Student Problems

Students up to the age of 19 are invited to send solutions to either or both of the following problems to Lewis Roberts, Exeter Maths School, Rougemont House, Castle Street, Exeter, Devon EX4 3PU.

Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is not necessary to submit solutions to both. Entries should arrive by 20th January 2020 and solutions will be published in the March 2020 edition.

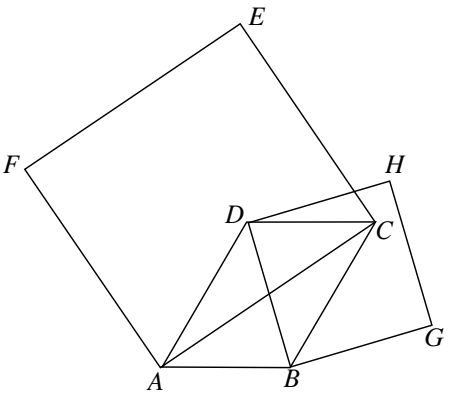
The Mathematical Association and the *Gazette* comply fully with the provisions of the 2018 GDPR legislation. Submissions **must** be accompanied by the SPC permission form which is available on the MA website

https://www.m-a.org.uk/the-mathematical-gazette

Note that if permission is not given, a pupil ***may still participate and will be eligible for a prize in the same way as others***.

Problem 2019.5 **(Dao Thanh Oai)**

Let $ABCD$ be a parallelogram with A and B fixed points in a plane. Quadrilaterals $ACEF$ and $BGHD$ are squares. Show that the midpoint of FG remains fixed when the line segment CD is translated in the plane.



Problem 2019.6 **(Chris Starr)**

For n odd, prove that

$$\left(\binom{n}{2} + 2\binom{n}{4} + 2^2\binom{n}{6} + \cdots\right)^2 + \left(\binom{n}{0} + \binom{n}{2} + 2\binom{n}{4} + 2^2\binom{n}{6} + \cdots\right)^2$$
$$= \left(\binom{n}{1} + 2\binom{n}{3} + 2^2\binom{n}{5} + \cdots\right)^2.$$

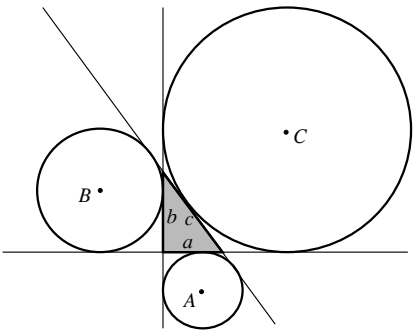
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Solutions to 2019.3 and 2019.4

Both problems were solved by Nathan Burn (Bishop Wordsworth's Grammar School), Patrick Luo (King's College London Mathematics School), James Tan (Queen Elizabeth's School Barnet) and Ping Chuan Zhu (St. Paul's School). Problem 2019.3 was solved by Lucas Boo Tse Yang (Raffles Institution, Singapore) and Student H (ATL Township, Bhupiamau). Problem 2019.4 was solved by William Armstrong (Bishop Wordsworth's Grammar School), Karthik Prabhu Bantwal (The Learning Centre, Mangaluru), Sam James (Bishop Wordsworth's Grammar School), Zhizhen Lianne Liu (King's College London Mathematics School) and Stavan Shah (SMCP Sanskar Vidya, Bhavan, Gujarat).

Problem 2019.3 (**Paul Stephenson**)

The right-angled triangle shown shaded in the diagram has sides of lengths a, b and (hypotenuse) c. Also shown in the diagram are the three excircles of the shaded triangle and the excentres A, B and C. Find the area of triangle ABC.

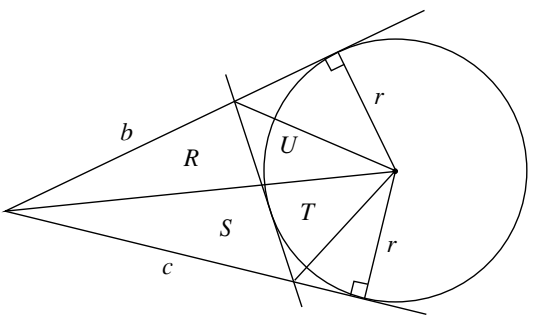


Solution

Let Δ denote the shaded triangle and let A', B' and C' be the vertices of Δ which are opposite the edges of lengths a, b and c respectively.

There were several very different but equally excellent solutions to this problem. Nathan, Lucas and Ping started by observing that both B and C lie on the bisector of the external angle to Δ at A'. Therefore A' lies on the line BC and, similarly, B' lies on CA and C' lies on AB.

Student H now quoted an easily-proved result about the radius of the excircle of any triangle. For *any* triangle with area Δ and edge-lengths a, b and c the radius of the excircle on the edge of length a can be found as follows.



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Let R, S, T and U be the areas of four triangles, as shown. Then the simple formula "half base times perpendicular height" gives:-

$$R + U = \tfrac{1}{2}br,\ S + T = \tfrac{1}{2}cr,\ R + S = \Delta,\ T + U = \tfrac{1}{2}ar.$$

Then $\frac{1}{2}(b + c)r = \Delta + \frac{1}{2}ar$ and $r = \frac{2\Delta}{b+c-a}$.

The area of triangle ABC can now be found as the sum of four areas

$$\begin{aligned}&\tfrac{1}{2}ab + \tfrac{1}{2}a\left(\frac{ab}{b + c - a}\right) + \tfrac{1}{2}b\left(\frac{ab}{c + a - b}\right) + \tfrac{1}{2}c\left(\frac{ab}{a + b - c}\right)\\ &= \tfrac{1}{2}ab + \tfrac{1}{4}a(c + a - b) + \tfrac{1}{4}b(b + c - a) + \tfrac{1}{4}c(a + b - c)\\ &= \tfrac{1}{2}c(a + b + c).\end{aligned}$$

The most elegant and simplest answer was given by James who noted that CC' is perpendicular to the base AB of triangle ABC. The problem is then solved by finding a few simple lengths in terms of a, b and c.

Problem 2019.4 **(Stan Dolan)**

If a 5-digit number, n, is a multiple of 271 then so are all numbers given by cyclic permutations of the digits of n.

For example, 91327 is a multiple of 271 and therefore so are 79132, 27913, 32791 and 13279.

Explain this property of 271 and generalise your result.

Solution

The solutions all followed the same lines. The solutions of Karthik, Lianne, James and Ping were especially good in terms of clarity and a well-expressed generalisation.

We can define n to be the number

$$10000A + 1000B + 100C + 10D + E$$

which is assumed to be a multiple of 271.

It is sufficient to consider one cyclic permutation:-

$$n' = 10000E + 1000A + 100B + 10C + D.$$

Then $10n' = n + 99999E$ and, since 271 is a factor of both n and 99999, it is also a factor of $10n'$ and thus of n'.

In general, for a number n with m digits, any common factor of n and $10^m - 1$ is also a factor of the cyclic permutations of the digits of n.

The first prize of £25 is awarded to James Tan. The second prize of £20 is awarded to Nathan Burn.

10.1017/mag.2019.130 STAN DOLAN

Note from the Editor

See *www.m-a.org.uk/the-mathematical-gazette* for the SPC questions.

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