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# BOUNDING THE STATIONARY DISTRIBUTION OF THE *M/G/*1 QUEUE SIZE

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We start with a simple derivation of an identity connecting the conditional expected residual service time as seen by an arrival and the steady-state tail distribution function of the number of customers in the system, which was previously proven by Mandelbaum and Yechiali. We then show how to use it to obtain bounds on the the stationary distribution of the number of customers in the $M/G/1$ queue.

## 1. INTRODUCTION

Let $X$ denote the steady-state number of customers in the system for the first come–first served $M/G/1$ queue. Imagine that, for a specified $n$, the $n$th oldest customer in the system (if there are at least $n$ in the system) pays us 1 per unit time. Then, with $\lambda$ being the Poisson arrival rate and with $S$ being a generic service time random variable, we obtain, upon applying the cost identity,

$$\text{Average rate at which we earn} = \lambda \times \text{Average amount a customer pays},$$

along with the PASTA principle, that when $n = 1$

$$P(X \geq 1) = \lambda E[S] \tag{1}$$

and for $n > 1$,

$$P(X \geq n) = \lambda[P(X \geq n)E[S] + P(X = n-1)E[R_{n-1}]],$$

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where $R_j$ is the remaining time in service of the person being served when a customer arrives to find $j$ others in the system.

Thus, with

$$K_j = \frac{\lambda E[R_j]}{1 - \lambda E[S]}, \quad j \geq 1$$

we see that

$$\begin{aligned} P(X \geq n) &= K_{n-1} P(X = n-1), \qquad n > 1 \\ &= K_{n-1}[P(X \geq n-1) - P(X \geq n)], \qquad n > 1. \end{aligned}$$

Thus,

$$P(X \geq n) = \frac{K_{n-1}}{1 + K_{n-1}} P(X \geq n-1), \qquad n > 1,$$

giving

$$P(X \geq n) = \lambda E[S] \prod_{j=1}^{n-1} \frac{K_j}{1 + K_j} = \lambda E[S] \prod_{j=1}^{n-1} \frac{\lambda E[R_j]}{1 + \lambda E[R_j] - \lambda E[S]}, \qquad n > 1. \tag{2}$$

*Remarks:*

1. Equations (1) and (2) are neatly summarized in terms of the failure rate function $r(n) = P(X = n | X \geq n) = 1 - P(X \geq n+1 | X \geq n)$. With $E[R_0] = E[S]$,

$$r(n) = \frac{1 - \lambda E[S]}{1 + \lambda E[R_n] - \lambda E[S]}, \qquad n \geq 0,$$

with

$$P(X \geq n) = \prod_{j=0}^{n-1} (1 - r(j)), \qquad n > 0.$$

2. Equation (2) was previously proven by Mandelbaum and Yechiali [1] by a more complicated argument involving transforms. It was also proved by Fakinos [2] by an argument essentially equivalent to the one we have used.

## 2. BOUNDS

If we let

$$m_1 = \inf_t E[S - t | S \geq t], \qquad m_2 = \sup_t E[S - t | S \geq t],$$

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then

$$m_1 \le E[R_j] \le m_2.$$

The preceding bounds can be particularly useful in cases where we only have partial information about $G$.

*Example 1:* If $S$ has the new worse than used in expectation property (NWUE) that

$$E[S - t | S \ge t] \ge E[S],$$

then $m_1 = E[S]$, showing that

$$P(X \ge n) \ge (\lambda E[S])^n.$$

In other words, $X$ is stochastically larger (in fact, hazard rate larger) than when the service distribution is exponential with mean $E[S]$. Similarly, if $S$ has the new better than used in expectation property that $E[S - t | S \ge t] \le E[S]$, then $m_2 = E[S]$, showing in this case that

$$P(X \ge n) \le (\lambda E[S])^n.$$

*Example 2:* Suppose that all that is known about the service distribution is its mean and that it is a mixture of exponential random variables with respective means $\theta_1 > \theta_2 > \cdots > \theta_m$. Because $S$ is NWUE, and the conditional distribution of $S$ given that it exceeds $t$ is a mixture of the exponentials whose weight on $\theta_1$ goes to 1 as $t \to \infty$, we have that $m_1 = E[S]$ and $m_2 = \theta_1$. Therefore,

$$(\lambda E[S])^n \le P(X \ge n) \le \lambda E[S] \left( \frac{\lambda \theta_1}{1 + \lambda \theta_1 - \lambda E[S]} \right)^{n-1}.$$

*Example 3:* Suppose that $S$ is a convolution of exponential random variables with respective means $\theta_1 > \theta_2 > \cdots > \theta_m$. Because $S$ is an increasing failure rate random variable whose limiting failure rate is $1/\theta_1$, it follows that $m_1 = \theta_1$. Thus, provided $\theta_1$ and $E[S]$ are known, we have

$$P(X \ge n) \ge \lambda E[S] \left( \frac{\lambda \theta_1}{1 + \lambda \theta_1 - \lambda E[S]} \right)^{n-1}.$$

To obtain stronger bounds than those given in the preceding, let $S, W_1, W_2, \ldots$ be independent, with $S$ having the service time distribution and the $W_i$ being exponential random variables with rate $\lambda$. Let $Y_j = \sum_{i=1}^{j} W_i$ and let

$$\alpha_j = E[S - Y_j | S > Y_j].$$

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Then, with $F_j$ denoting the distribution function of $Y_j$ (viz. gamma with parameters $j$ and $\lambda$), a simple computation yields

$$\alpha_j = \frac{E[SF_j(S)] - j/\lambda + E[Y_j G(Y_j)]}{E[F_j(S)]}, \qquad j \geq 1,$$

with the formula for $\alpha_1$ simplifying to

$$\alpha_1 = \frac{E[S]}{1 - E[e^{-\lambda S}]} - \frac{1}{\lambda}.$$

Now consider a customer, call it $*$, that finds $j$, $j > 0$, in the system upon arrival. Letting $M$ denote the number of customers (including $*$) that have so far arrived while the current service has been in progress, then the remaining service time is distributed as the conditional distribution of $S - Y_M$ conditional on the event that $S > Y_M$. Conditioning on $M$ shows that $E[R_j]$ is a weighted average of the values $\alpha_1, \ldots, \alpha_j$, yielding the bounds

$$\min(\alpha_1, \ldots, \alpha_j) \leq E[R_j] \leq \max(\alpha_1, \ldots, \alpha_j).$$

Now suppose that $S$ has the decreasing mean residual life time (DMRL) property that $E[S - t | S > t]$ decreases in $t$. Because the exponential distribution has a decreasing reversed hazard rate, it follows that the conditional distribution of $Y_j$, given that $Y_j < S$, is stochastically increasing in $j$ (see [3]). Consequentially, it follows from the DMRL property of $S$ that $\alpha_j$ is decreasing in $j$, showing that

$$\alpha_j \leq E[R_j] \leq \alpha_1,$$

which implies

$$\frac{1 - \lambda E[S]}{1 + \lambda\alpha_1 - \lambda E[S]} \leq r(j) \leq \frac{1 - \lambda E[S]}{1 + \lambda\alpha_j - \lambda E[S]}.$$

Similarly, if $S$ has the increasing mean residual life time property that $E[S - t | S > t]$ increases in $t$, then

$$\alpha_1 \leq E[R_j] \leq \alpha_j.$$

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