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How can one solve exactly some problems in plasma theory

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Abstract. A review of a technique for the application of a new Lambert W function for solving transcendental equations arising in classical problems of plasma and electron beam theories is presented, and the basic mathematical properties of the Lambert W function are outlined. The problems discussed include equilibrium electric charge of a solitary body in plasma, nonlinear ion-acoustic waves, the equilibrium configuration of a neutralized electric beam, the dynamics of shaping virtual cathodes in a medium with viscous friction, and the study of the dispersion equation for electron plasma waves in the framework of generalized Boltzmann kinetics.

1. Introduction

When solving mathematical problems of plasma physics, one can meet transcendental equations, and solving these equations looks impossible. These equations can arise in different ways: in finding the maximum or minimum of a function, in substituting the boundary conditions into general solutions of differential equations, in expressing an implicitly given function in explicit form and in searching for the eigenvalue of spectral problems. Dispersion relations of waves and instabilities can also be given as transcendental equations.

It is almost impossible to solve them exactly, and frequently approximate numerical or graphical methods are required. Sometimes, it is possible to find an approximate solution. However, theory requires exact solutions that can be substituted into new theories.

The literature presents some methods of exact solutions that allow the exact solution of any transcendental equation in an explicit form. For example, Burniston and Siewert (1973) offer a method connected with the Riemann boundary problem. Another method (Luck and Stevens 2002) is based on the known Cauchy theorem that determines the value of an integral along a closed contour that covers a simple pole on a complex plane. However, exact expressions for complex roots, even of the simplest equations obtained by these methods, look very bulky and inconvenient for the analysis. In this work, we present a method based on the use of a new special function—the Lambert W function. We familiarize the reader with this function and demonstrate the method's features in solving well-known classical problems of plasma theory.

Although the method described here cannot be called universal, it provides formulas that are simple in form and convenient for analysis.

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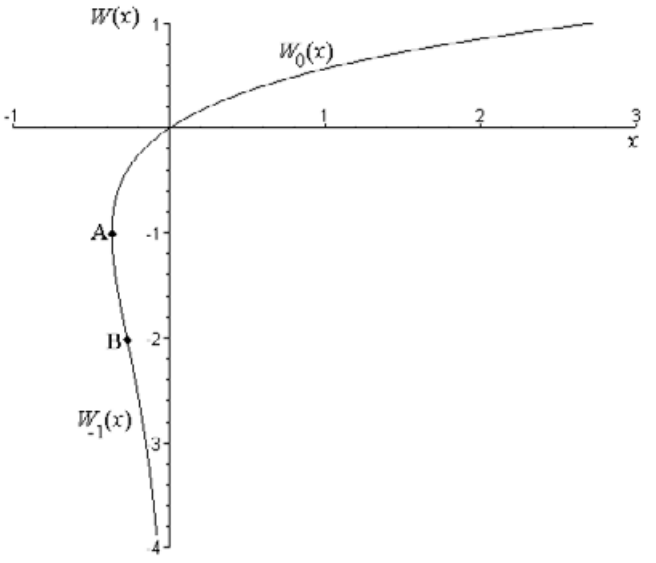


Figure 1. Graph of the Lambert W function real branches.

The Lambert W function was invented by Corless et al. (1996), and has already become popular among mathematicians and physicists. The review by Valluri et al. (2000) presents a technique for its application to the solution of various problems in physics. Among the examples considered there, we will single out a classical problem of an electric potential distribution in a plane capacitor with finite sizes (Rogovsky problem) and an exact deduction of the Wien displacement law from the Planck formula for heat radiation.

Other examples of the successful application of the Lambert W function in mathematical physics are known; for example, the exact solution of a problem concerning an electron longitudinal distribution in a storage ring with consideration made for a space charge (Thomas et al. 2002) and a problem of solar-wind hydrodynamics (Cranmer 2004). The work by Bulygin (2002) provides an exact solution of the problem of a capacitor discharge against a resistor, the resistance of which depends on temperature. Its solution describes the initial stage of the current flow process over an exploding wire. A number of these examples will increase in time. Thus, we will determine the real Lambert W function for real x values as a solution of the functional equation

$$W(x)\exp(W(x)) = x. \tag{1}$$

In other words, the Lambert W function is the function inverse to the function $W = x\exp x$. It allows us to imagine the appearance of a graph of a function (Fig. 1) and to specify its simplest properties.

The Lambert W function is neither even nor odd. It is determined within the range $[-1/\mathrm{e};\infty)$, where it takes values from $-\infty$ to ∞; for negative x the function is double-valued. The point A with coordinates $(-1/\mathrm{e};\infty)$ splits the graph of the function into two branches, the top $W_0(x)$ and the bottom $W_{-1}(x)$, in such a way that both branches have a vertical tangent at A. The top branch, $W_0(x)$, which is often called the principal, goes through the point of origin and no longer has

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specific features. The bottom branch, $W_{-1}(x)$, has the inflection B with coordinates $(-2/e^2; -2)$ and a vertical asymptote at $x = 0$.

