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# Injective stability for unitary $K_1$, revisited

by

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**Abstract**

We prove the injective stability theorem for unitary $K_1$ under the usual stable range condition on the ground ring. This improves the stability theorem of A. Bak, V. Petrov and G. Tang where a stronger $\Lambda$-stable range condition was used.

*Key Words:* Unitary groups, parabolic subgroups, stable range conditions, Dennis-Vaserstein type decompositions.

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## 1. Introduction

Let $(R,\Lambda)$ be an associative form ring with unit. For a column $u \in R^n$ denote by $I(u)$ the left ideal spanned by its components. Recall that $u$ is called unimodular if $I(u) = R$. One says that the stable rank of $R$ equals $n$ if for any unimodular column $u \in R^{n+1}$ there exist elements $x_1,\dots,x_n \in R$ such that the column

$$(u_1 + x_1u_{n+1},\dots,u_n + x_nu_{n+1})^t \in R^n$$

is unimodular and $n$ is the minimal natural number for which this condition holds.

Let $G = \mathrm{U}(2n,R,\Lambda)$ be the hyperbolic unitary group and $E = \mathrm{EU}(2n,R,\Lambda)$ be its elementary subgroup. Set $\mathrm{KU}_{1,n}(R,\Lambda) = G/E$. Throughout the article we shall consider this object as a coset space, but it is known to be a group under certain assumptions on $R$, see Theorem 3.3 of [17], Proposition 8 of [10], Theorem 1.1 of [4]. The stabilization map $\mathrm{KU}_{1,n}(R,\Lambda) \to \mathrm{KU}_{1,n+1}(R,\Lambda)$ is induced by the embedding

$$\mathrm{U}(2n,R,\Lambda) \hookrightarrow \mathrm{U}(2n+2,R,\Lambda), \quad g \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is classically known that under certain stability assumptions on the ground ring $R$, the stabilization map is an isomorphism. Many results in this spirit were

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obtained since late 1960ies. Some of them are contained in classical textbooks ([6] 9.1.10, [9] VI.4.7.1). The strongest such result is the following theorem due to Anthony Bak, Victor Petrov and Guoping Tang ([2] 1.1).

**Theorem 1** *Suppose that $R$ satisfies the $\Lambda S_m$ condition for $m \geq 2$. Let $n \geq m+1$, then $\mathrm{KU}_{1,n}(R,\Lambda)$ is a group and the stabilization map*

$$\mathrm{KU}_{1,n-1}(R,\Lambda) \to \mathrm{KU}_{1,n}(R,\Lambda)$$

*is surjective. When $n \geq m+2$ the stabilization map*

$$\mathrm{KU}_{1,n-1}(R,\Lambda) \to \mathrm{KU}_{1,n}(R,\Lambda)$$

*is injective and thus is an isomorphism.*

The condition $\Lambda S_m$ of Anthony Bak says that $\mathrm{sr}(R) \leq m$ and for any two columns $u,v \in R^{m+1}$ such that $I(u)+I(v)=R$ there exist a matrix $a \in \mathrm{AH}(m+1,R,\overline{\Lambda})$ such that $u+av \in R^{m+1}$ is unimodular.

More information regarding the $\Lambda S_m$ condition and its precise relation with several other notions of ring dimension can be found in section 3 of [3]. See also section 9 of [8] for a more detailed survey on the history of the problem.

The main goal of the present paper is to show that the injectivity part of the theorem 1.1 can be proved under the usual stable range condition rather than the stronger condition $\Lambda S_m$. More precisely, our main result is the following analogue of Dennis-Vaserstein decomposition (cf. [14] 2.5, [2] 4.3).

