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# Entropy formula for random dynamical systems: relations between entropy, exponents and dimension

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*Abstract*. In this paper we consider random dynamical systems generated by compositions of one-sided independent and identically distributed random diffeomorphisms of class $C^2$ on a compact manifold. We prove an entropy formula for such random dynamical systems without assuming the SRB condition. This result is the random version of the main result obtained by Ledrappier and Young (The metric entropy of diffeomorphisms. Part II: Relations between entropy, exponents and dimension. *Ann. Math*. **122** (1985), 540–574).

## 1. *Introduction*

In smooth ergodic theory of deterministic dynamical systems, a remarkable result concerning the relationship between entropy and Lyapunov exponents is Pesin's entropy formula. It states that, if $f$ is a $C^2$ diffeomorphism on a compact Riemannian manifold $M$ preserving an SRB measure $\mu$, then Pesin's identity

$$h_\mu(f) = \int \sum_i \lambda_i(x)^+ m_i(x)\, d\mu,$$

holds, where $\lambda_i(x), 1 \le i \le r(x)$, denote the Lyapunov exponents of $f$ at the point $x$, with $m_i(x), 1 \le i \le r(x)$ their multiplicities and $h_\mu(f)$ the entropy of the system $f : (M, \mu) \circlearrowleft$. Pesin [**10**] first established the identity assuming the absolute continuity of $\mu$ with respect to the Lebesgue measure on $M$. (See [**8**] for an alternative proof.)

Without the SRB condition, Ledrappier and Young [**4**] proved a generalized entropy formula for the above system. The formula looks like Pesin's, and is the following:

$$h_\mu(f) = \int \sum_i \lambda_i(x)^+ \gamma_i(x)\, d\mu,$$

where the $\lambda_i(x)$'s are as before, but $\gamma_i(x)$ denotes, roughly speaking, the dimension of $\mu$ in the 'stretching (or unstable) directions' corresponding to $\lambda_i(x)$ at the point $x$; the $\gamma_i$'s are called the transverse dimensions.

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The question naturally arises if a similar result holds true for random dynamical systems (RDSs). We give an affirmative answer by proving in this paper that the entropy of an independent and identically distributed (i.i.d.) RDS associated with an invariant measure $\mu$ is indeed

$$\int \sum_i \lambda_i(x)^+ \gamma_i(x)\, d\mu,$$

where the $\gamma_i$'s have geometrical meanings similar to those in the deterministic case (see §2 for the explicit definition for the $\gamma_i$'s).

Ledrappier and Young mentioned these $\gamma_i$'s in [**5**]. Their result concerns the relations between the (pointwise) dimension of the sample measures and Lyapunov dimension under some absolute continuity conditions on the invariant measure. We believe that the above random version of the generalized entropy formula is known to them, although they did not mention this clearly. In view of the importance of such a formula, we think that a rigorous proof should exist somewhere. We provide this in this paper.

The proof of our result follows essentially the idea presented by Ledrappier and Young [**4**]. Also it involves tedious techniques that are parallel to those obtained by Ledrappier and Young [**6**] for RDSs. So we will concentrate upon the necessary modifications and omit the proof of most of the parallel arguments, for which we refer the reader to Liu and Qian [**7**].

The paper is organized in the following way. Section 2 is devoted to fixing required notions and to the formulation of the main result. In §3, we give some preliminary results. Sections 4 and 5 consist of the proof of the main result in the ergodic case for the unique two-sided extension of our one-sided system. Section 6 completes the proof without the ergodic assumption. Our result for the one-sided system then follows readily.

## 2. *Required notions and statement of the main result*

2.1. *The general setting.* In this paper, $M$ will always be an $m$-dimensional smooth and compact Riemannian manifold without boundary, with a norm $|\cdot|$ on $TM$. Let $\Sigma := \mathrm{Diff}^2(M)$ be the space of all $C^2$ diffeomorphisms on $M$ equipped with the $C^2$ topology and $v$ a Borel probability measure on $\Sigma$. In what follows we will denote by $\mathcal{B}(X)$ the Borel $\sigma$-algebra of a topological space $X$. Put $(\Omega, \mathcal{F}, v^{\mathbb{Z}^+}) := \prod_0^{+\infty}(\Sigma, \mathcal{B}(\Sigma), v)$. Thus $v^{\mathbb{Z}^+}$ is a probability on $(\Omega, \mathcal{F})$ which is invariant under the left shift operator $\theta$ on $\Omega$. For each $\omega \in \Omega$ write $\omega = (f_0(\omega), f_1(\omega), \dots)$ and define

$$f_\omega^n = \begin{cases} f_{n-1}(\omega) \circ \cdots \circ f_0(\omega) & \text{if } n > 0, \\ \mathrm{id} & \text{if } n = 0. \end{cases} \tag{2.1}$$

This set-up will be referred to as $\mathcal{X}^+(M, v)$ in the rest of this paper.

Define

$$(\Omega^*, \mathcal{F}^*, v^{\mathbb{Z}}) := \prod_{-\infty}^{+\infty}(\Sigma, \mathcal{B}(\Sigma), v)$$

and write $\omega = (\dots, f_{-1}(\omega), f_0(\omega), f_1(\omega), \dots)$ for $\omega \in \Omega^*$. The left shift operator on $\Omega^*$ will still be denoted by $\theta$. We also define $f_\omega^n$ for $\omega \in \Omega^*$ and $n \in \mathbb{Z}$ similar to (2.1), only

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adding that

$$f_\omega^n := (f_{\theta^n\omega}^{-n})^{-1}, \quad \text{if } n < 0.$$

Also there are two skew product maps,

$$\Theta : \Omega \times M \to \Omega \times M, \quad (\omega, x) \mapsto (\theta\omega, f_0(\omega)x)$$

and

$$\Theta : \Omega^* \times M \to \Omega^* \times M, \quad (\omega, x) \mapsto (\theta\omega, f_0(\omega)x),$$

associated with $\Omega \times M$ and $\Omega^* \times M$ respectively.

2.2. *Invariant measures and sample measures.* Let $\mathcal{X}^+(M, v)$ be given. We denote by $\sigma$ the $\sigma$-algebra generated by $\{A \times M : A \in \mathcal{F}^*\}$.

*Definition 2.1.* An *invariant measure* of $\mathcal{X}^+(M, v)$ is defined as a probability $\mu$ on $M$ such that

$$\int_\Sigma f\mu \, dv(f) = \mu.$$

Such invariant measures always exist. We denote by $\mathcal{M}(\mathcal{X}^+(M, v))$ the set of all invariant measures of $\mathcal{X}^+(M, v)$. Clearly a measure $\mu$ is invariant with respect to $\mathcal{X}^+(M, v)$ if and only if $v^{\mathbb{Z}^+} \times \mu$ is $\Theta$-invariant. For each $\mu \in \mathcal{M}(\mathcal{X}^+(M, v))$, there exists a unique $\Theta$-invariant measure $\mu^*$ on $\Omega^* \times M$ which has marginal measure $v^{\mathbb{Z}^+} \times \mu$ on $\Omega \times M$. This measure $\mu^*$ has marginal measure $v^{\mathbb{Z}}$ on $\Omega^*$. (See [**6**] or [**7**, p. 25].)

Since $\mathrm{Diff}^2(M)$ is a Polish space (see Hirsch [**2**]), $(\Omega^* \times M, \mu^*)$ with the $\mu^*$-completion of $\mathcal{F}^* \times \mathcal{B}(M)$ constitutes a Lebesgue space. According to Rokhlin [**11**], one can speak of the conditional measure $\mu_\omega^*$ of $\mu^*$ on $\{\omega\} \times M$ (identified with $M$) for $v^{\mathbb{Z}}$-a.e. $\omega$. $\{\mu_\omega^*\}_{\omega\in\Omega^*}$ is $v^{\mathbb{Z}}$-mod 0 uniquely defined and is called the family of *sample measures* of $\mu^*$. Clearly the sample measures have the invariance property $f_\omega^n \mu_\omega^* = \mu_{\theta^n\omega}^*$ for any $n \in \mathbb{Z}$ and $v^{\mathbb{Z}}$-a.e. $\omega$.

From now on $\mathcal{X}^+(M, v)$ associated with an invariant measure $\mu$ will be referred to as $\mathcal{X}^+(M, v, \mu)$ with its unique extension on $\Omega^* \times M$ denoted by $\mathcal{X}(M, v, \mu^*)$.

*Definition 2.2.* $\mu \in \mathcal{M}(\mathcal{X}^+(M, v))$ is said to be *ergodic* if $\Theta : (\Omega^* \times M, \mu^*) \hookleftarrow$ is ergodic.

2.3. *Lyapunov exponents.* Throughout this paper, it is always assumed that the probability $v$ on $\mathrm{Diff}^2(M)$ satisfies

$$\int (\log^+ |f|_{C^2} + \log^+ |f^{-1}|_{C^2}) \, dv(f) < +\infty, \tag{2.2}$$

where $|g|_{C^2}$ denotes the usual $C^2$ norm of $g \in \mathrm{Diff}^2(M)$ (see [**9**] or [**7**, p. 23]).

Let $\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_{r(x)}$ be the distinct Lyapunov exponents of $\mathcal{X}^+(M, v, \mu)$ at a regular (in the sense of Oseledec) point $(\omega, x) \in \Omega \times M$, with multiplicities $m_1(x), m_2(x), \ldots, m_{r(x)}(x)$. Note that these numbers are independent of $\omega \in \Omega$. By (2.2), all Lyapunov exponents are finite numbers.

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Now we apply the Oseledec multiplicative ergodic theorem to the two-sided RDS $\mathcal{X}(M, v, \mu^*)$, yielding the following. There exists a $\Theta$-invariant Borel set $\Lambda^* \subset \Omega^* \times M$ with full $\mu^*$-measure. For each $(\omega, x) \in \Lambda^*$, there is a splitting of $T_x M$,

$$T_x M = E_1(\omega, x) \oplus E_2(\omega, x) \oplus \cdots \oplus E_{r(x)}(\omega, x), \tag{2.3}$$

such that $\dim E_i(\omega, x) = m_i(x)$ for $i = 1, \ldots, r(x)$ and

$$\lim_{n \to \pm\infty} \frac{1}{n} \log |T_x f_\omega^n \xi| = \lambda_i(x), \quad \text{for all } \xi \in E_i(\omega, x) \setminus \{0\}.$$

Let $u(x), c(x)$ and $s(x)$ be the number of positive, neutral and negative Lyapunov exponents at $(\omega, x) \in \Lambda^*$ respectively, i.e.

$$u(x) := \#\{1 \le j \le r(x) : \lambda_j(x) > 0\}, \tag{2.4}$$
$$c(x) := \#\{1 \le j \le r(x) : \lambda_j(x) = 0\}, \tag{2.5}$$
$$s(x) := \#\{1 \le j \le r(x) : \lambda_j(x) < 0\}. \tag{2.6}$$

When $\mu^*$ (or equivalently $\mu$) is an ergodic measure, all numbers $r(x), u(x), c(x), s(x)$ and $\{\lambda_i(x), m_i(x) : 1 \le i \le r(x)\}$ will be constants for $\mu^*$-a.e. $(\omega, x) \in \Lambda^*$. In this case, we omit the $x$'s for simplicity of notation.

2.4. *Pointwise dimensions and transverse dimensions.*

*Definition 2.3.* $W^i(\omega, x) := \{y \in M : \limsup_{n \to +\infty} (1/n) \log d(f_\omega^{-n} x, f_\omega^{-n} y) \le -\lambda_i(x)\}$ is called the $i$th unstable manifold of $\mathcal{X}(M, v, \mu^*)$ at $(\omega, x)$, where $(\omega, x) \in \Lambda^*$ and $1 \le i \le u(x)$.

The $W^i(\omega, x)$'s are all $C^{1,1}$ immersed submanifolds of $M$ tangent at $x$ to $\bigoplus_{j=1}^i E_j(\omega, x)$ respectively. Hence each $W^i(\omega, x)$ inherits a Riemannian structure from $M$. This gives rise to a Riemannian metric, written $d^i_{(\omega,x)}(\cdot, \cdot)$, on each leaf of $W^i(\omega, x)$.

*Definition 2.4.* A measurable partition $\eta$ of $\Omega^* \times M$ is said to be *subordinate to the $W^i$-manifolds* if for $\mu^*$-a.e. $(\omega, x)$, $\eta_\omega(x) := \{y : (\omega, y) \in \eta(\omega, x)\} \subset W^i(\omega, x)$ and $\eta_\omega(x)$ contains an open neighborhood of $x$ in $W^i(\omega, x)$.

We have included in §3.3 an outline of the construction of such partitions. See also [**7**, pp. 94–101] for a similar construction.

*Definition 2.5.* A measurable partition $\eta$ is said to be *increasing* if $\Theta^{-1}\eta > \eta$, and to be a generator if $\mathcal{B}(\eta_0^n) \to \mathcal{B}(\Omega^* \times M)$, ($\mu^*$-mod 0) as $n \to +\infty$, where $\eta_0^n := \bigvee_{k=0}^n \Theta^{-k}\eta$ and $\mathcal{B}(\eta)$ denotes the $\sigma$-algebra generated by measurable $\eta$-sets.

In what follows we will define notions of transverse dimensions along unstable manifolds.

Let $\varepsilon > 0$. For each $(\omega, x) \in \Lambda^*$, put

$$B^i(\omega, x; \varepsilon) := \{y \in W^i(\omega, x) : d^i_{(\omega,x)}(x, y) < \varepsilon\}.$$

Let $\eta_1 > \eta_2 > \cdots > \eta_u$ be a sequence of measurable partitions of $\Omega^* \times M$ with each $\eta_i$ subordinate to the corresponding $W^i$-manifolds. The canonical system of conditional measures of $\mu^*_\omega$ associated with $\eta_{i,\omega}$ is denoted by $\{(\mu^*_\omega)_x^{\eta_{i,\omega}}\}$.

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We define the *lower* and *upper pointwise dimension of $\mu^*$ along $W^i$-manifolds* at $(\omega, x) \in \Lambda^*$ with respect to the partition $\eta_i$ by

$$\underline{\delta}_i(\omega, x; \eta_i) := \liminf_{\varepsilon \to 0} \log(\mu^*_\omega)^{\eta_{i,\omega}}_x(B^i(\omega, x; \varepsilon))/\log \varepsilon, \tag{2.7}$$

$$\overline{\delta}_i(\omega, x; \eta_i) := \limsup_{\varepsilon \to 0} \log(\mu^*_\omega)^{\eta_{i,\omega}}_x(B^i(\omega, x; \varepsilon))/\log \varepsilon. \tag{2.8}$$

Sometimes we write $\underline{\delta}_i(\omega, x; \eta_i, \mu^*) := \underline{\delta}_i(\omega, x; \eta_i)$ to indicate the dependence of this quantity on $\mu^*$. Other notations have similar meanings.

The following proposition tells us that the lower and upper pointwise dimension of $\mu^*$ along $W^i$-manifolds coincide. We call this common value the *pointwise dimension of $\mu$ along $W^i$-manifolds*. It will be verified in §§5 and 6.

