Problems

B.1(*a*) For a single particle in a large volume V, show that the number of allowed $\mathbf{k}$-values in a small volume $\mathrm{d}^3\mathbf{k} = \mathrm{d}k_x\,\mathrm{d}k_y\,\mathrm{d}k_z$ of $\mathbf{k}$-space is

$$\frac{V}{(2\pi)^3}\mathrm{d}^3\mathbf{k}.$$

(*b*) Show that, for two particles (1) and (2), the wave-vector associated with the centre-of-mass motion is $\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2$ and with the relative motion is $\mathbf{k} = (m_1\mathbf{k}_2 - m_2\mathbf{k}_1)/(m_1 + m_2)$. Hence show

$$\mathrm{d}^3\mathbf{K}\,\mathrm{d}^3\mathbf{k} = \mathrm{d}^3\mathbf{k}_1\,\mathrm{d}^3\mathbf{k}_2$$

and that the number of $(\mathbf{K}, \mathbf{k})$ values with K in the range K, $K + \mathrm{d}K$ and k in the range k, $k + \mathrm{d}k$ is $(V^2/4\pi^4)K^2\,\mathrm{d}K\,k^2\,\mathrm{d}k$ if the particles are distinguishable, but $(V^2/8\pi^4)K^2\,\mathrm{d}K\,k^2\,\mathrm{d}k$ if the particles are identical.

Appendix C
Angular momentum

Students are referred to texts on quantum mechanics for the derivations of the results summarised in this appendix, which is intended as no more than an aide-mémoire.

C.1 Orbital angular momentum

In the shell model of both atomic and nuclear physics the single-particle Schrödinger equation, neglecting effects of the intrinsic spin of the particle, is of the form

$$H\psi = \left(-\frac{\hbar^2}{2M}\nabla^2 + V(r)\right)\psi(\mathbf{r}) = E\psi(\mathbf{r}), \tag{C.1}$$

where the potential energy $V(r)$ is spherically symmetric, a function of the radial coordinate r only. Because of spherical symmetry, the operator ∇^2 is most useful in spherical polar coordinates (r, θ, ϕ), in which the Schrödinger equation takes the form

$$-\frac{\hbar^2}{2M}\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \left(\frac{\mathbf{L}^2}{2Mr^2} + V(r)\right)\psi = E\psi, \tag{C.2}$$

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where $\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2$ and $\mathbf{L}$ is the *orbital angular momentum* operator,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times (-\mathrm{i}\,\hbar\nabla).$$

$\mathbf{L}$ acts only on the angular coordinates (θ, ϕ). For example,

$$L_z = -\mathrm{i}\,\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = -\mathrm{i}\,\hbar\frac{\partial}{\partial\phi}.$$

From the definition of $\mathbf{L}$, it is not difficult to obtain the commutation relations

$$\begin{aligned} &[L_x, L_y] = \mathrm{i}\hbar L_z, \quad [L_y, L_z] = \mathrm{i}\,\hbar L_x, \quad [L_z, L_x] = \mathrm{i}\,\hbar L_y; \\ &[\mathbf{L}^2, L_x] = [\mathbf{L}^2, L_y] = [\mathbf{L}^2, L_z] = 0. \end{aligned} \tag{C.3}$$

Because L_x, L_y, L_z do not commute, it is not generally possible for a wave-function to be simultaneously an eigenstate of any two of them, but it is always possible to construct simultaneous eigenstates of $\mathbf{L}^2$ and any one of L_x, L_y, L_z. It is conventional to choose $\mathbf{L}^2$ and L_z.

The simultaneous eigenstates of $\mathbf{L}^2$ and L_z are denoted by $Y_{lm}(\theta, \phi)$, where

$$\begin{aligned} &\mathbf{L}^2 Y_{lm} = l(l+1)\,\hbar^2 Y_{lm} \\ &L_z Y_{lm} = m\,\hbar Y_{lm}. \end{aligned} \tag{C.4}$$

The allowed values of l are the integers $l = 0, 1, 2, 3, \ldots$ and, for a given l, m takes one of the $(2l+1)$ values $-l, -l+1, \ldots, l-1, l$. The functions $Y_{lm}(\theta, \phi)$ are well-known *spherical harmonics*, and are normalised so that

$$\begin{aligned} \int Y^*_{l'm'} Y_{lm}\,\mathrm{d}\Omega &= \int_0^\pi \mathrm{d}\theta \sin\theta \int_0^{2\pi} \mathrm{d}\phi\, Y^*_{l'm'}(\theta, \phi) Y_{lm}(\theta, \phi) \\ &= \delta_{ll'}\delta_{mm'}. \end{aligned}$$

For example, $Y_{00} = 1/\sqrt{4\pi}$; a state of zero orbital angular momentum is spherically symmetric. Also

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$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\frac{x+\mathrm{i}y}{r} = -\sqrt{\frac{3}{8\pi}}\sin\theta\,\mathrm{e}^{\mathrm{i}\phi}$$
$$Y_{10} = \sqrt{\frac{3}{4\pi}}\frac{z}{r} = \sqrt{\frac{3}{4\pi}}\cos\theta$$
$$Y_{1-1} = \sqrt{\frac{3}{8\pi}}\frac{x-\mathrm{i}y}{r} = \sqrt{\frac{3}{8\pi}}\sin\theta\,\mathrm{e}^{-\mathrm{i}\phi}.$$

Note that these states with $l = 1$ can be formed from the components of the unit vector $(x/r, y/r, z/r)$.

