Teaching Notes

A quick simultaneous evaluation of $\sum k^3$ and $\sum k^2$

In this short Note I offer a "two for the price of one" evaluation of the standard sums in the title.

Expanding brackets, we have:

$$\left(\sum_{k=1}^{n} k\right)^2 = \sum_{k=1}^{n} k^2 + 2\sum_{k=2}^{n} \big(1 + 2 + \ldots + (k-1)\big)k$$

so, using $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$ on both sides,

$$\begin{aligned} \frac{1}{4}n^2(n+1)^2 &= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k^2(k-1) \qquad (1)\\ &= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} k^2, \end{aligned}$$

whence $\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2$.

But this means that we can rewrite (1) as:

$$\begin{aligned} \sum_{k=1}^{n} k^3 &= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} (k-1+1)^2(k-1)\\ &= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} (k-1)^3 + 2\sum_{k=1}^{n} (k-1)^2 + \sum_{k=1}^{n} (k-1) \end{aligned}$$

so, on combining terms with the same powers,

$$n^3 = 3\sum_{k=1}^{n} k^2 - 2n^2 + \tfrac{1}{2}n(n-1)$$

and $$\sum_{k=1}^{n} k^2 = \tfrac{1}{6}\left(2n^3 + 3n^2 + n\right) = \tfrac{1}{6}n(n+1)(2n+1).$$

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How to impress a chemist (again!)

This note is a sequel to [1] and – spoiler alert – it ends in the same place!

We start with a circular filter paper with a radial slit and, with a paper clip, form it into a right circular cone of adjustable height. If we maximise the volume of this cone, its vertical angle is, as in [1], $\cos^{-1}\left(-\frac{1}{3}\right) = 109.5°$, the bond angle of methane.

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The verification of this is a pleasant A-level exercise.

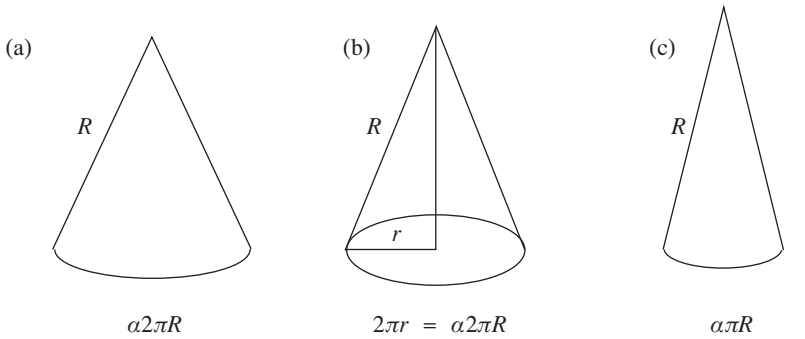


FIGURE 1: Cone (b) is formed by identifying edges of sector (a); it flattens to give sector (c).

If a fraction α, $0 < \alpha < 1$, of the circle of filter paper is used to make the cone then, from Figure 1(b), $2\pi r = \alpha 2\pi R$ so that $r = \alpha R$. The volume, V, of the cone is then

$$V = \tfrac{1}{3}\pi(\alpha R)^2\sqrt{R^2 - \alpha^2 R^2} = \tfrac{1}{3}\pi R^3\alpha^2\sqrt{1 - \alpha^2}.$$

This can be maximised by using differentiation, but it is also fun to use the AM-GM inequality:

$$\left[\alpha^4(1 - \alpha^2)\right]^{1/3} = \sqrt[3]{4}\left[\frac{1}{2}\alpha^2\frac{1}{2}\alpha^2(1 - \alpha^2)\right]^{1/3} \leqslant \frac{\sqrt[3]{4}}{3},$$

so that $V \leqslant \dfrac{1}{3}\pi R^3\sqrt{\dfrac{4}{27}} = \dfrac{2\sqrt{3}}{27}\pi R^3$ with equality if, and only if, $\frac{1}{2}\alpha^2 = 1 - \alpha^2$ or $\alpha = \sqrt{\frac{2}{3}}$. The vertical angle of the cone is then

$$2\sin^{-1}\left(\frac{r}{R}\right) = 2\sin^{-1}\alpha = \cos^{-1}(1 - 2\alpha^2) = \cos^{-1}\left(-\frac{1}{3}\right),$$

as claimed.

Incidentally, if we flatten any of the cones formed, as in Figure 1(c), the angle at the vertex increases from $2\sin^{-1}\alpha$ to $\alpha\pi$. We thus have a tactile proof that $2\sin^{-1}\alpha < \alpha\pi$ which is equivalent to Jordan's inequality $\frac{2}{\pi}x < \sin x$ for $0 < x < \frac{\pi}{2}$.

Reference

1. N. Lord, A 'maths bite': how to impress a chemist, *Math. Gaz.* **80** (November 1996) p. 584.

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