Other whole values of the index $k \neq 0, -1$ for the function $W_k(x)$ refer to complex-valued branches (Jeffrey et al. 1996). Furthermore, we will sometimes omit indices of function branches, if the properties under consideration are valid for every branch and if it does not cause confusion.

From the definition (1) it is easy to deduce the following simple identities:

$$\exp(W(x)) = \frac{x}{W(x)}, \quad \ln W(x) = \ln x - W(x). \tag{2}$$

We now present rules for the differentiation and integration of the Lambert W function. Its derivative is easy to find using the inverse function differentiation rule:

$$W'(x) = \frac{1}{(1 + W(x))\exp(W(x))} = \frac{W(x)}{x(1 + W(x))}. \tag{3}$$

The last equality is valid for $x \neq 0$. However, for x tending towards zero the derivative is specified as the limit

$$\lim_{x \to 0} W_0'(x) = 1. \tag{4}$$

It is also possible to find the indefinite integral of the Lambert W function using the inverse function integration rule from Parker (1955):

$$\int_x f(x)\,dx = xf(x) - \int_{f(x)} g(y)\,dy. \tag{5}$$

Substituting the Lambert W function, we easily get (without the integration constant)

$$\int W(x)\,dx = x\left[W(x) - 1 + \frac{1}{W(x)}\right]. \tag{6}$$

We shall also consider a series expansion of the Lambert W function:

$$W(x) = x - x^2 + \frac{3}{2}x^3 - \frac{8}{3}x^4 + \frac{125}{24}x^5 - \frac{54}{5}x^6 + \mathrm{O}(x^7) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!}x^n. \tag{7}$$

Now we shall consider the examples of the application of the Lambert W function to classical problems of plasma physics.

2. The problem of an equilibrium solitary-body electric charge in a plasma

The problem of an equilibrium electric charge accumulated on a test body (or particle) under a floating potential in a gas-discharge plasma is very important. First, its solution is connected with the theory of a single probe, whereby temperature and plasma concentration can be measured. Second, the solution is the basis for the majority of dusty plasma theories.

Least difficult, this problem is stated for a Maxwell plasma (i.e. a plasma with Maxwell distribution of electrons and ions with temperatures T_e and T_i, respectively) in a so-called orbital limit. One could cite a large number of works about probes and grains of dust in plasma, wherein its definition is given.

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We shall present a definition of the problem in terms of the work by Matsoukas and Russell (1995). The electron and ion currents per the spherical part of the radius R are

$$I_{\mathrm{e}}(q) = \begin{cases} k' \exp(+\beta_{\mathrm{e}} q) & \text{for } q \leqslant 0, \\ k'(1+\beta_{\mathrm{e}} q) & \text{for } q > 0, \end{cases} \tag{8}$$

$$I_{\mathrm{i}}(q) = \begin{cases} pk'(1-\tau\beta_{\mathrm{e}} q) & \text{for } q \leqslant 0, \\ pk' \exp(-\tau\beta_{\mathrm{e}} q) & \text{for } q > 0, \end{cases} \tag{9}$$

where

$$k' = \pi R^2 n_{\mathrm{e}} \left(\frac{8kT_{\mathrm{e}}}{\pi m}\right)^{1/2}, \quad \beta_{\mathrm{e}} = \frac{e^2}{4\pi\varepsilon_0 RkT_{\mathrm{e}}}, \quad p = \frac{1}{\nu}\left(\frac{\mu}{\tau}\right)^{1/2},$$

$$\nu = \frac{n_{\mathrm{e}}}{n_{\mathrm{i}}}, \quad \tau = \frac{T_{\mathrm{e}}}{T_{\mathrm{i}}}, \quad \mu = \frac{m}{M},$$

n_{e} and n_{i} are the electron and ion concentrations, m and M are the electron and ion masses, k is the Boltzmann constant and ε_0 is the dielectric constant. An equilibrium electric charge accumulated by a particle is specified by the exact equality of electron and ion currents:

$$I_{\mathrm{e}}(q) = I_{\mathrm{i}}(q). \tag{10}$$

Equation (10) is transcendental. Earlier, it was solved numerically (Xie 2002) or the solution was expressed in an approximate form. For example, in Matsoukas and Russell (1995) for the case $q < 0$ the following approximation is given:

$$q \approx C \frac{4\pi\varepsilon_0 RkT_{\mathrm{e}}}{e^2} \ln \frac{(\mu\tau)^{1/2}}{\nu}, \tag{11}$$

where C is some constant.

