7 Nonperturbative Methods in Weak Decays

7.1 General View

We have seen in previous chapters that the calculations of short-distance contributions to various decays of mesons, represented by Wilson coefficients, reached already within the SM a very satisfactory level. But to obtain the values of various decay rates and branching ratios, also long-distance (LD) contributions have to be included. We have seen in Chapter 3 that these effects depend strongly on the decays considered.

In leptonic decays of mesons the dominant LD effects are encoded in weak decay constants, like F_K, F_{B_d}, F_{B_s}. By now they are calculated with respectable precision by lattice QCD (LQCD). We collected their values in Table D.3. See also Table 7.1.

In semileptonic decays various formfactors have to be calculated. For K meson decays, LQCD and chiral perturbation theory (ChPT) appear presently to provide the most accurate results. In particular calculations of strong isospin breaking corrections and of QED effects are dominated by ChPT.

The most difficult are the calculations of LD effects in nonleptonic decays and transitions like $K^0-\bar{K}^0$ mixing that are dominantly contained in the matrix elements of local operators. Here still significant progress has to be made.

In $K \to \pi\pi$ decays and $K^0 - \bar{K}^0$ mixing, the so-called Dual QCD (DQCD) approach provided already in the 1980s first results, and these calculations have been improved recently. Among analytic approaches DQCD remains still the leading approach for the calculation of hadronic matrix elements in K meson system in QCD. This is related to the fact that it is presently the only analytic approach that allows to match the LD QCD effects in hadronic matrix elements to SD ones encoded in the Wilson coefficients. Such matching is presently not possible in ChPT, implying very large uncertainties in the calculations of $K \to \pi\pi$ decays in this framework. This deficiency of ChPT precludes also an accurate estimate of strong isospin breaking corrections to $K \to \pi\pi$ within this approach. A better estimate should be available in 2020 from DQCD and in a few years from LQCD.

LQCD provides already now the best results for hadronic matrix elements relevant for $K^0-\bar{K}^0$, $B^0_{s,d}-\bar{B}^0_{s,d}$, and $D^0-\bar{D}^0$ mixings with some competition only from DQCD but only for $K^0-\bar{K}^0$ mixing. It is satisfying that in the latter case, the results from LQCD and DQCD agree well with each other. Moreover, DQCD provides some insight into the dynamics behind the lattice values that are obtained by very demanding numerical simulations.

In fact DQCD turned over three decades by now to be an efficient approximate method for obtaining results for nonleptonic decays, years and even decades before useful results

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from numerically sophisticated lattice calculations could be obtained. But recently LQCD made impressive progress not only for $K^0 - \bar{K}^0$ mixing but also for $K \to \pi\pi$ decays, and a better comparison of the LQCD and DQCD results is now possible. This is important because also in the case of the latter decays DQCD provides some insight into the LQCD results.

The situation with nonleptonic decays of heavy mesons, in particular very important two-body B decays, is very different. Here DQCD has nothing to say, and LQCD can only help other approaches based on the so-called QCD-factorization (QCDF) approach, heavy quark expansions, and heavy quark effective theory by calculating simpler objects than hadronic matrix elements of four-quark operators present in these methods.

This chapter describes in some detail the DQCD approach (Section 7.2), which is easy for me as I took part in developing the applications of this approach to $K^0 - \bar{K}^0$ mixing and $K \to \pi\pi$ decays. We will also summarize the results of LQCD (Section 7.3) and present the general structure of the QCDF approach to two-body B decays (Section 7.4). Some general statements on heavy quark expansions and heavy quark effective theory will be given in Section 7.5. Finally, we will provide a few references to reviews and lectures in which other nonperturbative methods are presented.

In this chapter we will summarize only the results relevant for the SM. The ones obtained in the extensions of the SM, in which new operators are present, will be discussed in Chapter 13.

7.2 Dual QCD Approach

7.2.1 Preliminaries

This approach is based on the conjecture of 't Hooft [217, 218] and subsequently Witten [219, 220] that QCD (the theory of quarks and gluons) is for a large number of colors N equivalent to a theory of weakly interacting mesons with a quartic meson coupling being $O(1/N)$. This allows us to formulate a dual representation of the strong dynamics in terms of hadronic degrees of freedom. In the large N limit, this representation becomes exact, and a full description of the physics can be achieved using an infinite set of interacting meson fields. For large N QCD simplifies significantly, in particular in the strict large N limit QCD is dual to a free theory of mesons. In this section we will present the application of this dual QCD approach (DQCD) to the calculation of hadronic matrix elements of local operators as proposed in 1986 by Bardeen, Gérard, and myself [221] and developed by us in a series of papers since then. But let us first review the first attempts of using large N ideas in weak decays.

The first attempts to apply $1/N$ expansion to weak decays can be found in [222–224]. However, the first big step forward in the phenomenological applications of this expansion was made in [225] in the context of nonleptonic charm-meson decays, where it was realized that removing the $1/N$ Fierz terms from the so-called vacuum insertion approximation (VIA) softened the disagreement of the theory with both exclusive and inclusive data.

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This procedure has been motivated by the analysis in [226]. However, these authors did not attach it with a consistent application of the $1/N$ expansion.

VIA is just a method for evaluating hadronic matrix elements of four-quark operators by just inserting a vacuum or a single meson between the two currents in these operators, and then the product of the resulting matrix elements can easily be calculated. Moreover by making Fierz reordering of quarks, additional $1/N$ corrections to the matrix elements can be obtained.

In the strict large N limit the matrix elements of four-quark operators indeed factorize into a product of matrix elements of quark currents, and as done in [225] one could calculate all hadronic matrix elements for nonleptonic matrix elements in this manner. But it was soon realized that combining these results with the WCs of the contributing operators was problematic as the final result was renormalization-scale and renormalization-scheme dependent. As we have seen in previous chapters these dependences are present in the WCs and must be cancelled by those present in the hadronic matrix elements. But the matrix elements of conserved currents are free of such dependences, and consequently this is also the case of matrix elements calculated in this manner. Moreover, the $1/N$ corrections resulting in VIA from Fierz-reordering do not represent correct $1/N$ QCD corrections to hadronic matrix elements as we will stress later.

In what follows we will discuss the DQCD approach of [221, 227–229], which goes beyond the large N limit and thereby allows to remove the unphysical scale and renormalization-scheme dependences in question by including *nonfactorizable contributions*. We will discuss this approach in the context of K meson flavor physics, and readers not familiar with this subject should refresh the information gained in Sections 6.3 and 6.4 and look first briefly at Section 8.1 and Chapter 10 to get an idea what the following objects mean and why they are important:

$$\boxed{\hat{B}_K, \quad B_6^{(1/2)}, \quad B_8^{(3/2)}, \quad \Delta M_K, \quad \varepsilon_K, \quad \varepsilon'/\varepsilon, \quad \mathrm{Re}A_0, \quad \mathrm{Re}A_2.} \tag{7.1}$$

In particular the isospin amplitudes $A_{0,2}$ are defined in (8.29)–(8.31).

But dropping just $1/N$ terms in the VIA when calculating $\hat{B}_K$, $B_6^{(1/2)}$, and $B_8^{(3/2)}$ still gives correct results in the large N limit, and this simple philosophy of using $1/N$ expansion has been applied by Gérard and myself to $K \to \pi\pi$ decays, ΔM_K, and ε_K in [230]. The first leading order results for the matrix elements of operators relevant for these observables can be found in that paper. The most important results obtained there are $\hat{B}_K = 3/4$, also noticed in [231], and the realization that the removal of $1/N$ Fierz terms from VIA to current-current matrix elements suppresses $\mathrm{Re}A_2$, moving the theory in the direction of the data. In this paper also the first large N result for the matrix elements of the dominant QCD-penguin operator Q_6 can be found. In this case the problematic issue of scale dependence is practically absent because the Q_6 operator has a different structure than current-current operators considered in [225]. Q_6 is a density-density operator, and its factorized matrix element is scale dependent even in the large N limit. In fact, the product $C_6(\mu)\langle Q_6(\mu)\rangle$ is for $\mu \geq 1\,\mathrm{GeV}$ basically scale independent, small μ dependence coming from the mixing with other operators. We will demonstrate this later using the results from previous chapters.

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However, just dropping Fierz terms in the VIA results is really not a satisfactory procedure for the calculation of hadronic matrix elements and in particular does not provide the explanation of the $\Delta I = 1/2$ rule as given in (7.28). Moreover, as we will see later, this procedure fails in the case of operators built out of scalar currents, giving totally wrong results. What was still missing in these first papers was an effective Lagrangian describing the weak and strong interactions of mesons in the large N limit and allowing for a systematic evaluation of $1/N$ and higher-order corrections to this limit.

With these goals in mind Bardeen, Gérard, and myself developed in the second half of the 1980s an approach to $K^0 - \bar{K}^0$ mixing and nonleptonic K-meson decays [221, 227–229] based on an effective Lagrangian corresponding to the dual representation of QCD as a theory of weakly interacting mesons for large N [217–220] mentioned earlier. In the first application of this approach in [227, 228] we confirmed the leading order result for $\langle Q_6 \rangle$ [230]. Moreover, we have pointed out that the property of factorization of hadronic matrix elements in the large N limit is automatic in DQCD because in this limit QCD becomes a free theory of mesons. But the main virtue of this approach is the ability to combine consistently, even if approximately, hadronic matrix elements of four-quark operators with the renormalization group improved values of the corresponding WCs.

Our studies of the 1980s culminated in the formulation of the *meson evolution* [221, 229], to be discussed later, and the evaluation in this framework of $1/N$ corrections to $K \to \pi\pi$ amplitudes and to the parameter $\hat{B}_K$ relevant for $K^0 - \bar{K}^0$ mixing. These papers represent the first attempt at a consistent calculation of the weak matrix elements in the continuum field theory. Pedagogical summary of this work has been presented by us in various reviews and lectures [232–238] and more recently in [239, 240].

This approach provided, in particular, first results within QCD for the $K \to \pi\pi$ isospin amplitudes $\mathrm{Re}A_0$ and $\mathrm{Re}A_2$ in the ballpark of experimental values. In this manner, for the first time, the SM dynamics behind the $\Delta I = 1/2$ rule has been identified. In particular, it has been emphasized that at scales $O(1\,\mathrm{GeV})$ long-distance dynamics in hadronic matrix elements of current-current operators and not QCD-penguin operators, as originally proposed in [241], are dominantly responsible for this rule. Moreover, it has been demonstrated analytically why $\mathrm{Re}A_0$ is enhanced and why $\mathrm{Re}A_2$ is suppressed relative to the VIA estimates. In this context, we have emphasized that the Fierz terms in the latter approach totally misrepresent $1/N$ corrections to the strict large N limit for these amplitudes. In particular their sign is wrong.

Our approach, among other applications, allowed us to consistently calculate, for the first time within QCD, the nonperturbative parameters $\hat{B}_K$, $B_6^{(1/2)}$, and $B_8^{(3/2)}$ governing the corresponding matrix elements of the $\Delta S = 2$ SM current-current operator and $K \to \pi\pi$ matrix elements of the dominant QCD-penguin (Q_6) and electroweak penguin (Q_8) operators that we encountered in Section 6.4. As we will see in the next chapters these parameters are crucial for the evaluation of ε_K and ε'/ε within the SM and its various extensions. Also the $K \to \pi\pi\pi$ decays have been analyzed in [242] and the $K_L - K_S$ mass difference ΔM_K including long-distance contributions has been calculated [236, 243] within this approach. During the last two decades some of these calculations have been improved and extended, in particular in [239, 244, 245], which we will briefly review soon.

