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Lattice orbit distribution on $\mathbb{R}^2$

ARNALDO NOGUEIRA

Institut de Mathématiques de Luminy, 163, avenue de Luminy, Case 907, 13288 Marseille Cedex 9, France
(e-mail: nogueira@iml.univ-mrs.fr)

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Abstract. We study the distribution on $\mathbb{R}^2$ of the orbit of a vector under the linear action of $\mathrm{SL}(2, \mathbb{Z})$. Let $\Omega \subset \mathbb{R}^2$ be a compact set and $x \in \mathbb{R}^2$. Let $N(k, x)$ be the number of matrices $\gamma \in \mathrm{SL}(2, \mathbb{Z})$ such that $\gamma(x) \in \Omega$ and $\|\gamma\| \leq k$, $k = 1, 2, \ldots$. If Ω is a square, we prove the existence of an absolute error term for $N(k, x)$, as $k \to \infty$, for almost every x, which depends on the Diophantine property of the ratio of the coordinates of x. Our approach translates the question into a Diophantine approximation counting problem which provides the absolute error term. The asymptotical behaviour of $N(k, x)$ is also obtained using ergodic theory.

1. *Introduction*

Let $\mathrm{SL}(n, \mathbb{Z})$ be the group of integral $n \times n$-matrices of determinant 1. Let $z \in \mathbb{R}^n$ be a column vector. It is well known that $\mathrm{SL}(n, \mathbb{Z})z$ is a dense set in $\mathbb{R}^n$ if, and only if, z is not a multiple of an integral vector (one with integral coordinates). Otherwise, $\mathrm{SL}(n, \mathbb{Z})z$ is a discrete set. Let $\Omega \subset \mathbb{R}^n$ be a compact set and z not a multiple of an integral vector. It is natural to ask: what is the number of matrices $\gamma \in \mathrm{SL}(n, \mathbb{Z})$ such that $\gamma(z) \in \Omega$ and $\|\gamma\| \leq k$, as $k \to \infty$?

Here we consider this question for $n = 2$. We set $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. Our aim is to pursue the work in [**8**] on counting elements of orbits under the linear action of Γ inside a square. An error term is provided there for particular orbits, for example those which evolve a quadratic surd. Here we study the error term for all non-discrete orbits and all squares, which does not depend on the initial vector and the square. We provide an absolute error term for every vector $x = \binom{x_1}{x_2}$, where x_1/x_2 is an irrational which is not a Liouville number, so it works for Lebesgue-almost every vector x. If x_1/x_2 is a Liouville number no absolute error term can be provided. Moreover, we establish a relation between matrices in Γ and pairs of successive Farey fractions. The relation shows that counting elements of an orbit of the linear action of Γ equals counting Farey fractions close to a number, or primitive lattice points in a neighbourhood of a line on $\mathbb{R}^2$.

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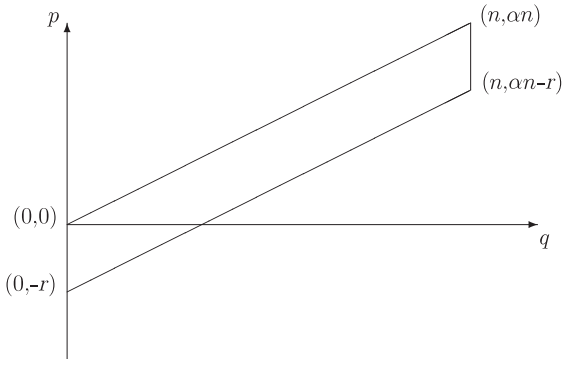


FIGURE 1. The parallelogram of vertices $(0, 0)$, $(0, -r)$, $(n, \alpha n - r)$ and $(n, \alpha n)$.

Before proceeding some definitions are needed. We define in Γ the norm $\|\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)\| = \max\{|a|, |b|, |c|, |d|\}$. We denote by $\mathcal{F}_n$ the *Farey sequence of order* $n \in \mathbb{N}$ (see Hardy and Wright [**3**, p. 23])

$$\mathcal{F}_n = \left\{ \frac{p}{q} : q,\ p \in \mathbb{Z},\ 1 \le q \le n,\ 0 \le p \le q,\ (q,\ p) = 1 \right\}, \tag{1.1}$$

where (q, p) is the greatest common divisor of q and p and $(q, p) = 1$ signifies that q and p are relatively prime. All fractions appearing throughout the paper are written in their reduced form. The cardinality of a finite set A is denoted by $|A|$.

Next we state our main results. The first theorem relates matrices in Γ to Farey fractions. Then (1.2) will be used to count the elements of an orbit, and to give an error term.

THEOREM 1.1. *Let* $x = \binom{\theta}{1}$, *where* $0 < \theta < 1$ *is irrational, and* $r > 0$. *Let* $n \in \mathbb{N}$. *Then*

$$|\{\gamma \in \Gamma : \gamma(x) \in [0, r]^2,\ \|\gamma\| \le n\}| - 2\left|\left\{ \frac{p}{q} \in \mathcal{F}_n : |p - q\theta| \le r \right\}\right| \tag{1.2}$$

is a bounded sequence and

$$\lim_{n\to\infty} \frac{1}{n} |\{\gamma \in \Gamma : \gamma(x) \in [0, r]^2,\ \|\gamma\| \le n\}| = \frac{6}{\pi^2} 4r. \tag{1.3}$$

The asymptotical limit (1.3) follows from (1.2) and a result proved by Watson [**11**] and Estermann [**2**]: let $\alpha > 0$ be irrational and $r > 0$. Let $N(r, n)$ be the number of primitive lattice points in the interior of the parallelogram (see Figure 1) in $\mathbb{R}^2$ whose vertices are the origin, $(0, -r)$, $(n, n\alpha - r)$ and $(n, n\alpha)$, for all $n \in \mathbb{N}$. Thus $\lim_{n\to\infty}(1/n)N(r, n) = (6/\pi^2)r$.