PROPOSITION 2.1. *If $\eta_i$ is an increasing generator subordinate to $W^i$-manifolds, then*

$$\underline{\delta}_i(\omega, x; \eta_i) = \overline{\delta}_i(\omega, x; \eta_i) =: \delta_i(\omega, x), \quad \mu^*\text{-a.e. } (\omega, x),$$

*where $\delta_i(\omega, x)$ is $\Theta$-invariant and depends only on $x$. So we write $\delta_i(x) := \delta_i(\omega, x)$. Furthermore, $\delta_i(\cdot)$ does not depend on the choice of such $\eta_i$.*

PROPOSITION 2.2. *Let $\delta_i(x), i = 1, \ldots, u(x)$ be introduced as above and define $\gamma_i(x) := \delta_i(x) - \delta_{i-1}(x)$ for $\mu^*$-a.e. $(\omega, x)$ and $1 \le i \le u(x)$ (where $\delta_0(x) := 0$). Then*

$$0 \le \gamma_i(x) \le m_i(x) \quad \text{for } i = 1, \ldots, u(x).$$

The number $\gamma_i(x)$ is called the *transverse dimension of $\mu$ on $W^i / W^{i-1}$* at $x$.

2.5. *Statement of the main result.* The main result of this paper is the following theorem.

THEOREM 2.3. *Let $\mathcal{X}^+(M, v, \mu)$ be given. Then the following entropy formula holds:*

$$h_\mu(\mathcal{X}^+(M, v)) = \int \sum_i \lambda_i(x)^+ \gamma_i(x)\, d\mu. \tag{2.9}$$

We state some corollaries of Theorem 2.3 that are known results.

COROLLARY 2.4. (Ruelle's inequality) *For any system $\mathcal{X}^+(M, v, \mu)$ we have*

$$h_\mu(\mathcal{X}^+(M, v)) \le \int \sum_i \lambda_i(x)^+ m_i(x)\, d\mu.$$

COROLLARY 2.5. *If $\mu$ is an SRB measure of $\mathcal{X}^+(M, v)$, then Pesin's entropy formula*

$$h_\mu(\mathcal{X}^+(M, v)) = \int \sum_i \lambda_i(x)^+ m_i(x)\, d\mu$$

*holds.*

Corollary 2.4 follows readily from Theorem 2.3 and Proposition 2.2. If $\mu$ is an SRB measure then sample measures $\{\mu^*_\omega\}_\omega$ have absolutely continuous conditional measures on $W^u$-manifolds (see [7]) and so $\delta_u(x) = \sum_{i=1}^u m_i(x)$, which implies that $\gamma_i(x) = m_i(x)$ for $1 \le i \le u(x)$. So Corollary 2.5 follows from the theorem.

We will prove Theorem 2.3 in §§3–6. Before this, we make in advance the assumption that $\mu$ (and hence $\mu^*$) is ergodic for simplicity of presentation. It is used only in §§3–5.

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## 3. *Preliminaries*

In this section, we state some preliminary results which will be very useful in subsequent sections. First we will introduce in §3.1 Lyapunov charts. In §3.2 we state some propositions about unstable manifolds, each of which is analogous to a certain statement in [**7**] and can be proved following the same lines. Hence the proofs are omitted. In view of these propositions one can compare the induced metric $d^i_{(\omega,x)}(\cdot,\cdot)$ on $W^i(\omega,x)$ with the original metric $d(\cdot,\cdot)$ on $M$ via Lyapunov charts. Then we include in §3.3 an outline of the construction of measurable partitions subordinate to $W^i$-manifolds. Some useful measurable partitions are also constructed. Finally in §3.4 two types of transverse metrics are built on quotient spaces.

3.1. *Lyapunov charts.* As usual, $|\cdot|$ denotes the norm on $T_xM$. We write $\mathbb{R}^m = \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_r}$ and for each $z \in \mathbb{R}^m$, let $(z_1, z_2, \ldots, z_r)$ be its coordinates with respect to this splitting. The usual standard norm of Euclidean space will always be denoted by $\|\cdot\|$. Since we want to employ the following so-called Lyapunov charts to obtain estimations on the unstable sets and center unstable sets, for $i = 1, \ldots, c+u$ write $\bar{\mathbb{R}}^{(i)} := \mathbb{R}^{m_1+\cdots+m_i}$, $\bar{\mathbb{R}}^{r-(i)} := \mathbb{R}^{m_{i+1}+\cdots+m_r}$ and put

$$\begin{aligned}
\bar{\mathbb{R}}^{(i)}(\rho) &:= \{z \in \mathbb{R}^{m_1+\cdots+m_i} : \|z\| \le \rho\},\\
\bar{\mathbb{R}}^{r-(i)}(\rho) &:= \{z \in \mathbb{R}^{m_{i+1}+\cdots+m_r} : \|z\| \le \rho\},\\
\bar{\mathbb{R}}(\rho) &:= \{z = (z_1, \ldots, z_r) \in \mathbb{R}^m : \|z_i\| \le \rho, 1 \le i \le r\}.
\end{aligned}$$

For each $z \in \mathbb{R}^m$ write $z^{(i)} := (z_1, \ldots, z_i)$ and $z^{r-(i)} := (z_{i+1}, \ldots, z_r)$; and define maximum norms $\|\cdot\|'$ and $\|\cdot\|'_i$ on $\mathbb{R}^m = \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_r}$ and $\mathbb{R}^m = \bar{\mathbb{R}}^{(i)} \times \bar{\mathbb{R}}^{r-(i)}$ by

$$\begin{aligned}
\|z\|' &:= \max_{1\le i\le r} \|z_i\|,\\
\|z\|'_i &:= \max(\|z^{(i)}\|, \|z^{r-(i)}\|).
\end{aligned}$$

Let $0 < \varepsilon < (1/(100m)) \min\{|\lambda_i - \lambda_j|, |\lambda_i| > 0 : i \ne j\}$ and $e^{-\lambda_u+10\varepsilon} + e^{5\varepsilon} < 2$. Put $\lambda_0 := \max\{|\lambda_i| : 1 \le i \le r\} + 2\varepsilon$. Now we introduce the Lyapunov charts, the analog of that given in [**4**] with slight modifications to meet the requirements of the following proposition; the proof is parallel and hence omitted (see for instance [**3**]).

PROPOSITION 3.1. *There exists a $\Theta$-invariant Borel set $\Lambda_0^* \subset \Lambda^*$ with full $\mu^*$-measure and a Borel function $l : \Lambda_0^* \to [1, +\infty)$ with $l(\Theta^{\pm1}(\omega,x)) \le e^{\varepsilon} l(\omega,x)$ for all $(\omega,x) \in \Lambda_0^*$ such that at each $(\omega,x) \in \Lambda_0^*$, there is an embedding $\Phi_{(\omega,x)} : \bar{\mathbb{R}}(\rho) \to M$ ($\rho = l(\omega,x)^{-1}$) satisfying the following.*

(1) *$\Phi_{(\omega,x)}(0) = x$ and $T_0\Phi_{(\omega,x)}$ takes $\mathbb{R}^{m_i}$ onto $E_i(\omega,x)$ for $i = 1, \ldots, r$ respectively. Furthermore, $\Phi_{(\omega,x)} = \exp_x \circ T_0\Phi_{(\omega,x)}|_{\bar{\mathbb{R}}(\rho)}$.*

(2) *Let $H_{(\omega,x)} := \Phi^{-1}_{\Theta(\omega,x)} \circ f^1_\omega \circ \Phi_{(\omega,x)}$ and $H^n_{(\omega,x)} := \Phi^{-1}_{\Theta^n(\omega,x)} \circ f^n_\omega \circ \Phi_{(\omega,x)}$ be defined wherever they make sense. Then for each $n \in \mathbb{Z}$, $1 \le i \le r$ and $z \in \mathbb{R}^{m_i}$,*

$$\exp(n\lambda_i - |n|\varepsilon)\|z\| \le \|T_0H^n_{(\omega,x)}z\| \le \exp(n\lambda_i + |n|\varepsilon)\|z\|.$$

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(3) *If* $\mathrm{Lip}(g)$ *denotes the Lipschitz constant of the function* $g$*, then*

$$\mathrm{Lip}(H_{(\omega,x)} - T_0 H_{(\omega,x)}) \le \varepsilon,$$
$$\mathrm{Lip}(H^{-1}_{(\omega,x)} - T_0 H^{-1}_{(\omega,x)}) \le \varepsilon$$

*and*

$$\mathrm{Lip}(T.H_{(\omega,x)}) \le \varepsilon l(\omega, x),$$
$$\mathrm{Lip}(T.H^{-1}_{(\omega,x)}) \le \varepsilon l(\omega, x).$$

(4) *For any* $z, z' \in \bar{\mathbb{R}}(\rho)$ *one has*

$$K_0^{-1} d(\Phi_{(\omega,x)}z, \Phi_{(\omega,x)}z') \le \|z - z'\| \le l(\omega, x) d(\Phi_{(\omega,x)}z, \Phi_{(\omega,x)}z'),$$

*for some universal constant* $K_0 > 0$*.*

The above $\{\Phi_{(\omega,x)}\}_{(\omega,x)} \in \Lambda_0^*$ will be referred to as a system of $(\varepsilon, l)$-charts.

3.2. *Some estimations on unstable manifolds.* Let $\triangle := \{(\omega, x) \in \Lambda_0^* : l(\omega, x) \le l_0\}$ be a non-empty set with $l_0 > 0$. Put $E^{(i)}(\omega, x) := \bigoplus_{j=1}^{i} E_j(\omega, x)$ and $E^{r-(i)}(\omega, x) := \bigoplus_{j=i+1}^{r} E_j(\omega, x)$ for $i = 1, \ldots, c + u$. Now fix an $i \le u$.

PROPOSITION 3.2. *For each* $n \in \mathbb{Z}^+$*, there exists a continuous family of* $C^1$ *embedded* $\left(\sum_{j=1}^{i} m_j\right)$*-dimensional disks* $\{W^i_{-n}(\omega, x)\}_{(\omega,x)\in\triangle}$ *in* $M$ *and numbers* $\alpha_n$*,* $\beta_n$ *and* $\gamma_n$*, which depend only on* $i$*,* $\varepsilon$ *and* $l_0$*, such that the following statements hold true for every* $(\omega, x) \in \triangle$*.*

(1) *There exists a* $C^{1,1}$ *map* $g^i_{(\omega,x),n}: O_n(\omega, x) \to E^{r-(i)}(\Theta^{-n}(\omega, x))$*, where* $O_n(\omega, x)$ *is an open subset of* $E^{(i)}(\Theta^{-n}(\omega, x))$ *which contains* $E^{(i)}(\Theta^{-n}(\omega, x)) \cap T_{f^{-n}_\omega x} M(\alpha_n)$*, such that:*
   (i) $g^i_{(\omega,x),n}(0) = 0$*;*
   (ii) $\mathrm{Lip}(g^i_{(\omega,x),n}) \le \beta_n, \mathrm{Lip}(T.g^i_{(\omega,x),n}) \le \beta_n$*;*
   (iii) $W^i_{-n}(\omega, x) = \exp_{f^{-n}_\omega x} \mathrm{Graph}(g^i_{(\omega,x),n})$ *and* $W^i_{-n}(\omega, x)$ *is tangent to* $E^{(i)}(\Theta^{-n}(\omega, x))$ *at* $f^{-n}_\omega x$*.*

(2) $f^{-1}_{\Theta^{-n}(\omega,x)} W^i_{-n}(\omega, x) \subset W^i_{-n-1}(\omega, x)$.

(3) $d^i_{\Theta^{-n-l}(\omega,x)}(f^{-l}_{\Theta^{-n}(\omega,x)}y, f^{-l}_{\Theta^{-n}(\omega,x)}z) \le \gamma_n \exp((-\lambda_i + 4\varepsilon)l) d^i_{\Theta^{-n}(\omega,x)}(y, z)$*, for* $y, z \in W^i_{-n}(\omega, x)$ *and* $l \in \mathbb{Z}^+$*, where* $d^i_{\Theta^{-n}(\omega,x)}(\cdot, \cdot)$ *is the distance along* $W^i_{-n}(\omega, x)$ *for* $n \in \mathbb{Z}^+$*.*

(4) $\alpha_n = \alpha_0 e^{-5n\varepsilon}$, $\beta_n = \beta_0 e^{7n\varepsilon}$, $\gamma_n = \gamma_0 e^{2n\varepsilon}$.

Let $\{\Phi_{(\omega,x)}\}_{(\omega,x)\in\Lambda_0^*}$ be a system of $(\varepsilon, l)$-charts. Sometimes it is necessary to reduce the size of the charts. Let $\delta \in (0, 1]$ be a reduction factor. For $(\omega, x) \in \Lambda_0^*$, define

$$S^{cu}_\delta(\omega, x) := \{z \in \bar{\mathbb{R}}(l(\omega, x)^{-1}) : \|H^{-n}_{(\omega,x)}z\| \le \delta l(\Theta^{-n}(\omega, x))^{-1}, n \ge 0\},$$

that is, $\Phi_{(\omega,x)} S^{cu}_\delta(\omega, x)$ consists of those points in $M$ whose backward orbit under actions $f^{-n}_{(\omega,x)}$, $n \ge 0$ stays well inside the domains of the charts at $f^{-n}_\omega x$ for all $n \ge 0$. It is called

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the center unstable set of $\mathcal{X}(M, v, \mu^*)$ at $(\omega, x)$ associated with $(\{\Phi_{(\omega,x)}\}_{(\omega,x)\in\Lambda_0^*}, \delta)$. On $S_\delta^{cu}(\omega, x)$, one clearly has

$$H_{(\omega,x)}^{-n} = H_{\Theta^{-n+1}(\omega,x)}^{-1} \circ \cdots \circ H_{(\omega,x)}^{-1}, n \geq 1.$$

We introduce the local $i$th-unstable manifolds of $\mathcal{X}(M, v, \mu^*)$ at $(\omega, x)$ associated with $(\{\Phi_{(\omega,x)}\}_{(\omega,x)\in\Lambda_0^*}, \delta)$. It is defined to be the component of $W^i(\omega, x)\cap\Phi_{(\omega,x)}\bar{\mathbb{R}}(\delta l(\omega, x)^{-1})$ that contains $x$. The $\Phi_{(\omega,x)}^{-1}$-image of this set is denoted by $W_{(\omega,x),\delta}^i(x)$. The following proposition characterizes $W_{(\omega,x),\delta}^i(x)$. See [**7**, pp. 146–147] for the proof of a similar result.