It is interesting and encouraging that most of our results have been confirmed by several recent LQCD calculations that we will specify in the next section. While the LQCD

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approach has a better control over the errors than our approach, it does not provide the physical picture of the dynamics behind the obtained numerical results. This is in particular seen in the case of the $\Delta I = 1/2$ rule, where our analytic approach offers a very simple picture of the dynamics behind this rule, as first pointed out in [221], more recently summarized in [239, 240], and described in this section. Moreover, to obtain reliable results for $K \to \pi\pi$ amplitudes in LQCD required already a quarter of a century, whereas all calculations performed in the DQCD taken together required less than three years and involved teams of two to three physicists compared to large LQCD collaborations. Yet, it should be stressed that LQCD will provide ultimately more precise results than is possible in our approach and has a broader spectrum of quantities for which the calculations can be done than using DQCD.

While until 2010 DQCD was ahead of lattice calculations, this changed in this decade for several quantities in view of the increased computer power and new ideas. Yet, as we will demonstrate in this section and in Chapter 13, DQCD still allows us to obtain an insight in the numerical LQCD results that cannot be gained by LQCD. Moreover, all of the hadronic matrix elements of BSM operators of dimension-six entering $K \to \pi\pi$ decays have been calculated recently in DQCD [246], while their LQCD values remain still unknown.

Other applications of large N ideas to $K \to \pi\pi$ and $\hat{B}_K$, but in a different spirit than our original approach, are reviewed in [50, 247]. We refer in particular to [248–256]. A review of SU(N) gauge theories at large N can be found in [257]. Here we present only our approach following closely our published work.

7.2.2 Basic Framework

General Structure

To describe our framework in explicit terms, let us return to $K \to \pi\pi$ decay amplitudes in the SM [123]

$$A(K \to \pi\pi) = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sum_{i=1}^{10} (z_i(\mu) + \tau y_i(\mu)) \langle \pi\pi | Q_i(\mu) | K \rangle, \qquad \tau \equiv -\frac{V_{td} V_{ts}^*}{V_{ud} V_{us}^*}, \tag{7.2}$$

with the operators Q_i discussed already at length in Section 6.4 and numerical values of the Wilson coefficients $z_i(\mu)$ and $y_i(\mu)$ given in Chapter 10. In our presentation only four of them will be relevant, namely the two current-current operators Q_1, Q_2, the QCD penguin operator Q_6, and the electroweak penguin operator Q_8. The operator Q_4 plays also some role as can be found in our papers, but we will not discuss it here.

In the DQCD it is useful to write these operators in the Fierz transformed form relative to the ones considered in previous chapters so that they are given as products of color singlet currents and densities, as follows

$$Q_1 = 4(\bar{s}_L \gamma_\mu d_L)(\bar{u}_L \gamma_\mu u_L), \qquad Q_2 = 4(\bar{s}_L \gamma_\mu u_L)(\bar{u}_L \gamma_\mu d_L), \tag{7.3}$$

$$Q_6 = -8(\bar{s}_L q_R)(\bar{q}_R d_L), \qquad Q_8 = -12 e_q (\bar{s}_L q_R)(\bar{q}_R d_L), \tag{7.4}$$

where e_q is the electric quark charge.

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We have already used this representation for Q_6 when calculating anomalous dimensions of QCD penguin operators in Section 6.4. Here $q_{R(L)} = (1/2)(1 \pm \gamma_5)q$ and sums over color indices and $q = u, d, s$ in Q_6 and Q_8 are understood. In this manner the leading, in $1/N$ expansion, contributions can be calculated as products of matrix elements of currents in the case of $Q_{1,2}$ and densities in the case of Q_6 and Q_8.

Because the operators Q_i in (7.3)–(7.4) are constructed from the light quark fields only, the full information about the heavy quark fields (c, b, t) is contained in the Wilson coefficients z_i and y_i. Correspondingly, the normalization scale μ in (7.2) is not completely arbitrary in our approach but must be chosen below the charm-quark mass. In our large N approach the structure of different contributions to physical amplitudes is then as follows. The physics contributions from scales above μ are fully contained in the coefficients $z_i(\mu)$ and $y_i(\mu)$, whereas the remaining contributions from the low-energy physics below μ (i.e., from μ to the factorization scale expected around m_π) are contained in the matrix elements $\langle \pi\pi|Q_i(\mu)|K\rangle$. It follows that for $\mu = O(1\,\text{GeV})$, the coefficients $z_i(\mu)$ and $y_i(\mu)$ can be calculated within a perturbative *quark-gluon picture* by means of renormalization group methods considered in previous chapters.

As far as the meson matrix elements are concerned, the ultimate goal is to compute them in a nonperturbative quark-gluon picture where mesons occur as bound states. This route is followed by lattice computations and in fact, as summarized in the next section, since our work appeared in 1986 impressive progress has been made in this manner. Yet this numerical route is very demanding as even after more than a quarter of a century of hard work by the LQCD community the present results for $K \to \pi\pi$ amplitudes are still not fully satisfactory. Moreover, it is much harder to understand the underlying physics than by means of our analytic approach as we already stressed earlier and will demonstrate in particular in Chapter 13.

Our proposal, summarized most explicitly in [221], was to apply instead the DQCD approach of 't Hooft [217, 218] and Witten [219, 220] to nonleptonic K decays and $K^0 - \bar{K}^0$ mixing. The fact that QCD can be formulated both as the theory of quarks and gluons on the one hand and as the theory of mesons on the other hand can now be used for these processes as follows. The main point is that the matrix elements of four-fermion operators governing these transitions can be written at leading order in large N as products of matrix elements of color singlet currents in the case of current-current operators and as products of matrix elements of quark densities in the case of penguin operators. This is evident from (7.3) and (7.4).

At the next-to-leading order one has two classes of contributions:

- $1/N$ corrections to the matrix elements of factorized operators
- Low-energy, nonfactorized matrix elements of two currents or two quark densities

The latter contributions can be written as an integral over the momentum flowing through the currents (densities) in the connected *planar amplitude*. One can then use our knowledge of both the high- and low-energy behavior of the integrand. At high momentum, these are just the short-distance contributions to the coefficient functions of the OPE, which can be computed perturbatively in the quark-gluon picture. While in principle this could also be done in the *meson picture*, such an analysis would be

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very complex requiring many meson states and complicated interactions. However the long-distance analysis is correspondingly simple as only lowest-lying meson states may be required, and the interactions are largely dictated by the chiral symmetry structure of the effective Lagrangian. Moreover the spontaneous breakdown of chiral symmetry $SU(3)_L \times SU(3)_R \to SU(3)_V$ can be proven to be true in QCD in its large N limit [258].

Our proposal in [221] was to use the meson theory to interpolate to the point where one can match the behavior of the integrand of the short-distance theory. If the amplitude is smooth enough then it may be sufficient to match the meson amplitude to the quark amplitude at an appropriate scale. In this manner one can achieve a consistent unified description of the physics by using the quark-gluon picture at short distances matched to the meson picture at long distances. The accuracy of the method depends on the interpolation of the integrand between short and long distances.

A full AdS/QCD description [259, 260] should be able to interpolate the meson amplitudes to arbitrarily short distance, and first attempts in this direction have been made in [261, 262]. In our approach the matching scale must presently be chosen around 1 GeV, implying approximate treatments in both pictures. In particular, the scheme dependence of the long-distance part comes when one subtracts the short-distance part of the integral using a particular scheme. This scheme dependence can be treated exactly if needed. In this context, calculating Wilson coefficients and the hadronic matrix elements in a momentum scheme in [239] we have made a significant progress relative to our previous papers.

Despite not being exact, this approach has several virtues. Indeed, the simplicity of this formulation lies in the fact that in the strict large N limit QCD becomes a free theory of mesons, and consequently the leading order contributions to any quantity are obtained by calculating tree diagrams with the propagated objects being mesons, not quarks or gluons. In this strict limit, also the factorization of hadronic matrix elements of four-quark operators into the product of matrix elements of quark currents or quark densities follows. Beyond this limit, one obtains $1/N$ expansion represented by a loop expansion in the meson theory. It should be emphasized that in this loop expansion $1/N$ never explicitly appears and is replaced by $1/F_\pi^2$ as discussed later.

Even if naively these loop corrections could be expected to be small, one should notice that one-loop contributions in the meson theory represent, in fact, the leading term in the $1/N$ expansion for observables like the $\pi^+ - \pi^-$ electromagnetic mass difference or the K^0 decay into two neutral pions. In particular, they have to be sizable if one wants to explain why the subleading $K^0 \to \pi^0\pi^0$ decay amplitude turns out to be almost equal to the $K^0 \to \pi^+\pi^-$ leading one, namely the so-called $\Delta I = 1/2$ rule discussed in detail later.

An important issue in this framework is the matching between the quark-gluon and meson theories. In the quark-gluon picture, the scale μ enters naturally as the normalization scale in the renormalization group improved perturbative QCD calculations

$$\mu^2 \frac{d}{d\mu^2} Q_i(\mu^2) = -\frac{1}{2}\hat{\gamma}_{ij} Q_j(\mu^2), \tag{7.5}$$

with $\hat{\gamma}$, the anomalous dimension matrix for the Q_i operators. In our formulation, μ serves as an infrared cutoff below which one should switch to the meson picture unless one wants to perform lattice computations. Now the truncated meson theory, involving a finite set

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of light pseudoscalar and vector mesons only, appears nonrenormalizable. In particular, if only lowest-lying pseudoscalar mesons are included without ultraviolet QCD completion, it exhibits a quadratic dependence on the cutoff which we will denote by M. This *physical* cutoff must be introduced to restrict the truncated meson theory to the long-distance domain or, in other words, to cutoff the high-mass and high-momentum contributions in the meson loops. Therefore, the physical cutoff introduced here should be distinguished from the usual cutoff regularization procedure in which M could be sent to arbitrarily large values, to disappear from observables after renormalization.

On the other hand, we know that QCD being renormalizable has a logarithmic dependence on the ultraviolet cutoff. While this difference from the quadratic dependence on M in the truncated meson theory has been in the 1980s a subject of criticism of our approach, one should emphasize that these two dependences are not inconsistent with each other. Indeed, the strict logarithmic cutoff dependence of QCD is valid only at short distances, whereas power counting supplemented with chiral symmetry requires quadratic dependence on the cutoff for the long-distance behavior of QCD. For high values of M, after the inclusion of vector mesons and heavier meson states, this quadratic dependence on M should smoothly turn into a logarithmic dependence as expected in the full meson theory. In fact, as demonstrated in [239], already the inclusion of vector mesons shows that this expectation is correct.

In the evaluation of the matrix elements $\langle \pi\pi|Q_i(\mu)|K\rangle$ the simplest choice one can make is $\mu = M$. This identification of μ with M is certainly an idealization in the approximate treatment used in our papers in 1980s, but has been improved in [239]. In particular, to relate μ to M in a meaningful manner, we went beyond the Fermi limit for the W-propagator and calculated at NLO the Wilson coefficients not in the usual NDR-$\overline{\text{MS}}$ scheme but in a momentum scheme. But as the Wilson coefficients in momentum scheme and NDR-$\overline{\text{MS}}$ scheme are related by simple one-loop shifts, one can finally use the latter scheme for phenomenology. Moreover, the inclusion of the lowest-lying vector mesons improved the matching between meson and quark pictures significantly. We refer also to a very nice earlier analysis in [263] in which this improved matching has been discussed first.

