}\left(\frac{|f(x)|}{\lambda}\right)dx.$$

This, in some sense, is an extension of inequality (1.1).

Remark 1.7 In the proof of Theorem 1.3, we will employ some ideas and estimates of Seeger [19]. The main difficult in our proof is that for $\beta \in (0, \infty)$, the inequality

$$\int_{\mathbb{R}^d} \Psi_\beta\left(\sum_{j=1}^{\infty} |f_j(x)|\right)dx \lesssim \sum_{j=1}^{\infty} \int_{\mathbb{R}^d} \Psi_\beta(|f_j(x)|)dx$$

is not true. Our observation is that, if b is the bad part in the Calderón–Zygmund decomposition of function f, then $T_\Omega b(x)$ can be decomposed to four terms, all

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of which enjoy suitable $L^2(\mathbb{R}^d)$ estimates. These estimates either lead to desired conclusion directly or lead to the desired estimate via the $L^1(\mathbb{R}^d)$ estimate and an interpolation argument.

Throughout this paper, C always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the symbol $A \lesssim B$ to denote that there exists a positive constant C, such that $A \le CB$. Specially, we use $A \lesssim_{d,p} B$ to denote that there exists a positive constant C depending only on d, p, such that $A \le CB$. Constant with subscript such as c_1 does not change in different occurrences. For any set $E \subset \mathbb{R}^d$, χ_E denotes its characteristic function. For a cube $Q \subset \mathbb{R}^d$ and $\lambda \in (0, \infty)$, we use $\ell(Q)$ to denote the side length of Q and λQ to denote the cube with the same center as Q and whose side length is λ times that of Q.

2 Proof of Theorem 1.3

We employ the ideas in [19, Section 2]. For $s > 3$, let $\mathfrak{E}^s = \{e_v^s\}$ be a collection of unit vectors on S^{d-1}, such that

(a) for $v \neq v'$, $|e_v^s - e_{v'}^s| > 2^{-s-10}d^{-1}$;
(b) for each $\theta \in S^{d-1}$, there exists an e_v^s satisfying that $|e_v^s - \theta| \le 2^{-s-1}$.

We may construct disjoint measurable set $E_v^s \subset S^{d-1}$ with $e_v^s \in E_v^s$, $\operatorname{diam}(E_v^s) \le 2^{-s}$, and $\cup_v E_v^s = S^{d-1}$.

Fix a number $\kappa \in (0, 1)$. Let $\psi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\psi \subset [-4, 4]$ and $\psi(t) \equiv 1$ when $t \in [-2, 2]$. Define the operator P_v^s by

$$\widehat{P_v^s}(\xi) = \psi(2^{s(1-\kappa)}\langle \xi, e_v^s\rangle/|\xi|).$$

Take a smooth radial non-negative function ϕ on $\mathbb{R}^d$, such that $\operatorname{supp}\phi \subset \{x : \frac{1}{2} \le |x| \le 2\}$ and $\sum_j \phi_j(x) = 1$ for all $x \in \mathbb{R}^d\backslash\{0\}$, where $\phi_j(x) = \phi(2^{-j}x)$. Let Ω be homogeneous of degree zero and $\Omega \in L^1(S^{d-1})$, $K_j(x) = \Omega(x')|x|^{-d}\phi_j(x)$,

$$K_{jv}^s(x) = K_j(x)\chi_{E_v^s}(x/|x|),\ \Gamma_j^s(x) = \sum_v P_v^s * K_{jv}^s(x). \tag{2.1}$$

Lemma 2.1 *Let Ω be homogeneous of degree zero and $\Omega \in L^\infty(S^{d-1})$, $\mathcal{Q}$ be a collection of cubes with disjoint interiors. Define $L(Q) = m$ if $2^{m-1} < \ell(Q) \le 2^m$ and let $\mathcal{Q}_m = \{Q \in \mathcal{Q} : L(Q) = m\}$. For each cube $Q \in \mathcal{Q}$, let f_Q be an integrable function supported in Q satisfying*

$$\|f\|_{L^1(\mathbb{R}^d)} \le |Q|.$$

Let $F_m = \sum_{Q\in\mathcal{Q}_m} f_Q$. Then, for $s \ge 3$,

$$\Big\| \sum_j \Gamma_j^s * F_{j-s} \Big\|_{L^2(\mathbb{R}^d)}^2 \le \|\Omega\|_{L^\infty(S^{d-1})}^2 2^{-s(1-\kappa)} \sum_Q \|f_Q\|_{L^1(\mathbb{R}^d)}.$$

This lemma is Lemma 2.1 in [19].