Meanwhile, using substitution, it is easy to make sure that (10) has the following exact solution:

$$q = \begin{cases} -\dfrac{1}{\beta_{\mathrm{e}}} W\left(\dfrac{1}{p\tau}\exp\dfrac{1}{\tau}\right) + \dfrac{1}{\beta_{\mathrm{e}}\tau} & \text{for } q \leqslant 0, \\ \dfrac{1}{\beta_{\mathrm{e}}\tau} W(p\tau\exp\tau) - \dfrac{1}{\beta_{\mathrm{e}}} & \text{for } q > 0. \end{cases} \tag{12}$$

Thus, the exact solution (12) is obtained for the equilibrium electric charge on a spherical particle in a Maxwell plasma that involves the Lambert W function and can be used in the theories of a probe and dusty plasma. This example refers to the case when transcendental equation (10) arises as a stationary solution of a differential equation of a particle charging.

3. The problem of nonlinear ion-acoustic waves

The problem of nonlinear ion-acoustic waves has been investigated in Vedenov et al. (1961) and Sagdeev (1966). The point of the method is in substituting the velocity particle distribution laws into a one-dimensional Poisson's equation. The nonlinear equation is the equation of a generalized oscillator in some pseudopotential. More

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recently, the so-called Sagdeev pseudopotential method $\Phi(\varphi)$, applied where the electrostatic potential φ plays the role of a coordinate and the wave initial object $\xi = x - ut$ the role of time, has become classical.

This method has found widespread application in the following theories: nonlinear ion-acoustic solitons (Ghosh and Sekar Iyengar 2000), double layers (Schamel 1982) and ion holes (Shukla and Eliasson 2003) in plasmas. In the past decade, it has become popular to use the pseudopotential method in dusty plasma theories: Bohm layers in dust plasma (Liu et al. 2003) and solitary dust-acoustic waves (Mahmood and Saleem 2003; Gupta et al. 2003).

However, in pseudopotential investigations transcendental equations arise almost always. We shall show in a simple example how the Lambert W function can help us.

We shall proceed from the following Poisson equation:

$$\varphi'' = \frac{\rho_0}{\varepsilon_0}\left[\exp\left(\frac{e\varphi}{kT}\right) - \frac{u}{\sqrt{u^2 - 2e\varphi/M}}\right], \tag{13}$$

where ρ_0 is the non-perturbed plasma concentration, ε_0 is the dielectric constant, e is the elementary charge, k is the Boltzmann constant, T is the electron temperature, u is the wave velocity, M is the ion mass, and prime means ξ differentiation.

Let us introduce some dimensionless variables: $\tilde{v} = v/u$, $\tilde{\rho} = \rho/\rho_0$, $\tilde{\varphi} = e\varphi/Mu^2$, $\tilde{\xi} = \omega_{\mathrm{i}}\xi/u$, and $\tilde{T} = kT/Mu^2$ (where ω_{i} is the ion plasma frequency). Omitting tildas over them, we get

$$\varphi'' = \exp\left(\frac{\varphi}{T}\right) - \frac{1}{\sqrt{1-2\varphi}}, \tag{14}$$

which is given by the equation of a nonlinear oscillator in the pseudopotential field. Integrating this equation once, we get the pseudopotential exterior:

$$\Phi(\varphi) = -\sqrt{1-2\varphi} - T\exp\left(\frac{\varphi}{T}\right) + T + 1. \tag{15}$$

The integration constant in (15) is chosen in such a way that $\Phi(0) = 0$. We shall investigate the pseudopotential structure. The state of oscillator equilibrium is found from the condition for minimum $\Phi(\varphi)$, which leads to the transcendental equation:

$$1 - 2\varphi = \exp(-2\varphi/T). \tag{16}$$

Earlier, this equation was solved graphically in an approximate manner, although it can be solved exactly. Equation (16) always has two real roots, one of which is trivial, and the other is expressed through the Lambert W function,

$$\varphi_1 = 0, \quad \varphi_2 = \frac{1}{2}\left\{TW_{0,-1}\left[-\frac{1}{T}\exp\left(-\frac{1}{T}\right)\right] + 1\right\}, \tag{17}$$

where $W_0(x)$ and $W_{-1}(x)$ are the principal and the bottom real branches of the Lambert W function, respectively. It is easy to see that at $T > 1$, when the solution is expressed through the principal branch of the Lambert W function, $\varphi_2 < 0$, and so φ_2 corresponds to the pseudopotential maximum and φ_1 corresponds to the minimum. In the case of $T < 1$, when the solution is expressed through the bottom branch, $\varphi_2 > 0$, and so φ_2 corresponds to the pseudopotential minimum and φ_1 corresponds to the maximum. Figure 2 presents the pseudopotential distribution for these two cases, and Fig. 3 presents the equilibrium pseudocoordinate φ_2 as a function of T.

