Should we play on your lawn or mine?

J. N. S. MATTHEWS

Background

Two friends, David and Hamish, were discussing on the telephone whether they should play a forthcoming croquet match at David's club or at Hamish's. Hamish said that he would toss a coin and David could call. David pointed out that this required a substantial degree of mutual trust, so Hamish suggested how they might use the published results of the following night's National Lottery draw. Hamish's proposal was, in fact, unfair and this article shows why and proposes a satisfactory alternative. There is no suggestion that the alternative proposed is a sensible way to solve David and Hamish's problem, but it does take us on a route that reinforces some important lessons. In particular it illustrates some of the advantages and disadvantages of the use of simulation, a technique widely used in more difficult problems. It also emphasises once more the importance in probability of not taking claims at face value.

Until recently the UK National Lottery (the 'Lottery') selected six numbers at random from the integers $1, 2, \ldots, 49$, without replacement (a seventh number, the bonus ball, is also selected but this does not concern us).* Neither Hamish nor David would see the draw live on television, so decided to rely on the published result. This would provide the six numbers, not in the order they were drawn but *arranged in ascending order*. Hamish pointed out that the smallest of the six numbers, let's call it N, must be in the range $1 \leqslant N \leqslant 44$. As such, N could take an equal number of odd and even values, so Hamish proposed that a 'Head' could be identified with the event $\{N \text{ is odd}\}$, whereas a 'Tail' would be declared if N was even. However, is this right? One way to decide, of course, would be to work out $\Pr(N \text{ is odd})$ analytically but first some other options are explored.

Simulation studies

If we use a computer to produce a value for N, then we could do this repeatedly and work out the proportion of times that N was odd. More formally, if we generate S independent values of N, say $N_1, \ldots, N_S$ then if $I_O(N)$ is 1 if N is odd and 0 otherwise, the Law of Large numbers (strong or weak version) indicates that the mean $S^{-1} \sum I_O(N_i)$ will tend to $\Pr(N \text{ is odd})$ as $S \to \infty$.

The Lottery can be simulated in many systems but the free statistical program R [1] is very suitable. In particular, entering the command `sample(49,6)` simulates the Lottery, giving output

* Note that from October 2015 the Lottery has been choosing six integers from $1, \ldots, 59$, a change which makes no essential difference to the points made in this article.

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```
sample(49,6)
[1] 33 23 37 17 39 42
```

It is also possible to combine R commands, so had the above been `min(sample(49,6))`, then the output would have been simply `17`. If we want to repeat the process, say 1000 times, we could use the following code:

```
results=vector(,1000)
for (i in 1:1000){
results[i]=min(sample(49,6))}
sum(results %% 2)/1000

[1] 0.527
```

The first line sets up a vector of length 1000 to hold the results, while the next two are a loop simulating the Lottery 1000 times and recording the value of the smallest ball on each occasion. The final line uses the R syntax that `a %% b` is the remainder of `a` modulo `b`, so `results %% 2` is a vector of length 1000, with elements 0 and 1 according to whether the corresponding element of `results` was even or odd. Consequently `sum(results %% 2)` is the number of odd values of N in the 1000 simulations and the printed result is simply the proportion of odd values, i.e. an estimate of Pr (N is odd).

The problem is how to interpret the value of 0.527. It is clearly not the exact value of Pr (N is odd) because repeating the simulation gave a value of 0.510. However, while these simulations do not provide the exact value of Pr (N is odd), do they exclude the possibility that Pr (N is odd) is $\frac{1}{2}$, the value required if Hamish's suggestion is to be fair? Clearly any interpretation of the simulations must take account of their inherent variability. One way to gain an impression of the variability of a simulation is to work out the proportion obtained from 1000 simulations when the answer is known. The R command `rbinom(1000,1,0.5)` provides 1000 values, each equally likely to be 0 or 1. Here the proportion of 1s is

