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Multi-level dynamo and opinion spreading†

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We consider the following multi-level opinion spreading model on networks. Initially, each node gets a weight, from the set $\{0,\ldots,k-1\}$, which measures the individual conviction of a new idea or product. Then, by proceeding in rounds, each node updates its weight according to those of its neighbours. We study k-dynamos that are initial assignments of weights leading each node to get the value $k-1$ – e.g. unanimous maximum level of acceptance – within a given number of rounds; the goal is to minimize the sum of the initial weights of the nodes. We determine lower bounds on the sum of the initial weights under the irreversible simple majority rules, where a node increases its weight if and only if the majority of its neighbours have a weight that is higher than its own. We study the relations among 2-dynamos and k-dynamos, with and without a bound on the number of rounds needed to reach the desired all-$(k-1)$ configuration. Moreover, we provide constructive tight upper bounds for some classes of regular topologies: rings, tori and cliques.

1. Introduction

New opinions and behaviours usually spread gradually through social networks. In 1966 a classical study showed how doctors' willingness to prescribe a new antibiotic diffused through professional contacts. A similar pattern can be detected in a variety of innovations: initially a few innovators adopt, then people in contact with the innovators get interested and then adopt, and so forth until eventually the innovation spreads throughout the society. A classical question is then how many innovators are needed, and how they need to be disposed, in order to get a fast unanimous adoption (Valente 1995). Disentangling the many factors behind persuasion dynamics is far from trivial. In fact, recent studies exploring the evolution of social phenomena on online social networks that long ties are relevant in the spreading of innovation and social movements. Just recently in Centola (2010), the author found evidence of an effect of reinforcement in the adoption when a user receive multiple stimuli about the same topic but from different sources. Such a phenomenon has been called *Complex Contagion*.

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In the wide set of the information spreading models, the first computational study about information diffusion (Granovetter 1985) used the *linear threshold model* where the threshold triggering the adoption of a new idea to a node is given by the majority of its active neighbours.

Recently, information spreading has been intensively studied also in the context of *viral marketing*, which uses social networks to achieve marketing objectives through self-replicating viral processes, analogous to the spread of viruses. The goal here is to create a marketing message that can initially convince a selected set of people and then spread to the whole network in a short period of time (Domingos and Richardson 2001). One problem in viral marketing is the *target set selection problem* which asks for identifying the minimal number of nodes which can activate, under some conditions, the whole network (Easley and Kleinberg 2010). The target set selection problem has been proved to be NP-hard through a reduction to the node cover problem (Kempe *et al.* 2003) and inapproximability results of opinion spreading problems have been presented in Chen (2009).

It is well known that in viral marketing it is quite important to spread information quickly. Indeed, research in Behavioural Economics shows that humans make decisions mostly on the basis of very recent events, even though they might remember much more (Alba *et al.* 1991). Therefore, it seems reasonable to study processes of information spreading that reach the desired goals within a fixed time bound (Flocchini *et al.* 2003) and (Cicalese *et al.* 2014).

Inspired by the notion of complex contagion, in this paper, we consider a weighted opinion spreading model, where the weight of an actor represents the level of acceptance of the opinion. The spreading process proceeds in synchronous rounds where each actor updates its weight depending on the opinions of its neighbours.

We want everyone to completely accept the new opinion within a given time bound while minimizing the initial convincing effort (sum of the initial node weights).

We notice that we are interested in the case in which the spreading is essentially a one-way process: once an agent has adopted an opinion (or behaviour, innovation, …), it sticks with it. These are usually referred as *irreversible* spreading processes.

1.1. *Dynamic monopolies and opinion spreading*

In a different scenario, spreading processes have been studied under the name of dynamic monopolies. Monopolies were initially introduced to deal with faulty nodes in distributed computing systems. A monopoly in a graph is a subset M of nodes such that each other node of the graph has a prescribed number of neighbours belonging to M. The problem of finding monopolies in graphs has been widely studied, see for example Bermond *et al.* (1996), Linial *et al.* (1993) and Mishra and Rao (2003) for connections with minimum dominating set problem.

Dynamic monopolies or shortly *dynamo* were introduced by Peleg (1998). A strong relationship between opinion spreading problems, such as the target set selection, and dynamic monopolies exists. Indeed, they can be used to model the irreversible spread of opinions in social networks.

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Dynamic monopolies have been intensively studied with respect to the bounds of the size of the monopolies, the time needed to converge into a fixed point, and topologies over which the interaction takes place (Bermond *et al.* 2003, 1998; Flocchini *et al.* 2003; Kulich 2011; Nayak *et al.* 1992). A survey on the subject is contained in Peleg (2002).

1.2. *Weighted opinion spreading*

We model the opinion spreading process considered in this paper, by means of weighted dynamos.

We extend the setting of dynamos from two possible weights, denoting whether a node has accepted the opinion or not, to k levels of opinion acceptance. Initially, each node is assigned a weight (which represents the node initial level of acceptance of the opinion) in the set $\{0,\ldots,k-1\}$. Then, each node updates its weight by increasing it of one unit if the weights of the simple majority of its neighbours is larger than its own.

We call k-dynamos, the initial weight assignments which lead each node in the network to have maximum weight $k-1$. We are interested in the minimum weight (i.e. the sum of the weight initially assigned to the nodes) of a k-dynamo. We focus on both the weight and the time (e.g. number of rounds needed to reach the final configuration); namely, we study k-dynamos of minimum weight which converge into at most t rounds.

2. The model

Let $G = (V,E)$ be an undirected connected graph. For each $v \in V$, we denote by $N(v) = \{u \in V \mid \{u,v\} \in E\}$ the neighbourhood of v and by $d(v)$ its cardinality (i.e. the degree of v).

We assume the nodes of G to be weighted by the set $[k] = \{0,1,\ldots,k-1\}$ of the first $k \geqslant 2$ integers. For each $v \in V$ we denote by $c_v \in [k]$ the weight assigned to a given node v.

Definition 1. A *k-configuration* $\mathcal{C}$ for G is a partition of V into k sets $\{V_0, V_1, \ldots, V_{k-1}\}$, where $V_j = \{v \in V \mid c_v = j\}$ is the set of nodes of weight j. The weight $w(\mathcal{C})$ of $\mathcal{C}$ is the weighted sum of its nodes

$$w(\mathcal{C}) = \sum_{j=0}^{k-1} j \times |V_j| = \sum_{v \in V} c_v.$$

Consider the following node, weighting game played on G using the set of weights $[k]$ and a threshold value λ (for some $0 < \lambda \leqslant 1$):

> In the initial configuration, each node has a weight in $[k]$. Then node weights are updated in synchronous rounds (i.e. node weights at round i only depend on the values at round $i-1$). Let $c_v(i)$ denote the weight of node v at the end of round $i \geqslant 0$; during round $i \geqslant 1$, each node updates its weight according to the weight of its neighbours at round $i-1$. Specifically, each node v

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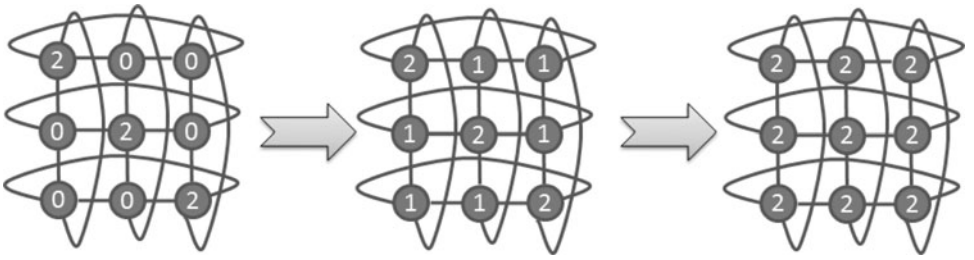


Fig. 1. A $(3, 2, 1/2)$-dynamo for a 3×3 Torus: starting from the initial configuration (left), two rounds are needed to reach the final all-(2)s configuration.

- first computes the number $n^+(v) = |\{u \in N(v) \mid c_u(i-1) > c_v(i-1)\}|$ of neighbours having a weight larger than its current one $c_v(i-1)$;
- then, it applies the following *irreversible rule:*

$$c_v(i) = \begin{cases} c_v(i-1) + 1 & \text{if } n^+(v) \geqslant \lceil \lambda d(v) \rceil \\ c_v(i-1) & \text{otherwise.} \end{cases} \tag{2.1}$$

We denote the initial configuration by $\mathcal{C}^0$ and the configuration at round i by $\mathcal{C}^i$.

We are interested into initial configurations that converge to the unanimous all-$(k-1)$s configuration – i.e. there exists a round t^* such that for each node v, it holds $c_v(t^*) = k-1$. Such configurations are named k-weights dynamic monopoly (henceforth *k-dynamo*).

Definition 2. A k-dynamo, with threshold λ, for a graph G is *optimal* if its weight is minimal among all the k-dynamos, with threshold λ , for the graph G.

A *(k, t, λ)-dynamo* is a k-dynamo, with threshold λ, which reaches its final configuration within t rounds, that is, $c_v(i) = k-1$ for each node $v \in V$ and $i \geqslant t$. An example of $(3, 2, 1/2)$-dynamo is depicted in Figure 1. Given a graph G, a set of weights $[k]$, a threshold λ, and an integer $t > 0$, we aim for a minimum weight (k, t, λ)-dynamo.

Definition 3. A (k, t, λ)-dynamo for a graph G is *optimal* if its weight is minimal among all the (k, t, λ)-dynamos for the graph G.

3. Time bounded dynamos

In this section, we present two preliminary results that characterize the structure of any optimal (k, t, λ)-dynamo. Then we provide a lower bound on the weight of a $(k, t, 1/2)$-dynamo.

3.1. *Preliminary results*

We can represent the spreading process on a graph $G = (V, E)$ – ruled by the Equation (2.1) – via a line chart that depicts for each node, its weight updating round by round. For instance, in Figure 2 (right), each line describes the spreading process of a node, i.e. its evolution. Since, the spreading process is irreversible we have that each line is monotone.