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Lemma 2.2 *Let Ω be homogeneous of degree zero and $\Omega \in L^\infty(S^{d-1})$. Let $\eta \in C^\infty(\mathbb{R}^d\backslash\{0\})$, such that $\operatorname{supp}\eta \subset \{1/2 \le |\xi| \le 2\}$ and $\sum_{k\in\mathbb{Z}} \eta^2(2^{-k}\xi) = 1$ for all $\xi \in \mathbb{R}^d\backslash\{0\}$, and L_k be defined by $\widehat{L_k}(\xi) = \eta(2^{-k}\xi)$. Let Q be a cube and $Q \in \mathcal{Q}_{j-s}$, b_Q be integrable, have integral zero, and supported on Q. Set*

$$m_{jv}^{sk}(\xi) = \eta(2^{-k}\xi)(1 - \widehat{P_v^s}(\xi))\widehat{K_{jv}^s}(\xi).$$

Then,

$$\left\| \sum_v \sum_k \mathcal{F}^{-1}[m_{jv}^{sk}] * L_k * b_Q \right\|_{L^1(\mathbb{R}^d)} \lesssim \|\Omega\|_{L^\infty(S^{d-1})} 2^{-\varepsilon s} \|b_Q\|_{L^1(\mathbb{R}^d)},$$

where $\varepsilon > 0$ is a universal constant depending on κ.

This is Lemma 2.2 in [19].

Proof of Theorem 1.3 By homogeneity, it suffices to prove inequality (1.8) for $\lambda = 1$. Applying the Calderón–Zygmund decomposition to f at level 1, we can obtain a collection of nonoverlapping closed dyadic cubes $\mathcal{S} = \{Q_l\}$, such that $\|f\|_{L^\infty(\mathbb{R}^d\backslash\cup_{Q\in\mathcal{S}} Q)} \lesssim 1$, and

$$\int_Q |f(x)|dx \lesssim |Q|, \quad \sum_{Q\in\mathcal{S}} |Q| \lesssim \|f\|_{L^1(\mathbb{R}^d)}.$$

Let

$$g(x) = f(x)\chi_{\mathbb{R}^d\backslash\cup_{Q\in\mathcal{S}} Q}(x) + \sum_{Q\in\mathcal{S}} \langle f\rangle_Q \chi_Q(x),$$

and

$$b(x) = \sum_{Q\in\mathcal{S}} b_Q(x), \text{ with } b_Q(x) = \big(f(x) - \langle f\rangle_Q\big)\chi_Q(x).$$

It is obvious that

$$\|g\|_{L^\infty(\mathbb{R}^d)} \lesssim 1, \quad \|g\|_{L^1(\mathbb{R}^d)} \lesssim \|f\|_{L^1(\mathbb{R}^d)}.$$

Thus, by the $L^2(\mathbb{R}^d)$ boundedness of U and T_Ω, we obtain that

$$\big|\{x \in \mathbb{R}^d : |UT_\Omega g(x)| > 1/2\}\big| \lesssim \|g\|_{L^2(\mathbb{R}^d)}^2 \lesssim \|f\|_{L^1(\mathbb{R}^d)}. \tag{2.2}$$

To estimate the term $UT_\Omega b$, we set $E = \cup_l 2^{200} Q_l$ and write

$$UT_\Omega b(x) = U\big(\chi_E T_\Omega b\big)(x) + U\big(\chi_{\mathbb{R}^d\backslash E} T_\Omega b\big)(x) =: \mathrm{I}_1 b(x) + \mathrm{I}_2 b(x).$$