```
sum(rbinom(1000,1,0.5))/1000
[1] 0.515
```

so even in a case where we know true probability is $\frac{1}{2}$, the simulated result departs somewhat from this value. Performing simulations for cases where we know the answer often helps in the interpretation of the wider study.

Of course, further help is available by using a more formal statistical assessment. The value of 0.527 is a *point* estimate of Pr (N is odd). We know that random variation from one set of simulations to another means that this estimate will not be exact, so it is useful to acknowledge this uncertainty by giving an *interval* estimate, often called a confidence interval. If we write p for the point estimate, then an interval estimate that

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has a 95% chance of covering the true value is $p \pm 1.96\sqrt{\frac{p(1-p)}{S}}$. For the first simulation we obtain (0.496, 0.558), whereas for the simulation using `rbinom` we get (0.484, 0.546). So both simulations are compatible with the probability of a 'Head' being a $\frac{1}{2}$. As the value of p is close to $\frac{1}{2}$, the 95% interval estimate has semi-width of approximately $S^{-1/2}$, so estimates of any given precision can be obtained by making S sufficiently large. Choosing $S = 10^6$ will give an estimate of $\Pr(N \text{ is odd})$ that is correct to within ± 0.001 (at least, with 95% confidence). Trying this gives

```
results=vector(,1000000)
for (i in 1:1000000){
results[i]=min(sample(49,6))}
sum(results %% 2)/1000000
[1] 0.532257
```

So the new estimate of $\Pr(N \text{ is odd})$ is 0.532, with a 95% interval estimate of (0.531, 0.533). By comparison, 10^6 simulations from `rbinom` gave 499714 'Heads', with a 95% interval (0.499, 0.501). Thus it seems that Hamish's claim is false and $\Pr(N \text{ is odd})$ is slightly greater than $\frac{1}{2}$. In essence this is unchanged even if more certainty about the location of $\Pr(N \text{ is odd})$ were required, as a 99.9% interval is (0.531, 0.534), to 3 decimal places.

Analytical approach

While the simulation was quick to implement and easy to understand, it has two disadvantages, both of which are usually shared by more complicated simulations. The first is perhaps less important, namely that the conclusion is not certain but has to be hedged by a confidence coefficient, albeit one which is under the control of the investigator. A more serious criticism is that while simulation has shown that $\Pr(N \text{ is odd}) > \frac{1}{2}$, it has not really provided any insight into why this is the case. To this end, two alternative approaches are now presented.

Counting the arrangements

Our first argument does not give a value for $\Pr(N \text{ is odd})$ but does explain why it exceeds $\frac{1}{2}$. Let A denote the set of all possible outcomes of the Lottery and let A_k be the set of all possible outcomes where the smallest value is k, $k = 1, \ldots, 44$, and let $|X|$ denote the number of elements in a set X. Consider an outcome $x \in A_2$, so x is a vector of length 6 with elements from the set $\{1, \ldots, 49\}$ with $\min(x) = 2$ and define x^* to have the same elements as x except that the value 2 is changed to 1. Clearly $x^* \in A_1$ and there is a one-to-one correspondence between A_2 and the subset of A_1 comprising those $y \in A_1$ where the second smallest element of y is 3 or more. As this is clearly a proper subset of A_1, we have that $|A_1| > |A_2|$ and dividing by $|A|$ yields $\Pr(N = 1) > \Pr(N = 2)$. An analogous

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argument shows that $\Pr(N = 3) > \Pr(N = 4)$ and so on. As

$$\Pr(N \text{ is odd}) = \Pr(A_1) + \Pr(A_3) + \ldots + \Pr(A_{43}),$$

then this sum is term by term larger than

$$\Pr(N \text{ is even}) = \Pr(A_2) + \Pr(A_4) + \ldots + \Pr(A_{44}).$$

Of course, a direct attack on the problem which evaluates $|A_k|$ is not difficult. The value of $|A|$ is $\binom{49}{6}$. The number of arrangements with $N = k$ amounts to counting the number of ways five balls can be drawn from the set $\{k + 1, \ldots, 49\}$, so $A_k = \binom{49 - k}{5}$, and therefore

$$\Pr(N \text{ is odd}) = \frac{\binom{48}{5} + \binom{46}{5} + \ldots + \binom{6}{5}}{\binom{49}{6}} = \frac{\sum_{j=1}^{22} \binom{49 - 2j + 1}{5}}{\binom{49}{6}}. \qquad (1)$$

This can be evaluated in R, using the function `choose(a,b)` to compute $\binom{a}{b}$, using the code