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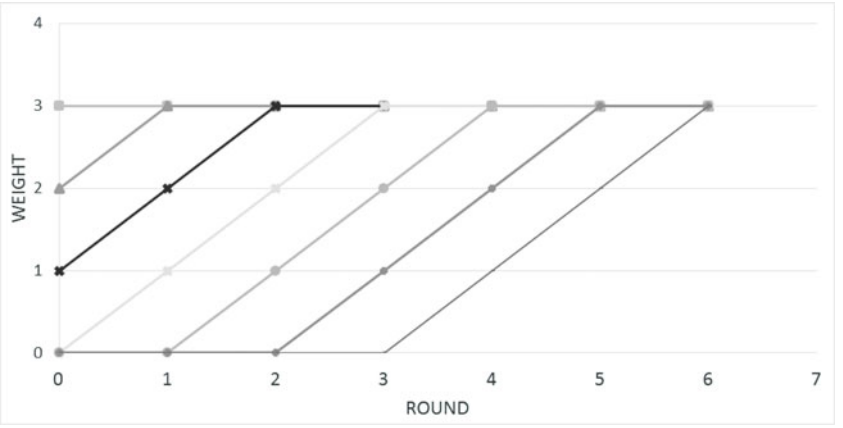


Fig. 2. The spreading process of a $(4, 5, 1/3)$-dynamo on a graph G having 4 nodes.

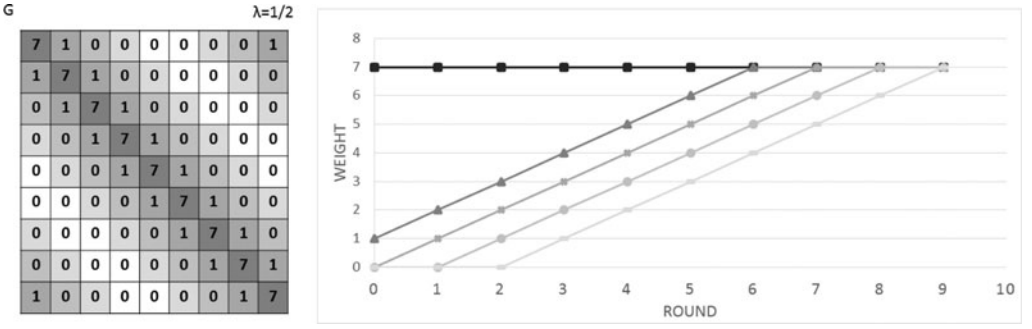


Fig. 3. The spreading process of a $(8, 9, 1/2)$-simple-monotone configuration on a 9×9 Torus. The evolution of the nodes in $X_7, X_1, X_0, X_{-1}, X_{-2}$ are illustrated.

We are particularly interested in some initial configurations, we call simple-monotone, which generate a spreading process of each node that is characterized by three phases, see Figure 3:

— An initial phase, where the node's weight is zero for some rounds. This phase can also be empty.
— A strictly monotone phase, where the node's weight increases until it reaches the maximum weight $(k - 1)$.
— Finally in the last phase the weight of the node remains at the maximum value (irreversible).

An example of simple-monotone configuration is depicted in Figure 3. It easy to see that, in a simple-monotone initial configuration, the evolution of a node is fully described by the round in which the node reaches the maximum weight. Hence, only $t + 1$ possible evolutions are available. Obviously, it is possible that two or more nodes evolves in the same way. In Figure 3, nodes having the same evolution are shown with the same colour, while each line describes the spreading process of a class of nodes.

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In the following, we provide a formal definition of (k,t,λ)-simple-monotone configuration. Then we show that any (k,t,λ)-simple-monotone configuration is a (k,t,λ)-dynamo (Lemma 3.1) and that any optimal (k,t,λ)-dynamo is a (k,t,λ)-simple-monotone configuration (Lemma 3.2).

Definition 4. Consider an undirected connected graph $G=(V,E)$. Let $k \geqslant 2$ and $t \geqslant 1$ be integers and $0 < \lambda \leqslant 1$. An initial k-configuration $\mathcal{C}$ for G is called (k,t,λ)-simple-monotone if V can be partitioned into $t+1$ sets $X_{k-t-1}, X_{k-t}, \ldots X_{k-1}$ where

1. $X_{k-1} \neq \varnothing$ and for each $v \in X_{k-1}$, $c_v(0) = k-1$;
2. for $i = k-t-1, k-t, \ldots, k-2$, and for each $v \in X_i$
   i. $c_v(0) = \max(i,0)$;
   ii. v has at least $\lceil \lambda d(v) \rceil$ neighbours in $\bigcup_{j=i+1}^{k-1} X_j$.

Each set X_i describes a class of nodes that evolves in the same way. In particular, every node in X_i reaches the weight $k-1$ into $k-1-i$ rounds.

In the following Lemma we show that the two properties above are enough to show that any simple-monotone configuration is a dynamo.

Lemma 3.1. Any (k,t,λ)-simple-monotone configuration for an undirected connected graph G is a (k,t,λ)-dynamo for G.

Proof. Let $s = t-k+1$, we show that for each $i = -s, -s+1, \ldots, k-1$ and $j = 0, \ldots, t$ and for each $u \in X_i$

$$c_u(j) = \begin{cases} \min(j+i, k-1) & \text{if } j+i > 0 \\ 0 & \text{otherwise.} \end{cases}$$

We prove this statement by induction on i from $k-1$ back to $-s$. For $i = k-1$ the nodes in X_{k-1} have weight $k-1$ from the initial configuration and since the weight of the nodes never decreases, the statement is trivially true for each round j.

Assume now that the statement is true for any $r > i$. For each $u \in X_i$, we know that u has at least $\lceil \lambda d(v) \rceil$ neighbours which belong to $\bigcup_{r=i+1}^{k-1} X_r$. By induction, each of this neighbour nodes, for each round j has a weight greater or equal to $\min(j+i+1, k-1)$ if $j + (i+1) > 0$.

Hence, u preserves its weight $c_u(j) = \max(i,0)$ until it increases its weight at each round j such that $j + (i+1) > 1$ (i.e. $j + i > 0$) and $c_u(j) < k-1$; as a result each node in X_i has weight $\min(j+i, k-1)$ whenever $j+i > 0$, for each $j = 0, 1, \ldots, t$.

The Lemma follows since at round t, $i + j = i + t \geqslant -s + t = k - 1 > 0$. Hence, all the nodes will have weight $\min(i+t, k-1) = k-1$. □

Lemma 3.2. Let $G = (V,E)$ be an undirected connected graph. Any optimal (k,t,λ)-dynamo for G is a (k,t,λ)-simple-monotone configuration for G.

Proof. Let $\mathcal{C}$ be an optimal (k,t,λ)-dynamo. We denote $s = t-k+1$. Moreover, we denote by X_i the set of nodes that, starting with configuration $\mathcal{C}$, reaches the weight $k-1$ at round $k-1-i$ exactly, for $i = -s, -s+1, \ldots, k-1$. Namely

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for each $i = -s, -s+1, \ldots, k-2$,

$$X_i = \{u \in V \mid c_u(k-2-i) \neq k-1, \text{ and } c_u(j) = k-1 \text{ for each } j \geqslant k-1-i\},$$

and

$$X_{k-1} = \{u \in V \mid c_u(j) = k-1 \text{ for each } j \geqslant 0\}.$$

Define a new configuration $\mathcal{C}'$ as follows:

for each $u \in X_i$ set $c'_u(0) = \max(i, 0)$.

In the following, we will show that $\mathcal{C}'$ is a (k,t,λ)-simple-monotone configuration for G and since $\mathcal{C}$ is optimal, then $\mathcal{C}$ coincides with $\mathcal{C}'$. Since, $\mathcal{C}$ is a (k,t,λ)-dynamo, $\{X_{-s}, X_{-s+1}, \ldots, X_{k-1}\}$ is a partition of V and $X_{k-1} \neq \varnothing$ (otherwise $\mathcal{C}$ cannot be a k-dynamo). Therefore, by construction $\mathcal{C}'$ satisfies point 1 and 2(i) of Definition 4. Moreover, for $i = -s, -s+1, \ldots, k-2$ and for each $u \in X_i$, we know that u in the configuration $\mathcal{C}$ reaches the weight $k-1$ at round $k-1-i$. Hence, at least $\lceil \lambda d(v) \rceil$ of its neighbours have weight $k-1$ at round $k-1-i-1 = k-1-(i+1)$, that is at least $\lceil \lambda d(v) \rceil$ of its neighbours belong to $\bigcup_{j=i+1}^{k-1} X_j$. Hence, point (ii) of Definition 4 also holds.

For each node $u \in V$ we have

a. if $u \in X_i$ for some index $i \leqslant 0$, then $c_u(0) \geqslant c'_u(0) = 0$;
b. if $u \in X_i$ for some index $i > 0$, then $c_u(0) \geqslant c'_u(0) = i$

(otherwise u cannot reach the final weight $k-1$ by round $k-1-i$, since the weight of a node increases by at most 1 at each round).

By using (a) and (b) above we have that for each node $u \in V$ it holds $c_u(0) \geqslant c'_u(0)$. Since, $\mathcal{C}$ is optimal, we also have $w(\mathcal{C}) \leqslant w(\mathcal{C}')$. Therefore, $c_u(0) = c'_u(0)$ for each $u \in V$ which means that $\mathcal{C} = \mathcal{C}'$. □

Thanks to Lemma 3.2, we know that optimal (k,t,λ)-dynamos are always (k,t,λ)-simple-monotone configurations but the opposite is not always true. In the following section, we provide a lower bound (Theorem 3.1) which allows to estimate how a given simple-monotone configuration is distant from the optimum. Moreover, the construction provided in the proof is also useful for building optimal dynamos (see Section 5).

3.2. *A lower bound*

In the following, we fix the value of λ to $1/2$ (that is we consider a simple majority rule). Henceforth, we write (k,t)-dynamo for $(k,t,1/2)$-dynamo.

Theorem 3.1. Consider an undirected connected graph $G = (V,E)$ and let $k \geqslant 2$ and $t \geqslant 1$ be integers. Any (k,t)-dynamo $\mathcal{C}$, has weight

$$w(\mathcal{C}) \geqslant \begin{cases} \frac{|V|}{2\rho(\ell+s+1)+1} \times (k-1+\rho\ell(\ell+1)) & \text{if } t \geqslant k-1 \\ \frac{|V|}{2\rho(\ell+s+1)+1} \times (k-1+\rho(\ell(\ell+1)-s(s+1))) & \text{otherwise,} \end{cases}$$

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where

$$\ell = \begin{cases} \left\lfloor \frac{\sqrt{(2\rho s+\rho+1)^2+4\rho(k-1)}-(2\rho s+\rho+1)}{2\rho} \right\rfloor & \text{if } t \geqslant k-1 \\ \left\lfloor \frac{\sqrt{4\rho(t+1)+(\rho-1)^2}-(2\rho s+\rho+1)}{2\rho} \right\rfloor & \text{otherwise,} \end{cases} \tag{3.2}$$

ρ is the ratio between the maximum and the minimum degree of the nodes in V and $s = t - k + 1$.