PROPOSITION 3.3. *Let $\{\Phi_{(\omega,x)}\}_{(\omega,x)\in\Lambda_0^*}$ be a system of $(\varepsilon, l)$-charts.*

(1) *If $0 < \delta \leq e^{-\lambda_0-\varepsilon}$ and $(\omega, x) \in \Lambda_0^*$, then:*
   (i) *$W_{(\omega,x),\delta}^i(x)$ is the graph of a $C^1$ function*

$$g_{(\omega,x)}^i : \bar{\mathbb{R}}^{(i)}(\delta l(\omega, x)^{-1}) \to \bar{\mathbb{R}}^{r-(i)}(\delta l(\omega, x)^{-1})$$

   *with $g_{(\omega,x)}^i(0) = 0$ and $\mathrm{Lip}(g_{(\omega,x)}^i) < 1$;*
   (ii) *$W_{(\omega,x),\delta}^1(x) \subset \cdots \subset W_{(\omega,x),\delta}^u(x) \subset S_\delta^{cu}(\omega, x)$.*

(2) *If $0 < \delta \leq e^{-2\lambda_0-2\varepsilon}$ and $(\omega, x) \in \Lambda_0^*$, then:*
   (i) *$H_{(\omega,x)}W_{(\omega,x),\delta}^i(x) \cap \bar{\mathbb{R}}(\delta l(\Theta(\omega, x))^{-1}) = W_{\Theta(\omega,x),\delta}^i(f_\omega^1 x)$;*
   (ii) *$S_\delta^{cu}(\omega, x) \cap \Phi_{(\omega,x)}^{-1}W^i(\omega, x) = W_{(\omega,x),\delta}^i(x)$.*

Fix $(\omega, x)\in\Lambda_0^*$. Let $\delta\in(0, \frac{1}{4}]$. Consider now $y\in\Phi_{(\omega,x)}S_\delta^{cu}(\omega, x)$ with $(\omega, y) \in \Lambda_0^*$. Let $W_{(\omega,x),2\delta}^i(y)$ be the $\Phi_{(\omega,x)}^{-1}$-image of the component of $W^i(\omega, y) \cap \Phi_{(\omega,x)}[\bar{\mathbb{R}}^{(i)}(2\delta l(\omega, x)^{-1}) \times \bar{\mathbb{R}}^{r-(i)}(4\delta l(\omega, x)^{-1})]$ that contains $y$. Then $\Phi_{(\omega,x)}W_{(\omega,x),2\delta}^i(y)$ contains an open neighborhood of $y$ in $W^i(\omega, y)$ and is also referred to as a local $i$th unstable manifold of $\mathcal{X}(M, v, \mu^*)$ at $(\omega, y)$ (although in general $\Phi_{(\omega,x)}W_{(\omega,y),2\delta}^i(y) \neq \Phi_{(\omega,x)}W_{(\omega,x),2\delta}^i(y)$). The following proposition holds analogously to Proposition 3.3.

PROPOSITION 3.4. *For each $(\omega, x) \in \Lambda_0^*$ and $1 \leq i \leq u$ the following hold.*

(1) *Let $0 < \delta \leq \frac{1}{4}e^{-\lambda_0-\varepsilon}$. If $y \in \Phi_{(\omega,x)}S_\delta^{cu}(\omega, x)$ with $(\omega, y) \in \Lambda_0^*$, then:*
   (i) *$W_{(\omega,x),2\delta}^i(y)$ is a graph of a $C^1$ function*

$$g_{(\omega,x),y}^i : \bar{\mathbb{R}}^{(i)}(2\delta l(\omega, x)^{-1}) \to \bar{\mathbb{R}}^{r-(i)}(4\delta l(\omega, x)^{-1})$$

   *with $\mathrm{Lip}(g_{(\omega,x),y}^i) < 1$;*
   (ii) *$W_{(\omega,x),2\delta}^1(y) \subset \cdots \subset W_{(\omega,x),2\delta}^u(y) \subset S_{4\delta}^{cu}(\omega, x)$.*

(2) *Let $0 < \delta \leq \frac{1}{4}e^{-2\lambda_0-2\varepsilon}$. If $y \in \Phi_{(\omega,x)}S_\delta^{cu}(\omega, x)$ with $(\omega, y) \in \Lambda_0^*$ and $f_\omega^1 y \in \Phi_{\Theta(\omega,x)}S_\delta^{cu}(\Theta(\omega, x))$, then*

$$H_{(\omega,x)}W_{(\omega,x),2\delta}^i(y) \cap [\bar{\mathbb{R}}^{(i)}(2\delta l(\Theta(\omega, x))^{-1}) \times \bar{\mathbb{R}}^{r-(i)}(4\delta l(\Theta(\omega, x))^{-1})] \subset W_{\Theta(\omega,x),2\delta}^i(f_\omega^1 y).$$

(3) *Let $0 < \delta \leq \frac{1}{4}e^{-2\lambda_0-2\varepsilon}$. For $\mu^*$-a.e. $(\omega, x) \in \Lambda_0^*$:*
   (i) *if $y \in \Phi_{(\omega,x)}S_\delta^{cu}(\omega, x)$ with $(\omega, y) \in \Lambda_0^*$, then*

$$S_{2\delta}^{cu}(\omega, x) \cap \Phi_{(\omega,x)}^{-1}W^i(\omega, y) \subset W_{(\omega,x),2\delta}^i(y) \subset S_{4\delta}^{cu}(\omega, x) \cap \Phi_{(\omega,x)}^{-1}W^i(\omega, y);$$

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(ii) *if* $y \in \Phi_{(\omega,x)} W^u_{(\omega,x),\delta}(x)$ *with* $(\omega, y) \in \Lambda_0^*$, *then*

$$W^i_{(\omega,x),2\delta}(y) \subset W^{i+1}_{(\omega,x),2\delta}(y) \quad \text{for } i = 1, \dots, u-1;$$

(iii) *if* $y, y' \in \Phi_{(\omega,x)} W^u_{(\omega,x),\delta}(x)$ *with* $(\omega, y), (\omega, y') \in \Lambda_0^*$, *then either*

$$W^i_{(\omega,x),2\delta}(y) = W^i_{(\omega,x),2\delta}(y')$$

*or otherwise the two terms in the above equation are disjoint.*

The following proposition describes the actions $\{H^n_{(\omega,x)}\}_{n\in\mathbb{Z}}$.

PROPOSITION 3.5. *Let* $0 < \delta \le e^{-\lambda_0-\varepsilon}$. *For each* $(\omega, x) \in \Lambda_0^*$ *and* $1 \le i \le u$ *the following hold.*

(1) *If* $z, z' \in \bar{\mathbb{R}}(e^{-\lambda_1-3\varepsilon} l(\omega,x)^{-1})$, *then* $H_{(\omega,x)}z$, $H_{(\omega,x)}z' \in \bar{\mathbb{R}}(l(\Theta(\omega,x))^{-1})$ *and*

$$\|H_{(\omega,x)}z - H_{(\omega,x)}z'\|' \le e^{\lambda_1+2\varepsilon}\|z - z'\|'.$$

(2) *If* $z, z' \in \bar{\mathbb{R}}(\delta l(\omega,x)^{-1})$ *and* $\|z - z'\|'_i = \|z^{(i)} - z'^{(i)}\|$, *then*

$$\|H_{(\omega,x)}z - H_{(\omega,x)}z'\|'_i = \|(H_{(\omega,x)}z)^{(i)} - (H_{(\omega,x)}z')^{(i)}\| \ge e^{\lambda_i-2\varepsilon}\|z - z'\|'_i.$$

(3) $\|H^{-1}_{(\omega,x)}z - H^{-1}_{(\omega,x)}z'\|'_i \le e^{-\lambda_i+2\varepsilon}\|z - z'\|'_i, \forall z, z' \in W^i_{(\omega,x),\delta}(x)$.

(4) $\|H^{-1}_{(\omega,x)}z - H^{-1}_{(\omega,x)}z'\|'_{c+u} \le e^{2\varepsilon}\|z - z'\|'_{c+u}, \forall z, z' \in S^{cu}_\delta(\omega,x)$.

(5) *Let* $0 < \delta \le \frac{1}{4}e^{-2\lambda_0-2\varepsilon}$. *For* $\mu^*$*-a.e.* $(\omega, x) \in \Lambda_0^*$, *if* $y \in \Phi_{(\omega,x)} S^{cu}_\delta(\omega,x)$ *with* $(\omega, y) \in \Lambda_0^*$, *then*

$$\|H^{-1}_{(\omega,x)}z - H^{-1}_{(\omega,x)}z'\|'_i \le e^{-\lambda_i+2\varepsilon}\|z - z'\|'_i, \quad \text{for all } z, z' \in W^i_{(\omega,x),2\delta}(y).$$

After an easy computation one has the following lemma which says that the metrics $d^i_{(\omega,x)}(\cdot,\cdot)$ and $d(\cdot,\cdot)$ are locally equivalent within $\Phi_{(\omega,x)} W^i_{(\omega,x),\delta}(x)$, a neighborhood of $x$ in $W^i(\omega,x)$.

LEMMA 3.6. *If* $y, z \in \Phi_{(\omega,x)} W^i_{(\omega,x),\delta}(x)$, *then*

$$d(y,z) \le d^i_{(\omega,x)}(y,z) \le 2K_0\|\Phi^{-1}_{(\omega,x)}y - \Phi^{-1}_{(\omega,x)}z\|'_i \le 2K_0 l(\omega,x) d(y,z).$$

3.3. *Related partitions.* First we include here an outline of the construction of partitions subordinate to $W^i$-manifolds, which follows the same lines presented in [**7**]. It is simpler because $\mu^*$ is now assumed to be ergodic.

Fix an $i \le u$. Let $l_0$ be large enough such that $\triangle$ (see §3.2) has positive measure. Then there exists an increasing sequence $\{\triangle_k\}_{k\in\mathbb{Z}^+}$ of compact subsets of $\triangle$ such that $\mu^*\big(\triangle \setminus \bigcup_k \triangle_k\big) = 0$. Fix a $k$ with $\mu^*(\triangle_k) > 0$. Let $\rho_0 > 0$ be a number such that the exponential map $\exp_x : T_xM(\rho_0) \to B(x,\rho_0)$ is a $C^\infty$ diffeomorphism for every $x \in M$.

PROPOSITION 3.7. *Let* $\{W^i_{\mathrm{loc}}(\omega,x)\}_{(\omega,x)\in\triangle_k}$ *be a continuous family of* $C^1$ *embedded* $\big(\sum_{j=1}^i m_j\big)$*-dimensional disks described in Proposition 3.2 with* $n = 0$ *and suitable* $\alpha_0$ *such that* $W^i_{\mathrm{loc}}(\omega,x) \subset \Phi_{(\omega,x)} W^i_{(\omega,x),\hat\delta}(x)$, *where* $\hat\delta = \frac{1}{4}e^{-\lambda_0+\varepsilon}$. *Then the following hold.*

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(1) *There exists* $A_k > 0$ *such that for all* $y, z \in W^i_{\mathrm{loc}}(\omega, x)$ *and* $n \geq 0$,

$$d^i_{\Theta^{-n}(\omega,x)}(f_\omega^{-n} y, f_\omega^{-n} z) \leq A_k \exp(-n(\lambda_i - 2\varepsilon)) d^i_{(\omega,x)}(y, z).$$

(2) *There exist* $\hat{r} \in (0, \rho_0/4), \hat{\varepsilon} \in (0, 1)$ *and* $\hat{d} \geq 2\hat{r}$ *such that for all* $\rho \in (0, \hat{r}]$ *and* $(\omega, x) \in \triangle_k$, *if* $(\omega', x') \in B_{\triangle_k}(\omega, x; \hat{\varepsilon}\rho) := \{(\hat{\omega}, \hat{x}) \in \triangle_k : d(\omega, \hat{\omega}) < \hat{\varepsilon}\rho, d(x, \hat{x}) < \hat{\varepsilon}\rho\}$, *then* $W^i_{\mathrm{loc}}(\omega', x') \cap B(x, \rho)$ *is connected with its* $d^i$*-diameter less than* $\hat{d}$*; and the map*

$$(\omega', x') \mapsto W^i_{\mathrm{loc}}(\omega', x') \cap B(x, \rho)$$

*is a continuous map from* $B_{\triangle_k}(\omega, x; \hat{\varepsilon}\rho)$ *to the space of subsets of* $B(x, \rho)$ *(endowed with the Hausdorff topology).*

(3) *Let* $\rho \in (0, \hat{r}]$ *and* $(\omega, x) \in \triangle_k$. *If* $(\omega', x'), (\omega', x'') \in B_{\triangle_k}(\omega, x; \hat{\varepsilon}\rho)$, *then either*

$$W^i_{\mathrm{loc}}(\omega', x') \cap B(x, \rho) = W^i_{\mathrm{loc}}(\omega', x'') \cap B(x, \rho)$$

*or otherwise the two terms in the above equation are disjoint. In the latter case, if it is assumed moreover that* $x'' \in W^i(\omega', x')$, *then*

$$d^i_{(\omega', x')}(y, z) > \hat{d} > 2\hat{r}$$

*for any* $y \in W^i_{\mathrm{loc}}(\omega', x') \cap B(x, \rho)$ *and* $z \in W^i_{\mathrm{loc}}(\omega', x'') \cap B(x, \rho)$.

(4) *There exists* $\hat{R} > 0$ *such that for each* $(\omega, x) \in \triangle_k$, *if* $(\omega', x') \in B_{\triangle_k}(\omega, x; \hat{\varepsilon}\rho)$ *and* $y \in W^i_{\mathrm{loc}}(\omega', x') \cap B(x, \hat{r})$, *then* $W^i_{\mathrm{loc}}(\omega', x')$ *contains the closed ball of center* $y$ *and* $d^i$*-radius* $\hat{R}$ *in* $W^i(\omega', x')$.

We now choose in $\triangle_k$ a density point $(\omega^*, x^*)$. For each $\rho \in [\hat{r}/2, \hat{r}]$, put

$$S_\rho := \bigcup_{(\omega,x) \in B_{\triangle_k}(\omega^*, x^*; \hat{\varepsilon}\rho)} \{\omega\} \times [W^i_{\mathrm{loc}}(\omega, x) \cap B(x^*, \rho)],$$

and let $\xi_\rho$ be the partition of $\Omega^* \times M$ into all sets $\{\omega\} \times [W^i_{\mathrm{loc}}(\omega, x) \cap B(x^*, \rho)]$, $(\omega, x) \in B_{\triangle_k}(\omega^*, x^*; \hat{\varepsilon}\rho)$ and the set $\Omega^* \times M \setminus S_\rho$. We now define a measurable function $\beta_\rho : S_\rho \mapsto \mathbb{R}^+$ by

$$\beta_\rho(\omega, y) := \inf_{n \geq 0} \left\{ \hat{R}, \frac{1}{2A_k} d(f_\omega^{-n} y, \partial B(x^*, \rho)) e^{n(\lambda_i - 2\varepsilon)}, \frac{\rho}{A_k} \right\}.$$

By arguments analogous to those in the proof of Proposition IV.2.1 in [**7**], we know that there exists $r' \in [\hat{r}/2, \hat{r}]$ such that $\beta_{r'} > 0$ $\mu^*$-a.e. on $S_i := S_{r'}$, and define

$$\xi_i := \xi^+_{r'} = \bigvee_{n=0}^{+\infty} \Theta^n \xi_{r'}.$$

Clearly, $\xi_i$ is an increasing generator subordinate to $W^i$-manifolds of $\mathcal{X}(M, v, \mu^*)$.

Let us introduce some more related partitions in order to make use of the geometry of Lyapunov charts in the evaluation of local entropy in §§4 and 5. Let $\{\Phi_{(\omega,x)}\}_{(\omega,x) \in \Lambda_0^*}$ be a system of $(\varepsilon, l)$-charts and $\delta \in (0, \frac{1}{16} e^{-2(\lambda_0 + \varepsilon)})$ be a reduction factor. Then there exists a measurable partition $\mathcal{D}$ of $\Omega^* \times M$ such that:

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(1) $\mathcal{D}^+_\omega(x) \subset \Phi_{(\omega,x)} S^{cu}_\delta(\omega, x)$ for $\mu^*$-a.e. $(\omega, x)$;
(2) $\sigma < \mathcal{D}$ and $H_{\mu^*}(\mathcal{D}|\sigma) < +\infty$;
(3) $\{S_i, \Omega^* \times M \setminus S_i\} < \mathcal{D}$ for $i = 1, \ldots, u$;
(4) $\{\hat{E}, \Omega^* \times M \setminus \hat{E}\} < \mathcal{D}$, where $\hat{E}$ will be specified later in §3.4.