```
k=seq(1,22)
sum(choose(50-2*k,5))/choose(49,6)
[1] 0.5322767
```

which is very close to the above simulation. By computing its value the latter approach shows that $\Pr(N \text{ is odd})$ is greater than $\frac{1}{2}$, but is it clear from the form of (1) that this will be so? In other words, does the direct evaluation provide more insight than the former argument, even if this did not yield a value for $\Pr(N \text{ is odd})$?

Insights and Alternatives

What insight a particular approach to a problem provides is often difficult to describe with precision, and may have a subjective component. It is felt that the approach based on the observation $|A_1| > |A_2|$ etc. suggests that while $N \in \{1, \ldots, 44\}$, there will be a tendency for the distribution of N to be shifted towards the lower values in the set and, as the odd numbers are on average rather smaller than the even values, this will lead to $\Pr(N \text{ is odd})$ being larger than $\frac{1}{2}$.

This way of thinking suggests some other properties for N which can be checked. For example, it would seem that in the Lottery the parity of the *largest* value would also be more likely to be odd. Another suggestion is that considering the parity of a less extreme ball, say the third smallest, might be provide a better solution to the problem faced by David and Hamish. It is the author's view that these hunches are more readily

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suggested by the first of the above arguments than by the expression (1). However, we ought to check if these hunches are correct.

First let's consider the largest value. If B_k is the set of Lottery outcomes where k is the largest value, $k \in \{6, \ldots, 49\}$, then arguing as above $|B_k| = \binom{k-1}{5}$, so if n is the largest value in the Lottery we get

$$\Pr(n \text{ is odd}) = \frac{\binom{6}{5} + \binom{8}{5} + \ldots + \binom{48}{5}}{\binom{49}{6}}$$

and this is identical to (1), so $\Pr(n \text{ is odd})$ does indeed exceed $\frac{1}{2}$.

Turning to the third smallest ball, which is denoted by T, $T \in \{3, \ldots, 46\}$. A simulation study of the parity of T can be performed using the R function `sort`, which puts its vector argument into ascending order, and the syntax `v[3]`, which is the third element of the vector `v`. The required code for 10^6 simulations is therefore