Another Look

Let us still formulate these ideas using the language of previous chapters. Our goal is the calculation of a decay amplitude, which one can write generally as follows

$$\mathcal{A} = \langle \vec{Q}^T(\mu)\rangle \vec{C}(\mu) \tag{7.6}$$

with $\mu < m_c$ in our approach but otherwise arbitrary. What is easy to do is

- The calculation of $\vec{C}(M_W)$, which can be done in ordinary perturbation theory.
- The calculation of $\langle \vec{Q}^T(0)\rangle$, corresponding to strict large N limit in which these matrix elements are calculated as products of quark currents or quark densities. Here to simplify the notation, we just replace the factorization scale by $\mu = 0$.

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We must next fill the gap between $\mu = 0$ and $\mu = M_W$ to include all physics contributions. Our idea is to divide the renormalization group evolution into a *short-distance evolution* from $\mathcal{M}_W$ to $\mu = O(1\,\text{GeV})$ to be called *quark-gluon evolution* described by

$$\vec{C}(\mu) = \hat{U}(\mu, M_W)\, \vec{C}(M_\text{W}), \tag{7.7}$$

and a *long-distance evolution* from scale 0 *up to* μ, to be called *meson evolution* as it is performed in the dual representation of QCD. The latter evolution is described by

$$\langle \vec{Q}^T(\mu)\rangle = \langle \vec{Q}^T(0)\rangle\, \hat{U}(0, \mu), \tag{7.8}$$

with $\hat{U}(0, \mu)$ being the evolution matrix in the long-distance regime. As

$$\hat{U}(0, \mu)\hat{U}(\mu, M_W) = \hat{U}(0, M_W), \tag{7.9}$$

we find inserting (7.7) and (7.8) into (7.6)

$$\mathcal{A} = \langle \vec{Q}^T(0)\rangle \hat{U}(0, M_W)\vec{C}(M_W), \tag{7.10}$$

so that all contributions from scales between $\mu = 0$ and $\mu = M_W$ are included in this manner.

It should be remarked that the relation (7.8) can be generalized to

$$\langle \vec{Q}^T(m_2)\rangle = \langle \vec{Q}^T(m_1)\rangle\, \hat{U}(m_1, m_2), \tag{7.11}$$

with $m_1 < m_2$ so that knowing the hadronic matrix elements at a given scale one can calculate them at a different scale. This is in particular useful if both scales are in perturbative regime and the standard renormalization group technology can be used.

Having these formulas at hand lets us make the following observation [123]. The evolution of $\vec{C}(\mu)$ in (7.7) is governed by $\hat{\gamma}^T$ with $\hat{\gamma}$ being the anomalous dimension matrix. On the other hand, the evolution of $\langle \vec{Q}(\mu)\rangle$ is governed by $\hat{\gamma}$. As this matrix is rather asymmetric, the structure of the evolution in (7.11) is rather different from the evolution in (7.7). Thus, whereas the evolution of $C_{1,2}(\mu)$ is unaffected by the presence of penguin contributions, the evolution of $\langle Q_{1,2}(\mu)\rangle$ depends on the size of $\langle Q_i(\mu)\rangle$, $i \neq 1, 2$. Conversely, whereas the evolution of $C_i(\mu)$, $i \neq 1, 2$ depends on the size of $C_{1,2}(\mu)$, the evolution of the matrix elements of penguin operators $\langle Q_i(\mu)\rangle$, $i \neq 1, 2$ is a sole penguin affair.

Basic Lagrangian of the Truncated Meson Theory

The explicit calculation of the contributions of pseudoscalars to hadronic matrix elements of local operators is based on a truncated chiral Lagrangian describing the low-energy interactions of the lightest mesons [227, 228, 264]

$$\boxed{L_\text{tr} = \frac{F^2}{8}\left[\text{Tr}(D_\mu U D^\mu U^\dagger) + r\,\text{Tr}(m(U + U^\dagger)) - \frac{r}{\Lambda_\chi^2}\text{Tr}(m(D^2 U + D^2 U^\dagger))\right],} \tag{7.12}$$

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where

$$U = \exp\left(i\sqrt{2}\frac{\Pi}{F}\right), \qquad \Pi = \sum_{a=1}^{8} \lambda^a \pi^a \tag{7.13}$$

is the unitary chiral matrix describing the octet of light pseudoscalars and transforming as $U \to g_L U g_R^\dagger$ under the chiral $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$. The parameter F is related to the weak decay constants $F_\pi \approx 130\,\mathrm{MeV}$ and $F_K \approx 156\,\mathrm{MeV}$ through

$$F_\pi = F\left(1 + \frac{m_\pi^2}{\Lambda_\chi^2}\right), \qquad F_K = F\left(1 + \frac{m_K^2}{\Lambda_\chi^2}\right). \tag{7.14}$$

It should be emphasized that to adjust the normalization of F_π to the one used in the rest of this book, we replaced the parameter f_π in our original DQCD papers by $F = \sqrt{2} f_\pi$.

The diagonal mass matrix m involving m_u, m_d, and m_s is such that

$$r(\mu) = \frac{2m_K^2}{m_s(\mu) + m_d(\mu)}, \tag{7.15}$$

with $r(1\,\mathrm{GeV}) \approx 3.75\,\mathrm{GeV}$ for $(m_s + m_d)(1\,\mathrm{GeV}) \approx 132\,\mathrm{MeV}$. The flavor-singlet η_0 meson decouples due to large mass m_0 generated by the nonperturbative $\mathrm{U}(1)_A$ anomaly. Consequently the matrix Π reads

$$\Pi = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta_8 & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta_8 & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta_8 \end{pmatrix}. \tag{7.16}$$

In (7.12), $D_\mu U$ is the usual weak covariant derivative acting on the U field, and m is the real and diagonal quark mass matrix. At $O(p^2)$ and in the isospin limit $m_u = m_d = m_{ud}$,

$$m_\pi^2 = r\, m_{ud}, \quad m_K^2 = \frac{r}{2}(m_s + m_{ud}), \quad m_8^2 = \frac{4}{3}m_K^2 - \frac{1}{3}m_\pi^2. \tag{7.17}$$

It should be emphasized that the chiral Lagrangian in (7.12) must not be viewed as a normal effective tree Lagrangian but instead must be used as a fully interacting field theory including loop effects. In this sense we are providing a bosonization of the fundamental quark theory, where all the quark currents and densities, presented later, have a valid representation in terms of the meson fields. But in the truncated version, the meson representation is valid only for a proper description of long-distance physics.

The parameter Λ_χ in (7.12) sets the scale of higher-order terms, which are always expected in a truncated theory. It should be emphasized that this scale is a hadronic scale different from Λ_{QCD}. As shown in [221, 264] and evident from (7.14), its value can be determined from the physical pseudoscalar masses and decay constants:

$$\Lambda_\chi^2 = F_\pi \frac{m_K^2 - m_\pi^2}{F_K - F_\pi} + O\left(\frac{1}{N}\right) \Rightarrow \Lambda_\chi \approx 1.1\,\mathrm{GeV}, \tag{7.18}$$

where we used the most recent lattice value for the ratio $F_K/F_\pi \approx 1.20$. The $1/N$ correction, calculated in [221], is positive and in the ballpark of 5–10% for the range of

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M considered. As this correction is only logarithmically dependent on this scale, Λ_χ is practically independent of M with variation in the range $0.6\,\text{GeV} \leq M \leq 0.8\,\text{GeV}$ of less than 2%.

As stressed in [221], this cutoff independence of Λ_χ results only if the cutoff dependence of $F(M^2)$ following from our Lagrangian is taken into account. Explicit formula can be found in [221]. As $1/F^2(M^2)$, being $O(1/N)$, is the actual expansion parameter and

$$\frac{\partial}{\partial M^2}\left(\frac{1}{F^2(M^2)}\right) < 0, \tag{7.19}$$

it is the meson picture analog of the QCD running coupling in the quark picture.

The chiral Lagrangian (7.12) contains only terms with a single trace over flavor indices, which reflects the large N structure of QCD. The leading N contributions to any quantity are simply obtained from the tree diagrams, whereas the leading $1/N$ corrections are found by calculating the one-loop contributions. More generally, the $1/N$ expansion corresponds to the loop expansion characterized by inverse powers of $(4\pi F)^2 (F^2 \sim N)$ with the strong interaction vertices given by the truncated Lagrangian in (7.12). As pointed out by John Donoghue, it is similar to an expansion in inverse powers of M_p^2 $(G_N = 1/M_p^2)$ if one treats general relativity as an effective field theory for gravity, which is modified above the Planck scale by new degrees of freedom. Other details on the Lagrangian in (7.12) and explicit results of the relevant loop calculations can be found in [221] and in the lecture notes [233, 236].

The Structure of Hadronic Matrix Elements

The resulting matrix elements of *current-current* operators in this approach have then the structure $(i = 1, 2)$

$$\langle \pi\pi | Q_i(\mu) | K \rangle = A_i \sqrt{N} \left[1 + \frac{K_i(\mu)}{N} + O\left(\frac{1}{N^2}\right) \right], \tag{7.20}$$

where A_i and K_i are N-independent numerical expansion coefficients, which, in our approach, are given in terms of the parameters of the truncated Lagrangian. Note that the μ dependence in the matrix elements of $Q_{1,2}$ appears as a $1/N$ correction. This is consistent with the μ dependence of the Wilson coefficients $z_{1,2}(\mu)$ and reflects the simple fact that the anomalous dimensions of $Q_{1,2}$ vanish in the large N limit. See (7.40).

On the other hand, for *penguin* operators Q_6 and Q_8 the matrix elements have the structure $(i = 6, 8)$

$$\langle \pi\pi | Q_i(\mu) | K \rangle = \tilde{A}_i(\mu) \sqrt{N} \left[1 + \frac{\tilde{K}_i(\mu)}{N} + O\left(\frac{1}{N^2}\right) \right]. \tag{7.21}$$

The important difference relative to (7.20) is the appearance of the μ dependence already in the leading term. Again, this is consistent with the μ dependence of $z_{6,8}(\mu)$ and $y_{6,8}(\mu)$ and reflects the fact that the anomalous dimensions of density-density operators do not vanish in the large N limit but are twice the anomalous dimension of the mass operator.[1]

[1] See Section 4.3.

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This fact allows a better matching of the truncated meson theory with the short-distance contributions of penguin operators than is possible for the current-current operators in the case of $K \to \pi\pi$ amplitudes. This will be clearer from explicit formulas for the matrix elements of penguin operators given later, and we will return to this point at the end of this section.

To calculate the matrix elements of the local operators in question, we need meson representation of quark currents and quark densities. They are directly obtained from the effective Lagrangian in (7.12) and are given, respectively, as follows

$$\boxed{\bar{q}_L^j \gamma_\mu q_L^i = i\frac{F^2}{8}\left\{(\partial_\mu U)U^\dagger - U(\partial_\mu U^\dagger) - \frac{r}{\Lambda_\chi^2}\left[m(\partial_\mu U^\dagger) - (\partial_\mu U)m^\dagger\right]\right\}_{ij}}, \tag{7.22}$$

$$\boxed{\bar{q}_R^j q_L^i = -\frac{F^2}{8} r\left[U - \frac{1}{\Lambda_\chi^2}\partial^2 U\right]_{ij}}. \tag{7.23}$$

At tree level, corresponding to leading order in $1/N$, using these representations one simply expresses the operators in terms of the meson fields and expands the matrix U in powers of $1/F$. For $K^0 - \bar{K}^0$ mixing the relevant contribution to hadronic matrix elements is read off from terms involving only K^0 and $\bar{K}^0$ and for $K \to \pi\pi$ decays from terms involving K^0 or $K^\pm$ and two pions. When expressing derivatives in terms of four-momenta one has to take care that all fields in the amplitude resulting from the Lagrangian are incoming. Therefore for $K^0 - \bar{K}^0$ mixing with $K^0 \to \bar{K}^0$ and general relation $i\partial_\mu = p_\mu$ we have

$$\partial_\mu K^0 = -ip_{K^0} K^0, \qquad \partial_\mu \bar{K}^0 = ip_{\bar{K}^0}\bar{K}^0, \tag{7.24}$$

and for $K \to \pi\pi$

$$\partial_\mu K^0 = -ip_{K^0} K^0, \qquad \partial_\mu \pi^\pm = ip_{\pi^\pm}\pi^\pm, \tag{7.25}$$

with analogous rules for K^+ and π^0.