Now define $\eta_i := \xi_i \vee \mathcal{D}^+$ for $i = 1, \ldots, u$. We have the following proposition.

PROPOSITION 3.8. *$\{\eta_i\}_{i=1}^u$ satisfy the following statements:*
(1) $\eta_1 > \eta_2 > \cdots > \eta_u$;
(2) *the $\eta_i$'s are increasing generators and $\eta_{i,\omega}(x) \subset \Phi_{(\omega,x)} W^i_{(\omega,x),\delta}(x)$ for $\mu^*$-a.e. $(\omega, x)$;*
(3) $h^\sigma_{\mu^*}(\Theta, \eta_i) = h^\sigma_{\mu^*}(\Theta, \xi_i)$ *for $i = 1, \ldots, u$, where*

$$h^\sigma_{\mu^*}(\Theta, \eta) := \lim_{n \to +\infty} \frac{1}{n} H_{\mu^*}(\eta_0^n | \sigma);$$

(4) *for $\mu^*$-a.e. $(\omega, x)$ and $2 \le i \le u$, if $(\omega, y) \in \eta_i(\omega, x) \cap \Lambda_0^*$, then*

$$\Phi_{(\omega,x)} W^{i-1}_{(\omega,x),2\delta}(y) \cap \eta_{i,\omega}(x) = \eta_{i-1,\omega}(y)$$

*and*

$$\Theta^{-1}(\eta_{i-1}(\omega, y)) = \eta_{i-1}(\Theta^{-1}(\omega, y)) \cap \Theta^{-1}(\eta_i(\omega, x)).$$

3.4. *Transverse metrics on $\eta_i(\omega, x)/\eta_{i-1}$ with $2 \le i \le u$.* Let $\{\Phi_{(\omega,x)}\}_{(\omega,x)\in\Lambda_0^*}$ be a system of $(\varepsilon, l)$-charts. Fix $(\omega, x) \in \Lambda_0^*$. Let $1 \le i \le u$ and $\hat{\delta} := \frac{1}{4}\exp(-\lambda_0 - \varepsilon)$. Denote by $L(\bar{\mathbb{R}}^{(i)}, \bar{\mathbb{R}}^{r-(i)})$ the space of all linear maps from $\bar{\mathbb{R}}^{(i)}$ to $\bar{\mathbb{R}}^{r-(i)}$. By Proposition 3.4(1)(i) we know that, if $y \in \Phi_{(\omega,x)} S^{cu}_{\hat{\delta}}(\omega, x)$ with $(\omega, y) \in \Lambda_0^*$, then there exists a unique $P^i_{(\omega,x),y} \in L(\bar{\mathbb{R}}^{(i)}, \bar{\mathbb{R}}^{r-(i)})$ with $\|P^i_{(\omega,x),y}\| < 1$ such that

$$T_y \Phi^{-1}_{(\omega,x)} E^{(i)}(\omega, y) = \mathrm{Graph}(P^i_{(\omega,x),y}).$$

Define

$$\mathcal{L}^i_{(\omega,x)} : \{y \in \Phi_{(\omega,x)} S^{cu}_{\hat{\delta}}(\omega, x) : (\omega, y) \in \Lambda_0^*\} \mapsto L(\bar{\mathbb{R}}^{(i)}, \bar{\mathbb{R}}^{r-(i)});$$
$$y \mapsto P^i_{(\omega,x),y}.$$

The following proposition says that the map $\mathcal{L}^i_{(\omega,x)}$ is Lipschitz for all $(\omega, x) \in \Lambda_0^*$.

PROPOSITION 3.9. *For each $(\omega, x) \in \Lambda_0^*$ and $1 \le i \le u$, $\mathcal{L}^i_{(\omega,x)}$ is a Lipschitz map and*

$$\mathrm{Lip}(\mathcal{L}^i_{(\omega,x)}) \le D_0 l(\omega, x)^2,$$

*where $D_0 > 0$ is a number depending only on the exponents and $\varepsilon$.*

Let $\eta_i$'s be introduced as above. For $2 \le i \le u$, we now define two metrics on the factor-space $\eta_i(\omega, x)/\eta_{i-1}$ for $\mu^*$-a.e. $(\omega, x)$. We shall actually deal with $\eta_i$'s restricted to a certain measurable set with full $\mu^*$-measure. Now we choose a $\Theta$-invariant measurable set $\Lambda_0' \subset \Lambda_0^*$ with $\mu^*(\Lambda_0') = 1$ such that for each $(\omega, x) \in \Lambda_0'$ the requirements of Proposition 3.8 are satisfied. We then put

$$\eta_i' := \eta_i|_{\Lambda_0'}, \quad i = 1, \ldots, u.$$

In what follows we define two transverse metrics on $\eta_i'(\omega, x)/\eta_{i-1}'$ for $\mu^*$-a.e. $(\omega, x) \in \Lambda_0'$.

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First we give a point-dependent definition. Let $(\omega, x) \in \Lambda_0'$. From Proposition 3.8, we know that for every $y \in \eta'_{i,\omega}(x)$, $W^{i-1}_{(\omega,x),2\delta}(y)$ intersects $\{0\} \times \mathbb{R}^{r-(i-1)}$ at exactly one point. We denote the $i$th coordinate of this point by $\zeta^i_y \in \mathbb{R}^{m_i}$. Clearly $\zeta^i_y$ is the $i$th coordinate of the point $(0, g^i_{(\omega,x),y}(0))$. For $y, y' \in \eta'_{i,\omega}(x)$, define

$$\hat{d}^i_{(\omega,x)}(y, y') := \|\zeta^i_y - \zeta^i_{y'}\|.$$

By Proposition 3.8, $\hat{d}^i_{(\omega,x)}(\cdot,\cdot)$ induces a metric on $\eta'_i(\omega, x)/\eta'_{i-1}$ for $i = 2, \ldots, u$.

We can express the above metric in another way using Lemma 8.3.1 in [**4**]. First, let

$$p^{(u)} : \mathbb{R}^m = \mathbb{R}^{m_1+\cdots+m_r} \to \bar{\mathbb{R}}^{(u)} = \mathbb{R}^{m_1+\cdots+m_u}$$

be the natural projection map. Then for each $(\omega, x) \in \Lambda_0'$ fixed, $p^{(u)}|_{W^u_{(\omega,x),\delta}(x)}$ is a lipeomorphism between $W^u_{(\omega,x),\delta}(x)$ and its image. For $i = 1, \ldots, u-1$, define foliations by

$$F^i_{(\omega,x)} := \{z \in p^{(u)} W^i_{(\omega,x),\delta}(y) : y \in \Phi_{(\omega,x)} W^u_{(\omega,x),\tau\delta}(x), (\omega, y) \in \Lambda_0'\},$$

where $\tau \in (0, \frac{1}{4}e^{-2(\lambda_0+\varepsilon)})$. By Proposition 3.9, these foliations satisfy the requirements of Lemma 8.3.1 in [**4**] with $\rho = \delta l(\omega, x)^{-1}$ and $C = D_0 l(\omega, x)^2$, providing $\delta$ and $\varepsilon$ small enough. Hence, according to Lemma 8.3.1 in [**4**], there exists a map $I_{(\omega,x)} = (I^1_{(\omega,x)}, \ldots, I^u_{(\omega,x)}) : \bar{\mathbb{R}}^{(u)}(\rho) \to \bar{\mathbb{R}}^{(u)}$ such that:

(1) $I_{(\omega,x)}$ is a homeomorphism between $\bar{\mathbb{R}}^{(u)}(\rho)$ and its image;

(2) for every $z, z' \in \bar{\mathbb{R}}^{(u)}(\rho)$, $I^j_{(\omega,x)}(z) = I^j_{(\omega,x)}(z')$ for $j = i+1, \ldots, u$ iff $z' \in F^i_{(\omega,x)}(z)$;

(3) both $I_{(\omega,x)}$ and $I^{-1}_{(\omega,x)}$ are Lipschitz with Lipschitz constant depending only on $C$.

Let $\pi_{(\omega,x)} = (\pi^1_{(\omega,x)}, \ldots, \pi^u_{(\omega,x)}) : W^u_{(\omega,x),\tau\delta}(x) \to \bar{\mathbb{R}}^{(u)}$ be given by

$$\pi_{(\omega,x)} := I_{(\omega,x)} \circ p^{(u)}.$$

We have $\zeta^i_y = \pi^i_{(\omega,x)} \circ \Phi^{-1}_{(\omega,x)} y$ for each $y \in \eta'_{i,\omega}(x)$ with $2 \le i \le u$. Clearly $\pi_{(\omega,x)}$ is a lipeomorphism between $W^u_{(\omega,x),\tau\delta}(x)$ and its image with $\mathrm{Lip}(\pi_{(\omega,x)})$, $\mathrm{Lip}(\pi^{-1}_{(\omega,x)}) \le N(\omega, x)$, where $N(\omega, x)$ depends only on $l(\omega, x)$ and the Lyapunov exponents. Moreover, $\pi_{(\omega,x)} W^i_{(\omega,x),\delta}(y)$ lies on a $\sum_{j=1}^i m_j$-dimensional plane parallel to $\bar{\mathbb{R}}^{(i)} \times \{0\} \times \cdots \times \{0\}$ and if $W^i_{(\omega,x),\delta}(y) \ne W^i_{(\omega,x),\delta}(y')$, then $\pi_{(\omega,x)} W^i_{(\omega,x),\delta}(y)$ and $\pi_{(\omega,x)} W^i_{(\omega,x),\delta}(y')$ lie on distinct planes.

Though $\hat{d}^i_{(\omega,x)}(\cdot,\cdot)$ is a metric on $\eta'_i(\omega, x)/\eta'_{i-1}$, in general $\hat{d}^i_{(\omega,x)}(\cdot,\cdot) \ne \hat{d}^i_{(\omega,x')}(\cdot,\cdot)$ for $x' \in \eta'_{i,\omega}(x)$ with $x' \ne x$. Now we need to rectify this situation to give a point-independent definition.

Let $(\omega^*, x^*)$ be as introduced in §3.3 corresponding to $\triangle_k$. Then there exist positive numbers $\tau_0$ and $s_0$ with $0 < \tau_0 < \frac{1}{4}e^{-2(\lambda_0+\varepsilon)}$ and a set

$$\hat{E} := B_{\triangle_k}(\omega^*, x^*; s_0/2)$$

such that the following (a) and (b) hold true (where $\hat{\rho} = \tau_0 \delta l_0^{-1}$).

(a) Let $(\omega, x) \in \triangle_k$ with $x \in \Phi_{(\omega^*,x^*)}\bar{\mathbb{R}}(\hat{\rho})$. For $i = 1, \ldots, u-1$, if $(\omega, y) \in \Lambda_0'$ and $y \in \Phi_{(\omega,x)} W^u_{(\omega,x),\tau_0\delta}(x) \cap \Phi_{(\omega^*,x^*)}\bar{\mathbb{R}}(\hat{\rho})$, then there exists a map

$$h^i_{(\omega,x),y} : \bar{\mathbb{R}}^{(i)}(2\hat{\rho}) \to \bar{\mathbb{R}}^{r-(i)}(2\hat{\rho})$$

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with $\mathrm{Lip}(h^i_{(\omega,x),y}) < \frac{1}{3}$ and

$$\mathrm{Graph}(h^i_{(\omega,x),y}) = \Phi^{-1}_{(\omega^*,x^*)} \circ \Phi_{(\omega,x)} W^i_{(\omega,x),\delta}(y) \cap \bar{\mathbb{R}}(2\hat{\rho}).$$

(b) For each $(\omega, x) \in \hat{E}$ and $1 \le i \le u-1$, define a foliation by

$$F^i_{(\omega^*,x^*),(\omega,x)} := \{p^{(u)}(\mathrm{Graph}(h^i_{(\omega,x),y})) : y \in \Phi_{(\omega,x)} W^u_{(\omega,x),\tau_0\delta}(x) \cap \Phi_{(\omega^*,x^*)}\bar{\mathbb{R}}(\hat{\rho})\}.$$

Then by Lemma 8.3.1 in [**4**], there exists a map

$$I_{(\omega^*,x^*),(\omega,x)} : \bar{\mathbb{R}}^{(u)}(\hat{\rho}) \to \bar{\mathbb{R}}^{(u)}$$

satisfying the requirements analogous to the above (1)–(3) for $I_{(\omega,x)}$.

We then define a map $\tilde{\pi} = (\tilde{\pi}^1, \ldots, \tilde{\pi}^u) : \bigcup_{n=0}^{+\infty} \Theta^n \hat{E} \to \bar{\mathbb{R}}^{(u)}$ as follows: for each $(\omega, y) \in \hat{E}$, suppose $y \in \Phi_{(\omega,x)} W^u_{(\omega,x),\tau_0\delta}(x) \cap \Phi_{(\omega^*,x^*)}\bar{\mathbb{R}}(\hat{\rho})$ with $(\omega, x) \in \triangle_k$, then

$$\tilde{\pi}(\omega, y) := I_{(\omega^*,x^*),(\omega,x)} \circ p^{(u)} \circ \Phi^{-1}_{(\omega^*,x^*)} y$$

and in general, let $\tilde{\pi}(\omega, y) := \tilde{\pi}(\Theta^{-n(\omega,y)}(\omega, y))$, where $n(\omega, y) := \inf\{k \ge 0 : \Theta^{-k}(\omega, y) \in \hat{E}\}$. Then for $i = 2, \ldots, u$ we define a point-independent metric on $\eta'_i(\omega, x)/\eta'_{i-1}$ by

$$\tilde{d}^i_{(\omega,x)}((\omega, y), (\omega, y')) := \|\tilde{\pi}^i(\omega, y) - \tilde{\pi}^i(\omega, y')\|, \quad \text{for all } (\omega, y), (\omega, y') \in \eta'_i(\omega, x).$$

The above metrics satisfy the following two propositions.

PROPOSITION 3.10. *Let $z \in W^u_{(\omega,x),\tau\delta}(x)$ with $0 < \tau < e^{-\lambda_1 - 3\varepsilon}$. Then for $1 \le i \le u$,*

$$\|\pi^i_{\Theta(\omega,x)} \circ H_{(\omega,x)} z\| \le e^{\lambda_i + 3\varepsilon} \|\pi^i_{(\omega,x)} z\|.$$

PROPOSITION 3.11. *There exists $N_0 > 0$ such that for all $y, y' \in \eta'_{i,\omega}(x)$ with $(\omega, x) \in \hat{E}$,*

$$N_0^{-1} \hat{d}^i_{(\omega,x)}((\omega, y), (\omega, y')) \le \tilde{d}^i_{(\omega,x)}((\omega, y), (\omega, y')) \le N_0 \hat{d}^i_{(\omega,x)}((\omega, y), (\omega, y')).$$

## 4. *Definition of local entropy along unstable manifolds*

In this section, we will define quantities named *local entropy along unstable manifolds*. These quantities play an important role in our arguments. In the ergodic case, the notion of local entropy along the $W^i$-manifolds is described by a number $h_i$ which measures the amount of randomness along the leaves of $W^i$-manifolds. As Ledrappier and Young have done in [**4**], although there are several equivalent definitions, we take a pointwise approach following [**1**].