In other terms

$$\lim_{n\to\infty} \frac{1}{n} |\{(q,\ p) \in \mathbb{N}^2 : q \le n,\ 0 \le q\theta - p \le r,\ (q,\ p) = 1\}| = \frac{6}{\pi^2} r. \tag{1.4}$$

Let θ be irrational, then its *continued fraction expansion* (see [**3**, Ch. 10]) is denoted by

$$[a_0; a_1, a_2, \ldots], \tag{1.5}$$

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where the integers $a_0, a_1, a_2, \ldots$ are its partial quotients. Its convergents p_n/q_n are defined inductively by

$$p_{-1} = 0, \quad p_0 = a_0, \quad p_k = a_k p_{k-1} + p_{k-2} \quad \text{for all } k \in \mathbb{N},$$
$$q_{-1} = 1, \quad q_0 = 1, \quad q_k = a_k q_{k-1} + q_{k-2}.$$

We say that θ is of *constant type*, if the sequence $(a_n)_{n\geq 1}$ is bounded (see Kuipers and Niederreiter [**6**, p. 122]). Otherwise, θ is of *type* η (see [**6**, Definition 3.4, p. 121]), if

$$\eta = \sup\Big\{\alpha : \liminf_q q^\alpha \{q\theta\}(1 - \{q\theta\}) = 0\Big\}.$$

Thus $\eta = \limsup_n(\log q_{n+1}/\log q_n)$, and $\eta \geq 1$. If $\eta = \infty$, θ is called a *Liouville number*.

The next theorem uses the asymptotical limit (1.3) and gives information about the error term.

THEOREM 1.2. *Let* $x = \binom{\theta}{1}$, *where* $0 < \theta < 1$ *is irrational, and* $r > 0$. *Let* $n \in \mathbb{N}$. *The error term*

$$E(\theta, r, n) = |\{\gamma \in \Gamma : \gamma(x) \in [0, r]^2, \|\gamma\| \leq n\}| - n\frac{6}{\pi^2}4r$$

satisfies the following.

(i) *If there exist* $k, l \in \mathbb{Z}$ *such that* $r = k\theta + l$, *then* $E(\theta, r, n) = O(\sqrt{n})$.

(ii) *If* θ *is of constant type, then* $E(\theta, r, n) = O(\sqrt{n}\log n)$.

(iii) *If* θ *is of finite type* η, *then, for every* $\epsilon > 0$, $E(\theta, r, n) = O(n^{1-1/(2\eta)+\epsilon})$.

Let $\beta \in \mathbb{R}$. We denote by $[\beta]$ its integer part and by $\{\beta\}$ its fractional part. Let $0 < \theta < 1$ be irrational. The error term $E(\theta, r, n)$ is related to the discrepancy $D_n(\theta)$ of the sequence $(\{q\theta\})_{1\leq q\leq n}$ (see [**6**, Definition 1.1, p. 88]). An explicit formula of $D_n(\theta)$ can be found in Schoissengeier [**9, 10**]. Let $a_1, a_2, \ldots$ be the partial quotients of θ. Let $n \in \mathbb{N}$ and p_k/q_k be the convergent such that $q_k \leq n < q_{k+1}$. Then [**9**, Corollary 2] gives

$$nD_n(\theta) = \sum_{i=0}^{k} a_{i+1}\{q_i n\theta\}(1 - \{q_i n\theta\}) + O(1) \leq \frac{1}{4}\sum_{i=1}^{k+1} a_i + O(1), \tag{1.6}$$

where the term $O(1)$ is absolute. Therefore, if θ is a Liouville number, its discrepancy is large, and no absolute error term can be written.

Let $\beta > 0$, and x and r be as in Theorems 1.1 and 1.2. We remark that $\gamma(\beta x) \in [0, r]^2$ if, and only if, $\gamma(x) \in [0, r/\beta]^2$. Theorems 1.1 and 1.2 give that

$$|\{\gamma \in \Gamma : \gamma(\beta x) \in [0, r]^2, \|\gamma\| \leq n\}| = n\frac{6}{\pi^2}4\frac{r}{\beta} + E\left(\theta, \frac{r}{\beta}, n\right).$$

Ledrappier [**7**] established the asymptotical limit (1.3) for discrete subgroups of $\mathrm{SL}(2, \mathbb{R})$ whose covolume is finite. His dynamical systems approach uses a theorem proved by Dani [**1**] on the ergodic properties of the horocycle flow. We draw attention to the fact that in [**7**, Theorems 1 and 2] a factor 2 is missing in the limit. As in [**7**], the theorem proved in [**2, 11**] does not provide an error term.

The article is organized as follows. Section 2 is dedicated to proving that the sequence (1.2) is bounded. Theorem 2.1 relates matrices in Γ satisfying $\gamma(x) \in [0, r]^2$

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to pairs of successive Farey fractions with a Diophantine property. In Theorem 2.3, we prove that the number of pairs of Farey fractions satisfying a simultaneously Diophantine property is almost the double of the number of fractions satisfying the Diophantine property. Section 3 concerns Farey fractions. We start by giving the cardinality of the set $\{p/q \in \mathcal{F}_n : |p/q - \theta| \le r/q\}$. Theorem 3.1 is the counterpart of Theorem 1.2 for Farey fractions. For completeness the proofs of Theorems 1.1 and 1.2 are summarized in §4. In §5, using the approach proposed in [**11**], we study the counting problem in the case of discrete orbits. We show that the asymptotical behaviour (1.3) is obtained as a limit of the discrete cases (see Corollary 5.3) which answers a question posed by Dani.

2. *Counting* $\mathrm{SL}(2, \mathbb{Z})$ *matrices*

Here we give the precise relation between matrices $\gamma \in \Gamma$ such that $\gamma(x) \in [0, r]^2$ and pairs of successive Farey fractions satisfying a Diophantine property.