**Theorem 2** *Let $n \geq 3$. Assume that $\mathrm{sr}(R) \leq n-2$. Then the Steinberg group $\mathrm{StU}(2n,R,\Lambda)$ can be decomposed as*

$$\mathrm{StU}(2n,R,\Lambda) = \mathrm{StUP}_n \cdot \mathrm{StUU}_n^- \cdot \mathrm{StUP}_1.$$

As a corollary we derive the following

**Theorem 3** *Let $n \geq 3$. Assume that $\mathrm{sr}(R) \leq n-2$. Then the map*

$$\mathrm{KU}_{1,n-1}(R,\Lambda) \to \mathrm{KU}_{1,n}(R,\Lambda)$$

*is injective.*

The even orthogonal group $\mathrm{SO}(2n,R)$ and the symplectic group $\mathrm{Sp}(2n,R)$ can be thought of as special cases of a unitary group $\mathrm{U}(2n,R,\Lambda)$. In this situation theorem 1.3 boils down to the special case of the injective stability theorem for $K_1$ of Chevalley groups ([14] 3.2). Although the condition $\Lambda S_m$ of 1.1 is equivalent to $\mathrm{sr}(R) \leq m$ for the symplectic case (and thus the assumptions on the ground ring are identical for all three stability theorems 1.1, 1.3 and [14] 3.2), it is not known

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whether the condition $\Lambda S_m$ is implied by $\mathrm{sr}(R) \leq m$ for the orthogonal case. Thus 1.3 is a uniform generalization of both 1.1 and [14] 3.2.

Theorem 1.2 is similar in spirit to the recent result of Andrei Smolensky, Balasubramanian Sury and Nikolai Vavilov. In [13] it has been proved that the elementary subgroup of a Chevalley group admits a Gauss type decomposition under the assumption $\mathrm{sr}(R) = 1$, which is an improvement over several previously known results where stronger assumptions on the ground ring were employed.

The proof of the theorem 1.2 is in fact a shortened version of Michael Stein's proof of injective stability for Chevalley groups, transcribed to the unitary case. As did the authors of [2], we also use some ideas from (unfortunately still unpublished) [12].

The present paper is organized as follows: in the sections 2 and 3 we collect the neccessary notations and definitions and in the section 4 we prove our main results.

## 2. Principal notation

Our notation closely follows [4]. Let $R$ be an associative ring with unit. Let $\alpha \mapsto \overline{\alpha}$ be an involution on $R$, i.e. an additive anti-automorphism of order two, $\overline{\alpha+\beta} = \overline{\alpha}+\overline{\beta}, \overline{\overline{\alpha}} = \alpha, \overline{\alpha\beta} = \overline{\beta}\overline{\alpha}$.

For natural numbers $m,n$ denote by $\mathrm{M}(m,n,R)$ the additive group of $m \times n$ matrices. In particular $\mathrm{M}(n,n,R) = \mathrm{M}(n,R)$ is the ring of matrices of degree $n$ over $R$. Let $e_n$ denote its identity element. For a matrix $x \in \mathrm{M}(m,n,R)$ we denote by $x_{ij}$ its entry in the position $(i,j)$. Denote by $x^* = \overline{x}^t \in \mathrm{M}(n,m,R)$ the Hermitian transpose matrix, i.e. the matrix which has $\overline{x_{j,i}}$ in the position $(i,j)$.

Fix an element $\lambda$ in the center of R such that $\lambda\overline{\lambda} = 1$ and set

$$\Lambda_{\min} = \{\alpha - \lambda\overline{\alpha} \mid \alpha \in R\}, \quad \Lambda_{\max} = \{\alpha \mid \alpha \in R, \alpha = -\lambda\overline{\alpha}\}.$$

A form parameter $\Lambda$ is an additive subgroup of $R$ such that
1) $\Lambda_{\min} \subseteq \Lambda \subseteq \Lambda_{\max}$,
2) $\alpha\Lambda\overline{\alpha} \subseteq \Lambda$ for all $\alpha \in R$.

