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A note on Watson transforms

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In (6) Plancherel considered linear, bounded transforms $W: L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ which satisfy

$$\frac{1}{\alpha} S_{1/\alpha} W = W S_\alpha \quad \text{for all} \quad \alpha \in \mathbb{R}_+, \tag{1}$$

where the linear, bounded transform $S_\alpha: L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is defined by

$$(S_\alpha f)(x) = f(\alpha x), \quad f \in L^2(\mathbb{R}_+), \quad \alpha \in \mathbb{R}_+.$$

Among other things he proved: if W_1, W_2 and W_3 are linear, bounded transforms satisfying (1), then

$$W_1 W_2 W_3 = W_3 W_2 W_1. \tag{2}$$

Plancherel showed this by using a representation as an integral for W in (1). He then remarks: 'Il serait intéressant d'avoir une démonstration directe de ce théorème, c'est à dire une démonstration n'utilisant pas la représentation analytique des transformations par les génératrices.'

In this note we shall characterize the linear, bounded transforms W satisfying (1) in a different manner. This leads to a representation for W, which shows that W is a Watson transform (to be defined precisely).

For such transforms Bochner and Chandrasekharan (1), Goldberg (2) and Leibowitz (5) (who used a result in (3)) proved in different ways that they can be factorized into the product of an 'elementary' Watson transform and a linear, bounded transform which commutes with S_α for all $\alpha \in \mathbb{R}_+$. This is however an easy consequence of (1).

If $K \in L^\infty(\mathbb{R})$ we define the linear, bounded transform $M[K]: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$(M[K]f)(x) = K(x) f(x), \quad f \in L^2(\mathbb{R}).$$

The translation operator T_a $(a \in \mathbb{R})$: $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined by

$$(T_a f)(x) = f(a+x), \quad f \in L^2(\mathbb{R}),$$

and the Fourier (–Plancherel) transform $F: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined by

$$(Ff)(x) = \frac{1}{\sqrt{(2\pi)}} \frac{d}{dx} \int_{\mathbb{R}} f(y) \frac{e^{ixy} - 1}{iy} dy \text{ a.e.}, \quad f \in L^2(\mathbb{R}).$$

We now state a result, which can be found in Bochner and Chandrasekharan (1).

THEOREM 1. *A linear, bounded transform* $B: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *commutes with all translations* T_a $(a \in \mathbb{R})$ *if and only if there exists a function* $K \in L^\infty(\mathbb{R})$ *such that*

$$B = F^{-1} M[K] F.$$

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Introducing the isometric isomorphism $U: L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ *by*

$$(Uf)(x) = e^{x/2} f(e^x), \quad f \in L^2(\mathbb{R}_+),$$

it is easily shown that the linear, bounded transform $\mathfrak{M}: L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ *defined by*

$$\mathfrak{M} = FU$$

is the Mellin transform, cf. (7).

With this the following theorem is a translation of Theorem 1.

Theorem 2. *A linear, bounded transform* $V: L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ *commutes with all* S_α $(\alpha \in \mathbb{R}_+)$ *if and only if there exists a function* $K \in L^\infty(\mathbb{R})$ *such that*

$$V = \mathfrak{M}^{-1} M[K] \mathfrak{M}.$$

Proof. Since $\mathfrak{M}U^{-1} = F$ and $(1/\sqrt{\alpha}) T_{\log \alpha} = US_\alpha U^{-1}$ for all $\alpha \in \mathbb{R}_+$, this follows directly from Theorem 1.

Let the linear, bounded transform $P: L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ be defined by

$$(Pf)(x) = \frac{1}{x} f\left(\frac{1}{x}\right), \quad f \in L^2(\mathbb{R}_+).$$

The next theorem characterizes all linear, bounded transforms $W: L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ which satisfy equation (1).