For d-regular graphs this result can be written as follows.

Corollary 3.1. Consider an undirected connected d-regular graph $G = (V, E)$. Let $k \geqslant 2$ and $t \geqslant 1$ be integers. Any (k, t)-dynamo $\mathcal{C}$, has weight

$$w(\mathcal{C}) \geqslant \begin{cases} \frac{|V|}{2\ell+2s+3} \times (k-1+\ell(\ell+1)) \text{ where } \ell = \lfloor \sqrt{t+1+s^2+s} \rfloor - (s+1) & \text{if } t \geqslant k-1 \\ \frac{|V|}{2\ell+2s+3} \times (k-1+\ell(\ell+1) - s(s+1)) \text{ where } \ell = \lfloor \sqrt{t+1} \rfloor - (s+1) & \text{otherwise,} \end{cases}$$

where $s = t - k + 1$.

Before proving Theorem 3.1, we would like to show how this result can be used to evaluate the quality of an initial configuration. For instance, we can use Corollary 3.1 to show that the $(8, 9, 1/2)$-simple-monotone configuration $\mathcal{C}$, on a 9×9 Torus is optimal (cf. Figure 3). In this particular case, we have that $k = 8, t = 9, s = 2$ and $w(\mathcal{C}) = 3 \times 9 + 2 \times 9 = 81$ (cf. Figure 3). On the other hand, using Corollary 3.1, we have that since $t \geqslant k - 1$, then $\ell = \lfloor \sqrt{t+1+s^2+s} \rfloor - (s+1) = 1$ and consequently the bound is $\frac{|V|}{2\ell+2s+3} \times (k-1+\ell(\ell+1)) = \frac{81}{9} \times (7+2) = 81$. Therefore, $\mathcal{C}$ is optimal.

Proof of Theorem 3.1.

By Lemma 3.2, we can restrict our attention to (k, t)-simple-monotone configurations for G. Therefore, the set V can be partitioned into $t+1$ subsets $X_{-s}, X_{-s+1}, \ldots, X_{k-1}$ where $s = t - k + 1$ and for $i = -s, -s+1, \ldots, k-1$, X_i denotes the set of nodes whose weight at round j is $\max(0, \min(j + i, k - 1))$. Henceforth, we denote the size of X_i by x_i and the sum of the degree of nodes in $A \subseteq V$ by $d(A)$.

In order to prove the theorem, we first show that, for each $i = -s, -s+1, \ldots, k-2$, it holds

$$x_i \leqslant 2\rho x_{k-1}. \tag{3.3}$$

Let $E(A, B) = |\{e = (u, v) \in E \ : \ u \in A \text{ and } v \in B\}|$ denote the number of edges between a node in A and one in B. Each node $v \in X_i$ must increase its weight for each round r such that $0 < r + i < k - 1$; hence, at round $r = \max(-i + 1, 0)$, node v must have at least $\lceil d(v)/2 \rceil$ neighbours which belong to $\bigcup_{j=i+1}^{k-1} X_j$. Overall the number of edges between X_i and $\bigcup_{j=i+1}^{k-1} X_j$ satisfies

$$E\left(X_i, \bigcup_{j=i+1}^{k-1} X_j\right) \geqslant \frac{d(X_i)}{2} \geqslant \frac{x_i d_{min}}{2}, \tag{3.4}$$

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where d_{min} represents the minimum degree of a node in G. Moreover, for each $i = -s, -s+1, \ldots, k-2$, the number of edges between X_i and $\bigcup_{j=i+1}^{k-1} X_j$ is

$$\begin{aligned} E\left(X_i, \bigcup_{j=i+1}^{k-1} X_j\right) &\leqslant \sum_{j=i+1}^{k-1} d(X_j) - 2E\left(X_{i+1}, \bigcup_{j=i+1}^{k-1} X_j\right) - 2E\left(X_{i+2}, \bigcup_{j=i+2}^{k-1} X_j\right) - \ldots \\ &\qquad \ldots - 2E\left(X_{k-2}, X_{k-2} \cup X_{k-1}\right) - 2E\left(X_{k-1}, X_{k-1}\right) \\ &\leqslant \sum_{j=i+1}^{k-1} d(X_j) - 2\left[d(X_{i+1})/2 + d(X_{i+2})/2 + \ldots + d(X_{k-2})/2\right] \\ &= d(X_{k-1}) \leqslant d_{max}|X_{k-1}| = d_{max} x_{k-1}, \end{aligned}$$

where d_{max} is the maximum node degree of a node in G. By this and Equation (3.4), recalling that $\rho = d_{max}/d_{min}$, we get Equation (3.3).

Define now $y_i = x_i/x_{k-1}$. By Equation (3.3), $0 \leqslant y_i \leqslant 2\rho$. Our goal is to minimize the weight function $w(\mathcal{C}) = \sum_{j=1}^{k-1} j x_j = x_{k-1}\left((k-1) + \sum_{j=1}^{k-2} j y_j\right)$ with $|V| = \sum_{j=-s}^{k-1} x_j = x_{k-1}\left(1 + \sum_{j=-s}^{k-2} y_j\right)$. Hence, $x_{k-1} = \frac{|V|}{1+\sum_{j=-s}^{k-2} y_j}$ and we can write

$$w(C) = |V| \times \frac{k-1+\sum_{j=1}^{k-2} j y_j}{1+\sum_{j=-s}^{k-2} y_j}. \tag{3.5}$$

We distinguish now two cases depending on whether $t \geqslant k-1$ or $t < k-1$.

Case I ($t \geqslant k-1$): In this case, it is possible to show that the rightmost term of Equation (3.5) is minimized when

$$y_i = \begin{cases} 2\rho & \text{if } -s \leqslant i \leqslant \ell \\ 0 & \text{if } \ell < i \leqslant k-2, \end{cases} \tag{3.6}$$

where $\ell = \left\lfloor \frac{\sqrt{(2\rho s+\rho+1)^2+4\rho(k-1)}-(2\rho s+\rho+1)}{2\rho} \right\rfloor$ is the floor of the positive root of the equation

$$\rho i^2 + (2\rho s+\rho+1)i - (k-1) = 0.$$

Let $f(y_{-s}, y_{-s+1}, \ldots, y_{k-2}) = \frac{k-1+\sum_{j=1}^{k-2} j y_j}{1+\sum_{j=-s}^{k-2} y_j}$. This function is decreasing in y_i for each $-s \leqslant i \leqslant 0$. Hence, since $0 \leqslant y_j \leqslant 2\rho$ for each j,

$$f(y_{-s}, y_{-s+1}, \ldots, y_0, y_1, \ldots, y_{k-2}) \geqslant f(2\rho, 2\rho, \ldots, 2\rho, y_1, \ldots, y_{k-2}).$$

Moreover, we show that:

$$f(2\rho, 2\rho, \ldots, 2\rho, y_i, \ldots, \ldots, y_{k-2}) \geqslant f(2\rho, 2\rho, \ldots, 2\rho, y_{i+1}, \ldots y_{k-2}), \tag{3.7}$$

for $1 \leqslant i \leqslant \ell$, and

$$f(2\rho, 2\rho, \ldots, 2\rho, y_{\ell+1}, \ldots y_i, 0, \ldots, 0) \geqslant f(2\rho, 2\rho, \ldots, 2\rho, y_{\ell+1}, \ldots y_{i-1}, 0, \ldots, 0), \tag{3.8}$$

for $\ell < i < k-2$. We first prove Equation (3.7). The inequality (3.7) is obtained by considering the following one for some $i \leqslant \ell$ (recalling that ℓ is the floor of the positive

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root of the equation $\rho i^2 + (2\rho s+\rho+1)i - (k-1)$)

$$f(2\rho,\ldots,2\rho,y_i,\ldots,y_{k-2})=\frac{A+iy_i}{B+y_i}\geqslant\frac{A+2\rho i}{B+2\rho}=f(2\rho,\ldots,2\rho,y_{i+1},\ldots,y_{k-2}), \tag{3.9}$$

where $A = k-1+\sum_{j=i+1}^{k-2} jy_j + \rho i(i-1)$ and $B = 1+\sum_{j=i+1}^{k-2} y_j + 2\rho(i+s)$.
We notice that Equation (3.9) is satisfied whenever $y_i(A - iB) \leqslant 2\rho(A - iB)$ and that for $i \leqslant \ell$

$$\begin{aligned} A - iB &= k-1+\sum_{j=i+1}^{k-2} jy_j + \rho i(i-1) - i\left(1+\sum_{j=i+1}^{k-2} y_j + 2\rho(i+s)\right) \\ &= k-1+\sum_{j=i+1}^{k-2}(j-i)y_j + \rho i^2 - \rho i - i - 2\rho i^2 - 2\rho is \\ &\geqslant -\rho i^2 - (2\rho s+\rho+1)i + k - 1 \geqslant 0. \end{aligned}$$

Hence, Equation (3.9) and consequently Equation (3.7) are satisfied. In order to get Equation (3.8), we show that for each $i > \ell$,

$$\begin{aligned} f(2\rho,\ldots,2\rho,y_{\ell+1},\ldots,y_i,0,\ldots,0) &= \frac{C+iy_i}{D+y_i} \\ &\geqslant \frac{C}{D}=f(2\rho,\ldots,2\rho,y_{\ell+1},\ldots,y_{i-1},0,\ldots,0), \end{aligned} \tag{3.10}$$

where $C = k-1+\sum_{j=\ell+1}^{i-1} jy_j + \rho\ell(\ell+1)$ and $D = 1+\sum_{j=\ell+1}^{i-1} y_j + 2\rho(s+\ell+1)$.
Since, Equation (3.10) is satisfied whenever $y_i(C - iD) \leqslant 0$ and since now $i > \ell$ we get

$$\begin{aligned} C - iD &= k-1+\sum_{j=\ell+1}^{i-1} jy_j + \rho\ell(\ell+1) - i\left(1+\sum_{j=\ell+1}^{i-1} y_j + 2\rho(s+\ell+1)\right) \\ &\leqslant k-1+\rho\ell^2+\rho\ell-(\ell+1)-2\rho(\ell+1)s-2\rho(\ell+1)\ell-2\rho(\ell+1) \\ &= -\rho\ell^2-(2\rho s+3\rho+1)\ell+k-2\rho s-2\rho-2\leqslant 0. \end{aligned}$$

Hence, Equation (3.10) and consequently Equation (3.8) are satisfied. Summarizing, we have that the minimizing values are

$$x_i = \begin{cases} \frac{|V|}{1+\sum_{j=-s}^{k-2} y_j} = \frac{|V|}{2\rho(\ell+s+1)+1}, & \text{for } i = k-1 \\ 2\rho x_{k-1} = \frac{2\rho|V|}{2\rho(\ell+s+1)+1} & \text{for } i = -s,-s+1,\ldots,\ell \\ 0 & \text{otherwise.} \end{cases} \tag{3.11}$$

Therefore,

$$\sum_{j=1}^{k-1} jx_j = \frac{|V|}{2\rho(\ell+s+1)+1}\left(k-1+2\rho\sum_{j=1}^{\ell} j\right) = \frac{|V|}{2\rho(\ell+s+1)+1}\times(k-1+\rho\ell(\ell+1)),$$

and we can conclude that $w(\mathcal{C}) \geqslant \frac{|V|}{2\rho(\ell+s+1)+1}\times(k-1+\rho\ell(\ell+1))$, when $t \geqslant k-1$.