THEOREM 2.1. *Let* $x = \binom{\theta}{1}$, *where* $0 < \theta < 1$ *is irrational, and* $r > 0$. *Then*

$$|\{\gamma \in \Gamma : \gamma(x) \in [0, r]^2, \|\gamma\| \le n\}| - \left|\left\{\left(\frac{p'}{q'}, \frac{p}{q}\right) \in \mathcal{F}_n^2 : qp' - pq' = 1, \left|\frac{p}{q} - \theta\right| \le \frac{r}{q}, \left|\frac{p'}{q'} - \theta\right| \le \frac{r}{q'}\right\}\right| \quad (2.1)$$

is a bounded sequence, for $n \in \mathbb{N}$.

In order to prove the above theorem we give the precise relation between matrices $\gamma \in \Gamma$ such that $\gamma(x) \in \mathbb{R}_+^2$ and pairs of successive Farey fractions.

Let $\Gamma_+ = \mathrm{SL}(2, \mathbb{Z})_+$ be the semigroup of Γ whose matrices have non-negative entries. To every $\gamma = \left(\begin{smallmatrix} q & p \\ q' & p' \end{smallmatrix}\right) \in \Gamma_+$ we associate the following matrices in Γ

$$\gamma_1 = \begin{pmatrix} q' & -p' \\ q & -p \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} q & -p \\ -q' & p' \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} -q & p \\ q' & -p' \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} -q' & p' \\ -q & p \end{pmatrix}.$$

We need the following technical lemma whose proof will be skipped.

LEMMA 2.2. *Let* γ *and* γ' *be distinct matrices in* Γ_+. *Then the following hold.*

(1) *The matrices* $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ *are pairwise distinct.*

(2) *The matrices* $\gamma_i' \ne \gamma_j$ *for all* i, j.

Conversely,

$$\{\gamma_i : \gamma \in \Gamma_+, i = 1, 2, 3, 4\} = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : ab \le 0, cd \le 0\right\}.$$

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let $r > 0$ and $x = \binom{\theta}{1}$, where $0 < \theta < 1$ is irrational. We will prove that the sequence (2.1) is bounded.

Let $a, b, c, d \in \mathbb{Z}$. Assume $|a\theta + b| \le r$. If a and b have the same sign, then $|a|\theta + |b| \le r$ implying $-r \le b \le r$ and $-r/\theta \le a \le r/\theta$. Therefore, if a and b have the same sign and c and d have also the same sign, there are only finitely many matrices $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $\gamma(x) \in [0, r]^2$. An upper bound for the number of such matrices

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depends only on r and θ. Next we will analyse the matrices $\gamma \in \Gamma$ such that both a and b and c and d have opposite signs and $\gamma(x) \in [0, r]^2$.

Let $\gamma = \left(\begin{smallmatrix} q & p \\ q' & p' \end{smallmatrix}\right) \in \Gamma_+$ be such that $|q\theta - p| \le r$ and $|q'\theta - p'| \le r$. As θ is irrational and p, q, p' and q' are non-negative, by Lemma 2.2 only one of the following cases happens.

(1) $q\theta - p > 0$ and $q'\theta - p' > 0$, thus $\gamma_i(x) \in [0, r]^2$ if, and only if, $i = 1$.
(2) $q\theta - p > 0$ and $q'\theta - p' < 0$, thus $\gamma_i(x) \in [0, r]^2$ if, and only if, $i = 2$.
(3) $q\theta - p < 0$ and $q'\theta - p' > 0$, thus $\gamma_i(x) \in [0, r]^2$ if, and only if, $i = 3$.
(4) $q\theta - p < 0$ and $q'\theta - p' < 0$, thus $\gamma_i(x) \in [0, r]^2$ if, and only if, $i = 4$.

We denote by γ' the unique matrix in $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ such that $\gamma'(x) \in [0, r]^2$.

Next we shall prove that there are at most $([r/(1-\theta)] + 1)^3 + [(r+1)/(1-\theta)]$ such matrices γ' with $p > q$ or $p' > q'$.

Assume $p \ge q$. We have $q\theta - p \le q(\theta - 1) \le 0$, as $0 < \theta < 1$. Thus $-r \le q\theta - p < 0$. Moreover, $p(1-\theta) \le r$, and as $p \le q$, they satisfy $0 \le q \le p \le r/(1-\theta)$. Therefore there are at most $([r/(1-\theta)] + 1)^3$ many matrices γ such that $|q\theta - p| \le r$ and $|q'\theta - p'| \le r$, where $p \ge q$ and $p' \ge q'$.

Otherwise, $p \ge q$ and $p' < q'$. We have $pq' + 1 = qp' < qq'$ implying $(q-p)q' > 1$ which is a contradiction, as $p \ge q \ge 0$ and $q' \ge 0$.

Assume $p' \ge q'$ and $q > p$. We have $qp' = pq' + 1 < qq' + 1$, thus $q(p' - q') < 1$ implying $p' = q'$. Moreover $\det(\gamma) = p'(q-p) = 1$ implying $p' = q' = 1$ and $q = p + 1$. On the other hand, $|(p+1)\theta - p| \le r$ implying $p \le (r+1)/(1-\theta)$. Therefore, in this case, there are at most $[(r+1)/(1-\theta)]$ matrices γ such that $|q\theta - p| \le r$ and $|q'\theta - p'| \le r$. This concludes the proof of our claim.

We have proved that associated to every matrix $\gamma = \left(\begin{smallmatrix} q & p \\ q' & p' \end{smallmatrix}\right) \in \Gamma_+$, except $([r/(1-\theta)] + 1)^3 + [(r+1)/(1-\theta)]$ many ones, the matrix γ' satisfies $\gamma'(x) \in [0, r]^2$ and $0 < p/q < p'/q'$ are consecutive terms of $\mathcal{F}_m$, where $m = \max\{q, q'\} = \|\gamma'\|$.