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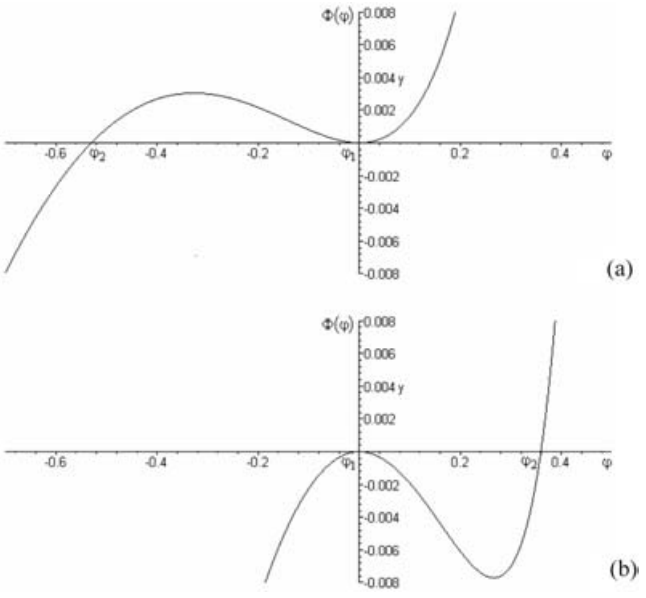


Figure 2. Pseudopotential $\Phi(\varphi)$ as a function of φ at various T: (a) $T > 1$; (b) $T < 1$.

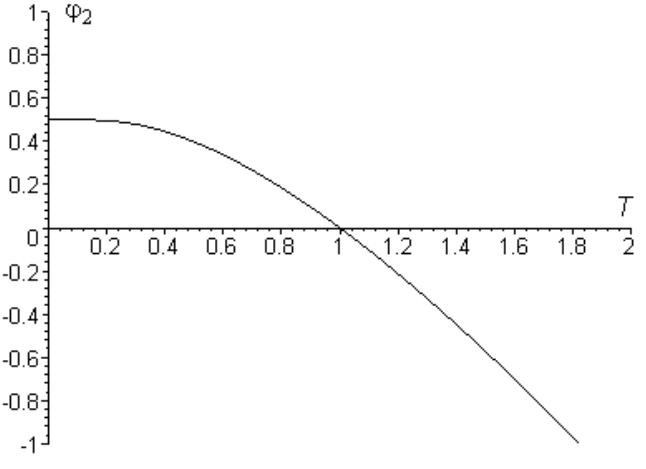


Figure 3. The second root of (16), φ_2, as a function of T.

The periodic motion of the oscillator in a pseudopotential well describes periodic ion-acoustic waves, and the motion of a separatrix, passing through the maximum, describes a solitary ion-acoustic wave.

However, the solution at $T > 1$ does not meet the boundary condition $\lim_{\xi \to \pm\infty} \varphi = 0$. Thus solitary waves exist only at $T < 1$, i.e. when their velocity exceeds the ion sound velocity, $u > c = \sqrt{kT/M}$. This inequality limits from below the range of possible velocities of the solitary wave. It is obvious that there is also

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a limitation from above, $\Phi(\varphi = 1/2) > 0$, otherwise umklapp will take place. The last inequality leads to

$$1 + T\left(1 - \exp\frac{1}{2T}\right) > 0. \tag{18}$$

Thus, the solitary wave velocity must be from the range $c < u < c\mathscr{M}$, where $\mathscr{M}$ is the Mach number, which is the solution of the next transcendental equation:

$$\mathscr{M}^2 + 1 = \exp\frac{\mathscr{M}^2}{2}. \tag{19}$$

It turns out that its solution can also be found exactly using the Lambert W function. It is given by

$$\mathscr{M} = \sqrt{-1 - 2W_{-1}[-\tfrac{1}{2}\exp(-\tfrac{1}{2})]} \approx 1.585\,201\,065\ldots, \tag{20}$$

and in Sagdeev (1966) it was $\mathscr{M} = 1.6$.

Thus, the problem of a possible value of the solitary ion-acoustic wave velocity is solved exactly, in an analytical manner. The example of the use of the transcendental equations is an example of the investigation of the function.