The pair $(R,\Lambda)$ is called a form ring.
Consider a free right $R$-module $V = R^{2n}$ of rank $2n$ with a fixed base

$$e_1,\ldots,e_n,e_{-n},\ldots,e_{-1}.$$

In the sequel the same numbering will be used for the rows and columns of square matrices of degree $2n$. We think of elements of $V$ as columns with components in $R$: $v = (v_1,\ldots,v_n,v_{-n},\ldots,v_{-1})^t \in R^{2n}$, where $v_i \in R$. In particular, $e_i$ is the column whose $i$-th coordinate is 1, while all others are zeroes. Denote by $p_n$ the

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matrix in the matrix ring $\mathrm{M}(n,R)$ which has 1's along its skew diagonal and zeroes elsewhere. Consider the sesquilinear form $f$ on $V$ defined by

$$f(u,v) = \overline{u}^t \begin{pmatrix} 0 & p_n \\ 0 & 0 \end{pmatrix} v = \overline{u_1} v_{-1} + \ldots + \overline{u_n} v_{-n}.$$

This form defines an even $\lambda$-Hermitian form $h(u,v) = f(u,v) + \lambda \overline{f(v,u)}$ and a $\Lambda$-quadratic form $q : V \to R/\Lambda$ by $q(v) = f(v,v) + \Lambda$. In other words, $h$ is the form with the Gram matrix $\begin{pmatrix} 0 & p_n \\ \lambda p_n & 0 \end{pmatrix}$, or what is the same,

$$h(u,v) = f(u,v) + \lambda \overline{f(v,u)} = \overline{u_1} v_{-1} + \ldots + \overline{u_n} v_{-n} + \lambda \overline{u_{-n}} v_n + \ldots + \lambda \overline{u}_{-1} v_1.$$

As usual, we denote by $\mathrm{GL}(n,R)$ the group of all two-sided invertible matrices of degree $n$ with entries from $R$. For a matrix $g$ we denote by $g^{-1}$ its inverse and by $g'_{ij}$ the entry of $g^{-1}$ in the position $(i,j)$. Denote by $\mathrm{E}(n,R)$ the elementary subgroup of $\mathrm{GL}(n,R)$, i.e. the subgroup generated by all elementary transvections

$$t_{ij}(\xi) = e + \xi e_{ij}, i,j = 1 \ldots n, i \neq j, \xi \in R.$$

Denote by $\mathrm{AH}(n,R,\Lambda)$ the set of $\Lambda$-antihermitian matrices which is defined as follows

$$\mathrm{AH}(n,R,\Lambda) = \{a \in \mathrm{M}(n,R) \mid a = -\lambda a^* \text{and } a_{ii} \in \Lambda \text{ for all } i = 1 \ldots n\}.$$

Let $\mathrm{U}(2n,R,\Lambda)$ be the group consisting of all elements of $\mathrm{GL}(V) \cong \mathrm{GL}(2n,R)$ which preserve the $\lambda$-Hermitian form $h$ and the $\Lambda$-quadratic form $q$. In other words, $g \in \mathrm{GL}(2n,R)$ belongs to $\mathrm{U}(2n,R,\Lambda)$ iff $h(gu,gv) = h(u,v)$ and $q(gu) = q(u)$ for $u,v \in V$.

**Lemma 4** *The following conditions are equivalent for a matrix*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2n,R).$$

1) $g$ *belongs to* $\mathrm{U}(2n,R,\Lambda)$,

2) $g^{-1} = \begin{pmatrix} pd^*p & \overline{\lambda} pb^*p \\ \lambda pc^*p & pa^*p \end{pmatrix}$ *and* $a^*pc, b^*pd \in \mathrm{AH}(n,R,\Lambda)$,

3) $g'_{ij} = \lambda^{(\mathrm{sign}(j)-\mathrm{sign}(i))/2} \overline{g_{-j,-i}}$ *and* $\sum_{k=1}^{n} \overline{g_{kj}} g_{-k,j} \in \Lambda$ *for all* $i,j = 1,\ldots,-1$.

For the proof of this fact see section 2.2 of [4].