Recall that U satisfies the estimate (1.7), and $|E| \lesssim \|f\|_{L^1(\mathbb{R}^d)}$. It then follows that

$$\begin{aligned}\int_E \Psi_\beta(|T_\Omega b(x)|)dx &\lesssim \int_0^\infty \big|\{x \in E : |T_\Omega b(x)| > s\}\big| d\Psi_\beta(s)\\ &\lesssim |E| + \int_e^\infty |\{x \in E : |T_\Omega b(x)| > 2s\}| d\Psi_\beta(s)\\ &\lesssim |E| + \int_e^\infty |\{x \in E : |T_\Omega(b\chi_{\{|b|>s\}})(x)| > s\}| d\Psi_\beta(s)\end{aligned}$$

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$$+\int_{\mathrm{e}}^{\infty}|\{x\in E:|T_\Omega(b\chi_{\{|b|\le s\}})(x)|>s\}|d\Psi_\beta(s).$$

Since T_Ω is bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$ (see [19]), we deduce that

$$\begin{aligned}&\int_{\mathrm{e}}^{\infty}|\{x\in E:|T_\Omega(b\chi_{\{|b|>s\}})(x)|>s\}|d\Psi_\beta(s)\\&\lesssim\int_{\mathrm{e}}^{\infty}\frac{1}{s}\int_{|b(x)|>s}|b(x)|dxd\Psi_\beta(s)\\&\lesssim\sum_{Q\in\mathcal{S}}\int_Q\Psi_{\beta+1}(|b_Q(x)|)dx\lesssim\int_{\mathbb{R}^d}\Psi_{\beta+1}(|f(x)|)dx,\end{aligned}$$

where in the last inequality, we have invoked Jensen's inequality, which tells us that

$$\langle|f|\rangle_Q\log^{\beta+1}(\mathrm{e}+\langle|f|\rangle_Q)\le\frac{1}{|Q|}\int_Q|f(y)|\log^{\beta+1}(\mathrm{e}+|f(y)|)dy.$$

On the other hand, we deduce from the $L^2(\mathbb{R}^d)$ boundedness of T_Ω that

$$\begin{aligned}&\int_{\mathrm{e}}^{\infty}|\{x\in E:|T_\Omega(b\chi_{\{|b|\le s\}})(x)|>s\}|d\Psi_\beta(s)\\&\lesssim\int_{\mathrm{e}}^{\infty}\frac{1}{s^2}\int_{|b(x)|\le s}|b(x)|^2dxd\Psi_\beta(s)\lesssim\|\Psi_\beta(|f|)\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

Therefore, we conclude the estimate of $\mathrm{I}_1b(x)$ as follows:

$$|\{x\in\mathbb{R}^d:|\mathrm{I}_1b(x)|>1/4\}|\lesssim\int_E\Psi_\beta(|T_\Omega b(x)|)dx\lesssim\int_{\mathbb{R}^d}\Psi_{\beta+1}(|f(x)|)dx.$$

By inequality (2.2), our proof of Theorem 1.3 is now reduced to proving that

$$|\{x\in\mathbb{R}^d:|\mathrm{I}_2b(x)|>1/4\}|\lesssim\int_{\mathbb{R}^d}\Psi_{\beta+1}(|f(x)|)dx.\tag{2.3}$$

To prove inequality (2.3), we will employ some estimates from [19]. Let

$$E_0=\{x'\in S^{d-1}:|\Omega(x')|\le1\}$$

and

$$E_k=\{x'\in S^{d-1}:2^{k-1}<|\Omega(x')|\le2^k\}$$

for $k\in\mathbb{N}$. Set

$$\Omega_0(x')=\Omega(x')\chi_{E_0}(x'),\ \Omega_k(x')=\Omega(x')\chi_{E_k}(x')\ (k\in\mathbb{N}).$$

For $j\in\mathbb{Z}$, let $K_j^i(x)=\frac{\Omega_i(x)}{|x|^d}\phi_j(x)$ and $\mathcal{S}_j=\{Q\in\mathcal{S},\ell(Q)=2^j\}$ and $B_j(x)=\sum_{Q\in\mathcal{S}_j}b_Q(x)$. Let $\Gamma_j^{i,s}$ be the same as in equation (2.1) with Ω replaced by Ω_i. Write

$$\begin{aligned}\chi_{\mathbb{R}^d\setminus E}(x)T_\Omega b(x)=\chi_{\mathbb{R}^d\setminus E}(x)\sum_{i=0}^{\infty}\sum_{100\le s\le Ni}\sum_{j\in\mathbb{Z}}K_j^i*B_{j-s}(x)\\+\chi_{\mathbb{R}^d\setminus E}(x)\sum_{i=0}^{\infty}\sum_{s>Ni}\sum_{j\in\mathbb{Z}}\Gamma_j^{i,s}*B_{j-s}(x)\end{aligned}$$