Conversely, let $0 < p/q < p'/q'$ be successive terms of $\mathcal{F}_m$ such that $|q\theta - p| \le r$ and $|q'\theta - p'| \le r$. Let $\gamma = \left(\begin{smallmatrix} q & p \\ q' & p' \end{smallmatrix}\right)$; thus $\gamma \in \Gamma_+$, and there exists an unique matrix γ_i such that $\gamma_i(x) \in [0, r]^2$. Moreover, $\|\gamma_i\| = m$.

The theorem is proved. □

The next result proves that the number of pairs of successive Farey fractions satisfying simultaneously a Diophantine property is almost the double of the number of fractions satisfying the property.

THEOREM 2.3. *Let* $0 < \theta < 1$ *be irrational and* $r > 0$. *Then*

$$\left|\left\{\left(\frac{p'}{q'}, \frac{p}{q}\right) \in \mathcal{F}_n^2 : qp' - pq' = 1, \left|\frac{p}{q} - \theta\right| \le \frac{r}{q}, \left|\frac{p'}{q'} - \theta\right| \le \frac{r}{q'}\right\}\right| - 2\left|\left\{\frac{p}{q} \in \mathcal{F}_n : \left|\frac{p}{q} - \theta\right| \le \frac{r}{q}\right\}\right| \tag{2.2}$$

is a bounded sequence, for $n \in \mathbb{N}$.

Proof. Let $0 < \theta < 1$ be irrational and $r > 0$. Let $n \in \mathbb{N}$ and $p/q \in \mathcal{F}_n$ with $q > 1$. Assume $|p/q - \theta| \le r/q$.

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We will prove that, for all $p/q \in \mathcal{F}_n$, except finitely many, there exists a unique pair of successive fractions in $\mathcal{F}_{q-1}$ such that each one, say p'/q', satisfies

$$\left|\frac{p'}{q'} - \theta\right| \leq \frac{r}{q'}.$$

The proof comes from an alternative definition of the Farey sequence. We can define the sequence $\mathcal{F}_n$ using recurrence in the following way. We set

$$\mathcal{F}_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}.$$

Once $\mathcal{F}_n$ is defined (see (1.1)), we write it in ascending order

$$\mathcal{F}_n = \left\{\frac{h_0}{k_0} = \frac{0}{1}, \ldots, \frac{h_i}{k_i}, \ldots, \frac{h_m}{k_m} = \frac{1}{1}\right\},$$

where $m = |\mathcal{F}_n| - 1$, $h_0 = 0$, $k_0 = 1$, $h_m = 1$, $k_m = 1$, h_i and k_i are positive integers and $h_i k_{i-1} - k_i h_{i-1} = 1$, for all $i = 1, \ldots, m$. The Farey sequence of order $n+1$ consists of all elements of $\mathcal{F}_n$, besides all the fractions $(h_{i-1} + h_i)/(n+1)$, where $1 \leq i \leq m$ and $k_{i-1} + k_i = n + 1$. We recall that, for all $j \geq 1$,

$$\frac{h_{j-1}}{k_{j-1}} < \frac{h_{j-1} + h_j}{k_{j-1} + k_j} < \frac{h_j}{k_j}.$$

Therefore given $p/q \in \mathcal{F}_n$, $q > 1$, there exists an unique pair $c/d < a/b$ of successive fractions in $\mathcal{F}_{q-1}$ such that

$$\frac{p}{q} = \frac{a+c}{b+d}.$$

Next we prove that for all p/q such that $|(p/q) - \theta| \leq r/q$, except finitely many,

$$\left|\frac{a}{b} - \theta\right| \leq \frac{r}{b} \quad \left|\frac{c}{d} - \theta\right| \leq \frac{r}{d}.$$

We have $a/b = p/q + 1/bq$ and $c/d = p/q - 1/dq$, thus

$$\left|\frac{a}{b} - \theta\right| = \left|\frac{1}{bq} + \frac{p}{q} - \theta\right| \leq \frac{1}{bq} + \frac{r}{q} \quad \text{and} \quad \left|\frac{c}{d} - \theta\right| \leq \frac{1}{dq} + \frac{r}{q}.$$

Hence $1/bq + r/q \leq r/b$ if, and only if, $1 + br \leq qr$ if, and only if, $b \leq q - 1/r$. Analogously, $1/dq + r/q \leq r/d$ if, and only if, $d \leq q - 1/r$. As $d = q - b$, we deduce

$$\frac{1}{r} \leq b \leq q - \frac{1}{r}.$$

Next we prove that, for only finitely many such p/q, the denominator b can be taken outside the interval $[1/r, q - 1/r]$. We have

$$\theta - \frac{r}{q} < \theta - \frac{r}{q} + \frac{1}{bq} \leq \frac{a}{b} \leq \theta + \frac{r}{q} + \frac{1}{bq} \leq \theta + \frac{r+1}{q}$$

and

$$\theta - \frac{r+1}{q} \leq \frac{c}{d} \leq \theta + \frac{r}{q}.$$

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Let p_k/q_k be the convergent of θ such that

$$\left|\theta - \frac{p_{k+1}}{q_{k+1}}\right| < \frac{r+1}{q} \leq \left|\theta - \frac{p_k}{q_k}\right|.$$

Let e/f be a fraction with positive denominator such that

$$\left|\frac{e}{f} - \theta\right| \leq \left|\theta - \frac{p_k}{q_k}\right|,$$

thus $f \geq q_k$. Therefore, for q sufficiently large, $q_k > 1/r$, thus implying $b, d > 1/r$. Hence, for q sufficiently large, we conclude that $1/r < b, d < q - 1/r$, where $d = q - b$.