Define the standard hyperbolic embedding $H : \mathrm{GL}(n,R) \to \mathrm{U}(2n,R,\Lambda)$ by

$$H(g) = \begin{pmatrix} g & 0 \\ 0 & p(g^*)^{-1}p \end{pmatrix}.$$

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## 3. Unitary Steinberg group

We consider the two following types of transformations in $\mathrm{U}(2n,R,\Lambda)$ which we call elementary unitary transvections. For $i,j=1\ldots,-1$, denote by $e_{i,j}$ the standard matrix units of the ring $\mathrm{M}(2n,R)$, i.e. the $2n\times 2n$ matrix with 1 in the $(i,j)$-position and 0 in all other positions. Unitary transvections $T_{ij}(\xi)$ correspond to the pairs $i,j=-1,\ldots,1$, $i\neq j$.

When $i\neq -j$, we set for any $\xi\in R$

$$T_{ij}(\xi)=e_{2n}+\xi e_{i,j}-\lambda^{(\mathrm{sign}(j)-\mathrm{sign}(i))/2}\overline{\xi}e_{-j,-i}.$$

On the other hand, for $j=-i$ and $\alpha\in\lambda^{-(\mathrm{sign}(i)+1)/2}\Lambda$, we set

$$T_{ij}(\alpha)=e_{2n}+\alpha e_{i,-i}.$$

In the above definition, $\alpha\in\overline{\Lambda}$ when $\mathrm{sign}(i)=1$ and $\alpha\in\Lambda$ otherwise.

Denote by $\mathrm{EU}(2n,R,\Lambda)$ the subgroup of $\mathrm{U}(2n,R,\Lambda)$ generated by all elementary unitary transvections $T_{i,j}(\xi), i\neq\pm j$, and $T_{i,-i}(\alpha)$.

Recall that for any two elements $x,y$ of a group $G$, one denotes by $[x,y]=xyx^{-1}y^{-1}$ their left-normed commutator. We also write $y^x$ for $x^{-1}yx$.

**Lemma 5** *The elementary transvections satisfy the following relations:*

R1) $T_{ij}(\xi)=T_{-j,-i}(-\lambda^{(\mathrm{sign}(j)-\mathrm{sign}(i))/2}\overline{\xi})$,

R2) $T_{ij}(\xi)T_{ij}(\zeta)=T_{ij}(\xi+\zeta)$,

R3) $[T_{ij}(\xi),T_{hk}(\zeta)]=e$, *for* $h\neq j,-i$ *and* $k\neq i,-j$,

R4) $[T_{ij}(\xi),T_{jh}(\zeta)]=T_{ih}(\xi\zeta)$, *for* $i,h\neq\pm j$ *and* $i\neq\pm h$,

R5) $[T_{ij}(\xi),T_{j,-i}(\zeta)]=T_{i,-i}(\xi\zeta-\lambda^{-\mathrm{sign}(i)}\overline{\zeta}\cdot\overline{\xi})$, *for* $i\neq\pm j$,

R6) $[T_{i,-i}(\alpha),T_{-i,j}(\xi)]=T_{ij}(\alpha\xi)T_{-j,j}(-\lambda^{(\mathrm{sign}(j)-\mathrm{sign}(-i))/2}\overline{\xi}\alpha\xi)$, *for* $i\neq\pm j$.

These relations can be found in the section 3.2 of [4].

Consider the free group with generators $X_{ij}(\xi)$ with $\xi$ as above (i.e. $\xi\in R$ when $i\neq -j$, otherwise $\xi\in\overline{\Lambda}$ or $\xi\in\Lambda$ depending on the sign of $i$). Define the unitary Steinberg group $\mathrm{StU}(n,R)$ as the factor group of the group above modulo relations R1–R6 with the $T_{ij}$'s replaced by the $X_{ij}$'s. There is a surjective map $\varphi:\mathrm{StU}(2n,R,\Lambda)\to\mathrm{EU}(2n,R,\Lambda)$ sending $X_{ij}(\xi)$ to $T_{ij}(\xi)$. Denote by $X_{ij}$ the subgroup consisting of all $X_{ij}(\xi)$. $X_{ij}$ is called a root subgroup.