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$$+\chi_{\mathbb{R}^d\backslash E}(x)\sum_{i=0}^{\infty}\sum_{s>Ni}\sum_{j\in\mathbb{Z}}(K_j^i-\Gamma_j^{i,s})*B_{j-s}(x)$$
$$:=\mathrm{D}_1b(x)+\mathrm{D}_2b(x)+\mathrm{D}_3b(x),$$

with $N\in\mathbb{N}$ a constant, such that $N>2\max\{\varepsilon^{-1},\,(1-\kappa)^{-1}\}$. Let M_{Ω_i} be the maximal operator defined as equation (1.9) with Ω replaced by Ω_i. The method of rotation of Calderón and Zygmund tells us that

$$\|M_{\Omega_i}h\|_{L^p(\mathbb{R}^d)}\lesssim\|\Omega_i\|_{L^1(S^{d-1})}\|h\|_{L^p(\mathbb{R}^d)},\ 1<p<\infty.$$

For each fixed $\lambda>0$, let $E_{Q,\lambda}=\{x\in\mathbb{R}^d:|b_Q(x)|>\lambda\}$ and set

$$\widetilde{b}_{Q,\lambda}=b_Q(x)\chi_{E_{Q,\lambda}}(x),\ \widetilde{b}_Q^{\lambda}(x)=b_Q(x)\chi_{\mathbb{R}^d\backslash E_{Q,\lambda}}(x),\tag{2.4}$$

and

$$\widetilde{b}_{\lambda}=\sum_{Q\in\mathcal{S}}b_{Q,\lambda},\ \widetilde{b}^{\lambda}=\sum_{Q\in\mathcal{S}}b_Q^{\lambda}.$$

Let $h\in L^2(\mathbb{R}^d\backslash E)$ with $\|h\|_{L^2(\mathbb{R}^d)}=1$ and $\operatorname{supp}h\subset\mathbb{R}^d\backslash E$. A trivial computation shows that

$$\begin{aligned}\Big|\int_{\mathbb{R}^d}\mathrm{D}_1\widetilde{b}^{\lambda}(x)h(x)dx\Big|&\le\sum_{i=0}^{\infty}\sum_{100\le s\le Ni}\sum_{j\in\mathbb{Z}}\sum_{Q\in\mathcal{S}_{j-s}}\int_{\mathbb{R}^d}\big|\widetilde{b}_Q^{\lambda}(x)K_j^i*h(x)\big|dx\\&\lesssim\sum_{i=0}^{\infty}\sum_{100\le s\le Ni}\int_{\mathbb{R}^d}\sum_{Q\in\mathcal{S}}|\widetilde{b}_Q^{\lambda}(y)|M_{\Omega_i}h(y)dy\\&\lesssim\sum_{i=0}^{\infty}\sum_{100\le s\le Ni}\|M_{\Omega_i}h\|_{L^2(\mathbb{R}^d)}\|\widetilde{b}^{\lambda}\|_{L^2(\mathbb{R}^d)}\\&\lesssim\|\Omega\|_{L\log L(S^{d-1})}\|\widetilde{b}^{\lambda}\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Therefore,

$$\|\mathrm{D}_1\widetilde{b}^{\lambda}\|_{L^2(\mathbb{R}^d)}\lesssim\|\widetilde{b}^{\lambda}\|_{L^2(\mathbb{R}^d)}.\tag{2.5}$$