```
results=vector(,1000000)
for (i in 1:1000000){
results[i]=sort(sample(49,6))[3]}
sum(results %% 2)/1000000
[1] 0.499336
```

So $\Pr(T \text{ is odd})$ does seem to be close to $\frac{1}{2}$, and a 99.9% interval estimate is (0.498, 0.501). But again, while simulation shows that using the parity of the third smallest ball is likely to be a fairer bet than using the smallest ball, it does not settle the question of whether the probability is exactly $\frac{1}{2}$. To do this we need to compute $\Pr(T \text{ is odd})$. The number of ways that $T = k$ can arise, for $k = 3, \ldots, 46$, is if one ball is at k, two are in the set $\{1, \ldots, k-1\}$ and three are in $\{k+1, \ldots, 49\}$, hence

$$\Pr(T = k) = \frac{\binom{k-1}{2}\binom{49-k}{3}}{\binom{49}{6}}.$$

As an exercise R could be used to sum this for $k = 3, \ldots, 46$ to check that the probabilities add to 1. The sum of the numerator over the odd values of k is

$$\binom{2}{2}\binom{46}{3} + \binom{4}{2}\binom{44}{3} + \ldots + \binom{42}{2}\binom{6}{3} + \binom{44}{2}\binom{4}{3}$$

and is 6992920. As $\frac{1}{2}\binom{49}{6}$ is 6991908, the $\Pr(T \text{ is odd})$ is not exactly $\frac{1}{2}$, and

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is, in fact, 0.5000724. It is unlikely that any coin Hamish had to hand would have $\Pr(\text{Head})$ so close to $\frac{1}{2}$, so using the parity of the third smallest ball should satisfy David and Hamish.

Other approximations

The probability that the smallest lottery ball is odd can be written as $\frac{1}{2} + \varepsilon$, and the probability that it is even is $\frac{1}{2} - \varepsilon$. Here ε is positive and, although small compared with $\frac{1}{2}$, it is not sufficiently small for David and Hamish's purpose. However, the Lottery is drawn twice per week, so if David and Hamish can muster enough patience to wait for two draws, then the probability that the minimum ball has the same parity in both draws (i.e. even & even or odd & odd) is

$$\Pr(\text{SP}) = \left(\frac{1}{2} - \varepsilon\right)^2 + \left(\frac{1}{2} + \varepsilon\right)^2 = \frac{1}{2} + 2\varepsilon^2. \tag{2}$$

where SP denotes the event of having the same parity. With $\varepsilon = 0.0322767$ we obtain $\Pr(\text{SP}) = 0.5020836$, which is notably closer to $\frac{1}{2}$ than $\Pr(N \text{ is odd})$. Of course, the probability that the two numbers have different parity, DP, is $\frac{1}{2} - 2\varepsilon^2$.

If David and Hamish can summon even more patience, then this approach can be repeated. If the results from two successive weeks are combined, then the probability that SP occurs in both the first and second week, or that DP occurs in both weeks, is

$$\left(\frac{1}{2} + 2\varepsilon^2\right)^2 + \left(\frac{1}{2} - 2\varepsilon^2\right)^2 = \frac{1}{2} + 8\varepsilon^4$$

which is 0.5000087.

Equation (2) shows how events which have probability close to $\frac{1}{2}$ might be combined to produce a further event with probability closer to $\frac{1}{2}$. Two events that each have probability $\frac{1}{2} + \varepsilon$ are, as we have seen, (i) that the smallest ball is odd and (ii) the largest ball is odd, and these events have the advantage for David and Hamish there is no need to wait for a second draw, as they are both available from a single draw. So it might be thought that the event 'the smallest and largest balls in a single draw have the same parity' will have a probability differing from $\frac{1}{2}$ by a term of order ε^2.

Unfortunately, the probabilities concerned cannot be combined as in (2) because this equation assumes that the events are independent, which is not the case for events determined by the smallest and largest balls in a single draw (to see this consider the unconditional probability that the largest ball is, for example, 20 and then find the conditional probability of the same event given that the smallest ball is 19). Consequently the probability that the smallest and largest balls have the same parity cannot be computed from probabilities relating to the smallest and largest balls individually and the *joint* distribution of these quantities needs to be evaluated. This is done by first counting the number of ways that a draw can have the smallest value r

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and largest s $(1 \leqslant r \leqslant 44, r + 4 < s \leqslant 49)$, which is $\binom{s - r - 1}{4}$ because there are $s - r - 1$ possible values for the intermediate four numbers. So the number of draws where both the minimum and maximum values are even is

$$\sum_{u=1}^{22} \sum_{v=u+3}^{24} \binom{2v - 2u - 1}{4}.$$

Again R can be used to evaluate this sum, using the `outer` function to produce pairwise differences, viz.