Next-to-leading corrections are then obtained from loop diagrams with weak vertices represented by the operators and strong vertices obtained from the effective Lagrangian in (7.12). Examples of the loop calculations with the list of the relevant integrals can be found in [239].

Here we give just one example of the parameter B_K, which is discussed in more detail in Section 7.2.4. Including both pseudoscalar and vector meson contributions, one finds [239]

$$B_K(M) = \frac{3}{4}\left\{1 - \frac{1}{(4\pi F_K)^2}\left[\frac{7}{8}M^2 + \frac{3}{8}m_V^2 \ln\left(1 + \frac{M^2}{m_V^2}\right) + \frac{3}{4}\frac{m_V^2 M^2}{(M^2 + m_V^2)}\right]\right\}, \tag{7.26}$$

where m_V is the vector meson mass and M the cutoff in the loop integral. If only pseudoscalar meson contributions are included $M \approx 0.7\,\text{GeV}$. With vector mesons taken into account, it can be increased to $M \approx 0.9\,\text{GeV}$. The important virtue of the inclusion of lightest vector mesons is the reduction of the coefficient in front of the quadratic

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term M^2. In the absence of vector meson contributions the factor 7/8 is increased to 2. This demonstrates that the inclusion of heavier vector mesons would eventually result in a purely logarithmic cutoff dependence for $M > 1\,\text{GeV}$ as expected from QCD at short-distance scales.

Comparison with Chiral Perturbation Theory

It is important to stress two major differences of this approach from the usual chiral perturbative calculations [50, 91, 265].

- First, the large N structure of the basic truncated low-energy Lagrangian provides a simplification over those effective Lagrangians used by chiral perturbation practitioners. In particular, within our ultraviolet quark-gluon completion, no $O(p^4)$ counter-terms are needed to absorb divergences generated by a dimensional regularization.
- More important, our loop calculations employ a cutoff regularization, and consequently our results exhibit a *quadratic* dependence on the *physical* cutoff M. This quadratic dependence is lost in the usual chiral perturbative calculations, which are based on the dimensional regularization. In effect, dimensional regularization makes extra infrared subtractions of quadratically divergent terms. These subtractions are not permitted in the full integration of the loop contributions in the truncated theory. As this quadratic dependence on the physical cutoff is usually a subject of criticism, we want to emphasize that it is an essential ingredient in the matching of the meson and quark-gluon pictures. Once again, it is required by power counting and chiral symmetry. Moreover, it stabilizes the $1/N$ expansion as exemplified through the cutoff independence of the hadronic scale Λ_χ. As discussed soon, it is also at the source of the $\Delta I = 1/2$ rule in our dual approach for QCD.

It is evident from these comments and from the review in [50] that in contrast to our $1/N$ approach, the chiral perturbation theory framework, while being very powerful in the determination of low-energy constants from experiment, cannot by itself address the issue of the dynamics behind the $\Delta I = 1/2$ rule and the evaluation of $\hat{B}_K$, $B_6^{(1/2)}$ and $B_8^{(3/2)}$.

With this brief formulation of DQCD at hand, we are ready to illustrate this approach on the example of the $\Delta I = 1/2$ rule and summarize the most important results obtained by us in [221, 227–230] and recently within the SM in [239, 244, 245]. The DQCD results for hadronic matrix elements of new operators present in the extensions of the SM will be discussed in Chapter 13.

7.2.3 The Dynamics behind the $\Delta I = 1/2$ Rule

Preliminaries

One of the puzzles of the 1950s was a large disparity between the measured values of the real parts of the isospin amplitudes A_0 and A_2 in $K \to \pi\pi$ decays, which on the basis of usual isospin considerations were expected to be of the same order. In 2020 we know the experimental values of the real parts of these amplitudes very precisely [35]

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$$\mathrm{Re}A_0 = 27.04(1) \times 10^{-8}\ \mathrm{GeV}, \quad \mathrm{Re}A_2 = 1.210(2) \times 10^{-8}\ \mathrm{GeV}. \tag{7.27}$$

As $\mathrm{Re}A_2$ is dominated by $\Delta I = 3/2$ transitions but $\mathrm{Re}A_0$ receives contributions also from $\Delta I = 1/2$ transitions, the latter transitions dominate $\mathrm{Re}A_0$, which expresses the so-called $\Delta I = 1/2$ rule [266, 267]

$$\mathrm{R} = \frac{\mathrm{Re}A_0}{\mathrm{Re}A_2} = 22.35. \tag{7.28}$$

In the 1950s QCD and Operator Product Expansion did not exist, and clearly one did not know that $W^\pm$ bosons exist, in nature, but using the ideas of Fermi [268], Feynman and Gell-Mann [269], and Marshak and Sudarshan [270], one could still evaluate the amplitudes $\mathrm{Re}A_0$ and $\mathrm{Re}A_2$ to find out that such a high value of R is a real puzzle.

In modern times we can reconstruct this puzzle by evaluating the simple $W^\pm$ boson exchange between the relevant quark's which after integrating out $W^\pm$ generates the current-current operator Q_2:

$$Q_2 = (\bar{s}u)_{V-A}\,(\bar{u}d)_{V-A}\,. \tag{7.29}$$

With only Q_2 contributing we have

$$\mathrm{Re}A_{0,2} = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \langle Q_2 \rangle_{0,2}. \tag{7.30}$$

Calculating the matrix elements $\langle Q_2 \rangle_{0,2}$ in the strict large N limit, which corresponds to factorization of matrix elements of Q_2 into the product of matrix elements of currents, we find

$$\langle Q_2 \rangle_0 = \sqrt{2}\langle Q_2 \rangle_2 = \frac{2}{3} F_\pi (m_K^2 - m_\pi^2), \tag{7.31}$$

and consequently

$$\mathrm{Re}A_0 = 3.59 \times 10^{-8}\,\mathrm{GeV}, \qquad \mathrm{Re}A_2 = 2.54 \times 10^{-8}\,\mathrm{GeV}\,, \qquad \mathrm{R} = \sqrt{2}, \tag{7.32}$$

in plain disagreement with the data in (7.27) and (7.28). It should be emphasized that the explanation of the missing enhancement factor of 15.8 in R through some dynamics must simultaneously give the correct values for $\mathrm{Re}A_0$ and $\mathrm{Re}A_2$. This means that this dynamic should suppress $\mathrm{Re}A_2$ by a factor of 2.1, not more, and enhance $\mathrm{Re}A_0$ by a factor of 7.5. This tells us that while the suppression of $\mathrm{Re}A_2$ is an important ingredient in the $\Delta I = 1/2$ rule, it is not the main origin of this rule. It is the enhancement of $\mathrm{Re}A_0$ as already emphasized in [241] even if, in contrast to this paper, as demonstrated next, the current-current operators are responsible dominantly for this rule and not QCD penguins as pointed out first in [221].

We will next describe how the QCD dynamics in the context of the DQCD allow to explain the dominant part of the $\Delta I = 1/2$ rule. We proceed in three steps.

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Step 1: Quark-Gluon Evolution

This step involves the calculation of the Wilson coefficients $z_{1,2}(\mu)$ of the current-current operators $Q_{1,2}$ at a low-energy scale $\mu = O(1\,\text{GeV})$ and was discussed in detail in previous chapters. This evolution starts already at scales $O(M_W)$ as opposed to the contribution of the QCD penguin operator Q_6 whose coefficient $z_6(\mu)$ is strongly suppressed by GIM mechanism for $\mu \geq m_c$ and will be included in Step 3. The contributions of the remaining operators, in particular electroweak penguin operators to $\text{Re}A_0$ and $\text{Re}A_2$ in the SM are negligible because of the very small Wilson coefficients $z_i(\mu)$ of these operators.

For the real parts of $K \to \pi\pi$ one finds then

$$\text{Re}A(K \to \pi\pi)_{cc} = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sum_{i=1,2} z_i(\mu) \langle \pi\pi | Q_i(0) | K \rangle, \tag{7.33}$$

where we indicate that only current-current operator contributions have been included. We have indicated that in this step the matrix elements are evaluated at $\mu \approx 0$ so that their values calculated in the strict large N limit can be used. As $z_i(\mu)$ are evaluated at $\mu = O(1\,\text{GeV})$, this is clearly inconsistent. But we do it here for pedagogical reasons, and we will improve on this in Step 2.

This first step corresponds in fact to the pioneering 1974 calculations in [119, 120] except that they were done at leading order in the renormalization group improved perturbation theory and now can be done at the NLO level. These 1974 papers have shown that the short-distance QCD effects enhance $\text{Re}A_0$ and suppress $\text{Re}A_2$. However, the inclusion of NLO QCD corrections to $z_{1,2}$ [71, 79] made it clear, as stressed in particular in [71], that the $K \to \pi\pi$ amplitudes without the proper calculation of hadronic matrix elements of Q_i are both scale and renormalization scheme dependent. For instance, setting $\mu = 0.8\,\text{GeV}$ we find for the ratio R in (7.28)[2]

$$R_{cc}(\text{NDR} - \overline{\text{MS}}) \approx 3.0, \qquad R_{cc}(\overline{\text{MOM}}) \approx 4.4, \tag{7.34}$$

where $\overline{\text{MOM}}$ is a momentum scheme, introduced in [239], which is particularly suited for the calculations of the amplitudes in the DQCD. In this scheme one finds then for $\mu = 0.8\,\text{GeV}$

$$\text{Re}A_0 = 7.1 \times 10^{-8}\,\text{GeV}, \qquad \text{Re}A_2 = 1.6 \times 10^{-8}\,\text{GeV}\,. \tag{7.35}$$

This is a significant improvement over the results in (7.32) bringing the theory closer to the data in (7.27) and (7.28). However, this result is scale and renormalization-scheme dependent. For the NDR $-$ $\overline{\text{MS}}$ scheme and $\mu \approx (2-3)\,\text{GeV}$ as used in LQCD calculations, this improvement is much smaller. But, even in the $\overline{\text{MOM}}$ scheme and at $\mu = 0.8\,\text{GeV}$, further enhancement of $\text{Re}A_0$ and further suppression of $\text{Re}A_2$ are needed to be able to understand the $\Delta I = 1/2$ rule. This brings us to Step 2.

[2] The subscript cc indicates that only contributions from current-current operators $Q_{1,2}$ have been taken into account.