Let $\varepsilon > 0$. For $(\omega, x) \in \Lambda^*_0$ and $n \in \mathbb{Z}^+$, put

$$B^i(\omega, x; n, \varepsilon) := \{y \in W^i(\omega, x) : d^i_{\Theta^k(\omega,x)}(f^k_\omega x, f^k_\omega y) < \varepsilon \text{ for } 0 \le k \le n\}.$$

Let $\eta$ be a measurable partition of $\Omega^* \times M$ subordinate to $W^i$-manifolds. Define

$$\underline{h}_i(\omega, x; \varepsilon, \eta) := \liminf_{n\to+\infty} -\frac{1}{n} \log(\mu^*_\omega)^{\eta_\omega}_x(B^i(\omega, x; n, \varepsilon)),$$

$$\overline{h}_i(\omega, x; \varepsilon, \eta) := \limsup_{n\to+\infty} -\frac{1}{n} \log(\mu^*_\omega)^{\eta_\omega}_x(B^i(\omega, x; n, \varepsilon)).$$

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Clearly these functions are measurable. Furthermore, we define the *lower* and *upper local entropy along $W^i$-manifolds* at $(\omega, x)$ with respect to $\eta$ by

$$\underline{h}_i(\omega, x; \eta) := \lim_{\varepsilon \to 0} \underline{h}_i(\omega, x; \varepsilon, \eta),$$
$$\overline{h}_i(\omega, x; \eta) := \lim_{\varepsilon \to 0} \overline{h}_i(\omega, x; \varepsilon, \eta).$$

These limits exist because $\underline{h}_i(\omega, x; \varepsilon, \eta)$ and $\overline{h}_i(\omega, x; \varepsilon, \eta)$ increase as $\varepsilon \downarrow 0$.

Let $\xi_i$ be an increasing generator subordinate to $W^i$-manifolds.

PROPOSITION 4.1. $\underline{h}_i(\omega, x; \xi_i) = \overline{h}_i(\omega, x; \xi_i) =: h_i = H_{\mu^*}(\xi_i | \Theta\xi_i \vee \sigma)$, $\mu^*$-*a.e.* $(\omega, x)$.

Proposition 4.1 tells us that the lower and upper local entropy along $W^i$-manifolds with respect to $\xi_i$ are coincident. From the proof below and the ergodic decomposition of $\mu^*$ in §6, we know that in general $h_i$ is a $\Theta$-invariant function independent of the choice of $\xi_i$. In view of Corollary VI.1.1 of [7], $h_i$ depends only on $x$ for $\mu^*$-a.e. $(\omega, x)$. So we write $h_i = h_i(x)$ and call it the *local entropy along $W^i$-manifolds* at $(\omega, x)$. This completes the definition of $h_i$.

Let us first introduce some facts and postpone the proof of Proposition 4.1 until the end of this section.

LEMMA 4.2. *Let $\alpha$ be a partition of $\Omega^* \times M$ with $H_{\mu^*}(\alpha|\sigma) < +\infty$. If $\xi$ is an increasing generator, then*

$$\lim_{n \to +\infty} -\frac{1}{n} \log(\mu^*_\omega)^{\xi_\omega}_x([(\alpha \vee \xi)^n_0]_\omega(x)) = H_{\mu^*}(\xi | \Theta\xi \vee \sigma), \quad \mu^*\text{-a.e. } (\omega, x) \in \Omega^* \times M.$$

LEMMA 4.3. *There exists a partition $\alpha$ of $\Omega^* \times M$ with $H_{\mu^*}(\alpha|\sigma) < +\infty$ such that*

$$[\alpha^n_0 \vee \xi_i]_\omega(x) \subset B^i(\omega, x; n, \delta), \quad n \geq n_0(\omega, x)$$

*for $\mu^*$-a.e. $(\omega, x)$, where $n_0 : \Omega^* \times M \mapsto \mathbb{Z}^+$ is a measurable function.*

*Proof of Lemma 4.2.* Define $I(\eta|\xi)(\omega, x) := -\log(\mu^*_\omega)^{\xi_\omega}_x(\eta_\omega(x))$. We have

$$\frac{1}{n} I((\alpha \vee \xi)^n_0 | \xi)(\omega, x) = \frac{1}{n} I(\alpha|\xi)(\omega, x) + \frac{1}{n} \sum_{k=0}^{n} I(\alpha \vee \xi | \Theta\xi \vee \alpha^{-1}_{-k})(\Theta^k(\omega, x)), \quad (4.1)$$

where $\alpha^l_{-k} := \bigvee_{j=-k}^{l} \Theta^{-j}\alpha$. Put $I_n := I(\alpha \vee \xi | \Theta\xi \vee \alpha^{-1}_{-n})$ and $I^* := \sup_{n \geq 1} I_n$. One can prove that $\int I^*(\omega, x)\, d\mu^* \leq H_{\mu^*}(\alpha \vee \xi|\sigma) + 1$ and that $\{I_n, \mathcal{B}(\alpha^0_{-n} \vee \xi \vee \Theta\xi \vee \sigma)\}$ is a supermartingale. Therefore

$$I_n \xrightarrow[\mu^*\text{-a.e.}]{L^1} I_\infty.$$

Hence the second term on the right-hand side of equation (4.1) tends $\mu^*$-a.e. to a $\Theta$-invariant Borel function $F \in L^1$. Then by the ergodicity of $\mu^*$, one has

$$\lim_{n \to +\infty} \frac{1}{n} I((\alpha \vee \xi)^n_0)(\omega, x) = \int F\, d\mu^* = \int I_\infty\, d\mu^*.$$

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Hence the limit function in (4.1) is constant almost everywhere and is therefore equal to

$$\lim_{n\to+\infty}\frac{1}{n}H_{\mu^*}((\alpha\vee\xi)_0^n|\xi\vee\sigma),$$

which can be written as

$$\lim_{n\to+\infty}\frac{1}{n}H_{\mu^*}(\xi_0^n|\xi\vee\sigma)+\lim_{n\to+\infty}\frac{1}{n}H_{\mu^*}(\alpha_0^n|\Theta^{-n}\xi\vee\sigma).$$

The first term is equal to $H_{\mu^*}(\xi|\Theta\xi\vee\sigma)$. The second term goes to 0 since $\Theta^{-n}\xi$ generates. □

*Proof of Lemma 4.3.* Without loss of generality, let $\xi_i$ and $S_i$ be as introduced in §3 and $0<\delta<\frac{1}{4}e^{-2\lambda_0-2\varepsilon}$. Let $\{\Phi_{(\omega,x)}\}_{(\omega,x)\in\Lambda_0^*}$ be a system of $(\varepsilon,l)$-charts. Put $\hat{S}_i:=S_i\cap\{(\omega,x)\in\Lambda_0^*:l(\omega,x)\le l_0\}$, where $l_0$ is large enough such that $\mu^*(\hat{S}_i)>0$.

First we define $n_+,n_-$ and $n_0:\hat{S}_i\to\mathbb{Z}^+$ by

$$\begin{aligned}n_+(\omega,x)&:=\inf\{n>0:\Theta^n(\omega,x)\in\hat{S}_i\},\\ n_-(\omega,x)&:=\inf\{n>0:\Theta^{-n}(\omega,x)\in\hat{S}_i\},\\ n_0(\omega,x)&:=\inf\{n\ge0:\Theta^n(\omega,x)\in\hat{S}_i\}.\end{aligned}$$

Then let $\psi:\Omega^*\times M\to\mathbb{R}$ be given by

$$\psi(\omega,x):=\begin{cases}\dfrac{\delta}{2K_0l_0}e^{-(\lambda_0+\varepsilon)\max(n_+(\omega,x),n_-(\omega,x))}, & \text{if } (\omega,x)\in\hat{S}_i,\\ \dfrac{\delta}{2K_0l_0}, & \text{otherwise.}\end{cases}$$

Finally we define $\psi_+$ by replacing $\max(n_+,n_-)$ in the definition of $\psi$ by $n_+$.

Since $\int-\log\psi\,d\mu^*<\infty$, there exists a measurable partition $\alpha$ of $\Omega^*\times M$ with $H_{\mu^*}(\alpha|\sigma)<\infty$ such that $\alpha_\omega(x)\subset B(x,\psi(\omega,x))$ for almost every $(\omega,x)$.

Let $(\omega,x)\in\hat{S}_i$. We will write $n_+=n_+(\omega,x)$ and $n_0=n_0(\omega,x)$ for simplicity of notation in the rest of this section. We assert that the following three claims are true.

CLAIM 1. *If $y\in\Phi_{(\omega,x)}W^i_{(\omega,x),\delta}(x)$ satisfies $\|\Phi^{-1}_{(\omega,x)}y\|'_i\le l_0\psi_+(\omega,x)$, then*

$$d^i_{\Theta^j(\omega,x)}(f^j_\omega x,f^j_\omega y)<\delta\quad\text{for } 0\le j\le n_+$$

*and $f^{n_+}_\omega y\in\Phi_{\Theta^{n_+}(\omega,x)}W^i_{\Theta^{n_+}(\omega,x),\delta}(f^{n_+}_\omega x)$.*

CLAIM 2. *If $y\in[\alpha_0^n]_\omega(x)\cap\Phi_{(\omega,x)}W^i_{(\omega,x),\delta}(x)$ for some $n\ge0$, then*

$$d^i_{\Theta^j(\omega,x)}(f^j_\omega x,f^j_\omega y)<\delta\quad\text{for } 0\le j\le n.$$

CLAIM 3. *If $y\in[\alpha_0^{n_0}\vee\xi_i]_\omega(x)$, then*

$$d^i_{\Theta^j(\omega,x)}(f^j_\omega x,f^j_\omega y)<\delta\quad\text{for } 0\le j\le n_0$$

*and $f^{n_0}_\omega y\in\Phi_{\Theta^{n_0}(\omega,x)}W^i_{\Theta^{n_0}(\omega,x),\delta}(f^{n_0}_\omega x)$.*

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Let us postpone the proof of the above claims and first proceed with the proof of Lemma 4.3. Consider now an arbitrary point $(\omega, x)$ with the property that $\Theta^n(\omega, x) \in \hat{S}_i$ infinitely often as $n \to \pm\infty$. If $y \in (\alpha_0^n \vee \xi_i)_\omega(x)$ with $n \geq n_0$, then by Claim 3,

$$d^i_{\Theta^j(\omega,x)}(f^j_\omega x, f^j_\omega y) < \delta \quad \text{for } 0 \leq j \leq n_0$$

and $f^{n_0}_\omega y \in \Phi_{\Theta^{n_0}(\omega,x)} W^i_{\Theta^{n_0}(\omega,x),\delta}(f^{n_0}_\omega x)$. Therefore one can apply Claim 2 to $f^{n_0}_\omega y$, yielding that

$$d^i_{\Theta^j(\omega,x)}(f^j_\omega x, f^j_\omega y) < \delta \quad \text{for } n_0 \leq j \leq n,$$

which implies $y \in B^i(\omega, x; n, \delta)$ for all $y \in [\alpha_0^n \vee \xi_i]_\omega(x)$. □

*Proof of Claim 1.* It follows from our assumptions on $y$ and Proposition 3.5(1) that $H^j_{(\omega,x)}\Phi^{-1}_{(\omega,x)} y \in W^i_{\Theta^j(\omega,x),\delta}(f^j_\omega x)$ and (note that $\lambda_0 \geq \lambda_1 + 2\varepsilon$)

$$\|H^j_{(\omega,x)}\Phi^{-1}_{(\omega,x)} y\|'_i \leq \|\Phi^{-1}_{(\omega,x)} y\|'_i \exp(j\lambda_0) \quad \text{for } j > 0,$$

provided that $\|\Phi^{-1}_{(\omega,x)} y\|'_i \exp(k\lambda_0) \leq e^{-\lambda_1 - 3\varepsilon} l(\Theta^k(\omega, x))^{-1}$ for all $0 \leq k < j$. This is guaranteed for $j \leq n_+$. Since $W^i_{\Theta^j(\omega,x),\delta}(f^j_\omega x)$ is a graph over $\bar{\mathbb{R}}^{(i)}(\delta l(\Theta^j(\omega, x))^{-1})$ with slope $< 1$, one has

$$d^i_{\Theta^j(\omega,x)}(f^j_\omega x, f^j_\omega y) \leq 2K_0 \|\Phi^{-1}_{\Theta^j(\omega,x)} f^j_\omega y\|'_i < \delta \quad \text{for } 0 \leq j \leq n_+.$$ □

*Proof of Claim 2.* First observe that if $y \in \alpha_\omega(x)$, then $\|\Phi^{-1}_{(\omega,x)} y\| \leq l_0\psi(\omega, x) \leq l_0\psi_+(\omega, x)$, so we have the desired conclusion for $0 \leq j \leq n_+$. Furthermore, if $n \geq n_+$ and $y \in [\alpha_0^n]_\omega(x)$, then $f^{n_+}_\omega y \in \alpha_{\theta^{n_+}\omega}(f^{n_+}_\omega x) \cap \Phi_{\Theta^{n_+}(\omega,x)} W^i_{\Theta^{n_+}(\omega,x),\delta}(f^{n_+}_\omega x)$. Therefore we can apply Claim 1 to $f^{n_+}_\omega y$ with $\Theta^{n_+}(\omega, x)$, $f^{n_+}_\omega x$ and $f^{n_+}_\omega y$ in place of $(\omega, x)$, $x$ and $y$ respectively. An inductive argument completes the proof of Claim 2. □

*Proof of Claim 3.* To prove this claim, let us assume that $(\omega, x) \notin \hat{S}_i$ and for simplicity of notation write $k = n_0 - n_-(\Theta^{n_0}(\omega, x))$. That is, $k$ is the largest integer $< 0$ such that $\Theta^k(\omega, y) \in \hat{S}_i$. Clearly $n_+(\Theta^k(\omega, x)) = n_0 - k = n_-(\Theta^{n_0}(\omega, x))$. Since $\Theta^{-n}\xi$ is increasing as $n \to +\infty$, we have $\Theta^k(\omega, y) \in \xi(\Theta^k(\omega, x))$ and $f^k_\omega y \in \Phi_{\Theta^k(\omega,x)} W^i_{\Theta^k(\omega,x),\delta}(f^k_\omega x)$ by our choice of $\hat{S}_i$. Also $\psi$ is chosen in such a way that $f^j_\omega(\alpha_{\theta^{n_0}\omega}(f^{n_0}_\omega x))$ lies well inside the charts at $f^{n_0-j}_\omega x$ for $j = 1, 2, \ldots, n_0 - k$. Hence

$$\begin{aligned}
\|\Phi_{\Theta^k(\omega,x)} f^k_\omega y\|'_i &= \|H^k_{(\omega,x)}\Phi^{-1}_{(\omega,x)} y\|'_i \\
&\leq \|H^{k+1}_{(\omega,x)}\Phi^{-1}_{(\omega,x)} y\|'_i \leq \cdots \\
&\leq \|H^{n_0}_{(\omega,x)}\Phi^{-1}_{(\omega,x)} y\|'_i = \|\Phi^{-1}_{\Theta^{n_0}(\omega,x)} f^{n_0}_\omega y\|'_i \\
&\leq l_0\psi(\Theta^{n_0}(\omega, x)) \leq l_0\psi_+(\Theta^k(\omega, x)).
\end{aligned}$$