On the other hand, we have

$$\|\mathrm{D}_1\widetilde{b}_{\lambda}\|_{L^1(\mathbb{R}^d)}\lesssim\sum_{i=0}^{\infty}\sum_{100\le s\le Ni}\sum_{j\in\mathbb{Z}}\sum_{Q\in\mathcal{S}_{j-s}}\|K_j^i*\widetilde{b}_{Q,\lambda}\|_{L^1(\mathbb{R}^d)}\lesssim\|\widetilde{b}_{\lambda}\|_{L^1(\mathbb{R}^d)},\tag{2.6}$$

and

$$\|\mathrm{D}_1b\|_{L^1(\mathbb{R}^d)}\lesssim\|b\|_{L^1(\mathbb{R}^d)}.\tag{2.7}$$

Estimate (2.7) now tells us that

$$\begin{aligned}\int_{\mathbb{R}^d}\Psi_\beta(|\mathrm{D}_1b(x)|)dx&=\int_{\{x:|\mathrm{D}_1b(x)|\le\mathrm{e}\}}\Psi_\beta(|\mathrm{D}_1b(x)|)dx\\&\quad+\int_{\{x:|\mathrm{D}_1b(x)|>\mathrm{e}\}}\Psi_\beta(|\mathrm{D}_1b(x)|)dx\\&\lesssim\|f\|_{L^1(\mathbb{R}^d)}+\int_{\{x:|\mathrm{D}_1b(x)|>\mathrm{e}\}}\Psi_\beta(|\mathrm{D}_1b(x)|)dx.\end{aligned}\tag{2.8}$$

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Write

$$\begin{aligned}\int_{\{|D_1 b(x)|>e\}} \Psi_\beta(|D_1 b(x)|)dx &\lesssim |\{x\in\mathbb{R}^d : |D_1 b(x)|>e\}| \\ &\quad + \int_e^\infty |\{x\in\mathbb{R}^d : |D_1 \widetilde{b}_\lambda(x)|>\lambda/2\}| d\Psi_\beta(\lambda) \\ &\quad + \int_e^\infty |\{x\in\mathbb{R}^d : |D_1 \widetilde{b}^\lambda(x)|>\lambda/2\}| d\Psi_\beta(\lambda) \\ &= E_1 + E_2 + E_3.\end{aligned}$$

It then follows inequality (2.6) that

$$\begin{aligned}E_2 &\lesssim \sum_Q \int_e^\infty \int_{\mathbb{R}^d} |\widetilde{b}_{Q,\lambda}(x)| dx \lambda^{-1} d\Psi_\beta(\lambda) \\ &\lesssim \sum_Q \int_{\mathbb{R}^d} |b_Q(x)| \int_e^{|b_Q(x)|} d\big(\log^{\beta+1}(e+\lambda)\big)dx \lesssim \int_{\mathbb{R}^d} \Psi_{\beta+1}(|f(x)|)dx.\end{aligned}$$

On the other hand, it follows from inequality (2.5) that

$$\begin{aligned}E_3 &\lesssim \sum_Q \int_e^\infty \int_{\mathbb{R}^d} |\widetilde{b}^\lambda_Q(x)|^2 dx \lambda^{-2} d\Psi_\beta(\lambda) \\ &\lesssim -\sum_Q \int_{\mathbb{R}^d} |b_Q(x)|^2 \int_{|b_Q(x)|}^\infty d\big(\lambda^{-1}\log^\beta(e+\lambda)\big)dx \lesssim \int_{\mathbb{R}^d} \Psi_\beta(|f(x)|)dx.\end{aligned}$$

Those estimates for E_2 and E_3, along with inequality (2.7), show that

$$\begin{aligned}\big|\{x\in\mathbb{R}^d : |U(D_1 b)(x)|>1/12\}\big| &\lesssim \int_{\mathbb{R}^d} \Psi_\beta(|D_1 b(x)|)dx \\ &\lesssim \int_{\mathbb{R}^d} \Psi_{\beta+1}(|f(x)|)dx.\end{aligned} \tag{2.9}$$

We now estimate $D_2 b$. An application of Lemma 2.1 gives us that

$$\Big\| \sum_j \Gamma_j^{i,s} * B_{j-s} \Big\|^2_{L^2(\mathbb{R}^d)} \lesssim 2^{-(1-\kappa)s} \|\Omega_i\|^2_{L^\infty(S^{n-1})} \sum_{Q\in\mathcal{S}} \|b_Q\|_{L^1(\mathbb{R}^d)}.$$