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Step 2: Meson Evolution

The renormalization group evolution down to the scales $O(1\,\text{GeV})$ just performed is continued as a short but fast meson evolution down to zero momentum scales at which the factorization of hadronic matrix elements is at work. Equivalently, starting with factorizable hadronic matrix elements $\langle Q_1\rangle_{0,2}$ and $\langle Q_2\rangle_{0,2}$ at $\mu \approx 0$ and evolving them to $\mu = O(1\,\text{GeV})$ at which $z_{1,2}$ are calculated, one is able to calculate the matrix elements of these two operators at $\mu = O(1\,\text{GeV})$ and properly combine them with $z_{1,2}$ calculated in the $\overline{\text{MOM}}$ scheme. Details of these calculations can be found in [221, 239], and there is no space for presenting them here. I just want to make a few comments:

- Our loop calculations in the meson theory with a cutoff $M = O(1\,\text{GeV})$ include the contributions from pseudoscalars and lowest-lying vector mesons, and the result can be cast in the form of evolution equations. It is remarkable that the structure of these evolution equations, in particular the anomalous dimension matrix in the meson theory, is very similar to the one in the quark-gluon picture. This allows to perform an adequate matching between the two evolutions in question thereby removing to a large extent scale and renormalization scheme dependences present in the results of Step 1.
- The inclusion of vector meson contributions in [239] in addition to pseudoscalar contributions calculated in [221] is a significant improvement over our 1986 analysis bringing the theory closer to data.
- The same comment applies to the matching between the quark-gluon and meson theory, which this time has been performed at NLO in QCD. In this manner we could justify equating the physical cutoff M of the truncated meson theory (pseudoscalars and lowest-lying vector mesons) with the renormalization scale μ in the quark-gluon theory.

The resulting values

$$\text{Re}A_0 \approx (13.3 \pm 1.0) \times 10^{-8}\,\text{GeV}, \qquad \text{Re}A_2 \approx (1.1 \pm 0.1) \times 10^{-8}\,\text{GeV} \tag{7.36}$$

show a very significant improvement over the results in (7.35), bringing the theory closer to the data in (7.27) and (7.28). In particular within the uncertainties of our approach we can claim that the experimental value of $\text{Re}A_2$ has been reproduced. The amplitude $\text{Re}A_0$ has been enhanced in this step by almost a factor of two relative to the result in (7.35), but it is still by a factor of two below the data. But whereas the calculation of $\text{Re}A_2$ has been completed in this step, to complete the calculation of $\text{Re}A_0$, we have to include QCD penguin contributions to this amplitude. This brings us to Step 3.

Step 3: QCD Penguins

As pointed out in [241], QCD penguin operators, of which the dominant is Q_6 in (7.4), could play an important role in enhancing the ratio R as in the isospin limit they do not contribute to A_2 and uniquely enhance the amplitude A_0. However, in 1975 the relevant matrix element $\langle Q_6\rangle_0$ was unknown within QCD, and its Wilson coefficient z_6 was poorly known. In the strict large N limit in which factorization applies one finds [227, 228, 230] the result in (7.44), where we have introduced the parameter $B_6^{(1/2)}$, which equals unity in the large N limit.

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While this matrix element is much larger than the matrix elements of $Q_{1,2}$, its Wilson coefficient z_6 is strongly GIM suppressed at scales $O(m_c)$ due to the fact that it results from the difference of QCD penguin diagrams with charm and up-quark exchanges. If these masses are neglected above $\mu = m_c$ then $z_6(m_c) = 0$ and its value is roughly by an order of magnitude smaller than $z_{1,2}$ at $\mu = 0.8\,\text{GeV}$. In [228] an additional (with respect to previous estimates) enhancement of the QCD penguin contributions to $\text{Re}A_0$ has been identified. It comes from an incomplete GIM cancellation above the charm-quark mass. But as the analyses in [221, 239] show, this enhancement is insufficient to reproduce fully the experimental value of $\text{Re}A_0$. We find that the Q_6 contribution to $\text{Re}A_0$ for $\mu \leq 1\,\text{GeV}$ is relevant as it is by a factor of 3 larger than $\text{Re}A_2$. Yet at $\mu = 0.8\,\text{GeV}$ it contributes at most at the level of 15% of the experimental value of $\text{Re}A_0$.

Summary of Results

Our final results for $K \to \pi\pi$ amplitudes obtained using the DQCD approach can be summarized as follows

$$\boxed{\text{Re}A_0 \approx (17.0 \pm 1.5) \times 10^{-8}\,\text{GeV}, \quad \text{Re}A_2 \approx (1.1 \pm 0.1) \times 10^{-8}\,\text{GeV}, \quad \text{R} \approx 16.0 \pm 1.5\,.} \tag{7.37}$$

Even if the result for $\text{Re}A_0$ is not satisfactory, it should be noted that the QCD dynamics identified by us were able to enhance the ratio R by an order of magnitude. We therefore conclude that QCD dynamics is dominantly responsible for the $\Delta I = 1/2$ rule. The remaining piece in $\text{Re}A_0$ could come from final state interactions (FSI) between pions as discussed later, neglected $1/N^2$ corrections and/or NP contributions as investigated in [271].

7.2.4 Results for $\hat{B}_K$, $B_6^{(1/2)}$, $B_8^{(3/2)}$, and $B_8^{(1/2)}$

$\hat{B}_K$

We will next discuss the parameter $B_K(\mu)$ that plays an important role in the calculation of the $K^0 - \bar{K}^0$ mass difference ΔM_K and of the parameter ε_K in the SM as discussed in subsequent parts of this book.

$B_K(\mu)$, introduced already in Section 6.3, is scale and renormalization scheme dependent. It is related to the relevant hadronic matrix element of the $\Delta S = 2$ operator

$$Q = (\bar{s}d)_{V-A}(\bar{s}d)_{V-A}, \tag{7.38}$$

as follows

$$\langle \bar{K}^0 | Q(\mu) | K^0 \rangle = B_K(\mu) \frac{4}{3} F_K^2 m_K. \tag{7.39}$$

More useful is the renormalization group invariant parameter $\hat{B}_K$ that is given by [137]

$$\hat{B}_K = B_K(\mu) \left[\alpha_s^{(3)}(\mu)\right]^{-d} \left[1 + \frac{\alpha_s^{(3)}(\mu)}{4\pi} J_3\right], \qquad d = \frac{9(N-1)}{N(11N-6)}. \tag{7.40}$$

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We have shown the N-dependence of the exponent d in the leading term to signal that d vanishes in the large N limit. The coefficient J_3 is renormalization-scheme dependent. This dependence cancels the one of $B_K(\mu)$.

As in the strict large N limit the exponent in (7.40) and the NLO term involving J_3 vanish, one finds [230] that independent of any renormalization scale or renormalization scheme for the operator Q

$$\hat{B}_K \to 0.75, \qquad \text{(in large } N \text{ limit, 1986)}. \tag{7.41}$$

It can be shown that including $1/N$ corrections suppresses $\hat{B}_K$ so that [272]

$$\hat{B}_K \leq 0.75, \qquad \text{(in DQCD)}. \tag{7.42}$$

Consistently with this bound our calculations in the 1980s [229] gave $\hat{B}_K = 0.67 \pm 0.07$. The latest analysis in our approach resulted in [239]

$$\hat{B}_K = 0.73 \pm 0.02, \qquad \text{(in DQCD)}, \tag{7.43}$$

where the error should not be considered as a standard deviation. Rather, this result represents the range for $\hat{B}_K$ we expect in our approach after the inclusion of NLO QCD corrections and the contributions of pseudoscalar and vector mesons as discussed in detail in [239]. As we will see in the next section, LQCD confirms this result within present uncertainties after efforts of many groups.

$B_6^{(1/2)}$, $B_8^{(3/2)}$, and $B_8^{(1/2)}$

We discuss next the matrix elements of the penguin operators Q_6 and Q_8, which are crucial for the evaluation of the ratio ε'/ε in the SM and in most extensions of this model. We will see this in Chapter 10 and subsequent chapters. The relevant matrix elements are conveniently parametrized by

$$\langle Q_6(\mu)\rangle_0 = -4h\left[\frac{m_K^2}{m_s(\mu)+m_d(\mu)}\right]^2 (F_K - F_\pi)\, B_6^{(1/2)} = -0.473\, h\, B_6^{(1/2)}\,\text{GeV}^3\,, \tag{7.44}$$

$$\langle Q_8(\mu)\rangle_2 = \sqrt{2}h\left[\frac{m_K^2}{m_s(\mu)+m_d(\mu)}\right]^2 F_\pi\, B_8^{(3/2)} = 0.862\, h\, B_8^{(3/2)}\,\text{GeV}^3, \tag{7.45}$$

$$\langle Q_8(\mu)\rangle_0 = \frac{h}{2}\left[\frac{2m_K^2}{m_s(\mu)+m_d(\mu)}\right]^2 F_\pi\, B_8^{(1/2)} = 1.219\, h\, B_8^{(1/2)}\,\text{GeV}^3, \tag{7.46}$$

with [230, 273]

$$B_6^{(1/2)} = B_8^{(3/2)} = B_8^{(1/2)} = 1, \tag{7.47}$$

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in the large N limit. We have introduced the factor h to emphasize different normalizations of these matrix elements present in the literature. In this book we will set $h = 1$. From (7.44)–(7.46) we obtain

$$\boxed{\frac{B_8^{(3/2)}(\mu)}{B_6^{(1/2)}(\mu)} = -0.55\frac{\langle Q_8(\mu)\rangle_2}{\langle Q_6(\mu)\rangle_0}, \qquad \frac{B_8^{(3/2)}(\mu)}{B_8^{(1/2)}(\mu)} = \sqrt{2}\frac{\langle Q_8(\mu)\rangle_2}{\langle Q_8(\mu)\rangle_0}.} \tag{7.48}$$

The standard renormalization group analysis shows that $B_6^{(1/2)}$, $B_8^{(3/2)}$, and $B_8^{(1/2)}$ exhibit a very weak scale dependence for $\mu \geq 1$ GeV [123]. But this is no longer true for $\mu < 1$ GeV in the case of $B_6^{(1/2)}$ and $B_8^{(3/2)}$ as demonstrated within DQCD in [244] with important implications for the ratio ε'/ε. The dimensionful parameters entering (7.44), (7.45) have been calculated at $\mu = m_c$ using [48]

$$m_K = 497.614\,\text{MeV}, \qquad F_\pi = 130.41(20)\,\text{MeV}, \qquad \frac{F_K}{F_\pi} = 1.194(5), \tag{7.49}$$

$$m_s(m_c) = 109.1(2.8)\,\text{MeV}, \qquad m_d(m_c) = 5.44(19)\,\text{MeV}. \tag{7.50}$$

Before giving the values of $B_6^{(1/2)}$, $B_8^{(3/2)}$, and $B_8^{(1/2)}$ beyond the large N limit, it should be reemphasized that the overall factor h in (7.44)–(7.46) depends on the normalization of the amplitudes $A_{0,2}$. In [123] and recent papers of the RBC-UKQCD collaboration [274–276] $h = \sqrt{3/2}$ is used, whereas in most recent phenomenological papers [50, 212, 239, 271], $h = 1$. Correspondingly, the experimental values quoted for $A_{0,2}$ differ by this factor.

The $1/N$ corrections to these three parameters have been investigated in [252] and more recently in [244]. We will describe now the results of the latter study. In the DQCD the large N limit in (7.47) corresponds to very low scales $O(m_\pi)$, where the factorization of the matrix elements into a product of matrix elements of quark densities is a good approximation. However, to combine these parameters with the Wilson coefficients $z_{6,8}$ and $y_{6,8}$ one has to calculate $B_6^{(1/2)}$, $B_8^{(3/2)}$, and $B_8^{(1/2)}$ at scales $O(1\,\text{GeV})$. This is achieved through the meson evolution from low-energy scales $O(m_\pi)$ to scales $O(1\,\text{GeV})$ as done in [244] with the result

$$B_6^{(1/2)}(M) = 1 - \frac{3}{2}\left[\frac{F_\pi}{F_K - F_\pi}\right]\frac{(m_K^2 - m_\pi^2)}{(4\pi F_\pi)^2}\ln\left(1 + \frac{M^2}{\tilde{m}_6^2}\right) = 1 - 0.66\,\ln\left(1 + \frac{M^2}{\tilde{m}_6^2}\right), \tag{7.51}$$

$$B_8^{(3/2)}(M) = 1 - 2\frac{(m_K^2 - m_\pi^2)}{(4\pi F_\pi)^2}\ln\left(1 + \frac{M^2}{\tilde{m}_8^2}\right) = 1 - 0.17\,\ln\left(1 + \frac{M^2}{\tilde{m}_8^2}\right), \tag{7.52}$$

$$B_8^{(1/2)}(M) = 1 + \frac{(m_K^2 - m_\pi^2)}{(4\pi F_\pi)^2}\ln\left(1 + \frac{M^2}{\tilde{m}_8^2}\right) = 1 + 0.08\,\ln\left(1 + \frac{M^2}{\tilde{m}_8^2}\right). \tag{7.53}$$

The pseudoscalar mass scale parameters $\tilde{m}_{6,8}$ are bounded necessarily by the effective cut-off around 1 GeV: $\tilde{m}_{6,8} \leq M$.