Therefore, by Claim 1, Claim 3 holds true. □

*Proof of Proposition 4.1.* It follows directly from the definition of $\overline{h}_i(\omega, x; \xi_i)$ together with Lemmas 4.2 and 4.3 that $\overline{h}_i(\omega, x; \xi_i) \leq H_{\mu^*}(\xi_i|\Theta\xi_i \vee \sigma)$ for $\mu^*$-a.e. $(\omega, x)$. What remains is to verify

$$\underline{h}_i(\omega, x; \xi_i) \geq H_{\mu^*}(\xi_i|\Theta\xi_i \vee \sigma), \quad \mu^*\text{-a.e. } (\omega, x) \in \Omega^* \times M. \tag{4.2}$$

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We know that

$$H_{\mu^*}(\xi_i|\Theta\xi_i \vee \sigma) = H_{\mu^*}(\Theta^{-1}\xi_i|\xi_i \vee \sigma) = \int -\log(\mu_\omega^*)_x^{\xi_{i,\omega}}((\Theta^{-1}\xi_i)_\omega(x))\,d\mu^*,$$

where the term after log is a conditional measure of the denoted set. Put

$$g(\omega, x) := -\log(\mu_\omega^*)_x^{\xi_{i,\omega}}((\Theta^{-1}\xi_i)_\omega(x)),$$
$$A_\delta := \{(\omega, x) \in \Omega^* \times M : B^i(\omega, x; \delta) \subset \xi_{i,\omega}(x)\}.$$

Since $\xi_i$ is an increasing generator subordinate to $W^i$-manifolds, one has $A_\delta \uparrow$ and $\mu^*(A_\delta) \uparrow 1$ as $\delta \downarrow 0$. Hence, given $\varepsilon > 0$, there exists $\delta' > 0$ such that for each $\delta \in (0, \delta')$,

$$\int_{\Theta^{-1}A_\delta} g(\omega, x)\,d\mu^* \geq H_{\mu^*}(\xi_i|\Theta\xi_i \vee \sigma) - \varepsilon.$$

Put $U^i(\omega, x; n, \delta) := \cap\{(\Theta^{-k}\xi_i)_\omega(x) : 0 \leq k \leq n, \Theta^k(\omega, x) \in A_\delta\}$. Then $B^i(\omega, x; n, \delta) \subset U^i(\omega, x; n, \delta)$. Hence

$$-\log(\mu_\omega^*)_x^{\xi_{i,\omega}}(B^i(\omega, x; n, \delta)) \geq -\log(\mu_\omega^*)_x^{\xi_{i,\omega}}(U^i(\omega, x; n, \delta)). \tag{4.3}$$

Furthermore, we will prove that for $\mu^*$-a.e. $(\omega, x)$,

$$-\log(\mu_\omega^*)_x^{\xi_{i,\omega}}(U^i(\omega, x; n, \delta)) \geq \sum_{k=0}^{n-1}(1_{\Theta^{-1}A_\delta} \cdot g)(\Theta^k(\omega, x)), \tag{4.4}$$

which combined with inequality (4.3) implies

$$-\frac{1}{n}\log(\mu_\omega^*)_x^{\xi_{i,\omega}}(B^i(\omega, x; n, \delta)) \geq \frac{1}{n}\sum_{k=0}^{n-1}(1_{\Theta^{-1}A_\delta} \cdot g)(\Theta^k(\omega, x)).$$

Then applying Birkhoff's ergodic theorem, one has for each $\delta \in (0, \delta')$

$$\underline{h}_i(\omega, x; \delta, \xi_i) \geq \int_{\Theta^{-1}A_\delta} g(\omega, x)\,d\mu^* \geq H_{\mu^*}(\xi_i|\Theta\xi_i \vee \sigma) - \varepsilon.$$

Now we return to the proof of the assertion (4.4). Let $\tau(\omega, x; n) := \max\{0 \leq k \leq n : \Theta^k(\omega, x) \in A_\delta\}$, where $\max\emptyset := +\infty$. For $\mu^*$-a.e. $(\omega, x)$ fixed, $\tau(\omega, x; n)$ is a finite number with sufficiently large $n$ by Poincaré's recurrence theorem. This together with the condition that $\xi_i$ is an increasing generator yields

$$\begin{aligned}
-\log(\mu_\omega^*)_x^{\xi_{i,\omega}}(U^i(\omega, x; n, \delta)) &= \sum_{k=0}^{\tau(\omega,x;n)-1} -\log(\mu_\omega^*)_x^{(\Theta^{-k}\xi_i)_\omega}((\Theta^{-k-1}\xi_i)_\omega(x)) \\
&\geq \sum_{k=0}^{\tau(\omega,x;n)-1} (1_{\Theta^{-1}A_\delta} \cdot g)(\Theta^k(\omega, x)) \\
&= \sum_{k=0}^{n-1}(1_{\Theta^{-1}A_\delta} \cdot g)(\Theta^k(\omega, x)). \qquad \square
\end{aligned}$$

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5. *Estimates of local entropy along unstable manifolds*

In this section we will estimate local entropy along unstable manifolds through Lyapunov exponents and transverse dimensions. To be explicit, we will prove in this section via Propositions 5.1, 5.3, 5.4 and 5.5 that

$$\overline{\delta}_i = \underline{\delta}_i =: \delta_i \quad \text{for } i = 1, \dots, u,$$

where $\overline{\delta}_i = \overline{\delta}_i(\omega, x)$ and $\underline{\delta}_i = \underline{\delta}_i(\omega, x)$, and

$$\gamma_i := \delta_i - \delta_{i-1} = (h_i - h_{i-1})/\lambda_i \le m_i, \quad 1 \le i \le u, \tag{5.1}$$

with $\delta_0 := 0$ and $h_0 := 0$. Thus the identity $h_\mu(\mathcal{X}^+(M, v)) = \sum_{i=1}^u \lambda_i \gamma_i$ follows from

$$h_u = h^\sigma_{\mu^*}(\Theta) = h_\mu(\mathcal{X}^+(M, v)).$$

5.1. *Estimate of local entropy $h_1$.* The local entropy $h_1$ measures the amount of randomness along the leaves of $W^1$-manifolds and can be formalized as follows.

PROPOSITION 5.1. *For $\mu^*$-a.e. $(\omega, x)$, $\overline{\delta}_1(\omega, x) = \underline{\delta}_1(\omega, x) =: \delta_1 =: \gamma_1$ and $h_1 = \lambda_1\gamma_1$. From the definition of $\overline{\delta}_1$ and $\underline{\delta}_1$, it is clear that $0 \le \gamma_1 \le m_1$.*

*Proof.* Let $\delta > 0$ be a sufficiently small number. We divide the proof into two parts.

(i) First for each $y \in B^1(\omega, x; \rho_n)$ (where $\rho_n = [\exp(-n(\lambda_1 + 2\varepsilon))/(2K_0 l(\omega, x))]\delta$), one has $y \in W^1_{\mathrm{loc}}(\omega, x)$ and $d^1_{(\omega,x)}(x, y) < \rho_n$. Hence, by Proposition 3.5 and Lemma 3.6, one has for each $k \le n$ $f^k_\omega y \in W^1_{\mathrm{loc}}(\Theta^k(\omega, x))$ and

$$\begin{aligned} d^1_{\Theta^k(\omega,x)}(f^k_\omega x, f^k_\omega y) &\le 2K_0 \|\Phi^{-1}_{\Theta^k(\omega,x)} \circ f^k_\omega x - \Phi^{-1}_{\Theta^k(\omega,x)} \circ f^k_\omega y\|'_1 \\ &= 2K_0 \|H^k_{(\omega,x)} \circ \Phi^{-1}_{(\omega,x)} y\|'_1 \\ &\le 2K_0 l(\omega, x) d(x, y) \exp(k(\lambda_1 + 2\varepsilon)) \\ &\le 2K_0 l(\omega, x) d^1_{(\omega,x)}(x, y) \exp(k(\lambda_1 + 2\varepsilon)) < \delta. \end{aligned}$$

Hence $B^1(\omega, x; \rho_n) \subset B^1(\omega, x; n, \delta)$ for $\mu^*$-a.e. $(\omega, x)$ and all $n \ge 0$. This implies $h_1 \le \lambda_1 \underline{\delta}_1$.

(ii) We then prove that $h_1 \ge \lambda_1 \overline{\delta}_1$. Let the $\xi_i$'s be as introduced in §3.3. By Lemma 4.3 there exists a partition $\alpha$ such that $(\alpha^n_0 \vee \xi_i)_\omega(x) \subset B^i(\omega, x; n, \delta)$ for all $n \ge n_0(\omega, x)$. Therefore, for each $y \in (\alpha^n_0 \vee \xi_i)_\omega(x)$, one has $f^n_\omega y \in W^i_{\mathrm{loc}}(\Theta^n(\omega, x))$ and $d^i_{\Theta^n(\omega,x)}(f^n_\omega x, f^n_\omega y) < \delta$. By Proposition 3.5(3)

$$\begin{aligned} d^i_{(\omega,x)}(x, y) &\le 2K_0 \|\Phi^{-1}_{(\omega,x)} y\|'_i = 2K_0 \|H^{-n}_{\Theta^n(\omega,x)} \circ \Phi^{-1}_{\Theta^n(\omega,x)} f^n_\omega y\|'_1 \\ &\le 2K_0 \exp(n(-\lambda_i + 2\varepsilon)) \|\Phi^{-1}_{\Theta^n(\omega,x)} f^n_\omega y\|'_1, \end{aligned}$$

where by Lemma 3.6, $\|\Phi^{-1}_{\Theta^n(\omega,x)} f^n_\omega y\|'_i \le l(\Theta^n(\omega, x)) d^i_{\Theta^n(\omega,x)}(f^n_\omega x, f^n_\omega y) < \delta l(\omega, x) e^{n\varepsilon}$. Hence $d^i_{(\omega,x)}(x, y) < 2K_0 \delta l(\omega, x) \exp(n(-\lambda_i + 3\varepsilon))$. This implies for $\mu^*$-a.e. $(\omega, x)$ that

$$(\alpha^n_0 \vee \xi_i)_\omega(x) \subset B^i(\omega, x; 2K_0 \delta l(\omega, x) e^{-n(\lambda_i - 3\varepsilon)}), \quad n \ge n_0(\omega, x). \tag{5.2}$$

Let $i = 1$. By Lemma 4.2 we have proved $h_1 \ge \lambda_1 \overline{\delta}_1$. □

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5.2. *Estimate of local entropy $h_i$ from below with $2 \le i \le u$.* In view of (5.2), Proposition 4.1 and Lemmas 4.2 and 4.3, the following lemma holds.

LEMMA 5.2. *Let $N$ be a positive integer. Let $\alpha$ be as introduced in Lemma 4.3. Then there exists a Borel function $\hat{n} : \Omega^* \times M \to \mathbb{Z}^+$ such that for $\mu^*$-a.e. $(\omega, x)$ there are infinitely many $n \ge \max(\hat{n}(\omega, x), 1/\varepsilon)$ satisfying the following (where $2 \le i \le u$):*

(1) $[\xi_i \vee \alpha_0^n]_\omega(x) \subset B^i(\omega, x; 2K_0\delta l(\omega, x)e^{-n(\lambda_i - 3\varepsilon)})$;

(2) $-(1/n)\log(\mu_\omega^*)_x^{\xi_{i-1,\omega}}([\alpha_0^n]_\omega(x)) \ge h_{i-1} - \varepsilon$;

(3) $-(1/n)\log(\mu_\omega^*)_x^{\xi_{i,\omega}}([\alpha_0^n]_\omega(x)) \le h_i + \varepsilon$;

(4) $L := B^{i-1}(\omega, x; 2K_0\delta N_1 e^{-n(\lambda_i - 3\varepsilon)}) \subset \xi_{i-1,\omega}(x)$;

(5) $\log(\mu_\omega^*)_x^{\xi_{i-1,\omega}}(L)/(-(n-1)(\lambda_i - 3\varepsilon)) \le \overline{\delta}_{i-1} + \varepsilon$;

(6) $\log(\mu_\omega^*)_x^{\xi_{i,\omega}}(B^i(\omega, x; 4K_0\delta N_1 e^{-n(\lambda_i - 3\varepsilon)}))/(-n(\lambda_i - 3\varepsilon)) \ge \overline{\delta}_i - \varepsilon$.

We then give a lower bound of the local entropy $h_i$ in terms of Lyapunov exponents and pointwise dimensions.

PROPOSITION 5.3. *For $2 \le i \le u$ and $\mu^*$-a.e. $(\omega, x)$, $(h_i - h_{i-1})/\lambda_i \ge \overline{\delta}_i - \overline{\delta}_{i-1}$.*

*Proof.* Let $\hat{\Gamma} := \{(\omega, x) : \hat{n}(\omega, x) \le N_1\}$. Then $\hat{\Gamma} \uparrow$ and $\mu^*(\hat{\Gamma}) \uparrow 1$ as $N_1 \to +\infty$. Put

$$A_{N_1} := \{(\omega, x) : (\mu_\omega^*)_x^{\xi_{i-1,\omega}}(L \cap \hat{\Gamma}_\omega) \ge \tfrac{1}{2}(\mu_\omega^*)_x^{\xi_{i-1,\omega}}(L) \quad \text{and} \quad l(\omega, x) \le N_1\}.$$

Obviously, $A_{N_1} \uparrow$ and $\mu^*(A_{N_1}) \uparrow 1$. Therefore, for $\mu^*$-a.e. $(\omega, x)$ fixed, one can assume that $(\omega, x) \in A_{N_1}$ for some $N_1$ large enough. According to Lemma 5.2(5),

$$(\mu_\omega^*)_x^{\xi_{i-1,\omega}}(L \cap \hat{\Gamma}_\omega) \ge \tfrac{1}{2}(\mu_\omega^*)_x^{\xi_{i-1,\omega}}(L) \ge \tfrac{1}{2}\exp(-n(\lambda_i - 3\varepsilon)(\overline{\delta}_{i-1} + \varepsilon)).$$

But by Lemma 5.2(2) one has $(\mu_\omega^*)_x^{\xi_{i-1,\omega}}([\alpha_0^n]_\omega(y)) = (\mu_\omega^*)_y^{\xi_{i-1,\omega}}([\alpha_0^n]_\omega(y)) \le e^{-n(h_{i-1}-\varepsilon)}$ for each $y \in L \cap \hat{\Gamma}_\omega$. Hence

$$\begin{aligned}\#\{[\alpha_0^n]_\omega(y) : y \in L \cap \hat{\Gamma}_\omega\} &\ge (\mu_\omega^*)_x^{\xi_{i-1,\omega}}(L \cap \hat{\Gamma}_\omega)/\exp(-n(h_{i-1} - \varepsilon)) \\ &\ge \tfrac{1}{2}\exp\{n[h_{i-1} - \varepsilon - (\lambda_i - 3\varepsilon)(\overline{\delta}_{i-1} + \varepsilon)]\}.\end{aligned}$$

On the other hand, according to Lemma 5.2(1), we have

$$[\xi_i \vee \alpha_0^n]_\omega(y) \subset B^i(\omega, y; 2K_0\delta N_1 e^{-n(\lambda_i - 3\varepsilon)}), \quad \text{for all } y \in \hat{\Gamma}_\omega.$$