Therefore,

$$\|D_2 b\|^2_{L^2(\mathbb{R}^d)} \lesssim \left(\sum_{i=0}^\infty \sum_{s>Ni} \Big\| \sum_j \Gamma_j^{i,s} * B_{j-s} \Big\|_{L^2(\mathbb{R}^d)} \right)^2 \lesssim \|b\|_{L^1(\mathbb{R}^d)}. \tag{2.10}$$

Chebyshev's inequality, along with estimate (2.10), yields

$$\big|\{x\in\mathbb{R}^d : |U(D_2 b)(x)|>1/12\}\big| \lesssim \|D_2 b\|^2_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^1(\mathbb{R}^d)}. \tag{2.11}$$

We turn our attention to $|U(D_3 b)(x)|$. Let $K_{jv}^{i,s}$, m_{jv}^{isk} be the same as in equation (2.1) with Ω replaced by Ω_i. Set

$$m_{jv}^{isk}(\xi) = \eta(2^{-k}\xi)(1-\widehat{P}_v^s(\xi))\widehat{H_{jv}^{is}}(\xi),$$

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and

$$K_j^i(y) - \Gamma_j^{i,s}(y) = \sum_v \sum_k \mathcal{F}^{-1}(m_{jv}^{isk}) * L_k(y).$$

For each fixed $\lambda > 0$, let $\widetilde{b}_{Q,\lambda}$, $\widetilde{b}_Q^\lambda(x)$ be the same as in equation (2.4), and set

$$b_{Q,\lambda}(x) = \big(\widetilde{b}_{Q,\lambda}(x) - \langle \widetilde{b}_{Q,\lambda}\rangle_Q\big)\chi_Q(x),\ b_Q^\lambda(x) = \big(\widetilde{b}_Q^\lambda(x) - \langle \widetilde{b}_Q^\lambda\rangle_Q\big)\chi_Q(x).$$

It then follows that

$$\begin{aligned}\mathrm{D}_3 b(x) &= \chi_{\mathbb{R}^d\setminus E}(x)\sum_{i=0}^{\infty}\sum_{s>Ni}\sum_v\sum_k\sum_{Q\in\mathcal{S}_{j-s}} \mathcal{F}^{-1}(m_{jv}^{isk}) * L_k * b_{Q,\lambda}(x)\\ &\quad + \chi_{\mathbb{R}^d\setminus E}(x)\sum_{i=0}^{\infty}\sum_{s>Ni}\sum_v\sum_k\sum_{Q\in\mathcal{S}_{j-s}} \mathcal{F}^{-1}(m_{jv}^{isk}) * L_k * b_Q^\lambda(x)\\ &:= \mathrm{D}_{3,\lambda} b(x) + \mathrm{D}_3^\lambda b(x).\end{aligned}$$

Let $b_\lambda = \sum_{Q\in\mathcal{S}} b_{Q,\lambda}$ and $b^\lambda = \sum_{Q\in\mathcal{S}} b_Q^\lambda$. Observe that both b_Q and $b_{Q,\lambda}$ have integrals zero. Applications of Lemma 2.2 lead to that

$$\|\mathrm{D}_3 b\|_{L^1(\mathbb{R}^d)} \lesssim \sum_{i=0}^{\infty}\|\Omega_i\|_{L^\infty(S^{d-1})}\sum_{s\geq Ni}2^{-\varepsilon s}\sum_j\|B_{j-s}\|_{L^1(\mathbb{R}^d)} \lesssim \sum_{Q\in\mathcal{S}}\|b_Q\|_{L^1(\mathbb{R}^d)}, \tag{2.12}$$

and

$$\|\mathrm{D}_{3,\lambda} b\|_{L^1(\mathbb{R}^d)} \lesssim \sum_Q \|b_{Q,\lambda}\|_{L^1(\mathbb{R}^d)}. \tag{2.13}$$