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We emphasize most important properties of these results:

- For $M = 0$, corresponding to strict large N limit and matrix elements evaluated at zero momentum, $B_6^{(1/2)} = B_8^{(3/2)} = B_8^{(1/2)} = 1$ in accordance with (7.47).
- With increasing M, the parameters $B_6^{(1/2)}$ and $B_8^{(3/2)}$ decrease below unity, and $B_6^{(1/2)}$ decreases faster than $B_8^{(3/2)}$. Consequently, at scales $O(1\,\text{GeV})$ relevant for the phenomenology, both $B_6^{(1/2)}$ and $B_8^{(3/2)}$ are predicted to be below unity and $B_6^{(1/2)} < B_8^{(3/2)}$.
- While the dependence of $B_6^{(1/2)}$ and $B_8^{(3/2)}$ on $M < 1\,\text{GeV}$ is stronger than their dependence on μ in the perturbative regime, these two properties of $B_6^{(1/2)}$ and $B_8^{(3/2)}$ are at the qualitative level consistent with the numerical analysis performed for $B_6^{(1/2)}$ and $B_8^{(3/2)}$ by means of the standard renormalization group running in [123]. Indeed, as seen in figures 11 and 12 of that paper, $B_6^{(1/2)}$ decreases with increasing μ, faster than $B_8^{(3/2)}$, albeit in this perturbative range the dependence of $B_6^{(1/2)}$ and $B_8^{(3/2)}$ on μ is very weak. It should be emphasized that in that paper, similar to other literature in the 1990s, $B_6^{(1/2)}(m_c) = B_8^{(3/2)}(m_c) = 1$ has been chosen, which is not correct as just explained. But the pattern of the μ dependence of these parameters is as given there.

  While the analysis in [123] includes NLO QCD and QED corrections, the inspection of the one-loop anomalous dimension matrix allows to see these properties explicitly. In particular Q_6 mixes with the linear combination (Q_4+Q_6), and we find for $\mu_1 \le \mu_2 \le m_c$

$$B_6^{(1/2)}(\mu_2) = B_6^{(1/2)}(\mu_1)\left[1 - \frac{\alpha_s(\mu_1)}{2\pi}\ln\left(\frac{\mu_2}{\mu_1}\right)\left(1 + \frac{\langle Q_4(\mu_1)\rangle_0}{\langle Q_6(\mu_1)\rangle_0}\right)\right]. \tag{7.54}$$

  As $|\langle Q_6(\mu_1)\rangle_0| > |\langle Q_4(\mu_1)\rangle_0|$, one finds that $B_6^{(1/2)}$ decreases with increasing μ. On the other hand, in the LO Q_8 runs only by itself, and the one-loop anomalous dimension matrix implies

$$B_8^{(1/2,3/2)}(\mu_2) = B_8^{(1/2,3/2)}(\mu_1), \tag{7.55}$$

  which follows from exact SU(3) symmetry imposed in SD calculations. The breakdown of SU(3) is only felt in the matrix elements of Q_8 making in the LD range $B_8^{(3/2)}$ dependent weakly on the scales involved. In view of this, the suppression of both $B_6^{(1/2)}$ and $B_8^{(3/2)}$ below the unity can be considered as a solid result, and our explicit calculation as well as different behavior of Q_6 and Q_8 under flavor SU(3) provide a strong support for $B_6^{(1/2)} < B_8^{(3/2)}$.
- $B_8^{(1/2)}$ is only mildly affected by meson evolution.

These results can be then summarized by the upper bounds on both $B_6^{(1/2)}$ and $B_8^{(3/2)}$ [244]

$$B_6^{(1/2)} < B_8^{(3/2)} < 1, \qquad (\text{DQCD}). \tag{7.56}$$

While this approach gives $B_8^{(3/2)}(m_c) = 0.80 \pm 0.10$, the result for $B_6^{(1/2)}$ is less precise, but there is a strong indication that $B_6^{(1/2)} < B_8^{(3/2)}$, with typical values $B_6^{(1/2)} \approx 0.6$ at scales $O(1\,\text{GeV})$, in agreement with the LQCD results discussed in the next section. Therefore the bound on $B_6^{(1/2)}$ in (7.56) should be considered as conservative. On the other hand a number of other large N approaches [255, 256, 277–279] violate strongly the bounds in (7.56) with

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$B_6^{(1/2)}$ in the ballpark of 3 and $B_8^{(3/2)} > 1$ in striking disagreement with the DQCD approach and LQCD results.

7.2.5 Final State Interactions

Already a long time ago the chiral perturbation theory practitioners put forward the idea that the amplitude $\mathrm{Re}A_0$, governed by the current-current operator $Q_2 - Q_1$ could be enhanced significantly through FSI [280–286] bringing the values of R in (7.37) close to its experimental value. Some support for this claim comes from the more recent reconsideration of the role of final state interactions (FSI) in the $\Delta I = 1/2$ rule within the DQCD approach [245].

However, the claim by the authors in [280–286] and recently in [247], that FSI would enhance $B_6^{(1/2)}$ above unity in a correlated manner with the $\mathrm{Re}A_0$ enhancement has not been confirmed in [245]. The main reason for the latter conclusion is the absence of the meson evolution of $B_6^{(1/2)}$ in the chiral perturbation analyses in question. According to [245] the suppression of $B_6^{(1/2)}$ through this evolution, seen in (7.51), overcompensates possible enhancements of $B_6^{(1/2)}$ considered in chiral perturbation papers so that the bound $B_6^{(1/2)} < 1$ should be valid in the presence of FSI. We will see in Chapter 10 that the values of $B_6^{(1/2)}$ and $B_8^{(3/2)}$ at scales $O(1\,\mathrm{GeV})$ are crucial for the SM prediction of the CP-violating ratio ε'/ε. Their values obtained by LQCD will be discussed in the next section.

7.2.6 More on the μ Dependence of $\langle Q_{6,8}\rangle$

The μ dependence of $B_6^{(1/2)}$ and $B_8^{(3/2)}$ is related to nonfactorizable contributions to the corresponding hadronic matrix elements in (7.44) and (7.45). But as seen in the latter formulas, these matrix elements depend strongly on μ through the μ dependence of quark masses, and for scales $\mu > 1\,\mathrm{GeV}$ this is by far the dominant μ dependence. This property should be contrasted with the matrix elements of $(V-A)\otimes(V-A)$ operators, which are μ-independent in the large N limit. Only through the inclusion of $1/N$ nonfactorizable corrections, corresponding to meson loops in the effective theory, can the proper μ-dependence of the latter matrix elements be generated.

The μ-dependence in $\langle Q_6\rangle_0$ and $\langle Q_8\rangle_2$ related to the μ dependence of quark masses is for $\mu > 1\,\mathrm{GeV}$ exactly canceled in the decay amplitudes by the diagonal evolution (no mixing) of the Wilson coefficients $y_6(\mu)$ and $y_8(\mu)$ taken in the large N limit. Indeed, the μ-dependence of $1/m_{s,d}^2(\mu)$ is governed in LO by $2\gamma_m^{(0)} = 12C_F$. On the other hand, the one-loop anomalous dimensions of $Q_{6,8}$, which govern the diagonal evolution of $y_{6,8}(\mu)$ are given by

$$\gamma_{66}^{(0)} = -2\gamma_m^{(0)} + \frac{2f}{3}, \qquad \gamma_{88}^{(0)} = -2\gamma_m^{(0)}. \tag{7.57}$$

For large N, $\gamma_m^{(0)} \sim O(N)$, we find indeed $\gamma_{66}^{(0)} = \gamma_{88}^{(0)} = -2\gamma_m^{(0)}$ in the large-N limit [273]. Going back to the respective evolutions of $m_{s,d}$ and $y_{7,8}(\mu)$ we indeed confirm the cancellation of the μ-dependence in question. This feature is preserved at the two-loop level as discussed in [122, 146].

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7.2.7 The Virtues of DQCD

Let us close this section by listing various virtues of DQCD even if it is really not QCD, the theory of quarks and gluons. But, as we have already seen, and we will see later, it is a successful low-energy approximation of QCD. Moreover, it has several virtues:

- It is an efficient approximate method for obtaining results for nonleptonic decays, years and even decades before useful results from numerically sophisticated and demanding lattice calculations could be obtained.
- It is the only existing method that allows us to study analytically the dominant dynamics below 1 GeV scale, represented by the meson evolution, which turns out to have the pattern of operator mixing, both for SM and BSM operators, to agree with the one found perturbatively at short-distance scales. This allows for satisfactory, even if approximate, matching between Wilson coefficients and hadronic matrix elements.
- It provides insight into the purely numerical results obtained by LQCD. We will see soon that the results for $B_6^{(1/2)}$, $B_8^{(3/2)}$, and $B_8^{(1/2)}$ in (7.51)–(7.53) explain their numerical values from LQCD given in the next section. Moreover, we will see in Chapter 13 that the pattern of LQCD results for $K^0 - \bar{K}^0$ hadronic matrix elements of BSM operators can also be explained as the result of meson evolution.
- Most important, as discussed in Chapter 14, DQCD allowed already in 2018 to evaluate $K \to \pi\pi$ matrix elements of all BSM operators, providing in this manner for the first time a global view on NP contributions to ε'/ε.

7.3 Lattice QCD Results

7.3.1 Preliminaries

LQCD plays an essential role in weak decays as it allows to calculate various nonperturbative quantities like weak decay constants, formfactors and hadronic matrix elements of local operators from first principles. Moreover, in contrast to DQCD, just discussed, it provides useful results not only for K meson decays but also for D and $B_{s,d}$ decays. However, these calculations are very tedious, and for certain quantities it required the efforts of various collaborations over decades to obtain satisfactory precision. But in the case of nonleptonic decays still the lattice calculations have to be improved by much before one could be satisfied.

It is not the purpose of this section to describe the lattice methods, simply because such a description is far beyond the skills of the present author. What, however, can be done here is the collection of most important results from LQCD that we will use in this book when discussing phenomenology of weak decays. The main source here will be the summary of lattice results from the very recent 2019 FLAG review [287]. They are summarized in Table 7.1. These numbers differ slightly from our input in Appendix D that was used in this book until 2019 FLAG review was available. As they will further change until 2020, when this book will appear, we decided not to update our calculations.