These examples illustrate a general rule: the parity of a state of given l is $(-1)^l$.

From (C.2), the eigenfunctions of the Schrödinger equation are of the form

$$\psi_{nlm} = u_{nl}(r)Y_{lm}(\theta,\phi) \tag{C.5}$$

where u_{nl} satisfies the ordinary differential equation

$$-\frac{\hbar^2}{2M}\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}(ru_{nl}) + \left(\frac{\hbar^2(l+1)}{2Mr^2} + V(r)\right)u_{nl}(r) = E_{nl}u_{nl}(r).$$

There are several examples of potentials $V(r)$ for which the radial functions $u_{nl}(r)$ are elementary, and many others for which the numerical solutions are easy to program on computers.

Note that the energy eigenstates (C.5) are also eigenstates of $\mathbf{L}^2$ and L_z. This is only possible because of the spherical symmetry of $V(r)$, which allows $\mathbf{L}^2$ and L_z (which act on θ and ϕ only) to commute with the energy operator.

C.2 Intrinsic angular momentum

A particle may have an *intrinsic angular momentum* or *spin* $\mathbf{s}$, satisfying the same commutation relations as (C.3). The eigenvalues of $\mathbf{s}^2$ are $s(s+1)\hbar^2$, and m_s can take $(2s+1)$ values from $-s$ to $+s$. In the case of orbital angular momentum treated above, l must be a positive integer. This condition stems from the single-valuedness of the wave-function in space. The quantum number s is not subject to this restriction, since the coordinates on which $\mathbf{s}$ acts are internal to the particle, and we require only that $(2s+1)$ should be an integer. Thus we may have $s = \frac{1}{2}$, as is the case with leptons and nucleons. For $s = \frac{1}{2}$ there are two eigenstates, cor-

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responding to $m_s = +\frac{1}{2}, m_s = -\frac{1}{2}$. We may denote these by $|+\frac{1}{2}\rangle$ and $|-\frac{1}{2}\rangle$. A general wave-function for the spin $\frac{1}{2}$ fermion is a superposition of 'spin-up' and 'spin-down' states of the form

$$\psi(\mathbf{r}, m_s) = \psi_+(\mathbf{r})|+\tfrac{1}{2}\rangle + \psi_-(\mathbf{r})|-\tfrac{1}{2}\rangle.$$

The two independent spin states $|+\frac{1}{2}\rangle$, $|-\frac{1}{2}\rangle$ may be represented by column vectors

$$|+\tfrac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\tfrac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

s_x, s_y, s_z are then represented by 2×2 matrices. It is convenient to take out a factor $(\hbar/2)$ and write $\mathbf{s} = (\hbar/2)\boldsymbol{\sigma} = (\hbar/2)(\sigma_x, \sigma_y, \sigma_z)$. It is easy to verify that the commutation relations are satisfied using the *Pauli matrices*:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$|+\frac{1}{2}\rangle$ and $|-\frac{1}{2}\rangle$ are eigenvalues of σ_z with eigenvalues $+1$ and -1.

C.3 Addition of angular momenta

The total angular momentum of a spin $\frac{1}{2}$ fermion is the sum of its orbital and intrinsic angular momenta:

$$\mathbf{J} = \mathbf{L} + \mathbf{s}.$$

It is easy to see that $\mathbf{J}$ satisfies commutation relations similar to (C.3), and also $[\mathbf{J}, \mathbf{L}^2] = 0$, $[\mathbf{J}, \mathbf{s}^2] = 0$. It is therefore possible to find states which are simultaneous eigenstates of $\mathbf{L}^2$, $\mathbf{s}^2$, $\mathbf{J}^2$ and j_z, specified by quantum numbers (l, s, j, j_z). These states have parity $(-1)^l$.

For a given value of l and $s = \frac{1}{2}$ there are $2 \times (2l+1) = 4l+2$ independent states, $Y_{lm}|\pm\frac{1}{2}\rangle$. We seek the linear combinations of these which are the eigenstates of $\mathbf{J}^2$ and J_z. Since $J_z = L_z + s_z$, the maximum value of j_z is $l + \frac{1}{2}$, corresponding to the state $Y_{ll}|+\frac{1}{2}\rangle$. This must also be the maximum value of j, and the state must also be an eigenstate of $\mathbf{J}^2$ corresponding to $j = l + \frac{1}{2}$, $j_z = l + \frac{1}{2}$.