On the other hand, as in the $L^2(\mathbb{R}^d)$ estimates for $\mathrm{D}_1 b$ and $\mathrm{D}_2 b$, we have that

$$\|\mathrm{D}_1 b^\lambda\|_{L^2(\mathbb{R}^d)} \lesssim \|b^\lambda\|_{L^2(\mathbb{R}^d)},\ \|\mathrm{D}_2 b^\lambda\|_{L^2(\mathbb{R}^d)}^2 \lesssim \|b^\lambda\|_{L^1(\mathbb{R}^d)},$$

which, together with the $L^2(\mathbb{R}^d)$ boundedness of T_Ω, gives us that for each fixed $\lambda > 0$,

$$\begin{aligned}\|\mathrm{D}_3^\lambda b\|_{L^2(\mathbb{R}^d)} &\leq \|T_\Omega b^\lambda\|_{L^2(\mathbb{R}^d)} + \|\mathrm{D}_1 b^\lambda\|_{L^2(\mathbb{R}^d)} + \|\mathrm{D}_2 b^\lambda\|_{L^2(\mathbb{R}^d)}\\ &\lesssim \left(\sum_{Q\in\mathcal{S}}\|b_Q^\lambda\|_{L^2(\mathbb{R}^d)}^2\right)^{\frac12} + \left(\sum_{Q\in\mathcal{S}}\|b_Q^\lambda\|_{L^1(\mathbb{R}^d)}\right)^{\frac12}.\end{aligned} \tag{2.14}$$

Let

$$\begin{aligned}\mathrm{F}_1 &= \int_{\mathrm{e}}^{\infty} |\{x\in\mathbb{R}^d : |\mathrm{D}_{3,\lambda} b(x)| > \lambda/2\}| d\Psi_\beta(\lambda),\\ \mathrm{F}_2 &= \int_{\mathrm{e}}^{\infty} |\{x\in\mathbb{R}^d : |\mathrm{D}_3^\lambda b(x)| > \lambda/2\}| d\Psi_\beta(\lambda).\end{aligned}$$

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It then follows inequality (2.13) that

$$\begin{aligned}\mathrm{F}_1 &\lesssim \sum_Q \int_{\mathrm{e}}^{\infty} \int_{\mathbb{R}^d} |b_{Q,\lambda}(x)| dx \lambda^{-1} d\Psi_\beta(\lambda) \\ &\lesssim \sum_Q \int_{\mathbb{R}^d} |b_Q(x)| \int_{\mathrm{e}}^{|b_Q(x)|} d\big(\log^{\beta+1}(\mathrm{e}+\lambda)\big) dx \lesssim \int_{\mathbb{R}^d} \Psi_{\beta+1}(|f(x)|) dx.\end{aligned}$$

While inequality (2.14) implies that

$$\begin{aligned}\mathrm{F}_2 &\lesssim \sum_Q \int_{\mathrm{e}}^{\infty} \int_{\mathbb{R}^d} |b_Q^\lambda(x)| dx \lambda^{-2} d\Psi_\beta(\lambda) + \sum_Q \int_{\mathrm{e}}^{\infty} \int_{\mathbb{R}^d} |b_Q^\lambda(x)|^2 dx \lambda^{-2} d\Psi_\beta(\lambda) \\ &\lesssim \sum_Q \int_{\mathbb{R}^d} |b_Q(x)| dx \int_{|b_Q(x)|}^{\infty} \lambda^{-1} d\Psi_\beta(\lambda) \\ &\quad + \sum_Q \int_{\mathbb{R}^d} |b_Q(x)|^2 dx \int_{|b_Q(x)|}^{\infty} d\big(\lambda^{-1} \log^\beta(\mathrm{e}+\lambda)\big) \\ &\lesssim \int_{\mathbb{R}^d} \Psi_{\beta+1}(|f(x)|) dx + \int_{\mathbb{R}^d} \Psi_\beta(|f(x)|) dx \lesssim \int_{\mathbb{R}^d} \Psi_{\beta+1}(|f(x)|) dx.\end{aligned}$$

As in the estimate for D_1, we obtain from estimates for F_1 and F_2 that

$$\int_{\mathbb{R}^d} \Psi_\beta(|\mathrm{D}_3 b(x)|) dx \lesssim \int_{\mathbb{R}^d} \Psi_{\beta+1}(|f(x)|) dx.$$