```
reven=2*(1:22); seven=2*(3:24)
ml=outer(seven,reven,"-") ; m2=(ml>4)
sum(choose(ml-1,4)*m2)
[1] 3056746
```

A similar computation shows that the number of draws where both the minimum and maximum balls are odd is 3959450, so that the correct probability that both the maximum and minimum values have the same parity is $(3056746 + 3959450)/13983916 = 0.5017333$, which is closer to $\frac{1}{2}$ than the value found from considering only the smallest ball from successive (independent) draws using (2).

Some elaborations

A general formula using a generalised hypergeometric function

Some of the above calculations can be put in the context of a more general lottery which chooses r distinct values from the set $\{1, \ldots, M\}$. Using an analogous argument to that above, the probability that the mth smallest ball is k, $p_m(k)$ say, is

$$p_m(k) = \frac{\binom{k-1}{m-1}\binom{M-k}{r-m}}{\binom{M}{r}}, \qquad k = m, \ldots, M - r + m. \tag{3}$$

The probability generating function of this distribution, $G(z) = \sum_{k=m}^{M-r+m} z^k p_m(k)$, can be found in terms of a generalised hypergeometric function, namely

$$\frac{{}_2F_1(m, r - M; m - M; z)}{{}_2F_1(m, r - M; m - M; 1)} z^m,$$

where

$${}_2F_1(a_1, a_2; b_1; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k}{(b_1)_k} \frac{z^k}{k!}$$

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and $(a)_k = a(a+1)\ldots(a+k-1)$. Note that in the present application this is a finite sum because $a_2 = r - M$ is a negative integer, so $(r-M)_k$ is zero for $k > M - r$.*

The probability generating function can be used to find the probability that the mth smallest ball is odd by noting that this is $\frac{1}{2}\big(G(1) - G(-1)\big)$. In R you can download 'libraries' which comprise code to do particular tasks and which extend the capabilities of the core R program. One such is the package `hypergeo` [2] and a function in this package, which is also called `hypergeo`, can be used to compute, for example, the probabilities that the smallest and third smallest balls in the Lottery are odd, as follows:

```
library(hypergeo)
m=1
0.5*(1-(-1)^m*hypergeo(m,6-49,m-49,-1)/
hypergeo(m,6-49,m-49,1))
[1] 0.5322767+0i
m=3
0.5*(1-(-1)^m*hypergeo(m,6-49,m-49,-1)/
hypergeo(m,6-49,m-49,1))
[1] 0.5000724+0i
```

confirming the values computed earlier (the `+0i` is just because `hypergeo` can handle complex values of z). This software allows the ready investigation of lotteries with M and r different from 49 and 6.

Odd and even again

Careful inspection of (3) shows that $p_m(k) = p_{r-m+1}\big(M+1-k\big)$. So, for example, putting $m = 1$ we see that the probability that the smallest ball is k is the same as the probability that the largest ball is $M + 1 - k$, a result which is intuitive from symmetry. From this it follows that the probability that the largest ball is odd is the same as the probability that the smallest ball is odd, provided that M is odd: if M is even the probability that the largest ball is odd equals the probability that the smallest ball is even. This fits well with the insight mentioned above – the largest ball will have a distribution shifted towards M, and this will bias n towards even values if M is even and odd values if M is odd.

While David and Hamish really ought to be satisfied with a probability of 0.5000724, it has to be conceded that an event, based on the ordered outcomes from a single draw of the Lottery, which has a probability of exactly $\frac{1}{2}$ has still to be identified. A step in this direction comes from the observation that if M is even, 2ℓ, say, and r is odd, $2s - 1$, say, then

* In this case b_1 is also a negative integer but as $b_1 \leqslant a_2$ the coefficient of z^k is deemed to be zero for $k > M - r$ as the numerator vanishes before the denominator. The case $a_2 = b_1$, i.e. $m = r$ causes no technical difficulty but does undermine the R function `hypergeo`.

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$p_s(k) = p_s(2\ell + 1 - k)$, $k = s, \ldots, 2\ell + 1 - s$. In this lottery the sth smallest ball is the 'middle' value and the probability that it is odd is the sum of $p_s(k)$ over the odd values of k in the range $s, \ldots, 2\ell + 1 - s$. However, as k runs through the odd values in this range, $2\ell + 1 - k$ goes through the even values in the same range, so the sum of $p_s(k)$ over odd values of k coincides with the sum over even values, so for a lottery with M even and r odd, the 'middle' ball is equally likely to be odd or even. Again this fits well with our intuition.

But how does it help Hamish and David, who are stuck with a lottery with odd M and even r? The following is perhaps the most cumbersome way of settling matters completely fairly. Suppose that the six lottery balls have numbers $n_1 < n_2 < n_3 < n_4 < n_5 < n_6$. The idea is that these values will be distributed symmetrically across the numbers $1, \ldots, 49$, so

1. If n_1 is strictly closer to 1 than n_6 is to 49, i.e. if $n_1 + n_6 < 50$, declare a Head, if $n_1 + n_6 > 50$ declare a Tail. These outcomes are equally likely but there is a possibility that $n_1 + n_6 = 50$, when no result is declared. In this case, go to step 2.
2. If step 1 has failed, consider $n_2 + n_5$ and declare a Head or Tail as it is less than or greater than 50. If $n_2 + n_5 = 50$ go to step 3.
3. If steps 1 and 2 have failed, repeat step 2 using $n_3 + n_4$ in place of $n_2 + n_5$.

This process will declare a Head with probability p and a Tail with probability p, and while $2p$ will be close to 1, it will fall short by the positive probability that all of the above steps have failed, i.e.

$$n_1 + n_6 = n_2 + n_5 = n_3 + n_4 = 50$$

(it is not too hard to compute the probability, $1 - 2p$, of this happening). However, if steps 1 to 3 fail to give a result it is clear that $n_3 \leqslant 24$ and $n_4 > 24$. Therefore n_1, n_2, n_3 must be distributed as a draw from a lottery with $M = 24$ and $r = 3$. In such a lottery we know that n_2 is equally likely to be odd or even, so a final step of declaring a Head if n_2 is odd and a Tail if it is even (*provided* $n_1 + n_6 = n_2 + n_5 = n_3 + n_4 = 50$), can be added to steps 1 to 3 and this procedure will satisfy David and Hamish's requirement exactly!

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References

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The theme this *Nemo* is mathematics as solace. The quotations are to be identified by reference to author and work. Solutions are invited to the Editor by 31st May 2016.

1. Old problems in trigonometry were the pleasing relaxations of his mind, and complications of figures were a delight to him.
2. It is the very same taste which relishes a demonstration in geometry, that is pleased with the resemblance of a picture to an original, and touched with the harmony of musick.
3. Mathematics are your pastime: souls shall rise in their degree;
   Butterflies may dread extinction– you'll not die, it cannot be!
4. In broken mathematics
   We estimate our prize,
   Vast, in its fading ratio,
   To our penurious eyes.
5. 'Tis told by one whom stormy waters there
   With fellow-sufferers by the shipwreck spared,
   Upon a desert coast, that having brought
   To land a single volume, saved by chance,
   A treatise of Geometry, he wont,
   Although of food and clothing destitute,
   And beyond common wretchedness depressed,
   To part from company and take this book
   (Then first a self-taught pupil in its truths)
   To spots remote, and draw his diagrams
   With a long staff upon the sand, and thus
   Did oft beguile his sorrow, and almost
   Forget his feeling.
6. The idea of this early prodigy of erudition being carried off by the gipsies, like a second Adam Smith, was not to be tolerated; and accordingly, though the charge was contrary to all his habits of life, he readily undertook it, and might be seen stalking about with a mathematical problem in his head, and his eye upon a child of five years old, whose rambles led him into a hundred awkward situations.

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