Clearly $d^i_{(\omega,x)}(x, y) \le d^{i-1}_{(\omega,x)}(x, y)$ for all $y \in L$. Therefore

$$[\xi_i \vee \alpha_0^n]_\omega(y) \subset B^i(\omega, x; 4K_0\delta N_1 e^{-n(\lambda_i - 3\varepsilon)}), \quad \text{for all } y \in L \cap \hat{\Gamma}_\omega.$$

Hence

$$\begin{aligned}&\log(\mu_\omega^*)_x^{\xi_{i,\omega}}(B^i(\omega, x; 4K_0\delta N_1 e^{-n(\lambda_i - 3\varepsilon)})) \\ &\quad \ge \log\#\{[\xi_i \vee \alpha_0^n]_\omega(y) : y \in L \cap \hat{\Gamma}_\omega\} + \log\min_y(\mu_\omega^*)_x^{\xi_{i,\omega}}([\xi_i \vee \alpha_0^n]_\omega(y)) \\ &\quad \ge \log\#\{[\alpha_0^n]_\omega(y) : y \in L \cap \hat{\Gamma}_\omega\} + \log\min_y(\mu_\omega^*)_x^{\xi_{i,\omega}}([\alpha_0^n]_\omega(y)) \\ &\quad \ge -\log 2 - n[h_i - h_{i-1} + 2\varepsilon + (\lambda_i - 3\varepsilon)(\overline{\delta}_{i-1} + \varepsilon)].\end{aligned}$$

Comparing with Lemma 5.2(6), one has

$$(\overline{\delta}_i - \overline{\delta}_{i-1} - 2\varepsilon)(\lambda_i - 3\varepsilon) \le \varepsilon\log 2 + h_i - h_{i-1} + 2\varepsilon,$$

which implies Proposition 5.3 by letting $\varepsilon \to 0$. □

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5.3. *Estimate of local entropy $h_i$ from above with $2 \le i \le u$.* Let the number $N_0$, the map $\tilde{\pi}^i : \eta_i(\omega, x) \to \mathbb{R}^{m_i}$, and metrics $\tilde{d}^i_{(\omega,x)}(\cdot,\cdot)$ and $\hat{d}^i_{(\omega,x)}(\cdot,\cdot)$ on $\eta_i(\omega, x)/\eta_{i-1}$ be as introduced in §3. We denote $(\mu^*_\omega)^{\eta_{i,\omega}}_x$ by $\mu^i_{(\omega,x)}$ for simplicity of notation. Since $\sigma < \eta_i$, $\mu^i_{(\omega,x)}$ is also viewed as a Borel measure on $\eta_i(\omega, x) = \{\omega\} \times \eta_{i,\omega}(x)$ in this subsection. Write $\tilde{\pi}_i := (\tilde{\pi}^1, \ldots, \tilde{\pi}^i) : \eta_i(\omega, x) \to \bar{\mathbb{R}}^{(i)}$ and put

$$\hat{B}^i(\omega, x; \rho) := \{(\omega, y) \in \eta_i(\omega, x) : \tilde{d}^i_{(\omega,x)}((\omega, x), (\omega, y)) < \rho\}.$$

Then we define transverse dimensions as follows.

*Definition 5.1.*

$$\tilde{\gamma}_i(\omega, x) := \liminf_{\rho \to 0} \frac{\log \mu^i_{(\omega,x)}(\hat{B}^i(\omega, x; \rho))}{\log \rho}$$

is called the transverse dimension of $\eta_i/\eta_{i-1}$ at $(\omega, x)$.

*Definition 5.2.* For each $y \in \eta_{i,\omega}(x)$,

$$\hat{\gamma}_i(\omega, y; \omega, x) := \liminf_{\rho \to 0} \frac{\log \nu(\{z \in \bar{\mathbb{R}}^{(i)} : \|z_i - \tilde{\pi}^i(\omega, y)\| < \rho\})}{\log \rho}$$

is called the transverse dimension of $W^i(\omega, x)/W^{i-1}$ at $(\omega, y)$, where $\nu := \mu^i_{(\omega,x)} \circ \tilde{\pi}_i^{-1}$ is a Borel probability on $\bar{\mathbb{R}}^{(i)}$.

Now we introduce the main results for this subsection.

PROPOSITION 5.4. *Given $\beta \in (0, 1)$. We have for sufficiently small $\delta$ and $\mu^*$-a.e. $(\omega, x)$*

$$(\lambda_i + \beta)\tilde{\gamma}_i(\omega, x) \ge (1 - \beta)(h_i - h_{i-1} - \beta)$$

*and $0 \le \tilde{\gamma}_i(\omega, x) \le m_i$. Hence $h_i - h_{i-1} \le \lambda_i m_i$.*

PROPOSITION 5.5. *Given $\beta \in (0, 1)$. For $\mu^i_{(\omega,x)}$-a.e. $y \in \eta_{i,\omega}(x)$, one has $\tilde{\gamma}_i(\omega, x) = \hat{\gamma}_i(\omega, y; \omega, x)$ and*

$$\underline{\delta}_i - \underline{\delta}_{i-1} \ge \hat{\gamma}_i(\omega, y; \omega, x) \ge \frac{(1 - \beta)(h_i - h_{i-1} - \beta)}{\lambda_i + \beta}. \tag{5.3}$$

*Hence, by letting $\beta \to 0$, we obtain $(h_i - h_{i-1})/\lambda_i \le \underline{\delta}_i - \underline{\delta}_{i-1}$.*

*Proof of Proposition 5.4.* Let $e^{-(\lambda_i+\beta)\varepsilon} N_0^{4\mu^*(\hat{E})} < 1$ with $\beta \in (0, 1)$ (this holds true provided $\mu^*(\hat{E})$ is small enough). We will prove that for $\mu^*$-a.e. $(\omega, x)$

$$(\lambda_i + \beta) \liminf_{\rho \to 0} \frac{\log \mu^i_{(\omega,x)}(\hat{B}^i(\omega, x; \rho))}{\log \rho} \ge (1 - \beta)(h_i - h_{i-1} - \beta). \tag{5.4}$$

The first conclusion then follows immediately from this together with Lemma VI.6.4 in [**7**] and Proposition 3.8. The second conclusion follows from the definition of $\tilde{\gamma}_i$ and Lemma 4.1.4 in [**3**] since the point-independent metric $\tilde{d}^i(\cdot,\cdot)$ on $\eta_i(\omega, x)/\eta_{i-1}$ makes it isometric to a subset of $(\mathbb{R}^{m_i}, \|\cdot\|)$.

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Now we come to the proof of (5.4). First fix $\varepsilon \in (0, \beta/3)$. Let $\widetilde{\Lambda}_0'$ be a set as chosen in §3. Recalling that $\mu^*(\widetilde{\Lambda}_0') = 1$ and $\Theta\widetilde{\Lambda}_0' = \widetilde{\Lambda}_0'$, for the sake of presentation we may assume that $\widetilde{\Lambda}_0' = \Omega^* \times M$.

We divide the proof into four parts following the lines presented in [**7**].

(A) Before proceeding with the main argument, we record some estimates analogous to those in [**7**, p. 171]. For $\delta > 0$, define $g$, $g_\delta$ and $g_* : \Omega^* \times M \to \mathbb{R}$ by

$$g(\omega, y) := \mu^{i-1}_{(\omega,y)}((\Theta^{-1}\eta_i)(\omega, y)),$$

$$g_\delta(\omega, y) := \frac{1}{\mu^i_{(\omega,y)}(\hat{B}^i(\omega, y; \delta))} \int_{\hat{B}^i(\omega,y;\delta))} \mu^{i-1}_{(\omega,z)}((\Theta^{-1}\eta_i)(\omega, y))\, d\mu^i_{(\omega,y)}(z),$$

$$g_*(\omega, y) := \inf_{\delta \in Q} g_\delta(\omega, y),$$

where $Q := \{e^{-(\lambda_i+\beta)l} N_0^{2j} : l, j \in \mathbb{Z}^+\}$.

According to Proposition 3.8, one has $g(\omega, y) = \mu^{i-1}_{(\omega,y)}((\Theta^{-1}\eta_{i-1})(\omega, y))$ for $\mu^*$-a.e. $(\omega, y)$. For each $\delta > 0$, one can check that the functions $(\omega, y) \mapsto \mu^i_{(\omega,y)}(\hat{B}^i(\omega, y; \delta))$ and $(\omega, y) \mapsto \mu^{*\Theta^{-1}\eta_i}_{(\omega,y)}(\hat{B}^i(\omega, y; \delta))$ are measurable and $\mu^i_{(\omega,y)}(\hat{B}^i(\omega, y; \delta)) > 0$ for $\mu^*$-a.e. $(\omega, y)$. Since $H_{\mu^*}(\Theta^{-1}\eta_i|\eta_i) < +\infty$, one has $\mu^i_{(\omega,y)}((\Theta^{-1}\eta_i)(\omega, y)) > 0$ for $\mu^*$-a.e. $(\omega, y)$ and

$$\begin{aligned} g_\delta(\omega, y) &= \frac{\mu^i_{(\omega,y)}(\hat{B}^i(\omega, y; \delta) \cap (\Theta^{-1}\eta_i)(\omega, y))}{\mu^i_{(\omega,y)}(\hat{B}^i(\omega, y; \delta))} \\ &= \frac{\mu^{*\Theta^{-1}\eta_i}_{(\omega,y)}(\hat{B}^i(\omega, y; \delta))}{\mu^i_{(\omega,y)}(\hat{B}^i(\omega, y; \delta))} \cdot \mu^i_{(\omega,y)}((\Theta^{-1}\eta_i)(\omega, y)). \end{aligned}$$

Therefore $g_\delta$ is measurable for each fixed $\delta > 0$. The measurability of $g_*$ is obvious.

We assert that $g_\delta \to g$, $\mu^*$-a.e. on $\Omega^* \times M$ when $\delta \in Q$ and $\delta \to 0$ and that $-\int \log g_*\, d\mu^* < +\infty$. To see this, first consider one element of $\eta_i$ at a time. Fix $(\omega, x)$. Substitute $(\eta_i(\omega, x), \mu^i_{(\omega,x)})$ for $(X, m)$ in Lemma 6.3 of [**7**, p. 168], let $\pi = \tilde{\pi}_i : \eta_i(\omega, x) \to \bar{\mathbb{R}}^{(i)}$ and let $\alpha = (\Theta^{-1}\eta_i)|_{\eta_i(\omega,x)}$. Then we can conclude that $g_\delta(\omega, \cdot) \to g(\omega, \cdot)$, $\mu^i_{(\omega,x)}$-a.e. as $\delta \in Q$ and $\delta \to 0$ and that for $\mu^*$-a.e. $(\omega, x)$,

$$\begin{aligned} -\int \log g_*(\omega, z)\mu^i_{(\omega,x)}(dz) &\le -\int \log\Big(\inf_{\delta>0} g_\delta(\omega, z)\Big)\mu^i_{(\omega,x)}(dz) \\ &\le H_{\mu^i_{(\omega,x)}}(\Theta^{-1}\eta_i|\eta_i) + \log c\Big(\sum_{j=1}^{i} m_j\Big) + 1 < +\infty, \end{aligned}$$

where $c(\cdot)$ is as in Besicovich's covering theorem (see [**7**, p. 166]).

(B) The purpose of this step is to study the induced action of $\Theta$ on $\Theta^{-1}(\eta_i(\omega, x))/\eta_{i-1} \to \eta_i(\omega, x)/\eta_{i-1}$ with respect to the metrics $\tilde{d}^i_{\Theta^{-1}(\omega,x)}(\cdot, \cdot)$ and $\tilde{d}^i_{(\omega,x)}(\cdot, \cdot)$. Consider $(\omega, x) \in \Omega^* \times M$. The point $(\omega, x)$ will be subjected to a finite number of $\mu^*$-a.e. assumptions. Let $t_0 < t_1 < \cdots$ be the successive times $t$ when $\Theta^t(\omega, x) \in \hat{E}$ with $t_0 \le 0 < t_1$. Note that $t_0$ is constant on $\eta_i(\omega, x)$. For large $n$ and $0 \le k < n$, define $a(\omega, x; k)$ as follows: if $t_j \le k < t_{j+1}$, then

$$a(\omega, x; k) := \hat{B}^i(\Theta^k\omega, x; N_0^{2j} e^{-(\lambda_i+\beta)(n-t_j)}).$$

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We now claim that

$$a(\omega, x; k) \cap (\Theta^{-1}\eta_i)(\Theta^k(\omega, x)) \subset \Theta^{-1}a(\omega, x; k+1). \tag{5.5}$$

In fact, if $k \neq t_j - 1$ for any $j$, then $\Theta^k a(\omega, x; k) \cap \eta_i(\Theta^{k+1}(\omega, x)) = a(\omega, x; k+1)$ automatically since $\tilde{d}^i_{\Theta^k(\omega,x)}(\cdot,\cdot)$ and $\tilde{d}^i_{\Theta^{k+1}(\omega,x)}(\cdot,\cdot)$ are defined by pulling back to $\hat{E}$. The case when $k = t_j - 1$ for some $j$ reduces to the following consideration. Let $(\omega, y) \in \hat{E}$ and let $t > 0$ be the smallest integer such that $\Theta^t(\omega, y) \in \hat{E}$. Let $(\omega, z) \in (\Theta^{-t}\eta_i)(\omega, y)$. It suffices to show that

$$\tilde{d}^i_{\Theta^t(\omega,y)}(\Theta^t(\omega, y), \Theta^t(\omega, z)) \leq N_0^2 e^{t(\lambda_i+\beta)} \tilde{d}^i_{(\omega,y)}((\omega, y), (\omega, z)).$$

First $\hat{d}^i_{(\omega,y)}((\omega, y), (\omega, z)) \leq N_0 \tilde{d}^i_{(\omega,y)}((\omega, y), (\omega, z))$. Then for $k = 1, 2, \ldots, t$, Proposition 3.3 tells us that

$$\hat{d}^i_{\Theta^k(\omega,y)}(\Theta^k(\omega, y), \Theta^k(\omega, z)) \leq e^{k(\lambda_i+\beta)} \hat{d}^i_{(\omega,y)}((\omega, y), (\omega, z)).$$

We pick up another factor of $N_0$ when converting back to the $\tilde{d}^i$-metric at $\Theta^t(\omega, y)$ (see Proposition 3.11). What we claimed above is thus proved.