4. The problem of the equilibrium configuration of a neutralized electron beam

The paper by Lawson (1973) considers the various models of equilibrium configurations of neutralized relativistic electron beams. The general equilibrium condition obtained from the equality of the centrifugal force and the Lorentz force affecting a separate electron at r_1 radius from the side of the beam's magnetic field is given by

$$\frac{\gamma m\beta_\theta^2(r_1)c^2}{r_1} = e\beta_z(r_1)B_\theta(r_1) = \frac{4\pi e^2\beta_z(r_1)}{r_1}\int_0^{r_1} n(r)\beta_z(r)r\,dr, \tag{21}$$

where γ is the electron lorentz factor, and β_0, β_θ and β_z are the full, azimuthal and axial relative electron velocities, respectively. The equality $\beta_\theta^2 + \beta_z^2 = \beta_0^2 =$ constant is true. The model of uniformity in the beam cross-section electron density $n(r) =$ constant gives a solution of (21) in the form (see Lawson 1973; Lai 1976; Romanovsky 1979)

$$\frac{4\pi r_1^2 ne^2}{\gamma mc^2} = \frac{\beta_0^2}{\beta_z^2(r_1)} + 2\ln\frac{\beta_0}{\beta_z(r_1)} - 1. \tag{22}$$

For this solution written in implicit form, there is an approximation by Romanovsky (1979)

$$\beta_z(r_1) \approx \beta_0\left[K\left(\frac{\pi r_1^2 ne^2}{\gamma mc^2}\right) + 1\right]^{-1/2}, \tag{23}$$

where $K(x)$ is some monotonously growing function from 2 at $x = 0$ to 4 at $x \to \infty$. At the same time, (22) assumes the explicit form

$$\beta_z(r_1) = \beta_0\left\{W\left[\exp\left(\frac{4\pi r_1^2 ne^2}{\gamma mc^2} + 1\right)\right]\right\}^{-1/2}. \tag{24}$$

The given example demonstrates the applicability of the Lambert W function in expressing implicitly the given function in the explicit form.

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5. The problem of virtual cathode formation dynamics in an environment with viscous friction

It is known that when starting to inject a high-current electron beam into an equipotential cavity, starting from some value of current strength, the beam space charge will form a potential barrier in the cavity that prevents further electron injection. Part of the electron beam is reflected back by the potential barrier to the place of its injection, and the turning point, characterized in the hydrodynamic limit by infinite electron concentration, was given the name of a virtual cathode (VC).

Electron devices with a VC are becoming increasingly popular in vacuum high-current electronics, being implemented in powerful microwave and X-ray sources, ion accelerators, powerful switches and deep vacuum meters. Reviews of physics and VC engineering current status are given in Dubinov and Selemir (2002) and Dubinov et al. (2002).

Strict analysis of general cases related to the dynamics of VC formation and their existence in an electron beam currently seems unfeasible, since the existing methods cannot be applied under velocity ambiguity. Thus, the basic tool of theoretical study into such systems is computer modeling.

However, in some cases, it works well to undertake self-consistent analytical consideration of a problem related to the initial stage of VC formation in vacuum conditions for the most straightforward situations such as for a half-space (Poukey and Rostoker 1971; Dubinov 2000). Of practical interest is the consideration of specific features in VC formation dynamics under the conditions that exist when an equipotential cavity is filled with some medium. This example shows an analytical self-consistent solution of the problem related to the initial stage of VC formation during beam injection into a half-space, filled with a model medium, in which electrons are subjected to dispersion, and the average effect of this dispersion is described by the friction force proportional to the electron velocity. Such electron friction in a medium, which is similar to viscous friction under body motion in a liquid, can arise in certain conditions, for example in the motion of free electrons in semiconductors (Birdsall and Bridges 1966; Akimov and Schamel 2002) and gases (Schmieder 1979).

Let us consider the process of monochromatic electron beam injection into a half-space filled with a medium in which electrons are subjected to friction proportional to the velocity and constrained by a metal plane $z = 0$. It is unimportant whether the electrons are injected through a thin metal electrode that is transparent to them (in this case an equal positive charge is induced on it) or the electrons come into the half-space, when they are knocked out from the surface, for example by intensive photon flux (in this case a non-compensated charge of positive ions is left on the photoemitter surface). At each point of time in both cases, the positive charge is exactly equal to the full charge of the electron beam.