We have proved that for all $p/q \in \mathcal{F}_n$, except finitely many, p'/q' can be chosen equal to either a/b or c/d; thus the sequence

$$\left|\left\{\left(\frac{p'}{q'}, \frac{p}{q}\right) \in \mathcal{F}_n^2 : |qp' - pq'| = 1, q' < q, \left|\frac{p}{q} - \theta\right| \leq \frac{r}{q}, \left|\frac{p'}{q'} - \theta\right| \leq \frac{r}{q'}\right\}\right| - 2\left|\left\{\frac{p}{q} \in \mathcal{F}_n : |p - q\theta| \leq r\right\}\right|$$

is bounded. We notice that (2.2) takes into account the order the fractions appear, so the condition $q' < q$ is replaced by the condition $qp' - pq' = 1$. Thus (2.2) holds which concludes the proof of the theorem. □

Theorems 2.1 and 2.3 establish the equivalence of the results obtained, respectively, in [**2, 7, 11**] using different approaches. Using these two theorems, we conclude the proof of (1.2), the first part of Theorem 1.1.

3. *Counting Farey fractions*

First we prove a central theorem which regards Farey fractions (see (3.2)).

THEOREM 3.1. *Let* $0 < \theta < 1$ *be irrational and* $r > 0$. *Let* $n \in \mathbb{N}$. *Then*

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{\frac{p}{q} \in \mathcal{F}_n : \left|\frac{p}{q} - \theta\right| \leq \frac{r}{q}\right\}\right| = \frac{6}{\pi^2}2r. \tag{3.1}$$

Moreover, the error term

$$e(\theta, r, n) = \left|\left\{\frac{p}{q} \in \mathcal{F}_n : \left|\frac{p}{q} - \theta\right| \leq \frac{r}{q}\right\}\right| - n\frac{6}{\pi^2}2r$$

satisfies the following.

(i) *If there exist* $k, l \in \mathbb{Z}$ *such that* $r = k\theta + l$, *then* $e(\theta, r, n) = O(\sqrt{n})$.

(ii) *If* θ *is of constant type, then* $e(\theta, r, n) = O(\sqrt{n}\log n)$.

(iii) *If* θ *is of finite type* η, *then, for every* $\epsilon > 0$, $e(\theta, r, n) = O(n^{1-1/(2\eta)+\epsilon})$.

The proof of the above theorem uses three technical lemmas. The first one counts the number of primitive lattice points in the parallelogram whose vertices are clockwise $(0, r)$, $(0, -r)$, $(n, \alpha n + r)$ and $(n, \alpha n - r)$ (see Figure 2), which proves (3.1) using [**2, 11**].

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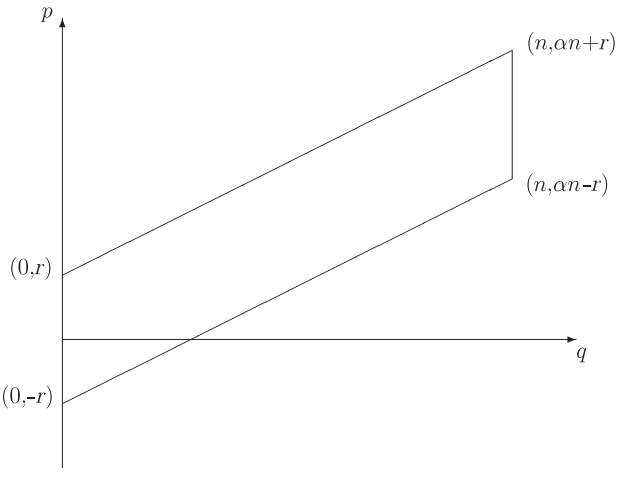


FIGURE 2. The parallelogram of vertices $(0, r)$, $(0, -r)$, $(n, \alpha n - r)$ and $(n, \alpha n + r)$.

LEMMA 3.2. *Let $\alpha > 0$ be irrational and $r > 0$ (see Figure 2). Then*

$$\lim_{n\to\infty} \frac{1}{n} |\{(q, p) \in \mathbb{N}^2 : q \le n, |q\alpha - p| \le r, (q, p) = 1\}| = \frac{6}{\pi^2} 2r.$$

Proof. Let $(q, p) \in \mathbb{N}^2$, $q \le n$ and $(q, p) = 1$. The points such that $0 \le q\alpha - p \le r$ are counted in (1.4). On the other hand, we have $0 \le p - q\alpha \le r$ if, and only if, $0 \le p(1/\alpha) - q \le r/\alpha$. By Figure 3

$$\left| \left\{ (q, p) \in \mathbb{N}^2 : p \le [n\alpha], 0 < p\frac{1}{\alpha} - q < \frac{r}{\alpha}, (q, p) = 1 \right\} \right|$$

is the number of primitive lattice points in the parallelogram whose vertices are clockwise $(0, 0)$, $(-r/\alpha, 0)$, $([\alpha n] - r/\alpha, [\alpha n])$ and $([\alpha n]/\alpha, [\alpha n])$. Comparing Figures 2 and 3 we verify that the sequence

$$\begin{aligned} &|\{(q, p) \in \mathbb{N}^2 : q \le n, 0 \le p - q\alpha \le r, (q, p) = 1\}| \\ &\quad - \left| \left\{ (q, p) \in \mathbb{N}^2 : p \le [n\alpha], 0 < p\frac{1}{\alpha} - q < \frac{r}{\alpha}, (q, p) = 1 \right\} \right| \end{aligned} \tag{3.2}$$

is bounded implying, by (1.4), that

$$\lim_{n\to\infty} \frac{1}{n} |\{(q, p) \in \mathbb{N}^2 : q \le n, |q\alpha - p| \le r, (q, p) = 1\}| = \frac{6}{\pi^2} 2r$$

which proves the lemma. □

Next for completeness a counting lemma is proved.