(C) It is easy to see that there exists a $\Theta$-invariant Borel set $\widetilde{\Delta} \subset \Omega^* \times M$ with full $\mu^*$-measure such that, if $(\omega, x) \in \widetilde{\Delta}$, then $\mu^i_{(\omega,x)}(\hat{B}^i(\omega, x; \delta)) > 0$ for all $\delta \in Q$. We now estimate $\mu^i_{(\omega,x)}(\hat{B}^i(\omega, x; e^{-(\lambda_i+\beta)(n-t_0(\omega,x))})) = \mu^i_{(\omega,x)}(a(\omega, x; 0))$ for $(\omega, x) \in \widetilde{\Delta}$ which will be subjected to a finite number of assumptions that hold almost everywhere. Write

$$\mu^i_{(\omega,x)}(a(\omega, x; 0)) = \prod_{k=0}^{T-1} \frac{\mu^i_{\Theta^k(\omega,x)}(a(\omega, x; k))}{\mu^i_{\Theta^{k+1}(\omega,x)}(a(\omega, x; k+1))} \cdot \mu^i_{\Theta^T(\omega,x)}(a(\omega, x; T)),$$

where $T = [n(1-\varepsilon)]$. First note that the last term $\leq 1$. For each $0 \leq k < T$, by the $\Theta$-invariance of $\mu^*$ and by the uniqueness of conditional measures one has

$$\frac{\mu^i_{\Theta^k(\omega,x)}(a(\omega, x; k))}{\mu^i_{\Theta^{k+1}(\omega,x)}(a(\omega, x; k+1))} = \mu^i_{\Theta^k(\omega,x)}(a(\omega, x; k)) \frac{\mu^i_{\Theta^k(\omega,x)}(\Theta^{-1}(\eta_i(\Theta^{k+1}(\omega, x))))}{\mu^i_{\Theta^k(\omega,x)}(\Theta^{-1}a(\omega, x; k+1))}.$$

This is less than or equal to

$$\frac{\mu^i_{\Theta^k(\omega,x)}(a(\omega, x; k))}{\mu^i_{\Theta^k(\omega,x)}((\Theta^{-1}\eta_i)(\Theta^k(\omega, x)) \cap a(\omega, x; k))} \cdot \mu^i_{\Theta^k(\omega,x)}((\Theta^{-1}\eta_i)(\Theta^k(\omega, x))) \tag{5.6}$$

by (5.5). If $g_\delta$ is defined as in (A), the first quotient in (5.6) is equal to

$$[g_{\delta(\omega,x;n,k)}(\Theta^k(\omega, x))]^{-1},$$

where $\delta(\omega, x; n, k) = e^{-(\lambda_i+\beta)(n-t_{j_k}(\omega,x))} N_0^{2j_k}$ with $j_k := \#\{0 < i \leq k : \Theta^i(\omega, x) \in \hat{E}\}$.

Write $I(\omega, x) := -\log \mu^i_{(\omega,x)}((\Theta^{-1}\eta_i)(\omega, x))$. Then the second term in (5.6) is equal to $e^{-I(\Theta^k(\omega,x))}$. Hence

$$\begin{aligned}\log \mu^i_{(\omega,x)}(\hat{B}^i(\omega, x; e^{-(\lambda_i+\beta)(n-t_0(\omega,x))})) \\ \leq -\sum_{k=0}^{T-1} \log g_{\delta(\omega,x;n,k)}(\Theta^k(\omega, x)) - \sum_{k=0}^{T-1} I(\Theta^k(\omega, x)).\end{aligned}$$

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Multiplying by $-1/n$ and taking lim inf on both sides of this inequality,

$$\begin{aligned}(\lambda_i+\beta)\liminf_{\rho\to 0}&\frac{\log\mu^i_{(\omega,x)}(\hat{B}^i(\omega,x;\rho))}{\log\rho}\\ &=(\lambda_i+\beta)\liminf_{n\to+\infty}\frac{\mu^i_{(\omega,x)}(\hat{B}^i(\omega,x;e^{-(\lambda_i+\beta)(n-t_0(\omega,x))}))}{\log e^{-(\lambda_i+\beta)n}}\\ &\geq\liminf_{n\to+\infty}\sum_{k=0}^{[n(1-\varepsilon)]}\log g_{\delta(\omega,x;n,k)}(\Theta^k(\omega,x))+\lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{[n(1-\varepsilon)]}I(\Theta^k(\omega,x)).\end{aligned}$$

The last limit $=(1-\varepsilon)H_{\mu^*}(\Theta^{-1}\eta_i|\eta_i)=(1-\varepsilon)h_i$ by Birkhoff's ergodic theorem. Hence Proposition 5.4 is proved if we show that

$$\limsup_{n\to+\infty}-\frac{1}{n}\sum_{k=0}^{[n(1-\varepsilon)]}\log g_{\delta(\omega,x;n,k)}(\Theta^k(\omega,x))\leq(1-\varepsilon)(h_{i-1}+2\varepsilon). \tag{5.7}$$

(D) We now prove the last assertion (5.7). It follows from (A) that there is a measurable function $\delta:\Omega^*\times M\to\mathbb{R}^+$ such that for $\mu^*$-a.e. $(\omega,x)$, if $\delta\in Q$ and $\delta\leq\delta(\omega,x)$, then $-\log g_\delta(\omega,x)\leq-\log g(\omega,x)+\varepsilon$. Since $\int-\log g_*\,d\mu^*<+\infty$, there is a number $\delta_*$ such that if $A=\{(\omega,x):\delta(\omega,x)>\delta_*\}$, then $\int_{\Omega^*\times M\setminus A}-\log g_*\,d\mu^*\leq\varepsilon$.

We claim that for $\mu^*$-a.e. $(\omega,x)$, if $n$ is sufficiently large, then $\delta(\omega,x;n,k)\leq\delta_*$ for all $k\leq n(1-\varepsilon)$. First by the Birkhoff ergodic theorem, there is a positive integer $N_1(\omega,x)$ such that for $n\geq N_1(\omega,x)$, $\#\{0\leq i<n:\Theta^i(\omega,x)\in\hat{E}\}\leq 2n\mu^*(\hat{E})$. If $n\geq N_1(\omega,x)$, then for each $k\leq n(1-\varepsilon)$ one has $t_{j_k}(\omega,x)\leq k\leq n(1-\varepsilon)$ and

$$\delta(\omega,x;n,k)=e^{-(\lambda_i+\beta)(n-t_{j_k}(\omega,x))}N_0^{2j_k}\leq e^{-(\lambda_i+\beta)\varepsilon n}N_0^{4n\mu^*(\hat{E})}.$$

Since $e^{-(\lambda_i+\beta)\varepsilon}N_0^{4\mu^*(\hat{E})}<1$, $\delta(\omega,x;n,k)$ is less than $\delta_*$ for sufficiently large $n$. Thus

$$\begin{aligned}&\sum_{k=0}^{[n(1-\varepsilon)]}-\log g_{\delta(\omega,x;n,k)}(\Theta^k(\omega,x))\\ &\quad\leq\sum_{\substack{k=0\\ \Theta^k(\omega,x)\in A}}^{[n(1-\varepsilon)]}(-\log g(\Theta^k(\omega,x))+\varepsilon)+\sum_{\substack{k=0\\ \Theta^k(\omega,x)\notin A}}^{[n(1-\varepsilon)]}-\log g_*(\Theta^k(\omega,x))\end{aligned}$$

and the lim sup we wish to estimate in (5.7) is bounded above by

$$(1-\varepsilon)\left[\int-\log g\,d\mu^*+\varepsilon+\int_{\Omega^*\times M\setminus A}-\log g_*\,d\mu^*\right].$$

Recalling that $g(\omega,x)=\mu^{i-1}_{(\omega,x)}((\Theta^{-1}\eta_{i-1})(\omega,x))$ for $\mu^*$-a.e. $(\omega,x)$, we have $\int-\log g\,d\mu^*=H_{\mu^*}(\Theta^{-1}\eta_{i-1}|\eta_{i-1})$ which is equal to $h^\sigma_{\mu^*}(\Theta,\xi_{i-1})=h_{i-1}$ by Proposition 3.8(3). This completes the proof. □

*Proof of Proposition 5.5.* It follows from the Lipschitz property of $\tilde{\pi}_i$ together with the definition of $\tilde{\gamma}_i$ and $\hat{\gamma}_i$ that for $\mu^*$-a.e. fixed $(\omega,x)$ and $\mu^i_{(\omega,x)}$-a.e. $y\in\eta_{i,\omega}(x)$,

$$\hat{\gamma}_i(\omega,y;\omega,x)=\tilde{\gamma}_i(\omega,x),$$

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if we assume $\mu^i_{(\omega,x)}\big(\bigcup_{n\geq 0}\Theta^n\hat{E}\big)=1$. Hence, from Proposition 5.4, we may assume that

$$\hat{\gamma}_i(\tilde{\pi}_i^{-1}z;\omega,x)\geq\frac{(1-\beta)(h_i-h_{i-1}-\beta)}{\lambda_i+\beta},\quad \nu\text{-a.e. } z\in\bar{\mathbb{R}}^{(i)}.$$

Consider now the partition of $\tilde{\pi}_i(\eta_i(\omega,x))\subset\bar{\mathbb{R}}^{(i)}=\bar{\mathbb{R}}^{(i-1)}\times\mathbb{R}^{m_i}$ into planes of the form $\{z=(z_1,z_2,\ldots,z_i)\in\bar{\mathbb{R}}^{(i)}:z_i=\text{constant}\}$. Using the Lipschitz property of $\tilde{\pi}_i$ and the definition of $\underline{\delta}_i$ and $\nu$, one can easily verify that at $\nu$-a.e. $z\in\bar{\mathbb{R}}^{(i)}$

$$\underline{\delta}_{i-1}(\omega,x)\leq\liminf_{\rho\to 0}\frac{\log\nu_{z_i}(\{w\in\bar{\mathbb{R}}^{(i-1)}:\|w-z^{(i-1)}\|<\rho\})}{\log\rho}$$

and

$$\underline{\delta}_i(\omega,x)=\liminf_{\rho\to 0}\frac{\log\nu(\{w\in\bar{\mathbb{R}}^{(i)}:\|w-z\|<\rho\})}{\log\rho},$$

where $\nu_c$ is the conditional measure of $\nu$ on $\{z=(z_1,\ldots,z_i)\in\bar{\mathbb{R}}^{(i)}:z_i=c\}$ (hence $\nu_c$ can be viewed as a Borel probability measure on $\bar{\mathbb{R}}^{(i-1)}$). Lemma 11.3.1 in [**4**] then tells us that inequality (5.3) holds for $\mu^i_{(\omega,x)}$-a.e. $y$. □

## 6. *The general case: without the ergodic assumption*

*Proof of Theorem 2.3 for the general case.* If $\mu^*$ is not ergodic, then according to [**7**, p. 132], there exists a unique ($\mu^*$-mod 0) measurable partition $\hat{\eta}_0$ of $\Omega^*\times M$ into a collection of disjoint sets $\{\hat{C}_\alpha\}_{\alpha\in\mathcal{A}}$ and $\Omega^*\times M\setminus\bigcup_{\alpha\in\mathcal{A}}\hat{C}_\alpha$ such that:

(i) $\bigcup_{\alpha\in\mathcal{A}}\hat{C}_\alpha$ has full $\mu^*$ measure for any $\mu\in\mathcal{M}(\mathcal{X}^+(M,v))$;

(ii) for each $\alpha\in\mathcal{A}$, $\Theta^{-1}\hat{C}_\alpha=\hat{C}_\alpha$, and there is a measure $\rho_\alpha\in\mathcal{M}(\mathcal{X}^+(M,v))$ such that its extension $\rho^*_\alpha$ is concentrated on $\hat{C}_\alpha$ and $\Theta:(\hat{C}_\alpha,\rho^*_\alpha)\circlearrowleft$ is ergodic;

(iii) for any $\mu\in\mathcal{M}(\mathcal{X}^+(M,v))$, $\{\rho^*_\alpha\}_{\alpha\in\mathcal{A}}$ is a canonical system of conditional measures of $\mu^*$ associated with $\hat{\eta}_0$;

(iv) for any $\mu\in\mathcal{M}(\mathcal{X}^+(M,v))$, the $\sigma$-algebra consisting of all measurable $\hat{\eta}_0$-sets equals $\mu^*$-mod 0 the $\sigma$-algebra $\mathcal{B}^I:=\{A\in\mathcal{B}_{\mu^*}(\Omega^*\times M):\Theta^{-1}A=A\}$.

By arguments analogous to Lemma IV.2.2 in [**7**], one can prove that

$$\mathcal{B}^I\subset\mathcal{B}^u,\quad \mu^*\text{-mod } 0,$$

where $\mathcal{B}^u:=\big\{B\in\mathcal{B}(\Omega^*\times M):B=\bigcap_{(\omega,x)\in B}\{\omega\}\times W^u(\omega,x)\big\}$. Hence one can easily check that $\xi_u\vee\sigma$ refines $\hat{\eta}_0$, where $\xi_u$ is a partition of $\Omega^*\times M$ subordinate to $W^u$-manifolds. Therefore the transitivity of conditional measures (see [**12**]) implies that

$$(\mu^*_\omega)^{\xi_{i,\omega}}_x=[(\rho^*_\alpha)_\omega]^{\xi_{i,\omega}}_x,\quad \text{for } \mu^*\text{-a.e. } (\omega,x)\in\hat{C}_\alpha,\quad \text{for all } \alpha\in\mathcal{A}.$$

Then results in §§4 and 5 tell us that for $\mu^*$-a.e. $(\omega,x)\in\hat{C}_\alpha$ and $\alpha\in\mathcal{A}$

$$\begin{aligned}\underline{h}_i(\omega,x;\xi_i,\mu^*)&=\underline{h}_i(\omega,x;\xi_i|_{\hat{C}_\alpha},\rho^*_\alpha)=h^\sigma_{\rho^*_\alpha}(\Theta|_{\hat{C}_\alpha},\xi_i|_{\hat{C}_\alpha})\\&=\overline{h}_i(\omega,x;\xi_i|_{\hat{C}_\alpha},\rho^*_\alpha)=\overline{h}_i(\omega,x;\xi_i,\mu^*)\end{aligned}$$

and

$$\begin{aligned}\underline{\delta}_i(\omega,x;\xi_i,\mu^*)&=\underline{\delta}_i(\omega,x;\xi_i|_{\hat{C}_\alpha},\rho^*_\alpha)=\overline{\delta}_i(\omega,x;\xi_i|_{\hat{C}_\alpha},\rho^*_\alpha)\\&=\overline{\delta}_i(\omega,x;\xi_i,\mu^*)=:\delta_i(\omega,x;\xi_i,\mu^*).\end{aligned}$$

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Since $\gamma_i(\omega, x; \mu^*) := \delta_i(\omega, x; \xi_i, \mu^*) - \delta_{i-1}(\omega, x; \xi_{i-1}, \mu^*)$, it follows from equation (5.1) that

$$\gamma_i(\omega, x; \mu^*) = \gamma_i(\omega, x; \rho_\alpha^*) = \frac{h^\sigma_{\rho_\alpha^*}(\Theta|_{\hat{C}_\alpha}, \xi_i|_{\hat{C}_\alpha}) - h^\sigma_{\rho_\alpha^*}(\Theta|_{\hat{C}_\alpha}, \xi_{i-1}|_{\hat{C}_\alpha})}{\lambda_i(x)}$$

holds for $\mu^*$-a.e. $(\omega, x) \in \hat{C}_\alpha$ and $\alpha \in \mathcal{A}$. Hence the $\gamma_i$'s are almost everywhere $\Theta$-invariant functions. In view of Corollary VI.1.1. of [**7**, p. 132], the $\gamma_i$'s are indeed functions well defined almost everywhere on $M$.

Furthermore, the entropy map $h^\sigma_{\mu^*}(\Theta)$ is affine with respect to $\mu^*$, i.e.

$$h^\sigma_{\mu^*}(\Theta) = \int h^\sigma_{\rho_\alpha^*}(\Theta)\, d\hat{\rho}(\alpha),$$

where $\hat{\rho}$ is a probability measure on $\mathcal{A}$. Hence, by $h_\mu(\mathcal{X}^+(M, v)) = h^\sigma_{\mu^*}(\Theta)$ and $h^\sigma_{\rho_\alpha^*}(\Theta) = h^\sigma_{\rho_\alpha^*}(\Theta|_{\hat{C}_\alpha}, \xi_u|_{\hat{C}_\alpha})$, Theorem 2.3 holds true. □

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