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Table 7.1 Values of some theoretical quantities extracted from FLAG 2019 [287].[a]

$F_\pi = *$	$F_K = 155.7(3)$ MeV	$\hat{B}_K = 0.717(18)(16)$
$F_{B_d} = 190.0(13)$ MeV	$F_{B_s} = 230.3(13)$ MeV	$F_{B^+} = (189.4(13)$ MeV
$F_\pi = 132(8)$	$F_K = 155.7(7)$ MeV	$\hat{B}_K = 0.7625(97)$
$F_{B_d} = 192.0(43)$ MeV	$F_{B_s} = 228.4(37)$ MeV	$F_{B^+} = (189.4(13)$ MeV
$\hat{B}_{B_d} = 1.30(10)$	$\hat{B}_{B_s} = 1.35(6)$	$\hat{B}_{B_s}/\hat{B}_{B_d} = 1.032(38)$
$F_{B_d}\sqrt{\hat{B}_{B_d}} = 225(9)$ MeV	$F_{B_s}\sqrt{\hat{B}_{B_s}} = 274(8)$ MeV	$\xi = 1.206(17)$

[a] The first two rows correspond to existing 2+1+1 calculations, the subsequent four after the horizontal line to 2+1 calculations. The $*$ means that no result was given in this review.

Indeed, due to significant progress in lattice calculations expected in coming years the numbers collected by us will be changing in the future, and potential readers should always check, by looking up FLAG's webpage, whether some modifications took place. This is similar to experimental data, which also in most cases change with time, in particular the uncertainties decrease. But in the case of many observables the achieved precision, both by LQCD and experiments, is such that any future modifications will only have small impact on phenomenology. In other cases, like hadronic matrix elements of four-quark operators, significant progress has still to be made to be able to reach clear-cut conclusions on the SM predictions and on the room left for NP contributions.

Before entering the details one could ask the question whether the results of DQCD presented in the previous section will become obsolete when the precision of LQCD will become satisfactory. We would like to emphasize once again that this will not be the case for reasons listed at the end of the previous section. In particular LQCD being a purely numerical method does not allow one to understand the dynamics responsible for the numerical values of hadronic matrix elements in $K \to \pi\pi$ decays and $K^0 - \bar{K}^0$ mixing. This insight can be obtained within DQCD as we have already demonstrated in the case of the $\Delta I = 1/2$ rule, the values of $B_6^{(1/2)}$, $B_8^{(3/2)}$, and $B_8^{(3/2)}$ and the values of BSM hadronic matrix elements in $K^0 - \bar{K}^0$ mixing discussed in Chapter 13.

7.3.2 Weak Decay Constants

During the last years very impressive progress has been made by LQCD in evaluating weak decay constants

$$\boxed{F_\pi, \quad F_K, \quad F_{B_d}, \quad F_{B_s}, \quad F_{B^+}.} \tag{7.58}$$

They play a very important role in the phenomenology, and we will see this explicitly in the next chapters. Their present values are collected in Appendix D and also in Table 7.1.

7.3.3 *B*-Physics Results

LQCD plays an important role in the calculations of form factors. It also provides the best results for hadronic matrix elements entering $B^0_{s,d} - \bar{B}^0_{s,d}$ mixings both in the SM

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and beyond. We will encounter them in the phenomenological sections later. However, the multitude of kinematically accessible hadronic final states prevents LQCD from being a useful tool for nonleptonic B decays. Possibly for B physics most useful is presently *QCD Factorization* method, discussed in the next section.

7.3.4 K-Physics Results

For K physics significant progress has been made recently by LQCD, and we will list selected results now.

The lattice world average for $\hat{B}_K$ from FLAG based on the calculations of various groups [288–293] reads for $N_f = 2 + 1$ calculations [48],

$$\hat{B}_K = 0.7625(97), \qquad (\text{LQCD}, N_f = 2 + 1). \tag{7.59}$$

While this result violates the bound in (7.42), it should be noted that a number of lattice groups among [288–293] published results with central values satisfying the bound in (7.42), but the errors did not allow for a clear-cut conclusion.

In fact, the most recent update from staggered quarks [294] quotes precisely $\hat{B}_K = 0.738 \pm 0.005$, but the additional systematic error of 0.037 does not allow for a definite conclusion. Similarly, the Rome group [295], the only group working with $N_f = 2 + 1 + 1$, finds basically the result in (7.43):

$$\hat{B}_K = 0.717(18)(16), \qquad (\text{LQCD}, N_f = 2 + 1 + 1). \tag{7.60}$$

We expect therefore that improved lattice calculations will eventually satisfy the bound in (7.42), and in a few years from now the LQCD average for $\hat{B}_K$ will read $\hat{B}_K \approx 0.74$.

It is interesting not only that the values of $\hat{B}_K$ obtained in the DQCD approach and in LQCD are basically equal to each other but also that the resulting value is so close to its large N limit value. While the lattice approach did not provide the explanation why $\hat{B}_K$ is so close to its large N limit 0.75, in DQCD the smallness of $1/N$ corrections follows from the approximate cancellation of negative pseudoscalar meson contributions by the positive vector meson contributions.

Next, the relevant hadronic $K \to \pi\pi$ matrix elements have been evaluated by RBC-UKQCD collaboration [275, 276]. From these results one can extract values of $B_6^{(1/2)}$, $B_8^{(3/2)}$, and $B_8^{(1/2)}$ for $\mu = m_c$. They read in 2015 [296]

$$B_6^{(1/2)}(m_c) = 0.57 \pm 0.19, \qquad B_8^{(3/2)}(m_c) = 0.76 \pm 0.05, \qquad B_8^{(1/2)}(m_c) = 1.0 \pm 0.2, \tag{7.61}$$

in agreement with the bounds from DQCD in (7.56). The improved results should be available in 2020. The implications for the ratio ε'/ε will be presented in Chapter 10.

The present result for the $\Delta I = 1/2$ rule from the RBC-UKQCD collaboration is [276]

$$\left(\frac{\text{Re}A_0}{\text{Re}A_2}\right)_{\text{lattice QCD}} = 31.0 \pm 11.1, \tag{7.62}$$

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and, in agreement with results from DQCD, this rule is governed by current-current operators. But the uncertainty is still very large, and it will be interesting to see whether LQCD will be able to come closer to the data than is possible using DQCD. See (7.37).

As far as LQCD results for formfactors are concerned, we will mention them in the phenomenological sections. Lattice QCD is also making progress in calculating long-distance effects in rare decays like $K^+ \to \pi^+ \nu\bar{\nu}$ [214, 297, 1337], $K_L \to \pi^0 \ell^+ \ell^-$ [298, 299], and the mass difference ΔM_K [373, 1330]. Also the progress in calculating QED corrections to decay amplitudes and leptonic decay rates should be mentioned [302–304].

The prospects for LQCD calculations relevant for flavor physics are summarized in [305–308]. In particular the projection on the reduction of errors in LQCD in the coming years is described well in section 11 of [305]. See in particular table 40 therein.

7.4 QCD Factorization for Exclusive B Decays

7.4.1 Introduction

We have seen that the DQCD approach could be successfully applied to the K meson system. In other meson systems one could apply the large N ideas for the calculation of hadronic matrix elements obtaining them as products of matrix elements of currents and quark densities. In fact, the first application of the large N method has been done for two-body charm decays in [225]. But to combine this calculation with the Wilson coefficients evaluated at scales $O(m_c, m_b)$ would require within the DQCD approach the meson evolution through the region of many resonances, which is clearly not doable. Here the so-called *QCD factorization* (QCDF) [309, 310] for exclusive hadronic B decays is much more powerful.

Indeed, the formulation of factorization theorems for exclusive hadronic B-meson decays in 1999 made an entire new class of processes accessible to systematic calculations of higher-order corrections in QCD [309, 310]. These processes include B decays into a pair of light mesons, the prototype of which is $B \to \pi\pi$, but also rare and radiative decays, such as $B \to K^*\gamma$ or $B \to K^* l^+ l^-$. In the heavy-quark limit, that is up to relative corrections of order $\Lambda_{\rm QCD}/m_b$, the problem of computing exclusive hadronic decay amplitudes simplifies considerably. In this limit the decay amplitudes can be written as hard-scattering kernels, which are *process dependent* but perturbatively calculable, multiplied by hadronic quantities such as $B \to \pi$ form factors, meson decay constants, and light-cone wave functions, which are nonperturbative but *process independent.* The decomposition into calculable hard contributions and universal hadronic quantities is in full analogy with the factorization of short-distance and long-distance terms that is the basis of any application of QCD to high-energy processes and has been discussed at length in the context of weak decays in previous chapters in this book. Correspondingly the framework is referred to as QCD factorization for exclusive hadronic B decays, or QCDF for short.

In the present section we review the factorization formula and give an overview of the NLO and NNLO calculations performed for exclusive B decays. To this end we give first

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the description of this approach by one of its fathers, Gerhard Buchalla, as presented in section 9 of [77]. Subsequently we will briefly describe recent developments.

7.4.2 Factorization Formula

The matrix element of an operator Q_i in the effective weak Hamiltonian for the decay of a $\bar{B}$ meson into a pair of light mesons $M_1 M_2$ is given by [309, 310]

$$\langle M_1 M_2 | Q_i | \bar{B} \rangle = F^{B\to M_1}(m_2^2) \int_0^1 du\, T_i^I(u)\, \Phi_{M_2}(u) + (M_1 \leftrightarrow M_2) + \int_0^1 d\xi du dv\, T_i^{II}(\xi, u, v)\, \Phi_B(\xi)\, \Phi_{M_1}(v)\, \Phi_{M_2}(u), \tag{7.63}$$

up to power corrections of order $\Lambda_{\rm QCD}/m_b$. $F^{B\to M_{1,2}}(m_{2,1}^2)$ are $B \to M_{1,2}$ form factors, where $m_{1,2}$ denote the light-meson masses, and Φ_M is the light-cone distribution amplitude for the quark-antiquark Fock state of the meson M. Here the light-cone distribution amplitudes are understood to include the decay constant F_M of the meson M in their normalization. These quantities define the nonperturbative input needed for the computation of the decay amplitudes in QCDF. They are simpler than the full matrix element on the l.h.s. of (7.63) and universal in the sense that they appear as well in many other processes, which are different from $\bar{B} \to M_1 M_2$. The $T_i^I(u)$ and $T_i^{II}(\xi, u, v)$ are the hard-scattering functions, which are process-specific and depend in particular on the operator Q_i. They are calculable by standard methods in perturbative QCD.

The formula (7.63) exhibits the factorization of the short-distance kernels T_i and the long-distance hadronic quantities $F^{B\to M}$ and Φ_M. The factorization of the latter takes, in general, the form of a convolution over the parton momentum fractions ξ, u, $v \in [0, 1]$. A graphical representation of (7.63) is given in Figure 7.1, where index j accounts for the possibility of more than a single $B \to M_1$ formfactor. The second term ($\sim T^{II}$) is distinguished from the first ($\sim T^I$) by the participation of the B-meson spectator quark in the hard interaction, indicated by the spectator line entering the kernel T^{II}. The spectator interaction requires the exchange of a hard gluon. T^{II} starts therefore at order α_s, whereas T^I is of order unity, schematically

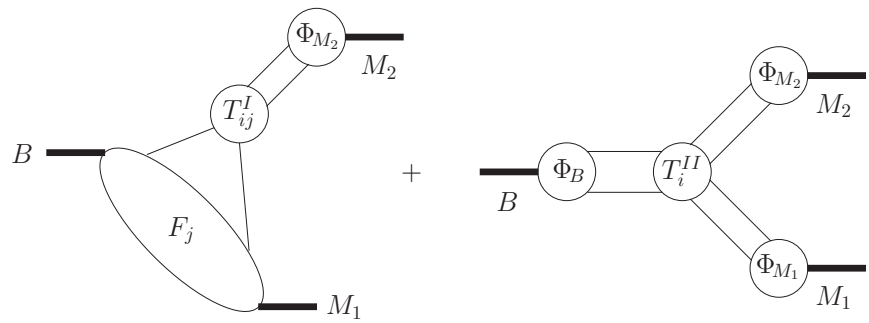


Figure 7.1 Factorization formula.