Case II ($t < k-1$): In this case, $w(\mathcal{C}) = \sum_{j=-s}^{k-1} jx_j$ and the rightmost term of Equation (3.5) become

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$$w(C) = |V| \times \frac{k-1+\sum_{j=-s}^{k-2} jy_j}{1+\sum_{j=-s}^{k-2} y_j}. \tag{3.12}$$

and is minimized when

$$y_i = \begin{cases} 2\rho & \text{if } -s \leqslant i \leqslant \ell \\ 0 & \text{if } \ell < i \leqslant k-2, \end{cases} \tag{3.13}$$

where $\ell = \left\lfloor \frac{\sqrt{4\rho(t+1)+(\rho-1)^2}-(2\rho s+\rho+1)}{2\rho} \right\rfloor$ is the floor of the positive root of the equation

$$\rho i^2 + i(2\rho s + \rho + 1) - (k - 1 - \rho s^2 - \rho s) = 0.$$

To wit, let $f(y_{-s}, y_{-s+1}, \ldots, y_{k-2}) = \frac{k-1+\sum_{j=-s}^{k-2} jy_j}{1+\sum_{j=-s}^{k-2} y_j}$. Recalling that $0 \leqslant y_j \leqslant 2\rho$, for each j, we show that

$$f(2\rho, 2\rho, \ldots, 2\rho, y_i, \ldots y_{k-2}) \geqslant f(2\rho, 2\rho, \ldots, 2\rho, y_{i+1}, \ldots y_{k-2}), \tag{3.14}$$

for $-s \leqslant i \leqslant \ell$, and

$$f(2\rho, 2\rho, \ldots, 2\rho, y_{\ell+1}, \ldots y_{i+1}, 0, \ldots, 0) \geqslant f(2\rho, 2\rho, \ldots, 2\rho, y_{\ell+1}, \ldots y_i, 0, \ldots, 0), \tag{3.15}$$

for $\ell < i < k-2$.

The inequality in Equation (3.14) is obtained by considering the following inequality for some $-s \leqslant i \leqslant \ell$

$$f(2\rho, \ldots, 2\rho, y_i, y_{i+1}, \ldots, y_{k-2}) = \frac{A+iy_i}{B+y_i} \geqslant \frac{A+2\rho i}{B+2\rho} = f(2\rho, \ldots, 2\rho, y_{i+1}, \ldots, y_{k-2}), \tag{3.16}$$

where $A = k-1+\sum_{j=i+1}^{k-2} jy_j + \rho(i(i-1) - s(s+1))$ and $B = 1+\sum_{j=i+1}^{k-2} y_j + 2\rho(i+s)$. We notice that Equation (3.16) is satisfied whenever $y_i(A - iB) \leqslant 2\rho(A - iB)$ and that for $i \leqslant \ell$,

$$\begin{aligned} A - iB &= k-1+\sum_{j=i+1}^{k-2} jy_j + \rho(i(i-1) - s(s+1)) - i\left(1+\sum_{j=i+1}^{k-2} y_j + 2\rho(s+i)\right) \\ &= k-1+\sum_{j=i+1}^{k-2} (j-i)y_j + \rho i^2 - \rho i - \rho s^2 - \rho s - i - 2\rho i^2 - 2\rho is \\ &\geqslant -\rho i^2 - i(2\rho s + \rho + 1) + k - 1 - \rho s^2 - \rho s \geqslant 0. \end{aligned}$$

Hence, Equation (3.16) and consequently Equation (3.14) are satisfied. In order to get Equation (3.15), we show that the following inequality, holds for each $i > \ell$

$$\begin{aligned} f(2\rho, \ldots, 2\rho, y_{\ell+1}, \ldots, y_i, 0, \ldots, 0) &= \frac{C+iy_i}{D+y_i} \\ &\geqslant \frac{C}{D} = f(2\rho, \ldots, 2\rho, y_{\ell+1}, \ldots, y_{i-1}, 0, \ldots, 0), \end{aligned} \tag{3.17}$$

where $C = k-1+\sum_{j=\ell+1}^{i-1} jy_j + \rho(\ell(\ell+1) - s(s+1))$ and $D = 1+\sum_{j=\ell+1}^{i-1} y_j + 2\rho(s+\ell+1)$.

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Since, Equation (3.17) is satisfied whenever $y_i(C - iD) \leqslant 0$ and since now $i > \ell$ we get

$$\begin{aligned} C - iD &= k - 1 + \sum_{j=\ell+1}^{i-1} j y_j + \rho(\ell(\ell+1) - s(s+1)) - i\left(1 + \sum_{j=\ell+1}^{i-1} y_j + 2\rho(s+\ell+1)\right) \\ &= k - 1 + \rho\ell^2 + \rho\ell - \rho s^2 - \rho s - i - 2\rho i s - 2\rho i\ell - 2\rho i - \sum_{j=\ell+1}^{i-1} (i-j) y_j \\ &\leqslant k - 1 + \rho\ell^2 + \rho\ell - \rho s^2 - \rho s - (\ell+1) - 2\rho(\ell+1)s - 2\rho(\ell+1)\ell - 2\rho(\ell+1) \\ &= k - 1 + \rho\ell^2 + \rho\ell - \rho s^2 - \rho s - \ell - 1 - 2\rho s\ell - 2\rho s - 2\rho\ell^2 - 2\rho\ell - 2\rho\ell - 2\rho \\ &= -\rho\ell^2 - (2\rho s + 3\rho + 1)\ell + k - 2 - \rho s^2 - 3\rho s - 2\rho \leqslant 0. \end{aligned}$$

Hence, Equation (3.17) and consequently Equation (3.15) are satisfied.

Using the Equation (3.11), we have that

$$\begin{aligned} \sum_{j=-s}^{k-1} j x_j &= \frac{|V|}{2\rho(\ell+s+1)+1}\left((k-1) + 2\rho \sum_{j=-s}^{\ell} j\right) \\ &= \frac{|V|}{2\rho(\ell+s+1)+1} \times (k - 1 + \rho(\ell(\ell+1) - s(s+1))), \end{aligned}$$

and we can conclude that $w(\mathcal{C}) \geqslant \frac{|V|}{2\rho(\ell+s+1)+1} \times (k-1 + \rho(\ell(\ell+1) - s(s+1)))$, when $t < k-1$. □

4. Relations among dynamos

4.1. *Relationship between (k, t)-dynamos and k-dynamos*

In this section, we study the minimum value of t for which an optimal (k, t, λ)-dynamo coincides with a k-dynamo with threshold λ. The following result holds.

Theorem 4.1. Let $G = (V, E)$ be an undirected connected graph and $0 < \lambda \leqslant 1$. Any optimal k-dynamo $\mathcal{C}$ with threshold λ contains only the weights 0 and $k-1$.

Proof. Let t be the time required by $\mathcal{C}$ to reach the final all-$(k-1)$ configuration. Hence, $\mathcal{C}$ is an optimal (k, t, λ)-dynamo. By Lemma 3.2, $\mathcal{C}$ is a (k, t, λ)-simple-monotone configuration, described by the partition $X_{-s}, X_{-s+1}, \ldots X_{k-1}$, $s = t - k + 1$. Assume by contradiction that $\mathcal{C}$ contains a node of weight $\notin \{0, k-1\}$. Let $c = \max\{i : X_i \neq \varnothing,\ \min(-s, 1) \leqslant i < k-1\}$.

Starting from $\mathcal{C}$ we can easily build a new configuration $\mathcal{C}'$ such that $\mathcal{C}'$ is a k-dynamo and $w(\mathcal{C}') < w(\mathcal{C})$.

Let $t' = t + c$ and $s' = t' - k + 1 = s + c \geqslant 0$ (since $c \geqslant -s$), $\mathcal{C}'$ is described by the partition $X'_{-s'}, X'_{-s'+1}, \ldots X'_{k-1}$ where

$$\forall i = -s', -s' + 1 \ldots, k-1,\ X'_i = \begin{cases} X_i & \text{if } i = k-1 \\ X_{i+c} & \text{if } i \leqslant 0 \\ \varnothing & \text{otherwise.} \end{cases}$$

In $\mathcal{C}'$ the initial weight of each $u \in X'_i$ is set to $c'_u(0) = \max(i, 0)$. Hence, $\mathcal{C}'$ contains only the weights 0 and $k-1$. Moreover, $X'_{k-1} = X_{k-1}$ (the set of nodes having initial weight

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$k-1$ is the same) and so $w(\mathcal{C}') < w(\mathcal{C})$. By construction, $\mathcal{C}'$ is a (k,t',λ)-simple-monotone configuration and we have the desired contradiction because this means that $\mathcal{C}'$ is a k-dynamo with threshold λ. □

In the light of Theorem 4.1, we would like to answer the following question:
Which is the smallest value of t such that the optimal dynamo contains only two weights?
For $\lambda = 1/2$, we are able to answer this question.

Corollary 4.1. Let $G = (V,E)$ be an undirected connected graph, $k \geqslant 2$, and $t \geqslant 1$. If $t > \frac{k(2\rho+1)-2}{2\rho} - 2$, then any optimal $(k,t,1/2)$-dynamo $\mathcal{C}$ contains only two weights (0 and $k-1$) and $w(\mathcal{C}) \geqslant \frac{|V|(k-1)}{2\rho(t-k+2)+1}$.