Denote by $\mathrm{StUP}_n$ the subgroup of $\mathrm{StU}(2n,R,\Lambda)$ generated by all root subgroups $X_{ij}$ with $i>0$. It admits a Levi decomposition

$$\mathrm{StUP}_n=\mathrm{StUU}_n\leftthreetimes\mathrm{StUL}_n,$$

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where $\mathrm{StUL}_n$ and $\mathrm{StUU}_n$ are generated by the subgroups $X_{ij}$ with $i,j > 0$ and $i > 0 > j$, respectively. Similarly one can define $\mathrm{StUU}_n^-$ as the subgroup spanned by the subgroups $X_{ij}$ for $i < 0 < j$. From R1 and R3, it follows that both $\mathrm{StUU}_n$ and $\mathrm{StUU}_n^-$ are abelian. Obviously, $\mathrm{StUL}_n$ is isomorphic to the linear Steinberg group $\mathrm{St}(n,R)$.

Denote by $\mathrm{EUP}_1$ the subgroup generated by all the elementary transvections $T_{ij}(\xi)$ with $i \neq -1$, $j \neq 1$. Replacing each $T_{ij}$ by $X_{ij}$, one can define its analogue in the Steinberg group. Denote it by $\mathrm{StUP}_1$.

From now on suppose $n \geq 3$. We need explicit notation for the following subgroups of $\mathrm{StUL}_n$

$$V^+ = \langle X_{i,n}, X_{i,n-1} \mid i = 1,\ldots,n-2\rangle, \qquad V^- = \langle X_{n,i}, X_{n-1,i} \mid i = 1,\ldots,n-2\rangle.$$

The images of these subgroups under $\varphi$ coincide with the images of the following subgroups of $E(n,R)$ embedded into $\mathrm{U}(2n,R,\Lambda)$ via $H$

$$\begin{pmatrix} e_{n-2} & * \\ 0 & e_2 \end{pmatrix}, \begin{pmatrix} e_{n-2} & 0 \\ * & e_2 \end{pmatrix}.$$

The following four subgroups will be used in the proof of the decomposition theorem

$$U^+ = \mathrm{StUU}_n \leftthreetimes V^+, \qquad U^- = \mathrm{StUU}_n^- \leftthreetimes V^-,$$

$$Y^+ = \langle X_{n,-n}, X_{n-1,-(n-1)}, X_{n-1,-n}\rangle \leq \mathrm{StUU}_n,$$

$$Y^- = \langle X_{-n,n}, X_{-(n-1),(n-1)}, X_{-(n-1),n}\rangle \leq \mathrm{StUU}_n^-.$$

**Lemma 6** $\mathrm{StU}(2n,R,\Lambda)$ *is generated by* $\mathrm{StUP}_n$ *and* $Y^-$.

*Proof:* Let $i = 1\ldots n-1$, $j = 1\ldots n-2$, $\xi \in R$, $\alpha \in \Lambda$. It follows from R4 and R6 that the elements

$$X_{-n,j}(\xi) = [X_{-n,n-1}(\xi), X_{n-1,j}(1)],$$

$$X_{-j,i}(\xi) = [X_{-j,-n}(1), X_{-n,i}(\xi)],$$

$$X_{-j,j}(\alpha) = X_{-n,j}(\alpha)[X_{-n,n}(-\alpha), X_{n,j}(1)]$$

belong to the subgroup spanned by $\mathrm{StUP}_n$ and $Y^-$. Thus, it contains all the generators of $\mathrm{StU}(n,R,\Lambda)$. □

Let $H_1\ldots,H_s$ be arbitrary subsets of a group $G$. Then $H_1\ldots H_s$ denotes their Minkowski product, i.e. $H_1\ldots H_s = \{h_1h_2\cdot\ldots\cdot h_s \mid h_i \in H_i\}$.