Let $N\ [\mathrm{cm}^{-2}\,\mathrm{c}^{-1}] = N(t)$ be the density of electrons emitted from the surface $z = 0$ at the time $t = 0$. By the time t, a spatial electron charge per unit surface area of

$$Q = -e\int_0^t N(\tau)\,d\tau \tag{25}$$

and a positive surface charge of the value $|Q|$ are formed. Therefore, the electrons within the injection range from τ to $\tau + d\tau$ are affected by the field $E(\tau)$, the value

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of which is determined only by the charge of the electrons issued earlier. Therefore

$$E(\tau) = 4\pi e \int_0^\tau N(\tau)\, d\tau. \tag{26}$$

The ratio (26) is valid only up to the point when a certain layer of electrons leaves behind the layer injected earlier. Below we find an exact expression for this point in time. In the particular case of a rectangular pulse with density $N(t) = N_0\Theta(t)$, where $\Theta(t)$ is the unit-step Heaviside function, we get

$$E(\tau) = 4\pi e N_0 \tau. \tag{27}$$

Then the equation of electron motion in Lagrangian variables (t,τ) can be written as

$$\ddot{z}(t,\tau) = -\frac{e}{m}E(\tau) - k\dot{z}(t,\tau) = -\frac{4\pi e^2 N_0}{m}\tau - k\dot{z}(t,\tau) \tag{28}$$

with the initial conditions

$$z(\tau,\tau) = 0, \quad \dot{z}(\tau,\tau) = v_0. \tag{29}$$

After the next normalization is introduced (time variables are normalized to $\tau' = mv_0/4\pi e^2 N_0$, the length to $v_0\tau'$, velocity to v_0, electron density to N_0, electric field to $mv_0/e\tau'$, potential to mv_0^2/e etc.; primes will be omitted) the system (28) and (29) is transformed to

$$\ddot{z}(t,\tau) = -\tau - k\dot{z}(t,\tau), \quad z(\tau,\tau) = 0, \quad \dot{z}(\tau,\tau) = 1. \tag{30}$$

The problem (6) is easily solved, with its solution being

$$v(t,\tau) = \frac{(\tau+k)\exp[-k(t-\tau)] - \tau}{k}, \tag{31}$$

$$z(t,\tau) = \frac{(\tau+k)\exp[-k(t-\tau)] - (\tau+k) + \tau k(t-\tau)}{k^2}. \tag{32}$$

Figure 4 presents the designed electron paths in the plane (z,t) for various values of the parameters k, from which the role of friction is clear. The VC forms earlier and closer to the injection plane, the greater k is.

The turn time t_r of various electron layers ($v = 0$) can be determined in the following way:

$$t_\mathrm{r} = \frac{\tau k - \ln(\tau/(\tau+k))}{k}, \tag{33}$$

and the recurrence time of t_p electrons to the plane can be determined from the equation $z(t_\mathrm{p},\tau) = 0$ (see (32)). This equation has two roots:

$$t_\mathrm{p} = \tau, \quad t_\mathrm{p} = \frac{1}{\tau k}\left\{\tau W_0\left[-\frac{\tau+k}{\tau}\exp\left(-\frac{\tau+k}{\tau}\right)\right] + \tau^2 k + \tau + k\right\}. \tag{34}$$

The first root corresponds to electron injection, and the second corresponds to electron recurrence to the plane $z = 0$.

We shall identify the point of time when the velocity unambiguity in the electron flux is violated, i.e. when the equality of (26) is no longer met. In the region of unambiguity we have

$$\frac{\partial}{\partial\tau}z(t,\tau) < 0. \tag{35}$$

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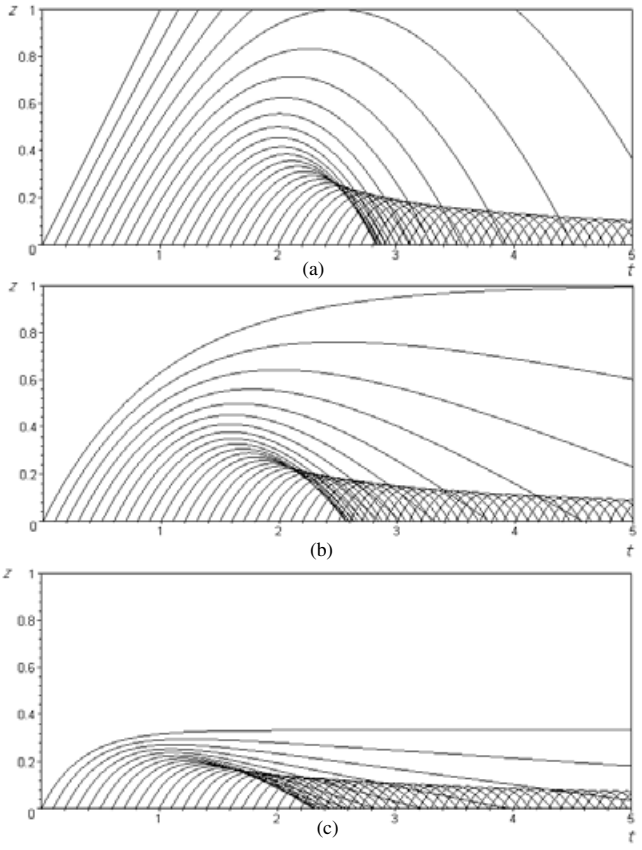


Figure 4. Trajectories of electrons in the plane (z, t) for various values of k parameter: (a) $k = 0$; (b) $k = 1$; (c) $k = 3$.