Proof. Let us notice that the value of ℓ in Theorem 3.1 (cf. Equation (3.2)) is non-increasing with t. Moreover, a closer look at the proof of Theorem 3.1 shows that in any optimal configuration $\mathcal{C}$ the sets $X_{\ell+1}, X_{\ell+2}, \ldots, X_{k-2}$ are empty; that is, when $\ell = 0$ then only two weights (0 and $k-1$) appear in the optimal configuration. By analyzing the value of ℓ in the case $t \geqslant k-1$ we have that if $t > \frac{k(2\rho+1)-2}{2\rho} - 2$ then $\frac{\sqrt{(2\rho s+\rho+1)^2+4\rho(k-1)}-(2\rho s+\rho+1)}{2\rho} < 1$ and therefore $\ell = 0$. As a consequence, only the weights 0 and $k-1$ appear in the optimal configuration. Using the same arguments of Theorem 3.1 the bound follows. □

Remark 1. Our result generalizes the one in *Flocchini et al. (2003)* with $k = 2$ and $\rho = 1$. Indeed, when $t \geqslant k-1 = 1$ by the above consideration we get $t > \frac{3}{2}k - 3 = 0$ and $\ell = 0$. Hence, $w(\mathcal{C}) \geqslant \frac{|V|(k-1)}{2\rho(t-k+2)+1} = \frac{|V|}{2t+1}$.

4.2. *Relationship between k-dynamos and 2-dynamos*

Since k-dynamos are a generalization of 2-dynamos, the problem of determining a 2-dynamo for a graph G is a sub-problem of determining a k-dynamo for G. In the following, we show that the two problems are strongly related when t is large (that is when ℓ defined in Theorem 3.1 is zero). On the other hand, when t is not big enough, we are still able to infer some relationships between the two problems, see Theorems 4.3 and 4.4.

Definition 5. Let $G = (V,E)$ be an undirected connected graph, and let $\mathcal{C}_2$ be a 2–configuration for G. A k-configuration $\mathcal{C}_k$ for G induced by $\mathcal{C}_2$ is obtained by replacing each 1 in $\mathcal{C}_2$ by $k-1$.

Easily one can show that $w(\mathcal{C}_k) = w(\mathcal{C}_2) \times (k-1)$ and if $\mathcal{C}_2$ is a 2-dynamo, then $t(\mathcal{C}_k) = t(\mathcal{C}_2) + k - 2$ (where $t(\mathcal{C})$ is the time needed to reach the final configuration).

Definition 6. Let $G = (V,E)$ be an undirected connected graph, and let $\mathcal{C}_k$ be a k-configuration for G. A 2-configuration $\mathcal{C}_2$ for G induced by $\mathcal{C}_k$ is obtained by replacing the weight i in $\mathcal{C}_k$ by the weight 1 if $i = k-1$, and by 0 otherwise.

Theorem 4.2. Let $G = (V,E)$ be an undirected connected graph, $\mathcal{C}_2$ be a 2-dynamo with threshold λ for G, and $\mathcal{C}_k$ be the k-configuration induced by $\mathcal{C}_2$, with $k \geqslant 2$. Then $\mathcal{C}_2$ is

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an optimal 2-dynamo with threshold λ if and only if $\mathcal{C}_k$ is an optimal k-dynamo with threshold λ.

Proof. By contradiction assume that $\mathcal{C}_2$ is an optimal 2-dynamo with threshold λ, while $\mathcal{C}_k$ is non-optimal. Let $\mathcal{C}'_k$ be the optimal k-dynamo configuration. We have $w(\mathcal{C}'_k) < w(\mathcal{C}_k)$. By Theorem 4.1, we have that $\mathcal{C}'_k$ contains only two weights (0 and $k-1$) and consequently the number of nodes weighted $k-1$ in $\mathcal{C}'_k$ is smaller than the number of nodes weighted $k-1$ in $\mathcal{C}_k$. Let t be the time required by $\mathcal{C}'_k$ to reach the final all-$(k-1)$ configuration. Clearly $t \geqslant k-1$ otherwise $\mathcal{C}'_k$ cannot reach the final all-$(k-1)$ configuration. Hence, $\mathcal{C}'_k$ is an optimal (k,t,λ)-dynamo. By Lemma 3.2, $\mathcal{C}'_k$ is a (k,t,λ)-simple-monotone configuration. Starting from $\mathcal{C}'_k$ we can build a configuration $\mathcal{C}'_2$ induced by $\mathcal{C}'_k$. Since, $\mathcal{C}'_k$ is (k,t,λ)-simple-monotone configuration which contains only two weights (0 and $k-1$) we have that $\mathcal{C}'_2$ is a $(2,t-k+2,\lambda)$-simple-monotone configuration and therefore $\mathcal{C}'_2$ is a 2-dynamo with threshold λ. Since, $w(\mathcal{C}'_k) < w(\mathcal{C}_k)$ we have that $w(\mathcal{C}'_2) < w(\mathcal{C}_2)$ and we get the desired contradiction because $\mathcal{C}_2$ was supposed to be optimal. As a consequence we deduce that $\mathcal{C}_k$ is an optimal k-dynamo with threshold λ.

Vice versa suppose that $\mathcal{C}_k$ is an optimal k-dynamo with threshold λ. By Theorem 4.1, we have that $\mathcal{C}_k$ contains only two weights (0 and $k-1$). Then, the number of $k-1$'s is minimal. There follows that the number of 1's in $\mathcal{C}_2$ is minimal and so $\mathcal{C}_2$ is an optimal 2-dynamo. □

When t is fixed and $\lambda = 1/2$, we provide a mapping which transforms an optimal k-configuration $\mathcal{C}_k$ into an optimal (within a different number of rounds) 2-configuration and vice-versa.

Theorem 4.3. Let $G = (V,E)$ be an undirected connected graph. Let $k > 2$ and $t \geqslant 1$ be integers. Let $\mathcal{C}_k$ be a (k,t)-dynamo for G that matches the lower bound in Theorem 3.1. Let $\mathcal{C}_2$ be the 2-configuration induced by $\mathcal{C}_k$. Then $\mathcal{C}_2$ is an optimal $(2, t-k+2+\ell)$-dynamo, where ℓ is defined in Theorem 3.1 (cf. Equation (3.2)).

Proof. By Lemma 3.2, $\mathcal{C}_k$ is a (k,t)-simple-monotone configuration. By construction then, $\mathcal{C}_2$ is $(2,t-k+2+\ell)$-simple-monotone configuration, and hence it is a $(2,t-k+2+\ell)$-dynamo. Since, $\mathcal{C}_k$ is a (k,t)-dynamo for G that matches the lower bound in Theorem 3.1, the number of $k-1$'s in $\mathcal{C}_k$ is exactly $|V|/(2\rho(\ell+s+1)+1)$ and we have that $w(\mathcal{C}_2) = \frac{|V|}{2\rho(\ell+s+1)+1}$ which matches the lower bound in Theorem 3.1, and hence $\mathcal{C}_2$ is optimal. □

Vice versa:

Theorem 4.4. Let $G = (V,E)$ be an undirected connected graph and $\mathcal{C}_2$ be a $(2,t(\mathcal{C}_2))$-dynamo for G that matches the lower bound in Theorem 3.1. Let $\mathcal{C}_k$ be the k-configuration induced by $\mathcal{C}_2$ with $2 < k < 2\rho t + 2$, where ρ is the ratio between the maximum and the minimum degree of the nodes in V. Then $\mathcal{C}_k$ is an optimal $(k, t(\mathcal{C}_2)+k-2)$-dynamo.

Proof. By Lemma 3.2, $\mathcal{C}_2$ is a $(2,t(\mathcal{C}_2))$-simple-monotone configuration. By construction then, $\mathcal{C}_k$ is $(k,t(\mathcal{C}_k))$-simple-monotone configuration, where $t(\mathcal{C}_k) = t(\mathcal{C}_2)+k-2$, and hence it is a $(2,t(\mathcal{C}_k)))$-dynamo. Since, $\mathcal{C}_2$ is a $(2,t(\mathcal{C}_2))$-dynamo for G that matches the lower

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bound in Theorem 3.1, we have $w(\mathcal{C}_2) = \frac{|V|}{2\rho t(\mathcal{C}_2)+1}$. Moreover, $w(\mathcal{C}_k) = w(\mathcal{C}_2) \times (k-1) = \frac{|V|(k-1)}{2\rho t(\mathcal{C}_2)+1} = \frac{|V|(k-1)}{2\rho(t(\mathcal{C}_k)-k+2)+1}$, where $s_k = t(\mathcal{C}_k) - k + 1 = t(\mathcal{C}_2) - 1$ which matches the lower bound in Theorem 3.1, since $k < 2\rho t(\mathcal{C}_2) + 2$ implies $t(\mathcal{C}_k) = t(\mathcal{C}_2) + k - 2 > \frac{k(2\rho+1)-2}{2\rho} - 2$ and by Corollary 4.1 we can conclude. □

4.3. *Relationship between time bounded k-dynamos and 2-dynamos*

The following definition extends the concept of k-configuration induced by 2-configuration described in Definition 5. For all $r = 0, 1, \ldots, t(\mathcal{C}_2)$, let $\mathcal{C}_2^r$ be the configuration $\mathcal{C}_2$ at round r. Clearly $\mathcal{C}_2^0 = \mathcal{C}_2$.

Definition 7. Let $G = (V, E)$ be an undirected connected graph, and let $\mathcal{C}_2$ be a $(2, t(\mathcal{C}_2), \lambda)$-simple monotone configuration for G. Let $k > 2$ be an integer, a k-configuration $\mathcal{C}_k$ for G induced by $\mathcal{C}_2$ ith offset f (where $0 \leqslant f \leqslant k-2$), is obtained by applying the following construction to $\mathcal{C}_2$.

— Assign the weight $k-1$ to all the nodes weighted by 1 in $\mathcal{C}_2^0$;
— For all $r = 1, 2, \ldots, t(\mathcal{C}_2)$, assign the weight $\max(f - r + 1, 0)$ to all the nodes which change their weight from 0 to 1 at round r in $\mathcal{C}_2$ (that is, they are weighted 0 in $\mathcal{C}_2^{r-1}$ and 1 in $\mathcal{C}_2^r$).