**Lemma 7** *One has*

$$Y^-U^+ \subseteq U^+Y^-Y^+.$$

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*Proof:* Let $x$ be an arbitrary element of $U^+$. We can express $x$ as a product $x = vy$, where $y \in Y^+$ and $v$ lies in the subgroup spanned by $X_{ij}(\xi)$ with $i \leq n-2$. Then from the Steinberg relations it follows that for any $g \in Y^-$ the element $gvg^{-1}$ lies in $U^+$. Thus $gx = gvg^{-1} \cdot g \cdot y \in U^+Y^-Y^+$ as desired. □

**Corollary 8** *One has the following inclusions*
1) $Y^+U^- \subseteq U^-Y^+Y^-$,
2) $Y^-U^+U^- \subseteq U^+U^-Y^+Y^-$.

## 4. Proof of the main results

**Lemma 9** *Let $m < n$ and $\mathrm{sr}(R) \leq n-m$. Let $v$ be a unimodular column of height $n$. Then there exists a matrix $b \in M(n-m,m,R)$ such that $(e_{n-m},b)v \in R^{n-m}$ is unimodular.*

*Proof:* The claim immediately follows from the definition of stable rank. One has to iterate the process of shortening of the column $v$. □

Denote by $\widetilde{Q}$ the set consisting of elements $g \in \mathrm{StUL}_n$, such that the matrix entries $\varphi(g)_{n-1,1}$ and $\varphi(g)_{n,1}$ are zero.

The following lemma is the only place in the proof which depends on stability conditions.

**Lemma 10** *Let $n \geq 3$. Assume that $\mathrm{sr}(R) \leq n-2$. Then for every $g \in \mathrm{StUL}_n$ there exist $x_g \in V^+$, $y_g \in V^-$ such that $y_g x_g g \in \widetilde{Q}$.*

*Proof:* Let $g \in \mathrm{StUL}_n$ and denote by $v$ the first column of $\varphi(g)$. By Lemma 9 there exists a matrix $b \in M(n-2,2,R)$, and thus $x_g \in V^+$, such that the first $n-2$ coordinates of

$$v' = \varphi(x_g)v = H\left(\begin{pmatrix} e_{n-2} & b \\ 0 & e_2 \end{pmatrix}\right)v$$

form a unimodular column. Now, we can find $c \in M(2,n-2,R)$, $y_g \in V^-$ such that $v''_n = v''_{n-1} = 0$, where

$$v'' = \varphi(y_g)v' = H\left(\begin{pmatrix} e_{n-2} & 0 \\ c & e_2 \end{pmatrix}\right)v'.$$

□

**Corollary 11** *One has*

$$\mathrm{StUU}_n \mathrm{StUU}_n^- \mathrm{StUL}_n \subseteq U^+U^-\widetilde{Q}.$$

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*Proof:* Since $x = x_g$ normalizes $\mathrm{StUU}_n^-$, we have

$$\mathrm{StUU}_n \mathrm{StUU}_n^- g = (\mathrm{StUU}_n x^{-1})(x \mathrm{StUU}_n^- x^{-1} \cdot {y_g}^{-1})(y_g \cdot xg) \subseteq U^+ U^- \widetilde{Q}.$$