This condition is met, if

$$t_m < \frac{W_{-1}[(-1-\tau k-k^2)\exp(-1-\tau k)]+2\tau k+1}{k} \quad \text{and}$$
$$t_m > \frac{W_0[(-1-\tau k-k^2)\exp(-1-\tau k)]+2\tau k+1}{k}. \tag{36}$$

Ratios (33), (34), and (36) determine the electron-beam states; the regions of these states in the plane (t, τ) are shown in Fig. 5. The region of electron motion forward is constrained by line 1 (injection line), curve 2 (electron stop and reflection line), and curve 3 (the line of contact between electrons and a plane). The colored region is the region of velocity ambiguity (many fluxes), where the main expression for the field (2) is violated and where our model is not valid. However, the boundaries of the multiflux region and the dynamics in the region of unambiguity remain valid even after VC formation and the origination of many fluxes.

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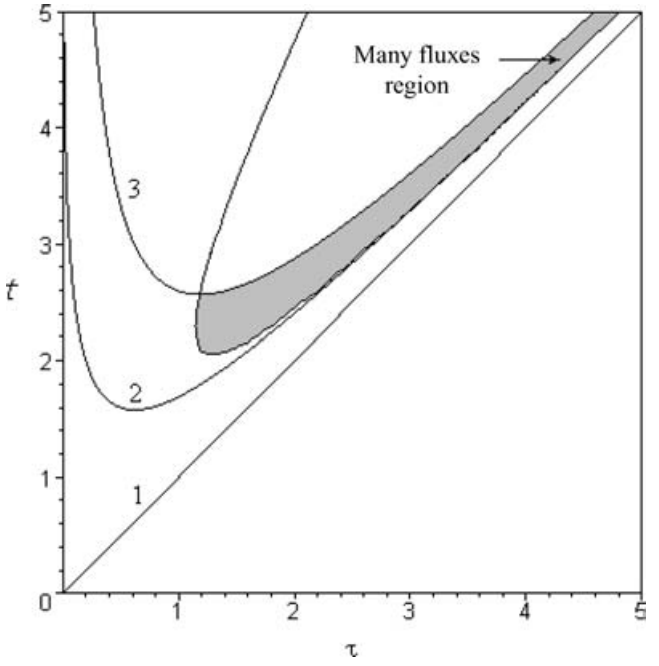


Figure 5. Regions of electron-beam states in the plane (t, τ) for $k = 1$.

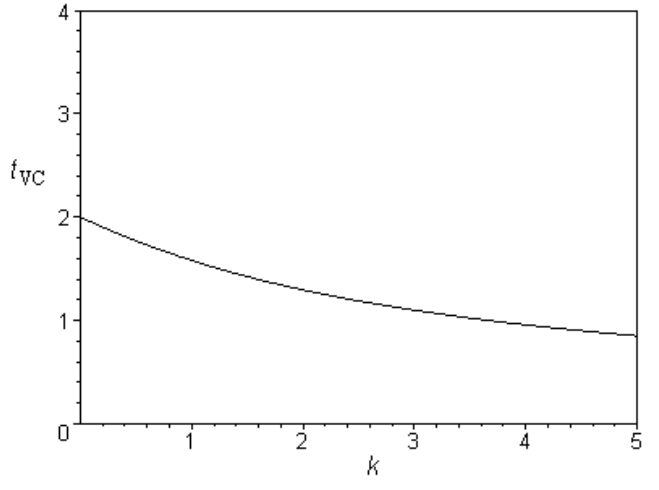


Figure 6. VC formation time t_{VC} as a function of the parameter k.

We will determine the duration of the VC formation process t_{VC} as the minimum value of the function $t_r(\tau)$ (33). Minimization provides

$$t_{VC} = -\frac{k - \sqrt{k^2+4}}{2} - \frac{1}{k} \ln\left(-\frac{k - \sqrt{k^2+4}}{k + \sqrt{k^2+4}}\right), \quad \{t_{VC} = 2\}. \tag{37}$$

The graph of VC formation time as a function of the parameter k is shown in Fig. 6.

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Thus, in this section we have built and investigated a self-consistent analytical model of the initial stage of the VC formation process in a medium in which electrons are subjected to friction proportional to velocity. In the investigation, two real branches of the Lambert W function are efficiently applied. In the next example, a problem will be considered with an infinite variety of complex W function branches.