Notice that for $f = 0$ we obtain the same construction provided in Definition 5.

Theorem 4.5. Let $G = (V, E)$ be an undirected connected graph. Let $\mathcal{C}_2$ be a $(2, t(\mathcal{C}_2))$-dynamo for G that matches the lower bound in Theorem 3.1. Let $k > 2$ be an integer, and $\mathcal{C}_k$ be the k-configuration induced by $\mathcal{C}_2$ with offset f. Let $t(\mathcal{C}_k) = t(\mathcal{C}_2) + k - 2 - f$, if $f = \ell$, as in Theorem 3.1, then $\mathcal{C}_k$ is an optimal $(k, t(\mathcal{C}_k))$-dynamo for G.

Proof. By construction, $\mathcal{C}_k$ is a $(k, t(\mathcal{C}_k))$-simple-monotone configuration partitioning V into $t(\mathcal{C}_k)+1$ sets $X_{-t(\mathcal{C}_k)+1}, X_{-t(\mathcal{C}_k)+2}, \ldots, X_1$. By Lemma 3.1 it is a $(k, t(\mathcal{C}_k))$-dynamo. Since, $\mathcal{C}_2$ is a $(2, t(\mathcal{C}_2))$-dynamo for G that matches the lower bound in Theorem 3.1, we have $w(\mathcal{C}_2) = \frac{|V|}{2\rho t(\mathcal{C}_2)+1}$ and for $j = -t(\mathcal{C}_k)+1, \ldots, 0$, $x_j = 2\rho x_1$, where x_j denotes the cardinality of X_j. By construction we have as follows:

$$w(\mathcal{C}_k) = \begin{cases} \frac{|V|}{2\rho(f+s+1)+1} \times (k-1+\rho f(f+1)) & \text{if } t(\mathcal{C}_k) \geqslant k-1 \\ \frac{|V|}{2\rho(f+s+1)+1} \times (k-1+\rho(f(f+1)-s(s+1))) & \text{otherwise.} \end{cases}$$

Therefore, $w(\mathcal{C}_k)$ matches the lower bound in Theorem 3.1 since $f = \ell$. □

5. Building (k, t)-dynamo

In this section, we provide several optimal (or almost optimal) (k, t, λ)-dynamo constructions for rings and tori ($\lambda = 1/2$) and cliques (any λ) using the results of Sections 3 and 4.

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5.1. *Rings*

A n-node ring $\mathcal{R}_n$ consists of n nodes and $n-1$ edges, where for $i = 0, 1, \ldots, n-1$ each node v_i is connected with $v_{(i-1) \bmod n}$ and $v_{(i+1) \bmod n}$.

A necessary condition for $\mathcal{C}(\mathcal{R}_n, k)$ to be a k-dynamo ($\lambda \leqslant 1/2$) is that at least one node of $\mathcal{R}_n$ is weighted by $k-1$. This condition is also sufficient.

Theorem 5.1. An optimal k-dynamo $\mathcal{C}(\mathcal{R}_n, k)$ as follows:

i. when $\lambda \leqslant 1/2$, has weight $w(\mathcal{C}(\mathcal{R}_n, k)) = k-1$, and reaches its final configuration within $t = k - 2 + \lceil \frac{n-1}{2} \rceil$ rounds;
ii. when $\lambda > 1/2$, has weight $w(\mathcal{C}(\mathcal{R}_n, k)) = (k-1) \times \lceil \frac{n-1}{2} \rceil$, and reaches its final configuration within $t = k-1$ rounds.

Proof.

i. Since $\mathcal{C}(\mathcal{R}_n, k)$ is a k-dynamo, then $w(\mathcal{C}(\mathcal{R}_n, k)) \geqslant (k-1)$. Let $\mathcal{C}$ be a k-configuration where one node v of $\mathcal{R}_n$ is weighted by $k-1$ and the remaining node are weighted 0 ($w(\mathcal{C}) = (k-1)$). Since, $\lambda \leqslant 1/2$ we have that for each node u at distance $0 < i \leqslant \lceil \frac{n-1}{2} \rceil$ from v, u changes its weight from 0 to 1 at round i, thereafter u increases its weight at each round until it reach the weight $k-1$. This is due to the fact that for each round $i \leqslant j \leqslant k-2+i$, u has at least one neighbour, at distance $i-1$ from v, having a weight greater than u. Summing up, since the diameter of $\mathcal{R}_n$ is $\lceil \frac{n-1}{2} \rceil$, after $t = k - 2 + \lceil \frac{n-1}{2} \rceil$ rounds, all the nodes reach the weight $k-1$.
ii. Since $\mathcal{C}(\mathcal{R}_n, k)$ is a k-dynamo and $\lambda > 1/2$, in the initial configuration each node having a weight smaller than $k-1$ must have both its neighbours weighted $k-1$. Hence, $w(\mathcal{C}(\mathcal{R}_n, k)) \geqslant (k-1) \times \lceil \frac{n-1}{2} \rceil$. Let $\mathcal{C}$ be a k-configuration where $\forall v_i \in \mathcal{R}_n$, $c_{v_i} = k-1$ if i is even and 0 otherwise. We have that each node u increases its weight at each round until it reaches the weight $k-1$. Hence, after $k-1$ rounds, all the nodes reach the weight $k-1$.

□

As noted in Section 4, when t is unbounded, the optimal initial configuration contains only nodes with weight 0 or $k-1$.

On the other hand, when t is bounded, the optimal configuration may also contain nodes that have a weight in $\{1, 2, \ldots, k-2\}$. In the following we consider the majority rule (i.e. $\lambda = 1/2$).

A $(k, t, 1/2)$-dynamo for a ring $\mathcal{R}_n$ is obtained by the following partition of V which defines the initial configuration (see Figure 4) $\mathcal{C}(\mathcal{R}_n, k, t)$: for $i = 0, 1, \ldots, n$,

$$\forall v_i \in \mathcal{R}_n, \quad v_i \in \begin{cases} X_{k-1} & \text{if } j = 0 \\ X_{\ell+1-j} & \text{if } 1 \leqslant j \leqslant \ell + s + 1 \\ X_{j-\ell-2s-2} & \text{if } \ell + s + 2 \leqslant j \leqslant 2\ell + 2s + 2, \end{cases}$$

where $s = t - k + 1$, $j = i \bmod (2\ell + 2s + 3)$ and $\ell = \lfloor \sqrt{t + 1 + s^2 + s} \rfloor - (s+1)$ if $t \geqslant k-1$ and $\ell = \lfloor \sqrt{t+1} \rfloor - (s+1)$ otherwise.

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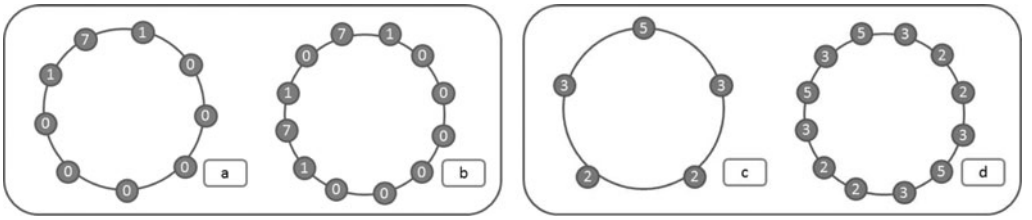


Fig. 4. $(k,t,1/2)$-dynamos for rings: (a) $\mathcal{C}(\mathcal{R}_9,8,9)$, a $(8,9,1/2)$-dynamo for $\mathcal{R}_9$ ($\ell=1$), in this particular case $n=2\ell+2s+3$; (b) $\mathcal{C}(\mathcal{R}_{12},8,9)$ a $(8,9,1/2)$-dynamo for $\mathcal{R}_{12}$ ($\ell=1$); (c) $\mathcal{C}(\mathcal{R}_5,6,3)$, a $(6,3,1/2)$-dynamo for $\mathcal{R}_5$ ($\ell=3$), in this particular case $n=2\ell+2s+3$; (d) $\mathcal{C}(\mathcal{R}_{12},6,3)$, a $(6,3,1/2)$-dynamo for $\mathcal{R}_{12}$ ($\ell=3$).

Theorem 5.2. (i) The configuration $\mathcal{C}(\mathcal{R}_n,k,t)$ is a $(k,t,1/2)$-dynamo for any value of n, $k\geqslant 2$ and $t\geqslant 1$. (ii) The weight of $\mathcal{C}(\mathcal{R}_n,k,t)$ is

$$w(\mathcal{C}(\mathcal{R}_n,k,t))\leqslant\begin{cases}\left\lceil\frac{n}{2\ell+2s+3}\right\rceil(k-1+\ell(\ell+1)) & \text{if } t\geqslant k-1\\ \quad\text{where } \ell=\lfloor\sqrt{t+1+s^2+s}\rfloor-(s+1)\\ \left\lceil\frac{n}{2\ell+2s+3}\right\rceil(k-1+\ell(\ell+1)-s(s+1)) & \text{otherwise.}\\ \quad\text{where } \ell=\lfloor\sqrt{t+1}\rfloor-(s+1)\end{cases}$$

Proof. (i) By construction $\mathcal{C}(\mathcal{R}_n,k,t)$ is $(k,t,1/2)$-simple-monotone, hence by Lemma 3.1, $\mathcal{C}(\mathcal{R}_n,k,t)$ is a $(k,t,1/2)$-dynamo. (ii) There are two cases to consider: if $t\geqslant k-1$, then starting from v_0 each set of $2\ell+2s+3$ nodes weights $k-1+2\sum_{i=1}^{\ell}i=k-1+\ell(\ell+1)$. Then the weight of $\mathcal{C}(\mathcal{R}_n,k,t)$ is smaller than the weight of $\mathcal{C}(\mathcal{R}_{\overline{n}},k,t)$ where $\overline{n}=\lceil\frac{n}{2\ell+2s+3}\rceil\times(2\ell+2s+3)$. Hence, $w(\mathcal{C}(\mathcal{R}_n,k,t))\leqslant w(\mathcal{C}(\mathcal{R}_{\overline{n}},k,t))=\lceil\frac{n}{2\ell+2s+3}\rceil(k-1+\ell(\ell+1))$. Similarly for $t<k-1$. □

By Corollary 3.1 and Theorem 5.2 we have the following Corollary.

Corollary 5.1. When $n/(2\ell+2s+3)$ is integer, $\mathcal{C}(\mathcal{R}_n,k,t)$ is an optimal $(k,t,1/2)$-dynamo.