LEMMA 3.3. *Let $0 < \theta < 1$ be irrational. Let $n \in \mathbb{N}$. Then*

$$\left| \left\{ \frac{p}{q} \in \mathcal{F}_n : \left| \frac{p}{q} - \theta \right| \le \frac{1}{2q^2} \right\} \right| = O(\log n).$$

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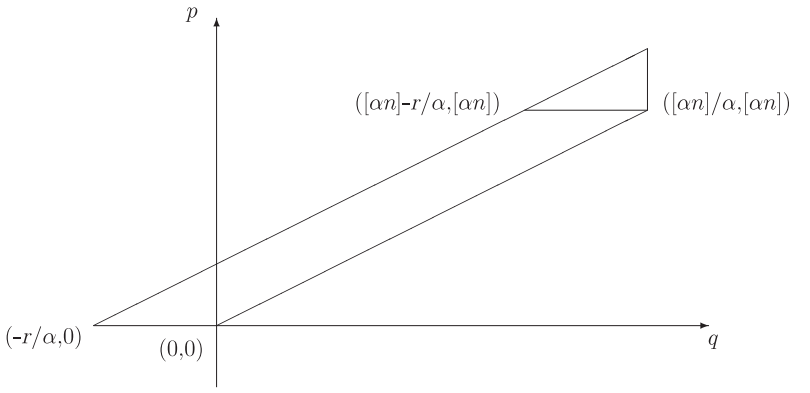


FIGURE 3. The parallelogram of vertices (0, 0), $(-r/\alpha, 0)$, $([\alpha n]/\alpha, [\alpha n])$ and $([\alpha n]/\alpha, [\alpha n])$.

Proof. We write $\theta = [0; a_1, a_2, \ldots]$ (see (1.5)), where $a_1, a_2, \ldots$ are natural numbers. Let $p/q \in \mathcal{F}_q$ be such that

$$\left|\frac{p}{q} - \theta\right| < \frac{1}{2q^2}.$$

Then, by [**3**, Theorem 184], p/q is a convergent of θ. Let $n \in \mathbb{N}$ and $k_n = k_n(\theta)$ be the number of positive convergents p/q of θ such that $q \le n$, which means

$$q_{k_n} \le n < q_{k_n+1}, \tag{3.3}$$

where $q_{-1} = 0$, $q_0 = 1$, $q_{i+1} = a_{i+1}q_i + q_{i-1}$, for all $i \ge 0$. We claim that among all irrational $0 < \theta < 1$, $\omega = (\sqrt{5} - 1)/2$ is the one which has the largest possible value k_n for all n, which means that, for all $i \ge 1$ and all irrational $0 < \theta < 1$,

$$q_i(\omega) \le q_i(\theta).$$

The number ω is the inverse of the golden number, and its continued fraction expansion is $[0; 1, 1, \ldots]$ (see (1.5)).

Now we will prove the claim. We have $q_1(\omega) = q_0(\omega) + q_{-1}(\omega) = 1$ which implies $q_1(\theta) = a_1(\theta) \ge 1 = q_1(\omega)$. Let $k \ge 1$ and assume $q_i(\theta) \ge q_i(\omega)$, for all $0 \le i \le k$, thus

$$q_{k+1}(\theta) = a_{k+1}(\theta)q_k(\theta) + q_{k-1}(\theta) \ge q_k(\omega) + q_{k-1}(\omega) = q_{k+1}(\omega),$$

which proves the claim. On the other hand we verify that

$$q_k(\omega) = \omega^{-k} - \omega^k.$$

Thus (3.3) reads $\omega^{-k} - \omega^k \le n < \omega^{-k-1} - \omega^{k+1}$, and as $\omega^k \to 0$, as $k \to \infty$, we have $k_n = O(\log n)$ which proves the lemma. □

In the next lemma we count the lattice points in $\mathbb{R}^2_+$ are close to a half-line.

LEMMA 3.4. *Let* $0 < \theta < 1$ *be irrational and* $r > 0$. *Let* $n \in \mathbb{N}$ *and*

$$f(r, n) = |\{(q, p) \in \mathbb{N}^2 : q \le n, 0 \le q\theta - p \le r\}|. \tag{3.4}$$

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(i) *The sequence* $(f(r,n)-rn)_n$ *is bounded if, and only if,* $r=k\theta+l$, *where* $k, l\in\mathbb{Z}$.
(ii) *Let* θ *be of constant type and* $0<r<1$, *then*

$$f(r,n)=rn+O(\log n).$$

(iii) *Let* θ *be of finite type* η, *then, for every* $\epsilon>0$,

$$f(r,n)=rn+O(n^{1-1/\eta+\epsilon}).$$

Moreover, in the above items the value of the error term is uniform, for example, for $0<r<1$.

Proof. Let $r=k\theta+l$, where $k, l\in\mathbb{Z}$. It follows, by [**2**], that

$$f(r,n)-rn$$

is a bounded sequence. The converse was proved by Kesten [**5**].

We assume that $r>0$ is not of the form $k\theta+l$ for $k, l\in\mathbb{Z}$. We write $r=[r]+\{r\}$; then the integer $[r]$ can be treated by the first paragraph of the proof. Thus we can assume $0<r<1$, moreover the line $\{(x,\theta x-r): x\in\mathbb{R}\}$ on $\mathbb{R}^2$ does not intersect the lattice $\mathbb{Z}^2$.