There are two independent states giving $j_z = l - \frac{1}{2}$, i.e. $Y_{l,l-1}|+\frac{1}{2}\rangle$, $Y_{l,l}|-\frac{1}{2}\rangle$. From these we must be able to construct the state correspond-

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ing to $j = l + \frac{1}{2}$, $j_z = l - \frac{1}{2}$. Another independent state can also be constructed; this clearly must correspond to $j = l - \frac{1}{2}$, $j_z = l - \frac{1}{2}$.

The value $j = l + \frac{1}{2}$ gives $2(l + \frac{1}{2}) + 1 = 2l + 2$ states; the value $j = l - \frac{1}{2}$ gives $2(l - \frac{1}{2}) + 1 = 2l$ states. Altogether, we have $(4l + 2)$ independent states, corresponding to the values $j = l + \frac{1}{2}$, $j = l - \frac{1}{2}$, and there can be no more allowed values of j. We can think of the intrinsic spin $\mathbf{s}$ of the particle as either aligned or anti-aligned with the orbital angular momentum vector $\mathbf{L}$, in so far as the uncertainty principle allows.

More generally, for two particles, or two systems, with angular momenta $\mathbf{J}_1$ and $\mathbf{J}_2$, we may form

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2.$$

By an extension of the argument above, it can be shown that, for given values of j_1 and j_2, the allowed values of j are

$$j = j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2|,$$

so that

$$|j_1 - j_2| \leqslant j \leqslant j_1 + j_2.$$

C.4 The deuteron

The total intrinsic spin $\mathbf{S}$ of two spin $\frac{1}{2}$ fermions is

$$\mathbf{S} = \mathbf{s}_1 + \mathbf{s}_2,$$

where from the rules above the quantum number S can take the values $S = 1$ and $S = 0$. Explicitly, the three $S = 1$ states $|S, S_m\rangle$ are found to be

$$\begin{aligned} |1, 1\rangle &= |+\tfrac{1}{2}\rangle_1 |+\tfrac{1}{2}\rangle_2 \\ |1, 0\rangle &= (|+\tfrac{1}{2}\rangle_1 |-\tfrac{1}{2}\rangle_2 + |-\tfrac{1}{2}\rangle_1 |+\tfrac{1}{2}\rangle_2)/\sqrt{2} \\ |1, -1\rangle &= |-\tfrac{1}{2}\rangle_1 |-\tfrac{1}{2}\rangle_2 \end{aligned} \tag{C.6}$$

and the $S = 0$ state is

$$|0, 0\rangle = (|+\tfrac{1}{2}\rangle_1 |-\tfrac{1}{2}\rangle_2 - |-\tfrac{1}{2}\rangle_1 |+\tfrac{1}{2}\rangle_2)/\sqrt{2}. \tag{C.7}$$

(The factors $\sqrt{2}$ are for normalisation.)

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The deuteron is a neutron–proton bound pair having total spin $\mathbf{J} = \mathbf{L} + \mathbf{S}$ with quantum number $j = 1$ and total intrinsic spin $\mathbf{S}$ with quantum number $S = 1$. Neglecting a small $l = 2$ wave component, its spatial wave-function is an $l = 0$ state. The wave-function of a deuteron at rest is therefore approximately of the form

$$u(r)|1, m_s\rangle,$$

where r is the distance between the two nuclei.

From (C.6), it will be seen that this wave-function is symmetric under the interchange of proton and neutron. Thus such a state is not accessible to two protons, or to two neutrons, since the wave-functions of two identical fermions must be anti-symmetric under particle interchange (§1.1). Although two nucleons with net intrinsic spin zero experience a strong attraction, this attraction is not sufficient to produce a bound state and the deuteron is the only bound state of two nucleons.

Problems

C.1 Show that $l = 0$ wave-functions $\psi(r)$ (functions only of the radial coordinate r) are also eigenstates of L_x, L_y, L_z.

C.2 Explain why the single particle states specified by (l, s, j, j_z) introduced in §C.3 have parity $(-1)^l$.

C.3(a) Show that the state $|0, 0\rangle$ given by equation (C.7) is an eigenstate of $S_x(= s_{1x} + s_{2x})$, S_y and S_z and hence that it has total spin zero.

(b) Show that

$$S_z|1, 1\rangle = \hbar|1, 1\rangle$$

and

$$S^2|1, 1\rangle = 2\hbar^2|1, 1\rangle.$$

Appendix D
Unstable states and resonances

In discussing unstable states, we have in mind a system like an excited nucleus, or a β-unstable nucleus. An unstable state of a system will decay, and often there are several alternative modes of decay. For example, an excited state of a nucleus can have several states of lower energy to which

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