By Theorem 4.3, when $n/(2\ell+2s+3)$ is integer, $\mathcal{C}_2$ induced by $\mathcal{C}(\mathcal{R}_n,k,t)$ is an optimal $(2,\ t-(k-2)+\ell)$-dynamo. For instance, $\mathcal{C}(\mathcal{R}_9,8,9)$ in Figures 4a induces an optimal 2-dynamo $\mathcal{C}_2$ with $t(\mathcal{C}_2)=9-(8-2)+1=4$ (i.e. $\mathcal{C}(\mathcal{R}_9,2,4)$ depicted in Figure 5a), and $\mathcal{C}(\mathcal{R}_5,6,3)$ in Figure 4c induces an optimal 2-dynamo $\mathcal{C}_2$ with $t(\mathcal{C}_2)=3-(6-2)+3=2$ (i.e. $\mathcal{C}(\mathcal{R}_5,2,2)$ depicted in Figure 5b).

Consider now the construction in Definition 7. This construction can be used to construct k-dynamos starting from 2-dynamos. For instance starting from $\mathcal{C}(\mathcal{R}_{10},2,2)$ and analyzing how it evolves round by round (see Figure 6 (top)) one can easily build the four 5-dynamos (one for each value of the offset, $0\leqslant f\leqslant 3$) depicted in Figure 6 (bottom).

Unfortunately, the configuration provided by the construction in Definition 7 are not always optimal. Theorem 4.5 can be used to identify optimal constructions. As an example, starting from $\mathcal{C}(\mathcal{R}_5,2,2)$ (see Figure 5b) which matches the bound in Theorem 3.1 and using the construction in Definition 7 with offset $f=3$, we obtain $\mathcal{C}(\mathcal{R}_5,6,3)$ (see Figure 4c). Since f verifies the condition of the theorem, we have that $\mathcal{C}(\mathcal{R}_5,6,3)$ is an optimal $(6,3)$-dynamo.

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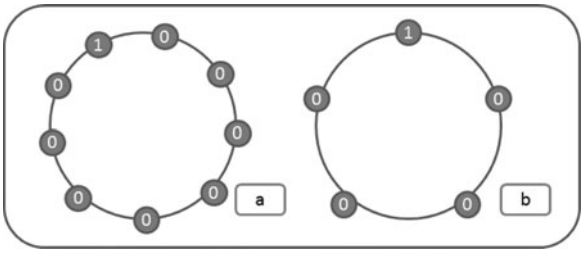


Fig. 5. $(2, t')$-dynamos induced by (k, t)-dynamos: (a) $\mathcal{C}(\mathcal{R}_9, 2, 4)$ induced by $\mathcal{C}(\mathcal{R}_9, 8, 9)$; (b) $\mathcal{C}(\mathcal{R}_5, 2, 2)$ induced by $\mathcal{C}(\mathcal{R}_5, 6, 3)$.

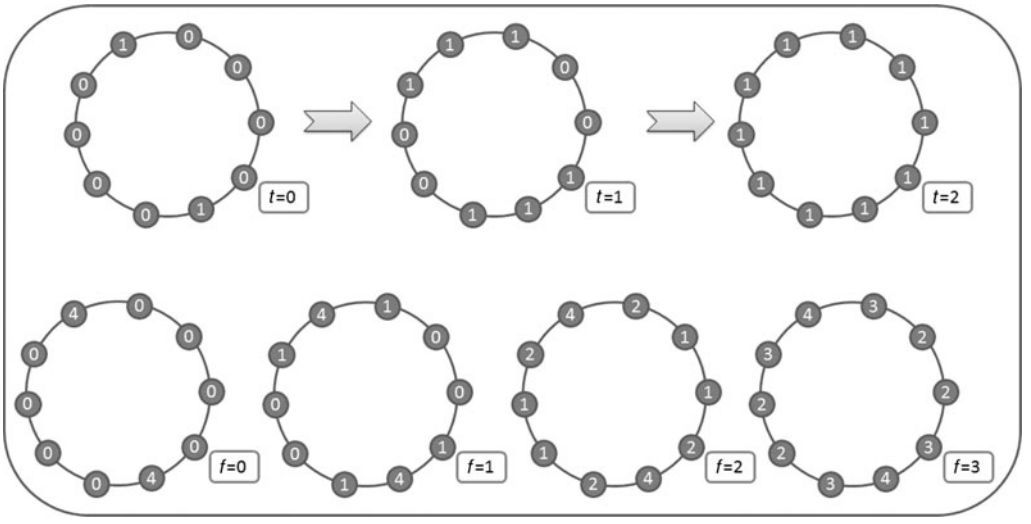


Fig. 6. (k, t')-dynamos induced by $(2, t)$-dynamos: (top) Starting from the initial configuration $\mathcal{C}(\mathcal{R}_{10}, 2, 2)$, two rounds are needed to reach the final all-(1)s configuration; (bottom) Four $(5, t')$-dynamos induced by $\mathcal{C}(\mathcal{R}_{10}, 2, 2)$ with offset $f \in \{0, 1, 2, 3\}$.

On the other hand, it may happen that, even if the conditions of Theorem 4.5 are not satisfied, the construction in Definition 7 still provides an optimal configuration. For instance, the configuration $\mathcal{C}_6$ (see Figure 7b) induced by $\mathcal{C}(\mathcal{R}_{12}, 2, 1)$ (see Figure 7a) with $f = 2$ is a $(6, 3)$-dynamo for $\mathcal{R}_{12}$. Despite the fact that the value of f does not satisfies the condition of Theorem 4.5, not only the configuration $\mathcal{C}_6$ is an optimal $(6, 3)$-dynamo for $\mathcal{R}_{12}$ but it is also better than $\mathcal{C}(\mathcal{R}_{12}, 6, 3)$ (see Figure 4d). Indeed $w(\mathcal{C}_6) = 36$, whereas $w(\mathcal{C}(\mathcal{R}_{12}, 6, 3)) = 38$.

5.2. *Tori*

A $n \times m$-node tori $\mathcal{T}_{n,m}$ consists of $n \times m$ nodes and $2(n \times m)$ edges, where for $i = 0, 1, \ldots, n-1$ and $j = 0, 1, \ldots, m-1$, each node $v_{i,j}$ is connected with four nodes: $v_{i,(j-1) \bmod m}$, $v_{i,(j+1) \bmod m}$, $v_{(i-1) \bmod n,j}$ and $v_{(i+1) \bmod n,j}$.

A $(k, t, 1/2)$-dynamo for $\mathcal{T}_{2\ell+2s+3,2\ell+2s+3}$ is obtained by weighting diagonals with the same order defined for dynamos on rings. Specifically, the configuration $\mathcal{C}(\mathcal{T}_{2\ell+2s+3,2\ell+2s+3}, k, t)$

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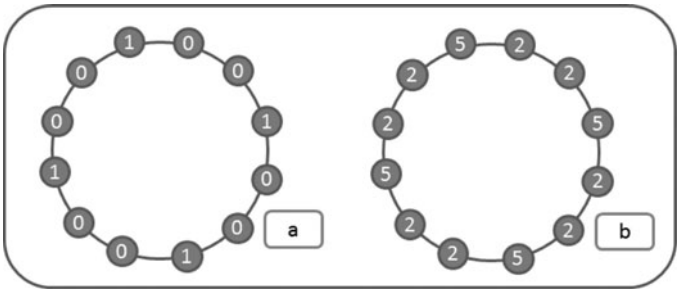


Fig. 7. A (k,t')-dynamos induced by a $(2,t)$-dynamo: (a) $\mathcal{C}(\mathcal{R}_{12},2,1)$, a $(2,1)$-dynamo for $\mathcal{R}_{12}$; (b) $\mathcal{C}_6$, a $(6,3)$-dynamo for $\mathcal{R}_{12}$ induced by $\mathcal{C}(\mathcal{R}_{12},2,1)$ with offset 2.

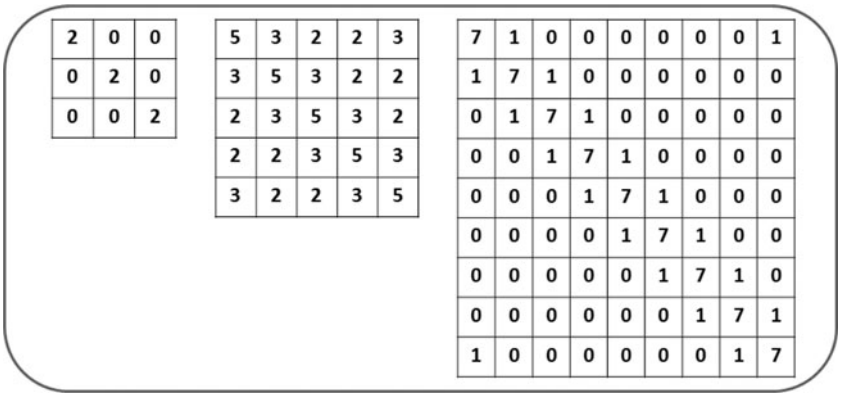


Fig. 8. Optimal $(k,t,1/2)$-dynamos for tori: (left) $\mathcal{C}(\mathcal{T}_{3,3},3,2)$, a $(3,2,1/2)$-dynamo for $\mathcal{T}_{3,3}$ (ℓ=0); (middle) $\mathcal{C}(\mathcal{T}_{5,5},6,3)$, a $(6,3,1/2)$-dynamo for $\mathcal{T}_{5,5}$ (ℓ=3); (right) $\mathcal{C}(\mathcal{T}_{9,9},8,9)$ a $(8,9,1/2)$-dynamo for $\mathcal{T}_{9,9}$ (ℓ=1).

is defined by the partition of V described as follows, let $D_i = \{v_{a,b} \ : \ i = (b-a) \bmod (2\ell + 2s + 3)\}$ denote the ith diagonal of $\mathcal{T}_{2\ell+2s+3,2\ell+2s+3}$, for $i = 0, 1, \ldots, 2\ell + 2s + 2$,

$$\forall v \in D_i, \quad v \in \begin{cases} X_{k-1} & \text{if } i = 0 \\ X_{\ell+1-i} & \text{if } 1 \leqslant i \leqslant \ell + s + 1 \\ X_{i-\ell-2s-2} & \text{if } \ell + s + 2 \leqslant i \leqslant 2\ell + 2s + 2, \end{cases}$$

where $s = t - k + 1$, $\ell = \lfloor \sqrt{t + 1 + s^2 + s} \rfloor - (s+1)$ if $t \geqslant k-1$ and $\ell = \lfloor \sqrt{t+1} \rfloor - (s+1)$ otherwise. Some examples are depicted in Figure 8.