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$$T^I = T^I_{(0)} + \frac{\alpha_s}{4\pi} T^I_{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 T^I_{(2)} + \cdots, \qquad T^{II} = \frac{\alpha_s}{4\pi} T^{II}_{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 T^{II}_{(2)} + \cdots \quad (7.64)$$

Such a description is only possible because in the two-body decay of the B meson the final-state particles are necessarily very energetic, with light-like four-momenta, in the rest frame of the B. Therefore, a meson emitted from the hard interaction, such as M_2 from T^I in Figure 7.1, is then described by its light-cone distribution amplitude. At leading power in $\Lambda_{\rm QCD}/m_b$ the amplitude is determined by the contribution from the light-cone wave function of leading twist, which corresponds to the simplest, two-particle Fock state. Higher Fock states give power-suppressed contributions and are therefore absent in the heavy-quark limit. For example, an additional energetic gluon, collinear to the light-like quark and antiquark in meson M_2 will generate an additional, far off-shell propagator when attached to the hard process T, which results in a power suppression. The properties of the light-cone wave functions, which vanish at the endpoints ($u = 0, 1$), also imply the suppression of highly asymmetric configurations where one parton carries almost the entire meson momentum and the other parton is soft.

At leading order in QCD, that is at $O(\alpha_s^0)$, the factorized matrix element in (7.63) reduces to a particularly simple result. The second term T^{II} is absent at this order, and $T^I(u)$ becomes a u-independent constant. Taking the matrix element of the operator $Q_2 = (\bar{u}b)_{V-A}(\bar{d}u)_{V-A}$ for $\bar{B} \to \pi^+\pi^-$ as an example, the factorization formula then states that

$$\langle \pi^+\pi^-|(\bar{u}b)_{V-A}(\bar{d}u)_{V-A}|\bar{B}\rangle = \langle \pi^+|(\bar{u}b)_{V-A}|\bar{B}\rangle \langle \pi^-|(\bar{d}u)_{V-A}|0\rangle = iF^{B\to\pi}(0) F_\pi m_B^2. \quad (7.65)$$

This corresponds to the prescription of factorizing the matrix element of the 4-quark operator into a product of matrix elements of bilinear quark currents, which has a long history in phenomenological applications [311]. Here such a prescription receives its proper justification in the context of QCDF. The approximation in (7.65) means that the emission of the π^- is independent of the remaining $\bar{B} \to \pi^+$ transition. The intuitive argument for this, namely that the energetic and highly collinear, color-singlet $\bar{u}d$ pair forming π^- has little interaction with the rest of the process, was described a long time ago in [312]. The factorization theorem (7.63) is the formal implementation of this intuitive idea, and it allows us very importantly to compute corrections systematically.

It is then not surprising that factorization also works for decays of the type $\bar{B} \to D^+\pi^-$ with a heavy and a light meson in the final state, if it is the light meson that is emitted from the hard interaction (meson M_2 in Figure 7.1). In this case spectator scattering is power suppressed, and the factorization theorem in the heavy-quark limit takes the form

$$\langle DM_2|Q_i|\bar{B}\rangle = F^{B\to D}(m_2^2) \int_0^1 du\, T_i^I(u)\, \Phi_{M_2}(u). \quad (7.66)$$

In fact the expression in (7.66) had already been used in [313] to compute the order-α_s corrections to the ratio of the $\bar{B} \to D\pi^-$ and $\bar{B} \to D^*\pi^-$ decay rates, prior to the systematic development of the QCDF.

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The factorization theorem can be formulated using soft-collinear effective theory (SCET) [314, 315]. This formalism is useful for proving factorization [316] and for disentangling the hard and hard-collinear scale in explicit terms. QCDF and SCET are theoretical concepts that are fully compatible with each other, but they refer to different aspects of the problem of B-decay matrix elements. In some sense the relation between QCDF and SCET is similar to the relation between the heavy-quark expansion (HQE) and heavy-quark effective theory (HQET) in their application to *inclusive* B decays.[3] QCDF [309, 310] refers to the separation of the matrix elements into simpler long-distance quantities and calculable hard interactions, where the long-distance form factors are defined in full QCD. SCET, on the other hand, is a general effective field theory formulation for the relevant QCD modes (hard, hard-collinear, collinear, soft) and allows a further separation of scales, for instance in the transition form factors. However, working with form factors in full QCD often seems preferable in practice. An excellent review of SCET can be found in [317]. Some aspects are also discussed in [112].

7.4.3 Beyond NLO and Outlook

Review of NLO and NNLO corrections to the factorization formula by Buchalla can be found in section 9 of [77]. It is clearly beyond the scope of this book to present such details here. Over the years QCDF has been applied to a large number of exclusive B decays, and the associated phenomenology is very rich.

Not all observables accessible in principle to a factorization calculation are equally useful in practice as at a certain level accurate estimates of power corrections are necessary, and these are still beyond our control. One expects typically 10–20% corrections. This makes it difficult to compute direct CP asymmetries because these are sensitive to the relatively small strong phases, of which only the perturbative (though formally leading) part is calculable. However, there are many cases where the level of precision attainable with QCDF provides the basis for accurate predictions and flavor-physics tests.

QCDF can be employed to estimate the size of SU(3) breaking in approaches that rely on flavor symmetries to determine CKM quantities from CP violation in hadronic B decays [318]. Moreover, it can be used to estimate QCD penguin contributions to nonleptonic B decays [319].

In the upcoming era of precision experiments with B mesons, QCDF in the heavy quark limit, at NLO and beyond, provides us with an important tool to control theory predictions at a level adequate for discoveries in flavor physics. A recent summary of the theory status in nonleptonic heavy meson decays can be found in [317]. See also [320]. An interesting application of SCET to decays of new very heavy resonances has been presented in [321, 322].

Finally, we should mention the perturbative QCD approach to exclusive B decays [323–325] with the most recent application presented in [326], where further references can be found.

[3] We will describe these approaches briefly at the end of this chapter.

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7.5 Heavy Quark Effective Theory (HQET) and Heavy Quark Expansions (HQE)

7.5.1 HQET

Since its advent in 1989 heavy quark effective theory (HQET) [327–331] has developed into an elaborate and well-established formalism, providing a systematic framework for the treatment of hadrons containing a heavy quark. HQET represents a static approximation for the heavy quark, covariantly formulated in the language of an effective field theory. It allows to extract the dependence of hadronic matrix elements on the heavy quark mass and to exploit the simplifications that arise in QCD in the static limit. The most important application of HQET has been to the analysis of exclusive semileptonic transitions involving heavy quarks, where this formalism allows to exploit the consequences of heavy quark symmetry to relate formfactors and provides a basis for systematic corrections to the $m \to \infty$ limit. There are several excellent reviews on this subject [332–335], and as there is no space to cover this topic in a satisfactory manner here, we ask the readers to look at these papers. A simple introduction to this subject can be found in chapter XV of [111].

7.5.2 HQE

The calculation of the decay rate difference $\Delta\Gamma_s$ is more involved than the one of ΔM_s.[4] In the box diagrams we have to take into account now only the internal up- and charm-quarks. Integrating out all heavy particles (in this case only the W boson) we are not left with a local $\Delta B = 2$ operator as in the case of ΔM_s, but with a bilocal object with up and charm quark propagators connecting two local vertices. To obtain a result in terms of local operators, whose matrix elements can be evaluated by lattice QCD, a second OPE is required. It relies on the smallness of the parameter Λ/m_b, where Λ is expected to be of the order of the hadronic scale $\Lambda_{\rm QCD}$. This so-called heavy quark expansion (HQE) has been developed in a number of papers [336–342]. The HQE applies in addition to $\Delta\Gamma_s$ also for lifetimes and totally inclusive decay rates of heavy hadrons. Nice reviews with some details are the ones in [335, 343–345], and a nice summary of the present situation including historical development can be found in [346]. It shows that after some difficult periods it is a successful framework. Technical aspects are discussed also in [112].

We cannot cover this topic here in any detail, but let us discuss just few examples. First according to the HQE, the off-diagonal element Γ_{12}^s of the $B_s^0 - \bar{B}_s^0$ mixing matrix, discussed in detail in the next chapter, can be expanded as a power series in the inverse of the heavy b-quark mass m_b and the strong coupling α_s:

$$\Gamma_{12}^s = \frac{\Lambda^3}{m_b^3}\left(\Gamma_3^{s,(0)} + \frac{\alpha_s}{4\pi}\Gamma_3^{s,(1)} + \cdots\right) + \frac{\Lambda^4}{m_b^4}\left(\Gamma_4^{s,(0)} + \cdots\right) + \cdots . \tag{7.67}$$

[4] See Section 8.1 for definitions of these quantities.

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Although Λ is assumed to be of the order of Λ_{QCD}, its actual value has to be determined by a nonperturbative calculation. Each of the $\Gamma_i^{s,(j)}$ is a product of perturbative Wilson coefficients and nonperturbative matrix elements. In Γ_3^s these matrix elements arise from dimension 6 four-quark operators, in Γ_4^s from dimension 7 operators, and so on.

The leading term in (7.67), $\Gamma_3^{s,(0)}$, was calculated a long time ago by [347–352]. NLO QCD corrections, i.e. $\Gamma_3^{s,(1)}$ in (7.67); were done for the first time in [353]; they turned out to be quite large. Subleading $1/m_b$ corrections were first calculated by [354], and they also turned out to be quite sizable. Both LQCD and QCD sum rules were used over the years to calculate hadronic matrix elements of the relevant operators. As we will discuss in the next chapter, the present theoretical value for $\Delta\Gamma_s$ agrees well with experiment.

Another known application of HQE is for *inclusive* decays of heavy mesons, like $B_{s,d}$-mesons. In an inclusive decay one sums over all (or over a special class) of accessible final states, and eventually one can show [340, 355–361] that the resulting branching ratio can be calculated in the expansion in inverse powers of m_b with the leading term described by the spectator model in which the B-meson decay is approximated by the decay of the b-quark with corrections $O(1/m_b^2)$.

Because the leading term in this expansion represents the decay of the quark, it can be calculated in perturbation theory or more correctly in the renormalization group improved perturbation theory. It should be realized that also here the basic starting point is the relevant effective Hamiltonian and that the knowledge of the Wilson coefficients $C_i(\mu)$ is essential for the evaluation of the leading term in this expansion. But there is an important difference relative to the exclusive case: The matrix elements of the relevant operators can be "effectively" evaluated in perturbation theory. This means, in particular, that their μ and renormalization scheme dependences can be evaluated and the cancellation of these dependences by those present in $C_i(\mu)$ can be investigated.

Clearly, to complete the evaluation of the rates nonleading terms have to be considered. These terms are of a nonperturbative origin, but fortunately they are suppressed often by two powers of m_b. They have been studied by several authors in the literature with the result that they affect various branching ratios by less than 10% and often by only a few percent. Consequently the inclusive decays were expected for decades to give generally more precise theoretical predictions than the exclusive decays. On the other hand their measurements are harder. Moreover with advances of lattice QCD calculations it should be expected that *exclusive* decays will take the lead in testing the theory as there the issues like hadron duality violations present in inclusive decays and introducing some systematic uncertainties are absent.

7.6 Other Nonperturbative Methods

There are other useful nonperturbative methods, in particular chiral perturbation theory [50, 91, 265, 362–365], QCD sum rules, reviewed in [366], and light-cone sum rules [367–369], but their description is outside the scope of this book.

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