We write (3.4) as

$$\begin{aligned} f(r,n) &= |\{q\in\mathbb{N}: q\le n, 0<\{q\theta\}<r\}|+O(1)\\ &= nr+(|\{q\in\mathbb{N}: q\le n, 0<\{q\theta\}<r\}|-nr)+O(1). \end{aligned} \tag{3.5}$$

Let $D_n(\theta)$ be the discrepancy of the suite $(\{q\theta\})_{1\le q\le n}$ (see (1.6)). First assume θ is of constant type, then $nD_n(\theta)=O(\log n)$. By (3.5) and the definition of the discrepancy, it follows that

$$f(r,n)=rn+O(\log n).$$

Let θ be of finite type η. Then, for every $\epsilon>0$, by [**6**, Theorem 3.2, p. 123], $D_n(\theta)$ satisfies

$$D_n(\theta)=O(n^{-(1/\eta)+\epsilon}).$$

Again, using (3.4) and the definition of the discrepancy, we obtain

$$f(r,n)=rn+O(n^{1-(1/\eta)+\epsilon}).$$

This concludes the proof of the lemma. □

A reference for the subject of counting lattice points inside a plane region is Huxley [**4**]. In particular, [**4**, Theorem 2.3.3] gives an upper bound for the number of lattice points inside a convex plane polygon. As we have a long and narrow polygon in (3.4), the error term is large.

Proof of Theorem 3.1. Let $0<\theta<1$ be irrational and $r>0$. The limit (3.1) is proved in Lemma 3.2.

Now we look at the cases where an absolute error term can be provided. We assume $0<r<1$. Let $\mu(\cdot)$ denote the Mobius function. Let $n\in\mathbb{N}$, by (e.g., [**3**, Theorem 263]), $\sum_{d|n}\mu(d)=1$, if $n=1$, otherwise $\sum_{d|n}\mu(d)=0$.

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In the same way as in the proof of Lemma 3.2, it suffices to count the number of elements in the set

$$\{(q,\ p) \in \mathbb{N}^2 : q \le n,\ 0 \le q\theta - p \le r,\ (q,\ p) = 1\}. \tag{3.6}$$

In order to get rid of the condition $(p, q) = 1$, we will introduce the Mobius function.

We define $g(q,\ p) = 1$, if $(q,\ p) \in \mathbb{N}^2$ and $0 \le q\theta - p \le r$, otherwise $g(q,\ p) = 0$; we obtain that

$$\sum_{1\le p\le q\le n} g(q,\ p) = |\{(q,\ p) \in \mathbb{N}^2 : q \le n,\ 0 \le q\theta - p \le r\}| = rn + e(\theta, r, n). \tag{3.7}$$

The cardinality of (3.6) is

$$\begin{aligned}\sum_{1\le p\le q\le n,(q,p)=1} g(q,\ p) &= \sum_{1\le p\le q\le n} \sum_{d|p,d|q} \mu(d) g(q,\ p)\\ &= \sum_{d=1}^{n} \mu(d) \sum_{1\le p\le q\le n, d|p, d|q} g(q,\ p)\\ &= \sum_{d=1}^{n} \mu(d) \sum_{1\le p'\le q'\le n/d} g(q'd,\ p'd). \end{aligned}\tag{3.8}$$

Now we consider the set

$$\{(q',\ p') : 1 \le p' \le q' \le n/d,\ |p' - q'\theta| \le r/d\}. \tag{3.9}$$

We take into account the set (3.9) in the definition of g and write (3.8) as a sum of two terms whose first one is the sum, by (3.7),

$$\sum_{d=1}^{\sqrt{2rn}} \mu(d) \sum_{1\le p'\le q'\le n/d} g(q'd,\ p'd) = \sum_{d=1}^{\sqrt{2rn}} \mu(d) \left(\frac{r}{d}\frac{n}{d} + e\left(\theta,\ \frac{r}{d},\ \frac{n}{d}\right)\right).$$

We have $\sum_{d=1}^{n} \mu(d)/d^2 = 6/\pi^2 + O(1/n)$ (see [**3**, Theorem 287 and (17.2.2)]).

Assume $r = k\theta + l$, where $k, l \in \mathbb{Z}$. By Lemma 3.4,

$$\sum_{d=1}^{\sqrt{2rn}} e\left(\theta,\ \frac{r}{d},\ \frac{n}{d}\right) = O(\sqrt{n}).$$

Assume that $0 < r < 1$ is not of the form $k\theta + l$ for $k, l \in \mathbb{Z}$. By Lemma 3.4, if θ is of constant type

$$\sum_{d=1}^{\sqrt{2rn}} e\left(\theta,\ \frac{r}{d},\ \frac{n}{d}\right) = O(\sqrt{n}\log n).$$

On the other hand, if θ is of finite type η, for every $\epsilon > 0$,

$$\sum_{d=1}^{\sqrt{2rn}} e\left(\theta,\ \frac{r}{d},\ \frac{n}{d}\right) = \sum_{d=1}^{\sqrt{2rn}} O((n/d)^{1-1/\eta+\epsilon}) = O(n^{1-1/(2\eta)+\epsilon}).$$

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Now we consider the case where $1 \le p \le q \le n/d$ and $|q\theta - p| \le r/d$, and $d \ge \sqrt{2rn}$. We notice that $d \ge \sqrt{2rn}$ if, and only if, $\sqrt{d/r} \ge \sqrt{2n/d}$, or $d/r \ge 2n/d$. Thus $d/r \ge 2q$ implying that

$$\left|\frac{p}{q} - \theta\right| < \frac{1}{2q^2}.$$

The second term of the sum on the right-hand side of (3.8) satisfies

$$\sum_{d=\sqrt{2rn}}^{n} \mu(d) \sum_{1 \le p' \le q' \le n/d} f(q'd,\, p'd) \le \left|\left\{\frac{p}{q}\mathcal{F}_n : \left|\frac{p}{q} - \theta\right| < \frac{1}{2q^2}\right\}\right|. \tag{3.10}$$

By (3.10) and Lemma 3.3, the total error term (see (3.2)) is

$$e(\theta, r, n) = \sum_{d=1}^{\sqrt{2rn}} e\left(\theta, \frac{r}{d}, \frac{n}{d}\right) + O(\log n)$$

which concludes the proof of the theorem. □

4. *Proofs of Theorems 1.1 and 1.2*

Using Theorems 2.1 and 2.3, Theorems 1.1 follows from (3.1). Theorems 1.2 follows completely from Theorems 1.1 and 3.1.