Theorem 5.3. The configuration $\mathcal{C}(\mathcal{T}_{2\ell+2s+3,2\ell+2s+3}, k, t)$ is an optimal $(k,t,1/2)$-dynamo for any $k \geqslant 2$, $t \geqslant 1$.

Proof. Let $\mathcal{C} = \mathcal{C}(\mathcal{T}_{2\ell+2s+3,2\ell+2s+3}, k, t)$. By construction $\mathcal{C}$ is $(k,t,1/2)$-simple-monotone, hence by Lemma 3.1, it is a $(k,t,1/2)$-dynamo. To show its optimality we distinguish two cases. If $t \geqslant k-1$, each row (resp. each column) corresponds to $\mathcal{C}(\mathcal{R}_{2\ell+2s+3}, k, t)$ and its weight is $k - 1 + \ell(\ell+1)$. Overall, $w(\mathcal{C}) = (2\ell + 2s + 3) \times (k - 1 + \ell(\ell+1))$ that matches the bound in Corollary 3.1. Similarly for $t < k - 1$. □

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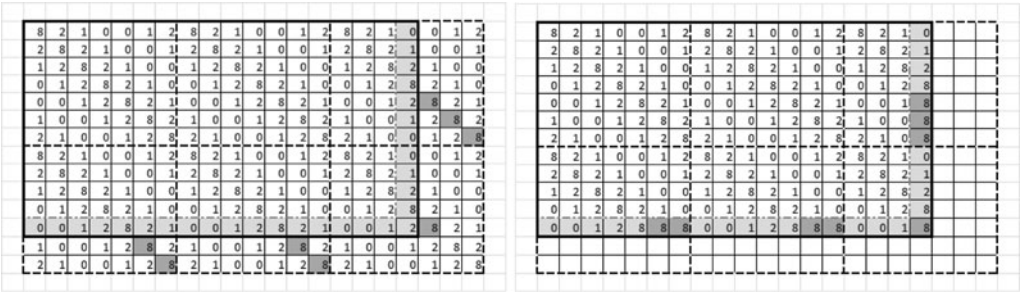


Fig. 9. $\mathcal{C}(\mathcal{T}_{12,18}, 9, 8)$, a $(9, 8, 1/2)$-dynamo for $\mathcal{T}_{12,18}$ ($\ell = 2$): (left) a grid 2×3 is filled with 6 configuration $\mathcal{C}(\mathcal{T}_{7,7}, 9, 8)$; (right) The exceeding parts i.e. the last two rows and the last three columns are removed. Finally, the last row and the last column are updated in order to obtain a configuration that satisfies Lemma 3.1.

A $(k, t, 1/2)$-dynamo for $\mathcal{T}_{n,m}$ is obtained by building a grid $\lceil \frac{n}{2\ell+2s+3} \rceil \times \lceil \frac{m}{2\ell+2s+3} \rceil$, where each cell is filled with a configuration $\mathcal{C}(\mathcal{T}_{2\ell+2s+3,2\ell+2s+3}, k, t)$ defined above. Then, the exceeding part is removed and the last row and the last column are updated. In particular, for each column (resp. row), if the removed part contains a $k-1$, then the element in the last row (resp. column) is given the value $k-1$ (see Figure 9). We call this configuration $\mathcal{C}(\mathcal{T}_{n,m}, k, t)$.

Theorem 5.4.

i. $\mathcal{C}(\mathcal{T}_{n,m}, k, t)$ is a $(k, t, 1/2)$-dynamo for any value of n, m, $k \geqslant 2$ and $t \geqslant 1$.
ii. The weight of $\mathcal{C}(\mathcal{T}_{n,m}, k, t)$ is

$$w(\mathcal{C}(\mathcal{T}_{n,m}, k, t)) \leqslant \begin{cases} \lceil \frac{n}{2\ell+2s+3} \rceil \lceil \frac{m}{2\ell+2s+3} \rceil (2\ell+2s+3)\,(k-1+\ell(\ell+1)) & \text{if } t \geqslant k-1 \\ \quad \text{where } \ell = \lfloor \sqrt{t+1+s^2+s} \rfloor - (s+1) & \\ \lceil \frac{n}{2\ell+2s+3} \rceil \lceil \frac{m}{2\ell+2s+3} \rceil (2\ell+2s+3)\,(k-1+\ell(\ell+1)-s(s+1)) & \text{otherwise.} \\ \quad \text{where } \ell = \lfloor \sqrt{t+1} \rfloor - (s+1). & \end{cases}$$

Proof.

i. By construction $\mathcal{C}(\mathcal{T}_{n,m}, k, t)$ is $(k, t, 1/2)$-simple-monotone (cf. Figure 9), hence by Lemma 3.1, $\mathcal{C}(\mathcal{T}_{n,m}, k, t)$ is a $(k, t, 1/2)$-dynamo.

ii. The grid contains $\lceil \frac{n}{2\ell+2s+3} \rceil \times \lceil \frac{m}{2\ell+2s+3} \rceil$ cells. If $t \geqslant k-1$, each cell has weight $w(\mathcal{C}(\mathcal{T}_{2\ell+2s+3,2\ell+2s+3}, k, t)) = (2\ell + 2s + 3) \times (k - 1 + \ell(\ell + 1))$.

Moreover, the nodes that change their weight take the weight of a removed element. Hence, the weight of $\mathcal{C}(\mathcal{T}_{n,m}, k, t)$ is upper bounded by the weight of the full grid which is $\lceil \frac{n}{2\ell+2s+3} \rceil \times \lceil \frac{m}{2\ell+2s+3} \rceil \times w(\mathcal{C}(\mathcal{T}_{2\ell+2s+3,2\ell+2s+3}, k, t))$. Similarly, for $t < k-1$. □

By Corollary 3.1 and Theorem 5.4 we have the following Corollary.

Corollary 5.2. If both n and m are multiples of $2\ell + 2s + 3$, $\mathcal{C}(\mathcal{T}_{n,m}, k, t)$ is an optimal $(k, t, 1/2)$-dynamo.

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Analogously to the case of rings, by the Corollary, Figure 8 illustrates three examples where $\mathcal{C}_2$ induced by $\mathcal{C}_k$ is an optimal $(2, t-(k-2)+\ell)$-dynamo, since $n = m = 2\ell + 2s + 3$. In particular, $2\ell + 2s + 3$ equals 3, $6 - 4 + 3 = 5$, and $2 + 4 + 3 = 9$, respectively.

Corollary 5.2 generalizes one of the result in Flocchini *et al.* (2004) in two directions: here, we have multi-level opinion with a bound on the number of rounds while (Flocchini *et al.* 2004) analyses binary dynamos without a bound on the number of rounds. In Flocchini *et al.* (2004) the authors characterize the weight of a 2-dynamo for a $\mathcal{T}_{n,m}$ tori, which is $\lceil \frac{n+m}{2} \rceil - 1$. In order to compare our results with their result we restrict ourself to the case $k = 2$ and $t = \max(n, m)/2$ (since our approach evolves by diagonals, the maximum influence time is given by the maximum diagonal distance between two nodes). Using such values we obtain $s = t + 1, \ell = 0$ and therefore the overall weight is $\lceil \frac{n}{2(t+1)+3} \rceil \lceil \frac{m}{2(t+1)+3} \rceil (2(t+1)+3) = \max(n, m) + 5$. Hence, when n and m are close, our result differs from the optimum $\lceil \frac{n+m}{2} \rceil - 1$ by a constant factor.

5.3. *Cliques*

Let K_n be the clique on n nodes. A necessary condition for a k-dynamo $\mathcal{C}(K_n, k)$ is that $\lceil \lambda(n-1) \rceil$ nodes are weighted by $k-1$. The condition is also sufficient and if the remaining $n - \lceil \lambda(n-1) \rceil$ nodes are weighted by 0, the k-dynamo is optimal and reaches its final configuration within $t = k - 1$ rounds. So, when $t \geqslant k - 1$ the optimal configuration is obtained by weighting $\lceil \lambda(n-1) \rceil$ nodes by $k - 1$ and the remaining nodes by 0. For $t < k - 1$, an optimal (k, t, λ)-dynamo is obtained by weighting $\lceil \lambda(n-1) \rceil$ nodes by $k - 1$ and by assigning weight $k - t - 1$ to all the remaining nodes. Clearly this configuration is optimal, if we assign a weight smaller than $k - t - 1$ to a node v, then v cannot reach the weight $k - 1$ within t rounds. Therefore:

Theorem 5.5. Let K_n be the clique on n nodes. An optimal $(k, t, 1/2)$-dynamo $\mathcal{C}(K_n, k, t)$ has weight $w(\mathcal{C}(K_n, k, t)) = (k-1) \times \lceil \lambda(n-1) \rceil + \max(k - t - 1, 0) \times (n - \lceil \lambda(n-1) \rceil)$.

6. Conclusion and open problems

We studied multi-level dynamos with respect to both weight and time. We derived lower bounds on the weight of $(k, t, 1/2)$-dynamo and provided constructive tight upper bounds for rings, tori and cliques. Multi-level dynamos extend the setting of 'classical' dynamos from two possible weights (denoting whether a node has accepted the opinion or not) to k levels so that our results generalize some of the results known for 2-dynamos. Although the latter is a special case of the former, we established some strong relationships between them which allow to construct k-dynamos starting from 2-dynamos optimal under certain conditions. Different meaningful generalizations might be considered concerning the topology, the updating rule, or the value to optimize. Constructions based on different graphs would be interesting to study. This challenge however is in general difficult due to the NP-hardness results (Kempe *et al.* 2003). Different updating rules could also be investigated: for instance, in Brunetti *et al.* (2011) the authors consider a different model, where a node updates its weight with the weight of the majority of its neighbours.

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Furthermore, in Ugander (2012) it was observed that the number of connected components in an individual's contact neighbourhood (representing separate social contexts for the user) is more influential than the size of the neighbourhood itself. According to this result, it is possible to devise an updating rule which takes into account the number, the size and the opinion of connected components in an individual's contact neighbourhood. Finally, instead of minimizing the sum of the weights of any (k, t)-dynamo, one could consider a weighted sum so addressing the case where not all the nodes have the same priority.

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