Theorem 3. *A linear, bounded transform* $W: L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ *satisfies equation* (1) *if and only if there exists a function* $K \in L^\infty(\mathbb{R})$ *such that*

$$W = \mathfrak{M}^{-1} M[K] \mathfrak{M} P.$$

Proof. If $K \in L^\infty(\mathbb{R})$ and $W = \mathfrak{M}^{-1} M[K] \mathfrak{M} P$, then W is linear and bounded. Since $P^2 = I$ it follows from Theorem 2 that WP commutes with S_α for all $\alpha \in \mathbb{R}_+$. Thus

$$S_{1/\alpha} WP = WPS_{1/\alpha} \quad \text{for all} \quad \alpha \in \mathbb{R}_+.$$

But $(1/\alpha) PS_{1/\alpha} = S_\alpha P$ and this shows

$$\frac{1}{\alpha} S_{1/\alpha} WP = WS_\alpha P$$

which is equivalent to the desired result.

Conversely, if W is a linear, bounded transform satisfying (1) then

$$S_{1/\alpha} WP = \alpha WS_\alpha P$$

or

$$\alpha S_\alpha WP = \alpha WPS_\alpha$$

and this shows that WP commutes with all S_α. Hence, according to Theorem 2 there exists a function $K \in L^\infty(\mathbb{R})$ such that $WP = \mathfrak{M}^{-1} M[K] \mathfrak{M}$, thus $W = \mathfrak{M}^{-1} M[K] \mathfrak{M} P$.

Definition. A Watson transform $W: L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is a linear, bounded transform, which satisfies equation (1).

With this definition and Theorem 3 we may state:

1. The product of an odd number of Watson transforms is again a Watson transform, cf. (6), Théorème 1, and (4), Satz 5.

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2. The product of an even number of Watson transforms is a transform of the type described in Theorem 2, cf. (4), Satz 5.

3. Watson transforms W_1, W_2 and W_3 satisfy equation (2), cf. (6), Théorème 4.

4. The transform P is the 'elementary' Watson transform: $K \equiv 1$.

5. Every Watson transform W can be factorized into the product of the 'elementary' Watson transform and a linear, bounded transform which commutes with all $S_\alpha, \alpha \in \mathbb{R}_+$, cf. (1), (2) and (5).

6. If B is a linear, bounded transform in $L^2(\mathbb{R}_+)$, which commutes with all $S_\alpha, \alpha \in \mathbb{R}_+$, then BP is a Watson transform.

7. Since the operators commuting with translations are known in general (cf. Ronald Larsen, *An Introduction to the Theory of Multipliers*, Springer–Verlag, 1971, Theorem 4.1.1) one can generalize Theorem 3 as follows:

If G is a locally compact Abelian group and $W: L^2(G) \to L^2(G)$ is a linear, bounded transform, then $WS\alpha = S_{1/\alpha} W$ for all $\alpha \in G$ if and only if there exists some $K \in L^\infty(\hat{G})$ such that $W = \mathfrak{M}^{-1} M[K] \mathfrak{M} P$, where $\mathfrak{M}: L^2(G) \to L^2(G)$ is the Plancherel transform and $P: L^2(G) \to L^2(G)$ is given by $(Pf)(x) = f(x^{-1})$).†

Of course the above remarks remain valid if the term 'Watson transform' is replaced by 'unitary Watson transform', since a Watson transform $W = \mathfrak{M}^{-1} M[K] \mathfrak{M} P$ with $K \in L^\infty(\mathbb{R})$ is unitary if and only if $|K| \equiv 1$.

For completeness' sake we recall the integral form of Watson transforms. If $K \in L^\infty(\mathbb{R})$, $W = \mathfrak{M}^{-1} M[K] \mathfrak{M} P$ and $g = Wf$, then

$$\int_0^x g(y)\, dy = \int_0^\infty \frac{\chi(xt)}{t} f(t)\, dt, \tag{3}$$

where the function χ_0, defined by $\chi_0(x) = \{\chi(x)\}/x$ $(x \in \mathbb{R}_+)$ is related to K by $\chi_0 = \mathfrak{M}^{-1} K_0$; here the function K_0 is defined by $K_0(x) = \{K(x)\}/\frac{1}{2} - ix$ $(x \in \mathbb{R})$, cf. (7).

We also mention the converse. If χ in (3) is such that to every $f \in L^2(\mathbb{R}_+)$ belongs a $g \in L^2(\mathbb{R}_+)$ such that equation (3) is satisfied, then this mapping $f \mapsto g$ defines a Watson transform W such that $g = Wf$. This has been shown by Kober ((4), Satz 2).

† I thank the referee for suggesting this remark.

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