5. *Counting n relatively prime to* $[n(p/q)]$

Here we show that the asymptotical behaviour proved in Theorem 1.1 (see (1.3)) can be derived as the limit of the number of elements of discrete orbits which are in the square. We follow the approach given in [**11**]. First we prove a technical lemma, where $\phi(\cdot)$ denotes the Euler function.

LEMMA 5.1. *Let* $q, a \in \mathbb{N}$. *Let* $N \in \mathbb{N}$ *be written as* $N = dq + l$, *where* $0 \le l < q$. *Then*

$$|\{a \le j < a + N : (q,\, j) = 1\}| = d\phi(q) + R(a, l, q), \tag{5.1}$$

where $0 \le R(a, l, q) = |\{a \le i < a + l : (q, i) = 1\}| \le \phi(q)$.

Proof. We have $|\{a \le j < a + q : (q,\, j) = 1\}| = \phi(q)$. Thus

$$|\{a \le j < a + dq : (q,\, j) = 1\}| = d\phi(q)$$

and

$$|\{a \le j < a + N : (q,\, j) = 1\}| = d\phi(q) + |\{a + dq \le j < a + dq + l : (q,\, j) = 1\}|.$$

On the other hand $0 \le l < d$, and therefore

$$|\{a + dq \le j < a + dq + l : (q,\, j) = 1\}| = |\{a \le j < a + l : (q,\, j) = 1\}| \le \phi(q),$$

which proves the lemma. □

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The next result is the counterpart of Theorem 3.1 when θ is replaced by a rational number p/q. We count the integers n relatively prime to $[np/q]$.

THEOREM 5.2. *Let* $p/q \in \mathcal{F}_q$ *and* $N \in \mathbb{N}$. *Then*

$$\left|\left\{1 \le n \le N : \left(n, \left[n\frac{p}{q}\right]\right) = 1\right\}\right| = \frac{N}{q} \sum_{1 \le i < q} \frac{\phi(i)}{i} + \epsilon(q, N), \tag{5.2}$$

where $1 \le \epsilon(N, q) \le 1 + \sum_{1 \le i < q} \phi(i)$. *The equation (5.2) shows that the density of the integers n relatively prime to* $[np/q]$ *is* $1/q \sum_{1 \le i < q} \phi(i)/i$. *Moreover,*

$$\frac{1}{q} \sum_{a \le i < q+a} \frac{\phi(i)}{i} \to \frac{6}{\pi^2} \quad \text{as } q \to \infty.$$

Proof. Let $n \in \mathbb{N}$. We write $n = bq + l$, where $0 \le l < q$; thus

$$\left[n\frac{p}{q}\right] = bp + \left[l\frac{p}{q}\right] = bp + l'.$$

Let $p'/q' \in \mathcal{F}_{q'}$ be such that $p'q - q'p = 1$; thus, for all integers u and v,

$$(u, v) = (pu - qv,\ p'u - q'v).$$

Therefore

$$\begin{aligned}\left(n, \left[n\frac{p}{q}\right]\right) &= (bq + l,\ bp + l') \\ &= (p(bq + l) - q(bp + l'),\ p'(bq + l) - q'(bp + l')) \\ &= (pl - ql',\ b + p'l - q'l').\end{aligned}$$

We have

$$pl - ql' = pl - q\left[l\frac{p}{q}\right] = q\left(l\frac{p}{q} - \left[l\frac{p}{q}\right]\right) = q\left\{l\frac{p}{q}\right\},$$

where $\{q\{lp/q\} : l = 0, \ldots, q-1\} = \{0, \ldots, q-1\}$. Thus

$$\left(n, \left[n\frac{p}{q}\right]\right) = (i,\ b + i'),$$

where $0 \le i < q$ and $i' \ge 1$ depends only on i. If $i = 0$, we verify that $i' = l$, which implies that

$$\sum_{0 \le b < N/q, (0, b+l) = 1} 1 = 1.$$

The left hand side of (5.2) equals

$$\sum_{1 \le n \le N, (n, [np/q]) = 1} 1 = 1 + \sum_{1 \le i < q} \sum_{0 \le b < N/q, (i, b+i') = 1} 1.$$

By Lemma 5.1, we obtain

$$\sum_{1 \le n < N, (n, [np/q]) = 1} 1 = 1 + \sum_{1 \le i < q} \left(\frac{N}{iq}\phi(i) + \epsilon_i(N, q)\right),$$

where $\epsilon_i(N, q) \le \phi(i)$, implying

$$= \frac{N}{q} \sum_{1 \le i < q} \frac{\phi(i)}{i} + \epsilon(N, q),$$

where $1 \le \epsilon(N, q) \le 1 + \sum_{1 \le i < q} \phi(i)$, which proves the theorem. □

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Next we apply the above result to prove that every infinite orbit of the linear action of Γ can be approached by discrete orbits. Relying on the methods introduced in the previous sections, we obtain a statement similar to (1.3) in Theorem 1.1. We have proved that the asymptotical behaviour (1.3) can be derived as an asymptotical limit of discrete orbits.

COROLLARY 5.3. *Let* $x = \binom{p/q}{1}$, *where* $0 < p/q < 1$ *and* $(q, p) = 1$. *Then*

$$\lim_{q\to\infty} \frac{1}{q} \lim_{n\to\infty} \frac{1}{n} |\{\gamma \in \Gamma : \gamma(x) \in [0, 1]^2, \|\gamma\| \le n\}| = 4\frac{6}{\pi^2}. \tag{5.3}$$

The case $r = 1$ is considered in Theorem 5.2; for any other r it works the same.

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