□

**Lemma 12** *One has*

$$Y^- U^+ U^- \widetilde{Q} \subseteq U^+ U^- \mathrm{StUL}_n \mathrm{StUP}_1.$$

*Proof:* Let $a$ be an element of $\widetilde{Q}$ and $x$ be an element of $(Y^-)^a$. Since $x$ lies in $\mathrm{StUU}_n^-$ we have that $\varphi(x) = \begin{pmatrix} e_n & 0 \\ p_n c & e_n \end{pmatrix}$ for some antihermitian $c$. By the very definition of $\widetilde{Q}$, the first column of $\varphi(a)$ remains unchanged if we multiply $\varphi(a)$ on the left by an element of $\varphi(Y^-)$. Hence, the first column of $\varphi(x)$ coincides with that of identity matrix and the first column of $p_n c$ is zero. Since $c$ is antihermitian this implies that the last row of $p_n c$ is also zero. The restriction of $\varphi$ to the subgroup $\mathrm{StUU}_n^-$ is an isomorphism, hence

$$x \in \mathrm{StUU}_n^- \cap \varphi^{-1}(\mathrm{EUU}_n^- \cap \mathrm{EUP}_1) \subseteq \mathrm{StUP}_1.$$

Now, the lemma follows from the following sequence of inclusions

$$Y^- U^+ U^- a \subseteq U^+ U^- Y^+ Y^- a \subseteq U^+ U^- a (Y^+)^a (Y^-)^a \subseteq U^+ U^- \mathrm{StUL}_n \mathrm{StUP}_1.$$

□

Now, we are able to finish the proof of our main results.

*Proof of the decomposition theorem:* Denote by $A$ the subset

$$\mathrm{StUP}_n \cdot \mathrm{StUU}_n^- \cdot \mathrm{StUP}_1.$$

Since $\mathrm{StUL}_n$ normalizes both $\mathrm{StUU}_n$ and $\mathrm{StUU}_n^-$, we have

$$A = \mathrm{StUU}_n \mathrm{StUL}_n \mathrm{StUU}_n^- \mathrm{StUP}_1 = \mathrm{StUU}_n \mathrm{StUU}_n^- \mathrm{StUL}_n \mathrm{StUP}_1.$$

We wish to prove that $A = \mathrm{StU}(2n, R, \Lambda)$. It suffices to prove that $A$ is stable under left multiplication by the generators of $\mathrm{StU}(2n, R, \Lambda)$. According to Lemma 6, this amounts to showing that $Y^- A \subseteq A$. Now the theorem follows from Corollary 11 and Lemma 12. Indeed

$$Y^- A = Y^- \mathrm{StUU}_n \mathrm{StUU}_n^- \mathrm{StUL}_n \mathrm{StUP}_1 \subseteq$$
$$\subseteq Y^- U^+ U^- \widetilde{Q} \mathrm{StUP}_1 \subseteq U^+ U^- \mathrm{StUL}_n \mathrm{StUP}_1 \subseteq A.$$

□

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*Proof of the stability theorem:* From the decomposition theorem it follows that an element $g$ of $\mathrm{U}(2n-2,R,\Lambda)\cap\mathrm{EU}(2n,R,\Lambda)$ can be expressed as a product $xuy$ where

$$x=\begin{pmatrix}x^{11} & x^{12}\\ 0 & x^{22}\end{pmatrix}\in\mathrm{EUP}_n,\quad u=\begin{pmatrix}e_n & 0\\ u^{21} & e_n\end{pmatrix}\in\mathrm{EUU}_n^-,\quad y\in\mathrm{EUP}_1.$$

Since the first column of $y$ coincides with that of identity matrix, we get that

$$x^{22}(u^{21})_{*1}=0.$$

Since $x^{22}$ is invertible it follows that $(u^{21})_{*1}$ is zero and $u\in\mathrm{EUP}_1$. Thus without loss of generality we can reduce to the case $g=xy$. Moreover, we can assume $x,y\in\mathrm{U}(2n-2,R,\Lambda)$.

Thus $x^{11}\in\mathrm{E}(n,R)\cap\mathrm{GL}(n-1,R)$. Since $\mathrm{sr}(R)\le n-2$ the well-known injective stability theorem for ordinary $K_1$ says that $x^{11}\in\mathrm{E}(n-1,R)$ ([5] V.4.2, [14] 3.2) and hence $g\in\mathrm{EU}(2n-2,R,\Lambda)$. □

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