This, in turn, leads to that

$$\begin{aligned}\big|\big\{x \in \mathbb{R}^d : |U(\mathrm{D}_3 b)(x)| > 1/12\big\}\big| &\lesssim \int_{\mathbb{R}^d} \Psi_\beta(|\mathrm{D}_3 b(x)|) dx \\ &\lesssim \int_{\mathbb{R}^d} \Psi_{\beta+1}(|f(x)|) dx. \qquad (2.15)\end{aligned}$$

Combining estimates (2.9), (2.11), and (2.15) leads to estimate (2.3) and then completes the proof of Theorem 1.3. ∎

Acknowledgment The authors would like to thank the referee for his/her valuable suggestions and helpful comments.

References

[1] C. Benea and F. Bernicot, *Conservation de certaines propriétés á travers un contrôle épars d'un opérateur et applications au projecteur de Leray–Hopf*. Preprint, 2020. arxiv:1703:00228
[2] A. P. Calderón, *Algebras of singular integral operators*. In: Singular integrals (Chicago, IL, 1966), Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 1967, pp. 18–55.
[3] A. P. Calderón, *Algebras of singular integral operators*. Proc. Int. Congr. Math. (Moscow, Russia, 1966), Mir, Moscow, Russia, 1968, pp. 393–395.
[4] A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*. Acta Math. **88**(1952), 85–139.
[5] A. P. Calderón and A. Zygmund, *On singular integrals*. Amer. J. Math. **78**(1956), 289–309.
[6] A. P. Calderón and A. Zygmund, *Algebras of certain singular operators*. Amer. J. Math. **78**(1956), 310–320.
[7] N. Carozza and A. Passarelli di Napoli, *Composition of maximal operators*. Publ. Mat. **40**(1996), 397–409.
[8] M. Christ, *Inversion in some algebras of singular integral operators*. Rev. Mat. Iberoam. **4**(1988), 219–225.

https://doi.org/10.4153/S0008439520000946 Published online by Cambridge University Press

[9] M. Christ and J. -L. Rubio de Francia, *Weak type (1, 1) bounds for rough operators, II*. Invent. Math. **93**(1988), 225–237.
[10] J. Duoandikoetxea and J. L. Rubio de Francia, *Maximal and singular integrals via Fourier transform estimates*. Invent. Math. **84**(1986), 541–561.
[11] L. Grafakos, *Modern Fourier analysis*. 3rd ed., Graduate Texts in Mathematics, 250, Springer, New York, 2014.
[12] G. Hu, *Weighted weak type endpoint estimates for the composition of Calderón–Zygmund operators*. J. Aust. Math. Soc. 109(2020), 320–339.
[13] G. Hu, X. Lai, and Q. Xue, *On the composition of rough singular integral operators*. J. Geom. Anal (2020). https://doi.org/10.1007/s12220-020-00374-6
[14] A. K. Lerner, *A weak type estimates for rough singular integrals*. Rev. Mat. Iberoam. **35**(2019), 1583–1602.
[15] B. Muckenhoupt and R. L. Wheeden, *Weighted norm inequalities for singular and fractional integrals*. Trans. Amer. Math. Soc. **161**(1971), 249–258.
[16] A. Nagel, F. Ricci, E. M. Stein, and S. Wainger, *Algebras of singular integral operators with kernels controlled by multiple norms*. Memoirs Amer. Math. Soc., 256, no. 1230, Amer. Math. Soc., Providence, RI, 2018.
[17] D. H. Phone and E. M. Stein, *Some further classes of pseudo-differential and singular integral operators arising in boundary-value problems I, composition of operators*. Amer. J. Math. **104**(1982), 141–172.
[18] F. Ricci and G. Weiss, *A characterization of* H^1 (S^{n-1}). In: S. Wainger and G. Weiss (eds.), Proc. Sympos. Pure Math., 35 I, Amer. Math. Soc., Providence, RI, 1979, pp. 289–294.
[19] A. Seeger, *Singular integral operators with rough convolution kernels*. J. Amer. Math. Soc. **9**(1996), 95–105.
[20] R. S. Strichartz, *Compositions of singular integral operators*. J. Funct. Anal. **49**(1982), 91–127.

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