6. A problem on the study into the dispersion equation for electron plasma waves under generalized Boltzmann kinetics

After linearization of the generalized Boltzmann equation with a collision term in Bhatnagar–Gross–Krook form (Alekseev 2000, 2001) the following dispersion equation for electron plasma waves can be obtained:

$$1 = -\frac{e^2}{\varepsilon_0 k}\left\{\frac{1}{m_\mathrm{e}}\int_{-\infty}^{+\infty}\frac{(\partial f_{0\mathrm{e}}/\partial u)[i-\tau_\mathrm{e}(\omega-ku)]}{i(\omega-ku)-\tau_\mathrm{e}(\omega-ku)^2-\nu_\mathrm{e}}\,du\right.$$
$$\left.+\frac{1}{m_\mathrm{i}}\int_{-\infty}^{+\infty}\frac{(\partial f_{0\mathrm{i}}/\partial u)[i-\tau_\mathrm{i}(\omega-ku)]}{i(\omega-ku)-\tau_\mathrm{i}(\omega-ku)^2-\nu_\mathrm{i}}\,du\right\}, \tag{38}$$

where $\tau_{\mathrm{e,i}}$ are the electron and ion relaxation times. Assuming that electrons have a Maxwellian distribution and the ions are at rest, the analysis of (38) in the given work is reduced to the analysis of the transcendental equation

$$-\exp(\gamma\cot\gamma)\sin\gamma = \frac{\pi}{2\beta^2}\gamma, \tag{39}$$

which was solved graphically by Alekseev (2000, 2001). It is denoted $\beta = 1 + r_\mathrm{D}^2k^2$ in (39) where r_D is the Debye radius and k is the wave number. It was also established that this equation has an infinite number of complex roots, each of which corresponds to its branch from a discrete vibration spectrum, and complex vibration frequencies can be calculated by the following equation:

$$\omega_n = \frac{1}{2}\sqrt{-\gamma_n\tan\frac{\gamma_n}{2}} - \frac{i}{2}\sqrt{-\gamma_n\cot\frac{\gamma_n}{2}}. \tag{40}$$

Equation (39) can be solved using the Lambert W function. For this purpose, we consider the algebraic equation

$$z + \alpha\exp(-z) = 0 \tag{41}$$

relative to a complex variable $z = x + iy$. We separate the real part from the imaginary part in (41), and obtain a system of equations in x and y:

$$x + \alpha\exp(-x)\cos y = 0, \tag{42}$$

$$y - \alpha\exp(-x)\sin y = 0. \tag{43}$$

Dividing (42) by (43) term by term, we have

$$x = -y\cot y. \tag{44}$$

Substituting (44) into (43), we get the equation for the imaginary part of the roots of equation (41):

$$\exp(y\cot y)\sin y = \frac{y}{\alpha}. \tag{45}$$

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This deduction of (45) from (41) is given in Pinney (1958). On the other hand, (45) completely agrees with (39), if $\alpha = -2\beta^2/\pi$. Thus, if we find the complex roots z_n of the equation

$$z - \frac{2\beta^2}{\pi}\exp(-z) = 0, \tag{46}$$

the roots of (39) will be $\gamma_n = \operatorname{Im} z_n$. Equation (41) is easily converted to the form of the functional equation (1), and thus the roots of (41) are

$$z_n = W_n\left(\frac{2\beta^2}{\pi}\right), \tag{47}$$

where n passes a natural set, and the exact solution of the dispersion equation (39) is

$$\gamma_n = \operatorname{Im}\left[W_n\left(\frac{2\beta^2}{\pi}\right)\right]. \tag{48}$$

Considering some exact values γ_n with $\beta = 1$,

$$\begin{aligned} \gamma_1 &= 1.360\,473\,393\pi, \quad \gamma_2 = 3.417\,140\,413\pi, \\ \gamma_3 &= 5.439\,124\,887\pi, \quad \gamma_4 = 7.451\,208\,982\pi, \\ \gamma_5 &= 9.458\,971\,675\pi, \quad \gamma_6 = 11.464\,427\,73\pi, \\ \gamma_7 &= 13.468\,495\,57\pi, \end{aligned} \tag{49}$$

from which it is possible to calculate complex frequencies. Thus, the Lambert W function has helped us to find an exact solution of an equation that seemed to be unsolvable.

7. Conclusion

There is an old and wise piece of advice: 'You'd rather teach a poor man how to get food than give him food'. Following this advice, we have presented the reader not with new plasma theories, but with a new simple mathematical tool from which he can develop theories himself.

We hope that by presenting this work we will have achieved our main goal—that of convincing the reader as to the usefulness of the Lambert W function for many mathematical problems. We will add that the Lambert W function has already been introduced into the known mathematical tools of the symbol algebra Maple and Mathematica, and now it is as simple to handle as an exponent, logarithm or sinus.

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