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Geometry of Interaction and linear combinatory algebras

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We present an axiomatic framework for Girard's Geometry of Interaction based on the notion of linear combinatory algebra. We give a general construction on traced monoidal categories, with certain additional structure, that is sufficient to capture the exponentials of Linear Logic, which produces such algebras (and hence also ordinary combinatory algebras). We illustrate the construction on six standard examples, representing both the 'particle-style' as well as the 'wave-style' Geometry of Interaction.

1. Introduction

Girard's Geometry of Interaction (Girard 1989; Girard 1990; Girard 1995) is a strikingly original interpretation of Linear Logic (Girard 1987) (and hence, *via* standard embeddings, of intuitionistic and classical logic), in which cut-elimination is modelled as a dynamical process of information flow. This interpretation was extensively studied in Danos and Regnier (1993) and Danos and Regnier (1995), applied to the analysis of optimal reduction by Gonthier, Asperti and Lévy (Gonthier *et al.* 1992), and to other execution mechanisms in Mackie (1998), and more recently used in the analysis of Elementary Linear Logic (Baillot and Pedicini 2000). However, some quite basic questions about the Geometry of Interaction have remained in need of clarification. We focus on two:

— What is the general axiomatic framework for the Geometry of Interaction, as opposed to the specific models considered in the literature?

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— In what sense exactly does the Geometry of Interaction yield a model of Linear Logic at all, given that, as already noted by Girard in Girard (1989), if a proof Π reduces to Π' under cut-elimination, it is *not* the case in general that Π and Π' will receive the same interpretation in the Geometry of Interaction.

1.1. *Axiomatics*

For the first question, Abramsky and Jagadeesan (1994b) has already given a model that clearly exemplifies the same general structure as the original Geometry of Interaction interpretation, while being of a significantly different nature. While the 'dynamic algebras' of Danos and Regnier do provide some axiomatic generality, they do not apply to the Abramsky–Jagadeesan-style models. Conversely, Abramsky and Jagadeesan did give a categorical axiomatics for their model in Abramsky and Jagadeesan (1994b), but this did not cover the Girard-style models. A key step was taken with the appearance of the paper on Traced Monoidal Categories (Joyal *et al.* 1996). The motivation for this paper was quite different, stemming from the axiomatics of categories of tangles (although the authors were aware of possible connections to iteration theories. In fact, similar axiomatics in the symmetric case, motivated by flowcharts and 'flownomials' had been developed some years earlier by Stefanescu (Stefanescu 2000).) However, the first author realised, following a stimulating discussion with Gordon Plotkin, that traced monoidal categories provided a common denominator for the axiomatics of both the Girard-style and Abramsky–Jagadeesan-style versions of the Geometry of Interaction, at the basic level of the multiplicatives. This insight was presented in Abramsky (1996), in which Girard-style GoI was dubbed 'particle-style', since it concerns information particles or tokens flowing around a network, while the Abramsky–Jagadeesan-style GoI was dubbed 'wave-style', since it concerns the evolution of a global information state or 'wave'. Formally, this distinction is based on whether the tensor product (that is, the symmetric monoidal structure) in the underlying category is interpreted as a coproduct (particle style) or as a product (wave style). This computational distinction between coproduct and product interpretations of the same underlying network geometry turned out to have been partially anticipated, in a rather different context, in a pioneering paper by E. S. Bainbridge (Bainbridge 1976), as observed by Dusko Pavlovic. These two forms of interpretation, and ways of combining them, have also been studied recently in Stefanescu (2000). He uses the terminology 'additive' for coproduct-based (that is, our 'particle-style') and 'multiplicative' for product-based (that is, our 'wave-style'); this is not suitable for our purposes, because of the clash with Linear Logic terminology.

A relevant observation for the wave-style interpretation of GoI, which was made independently by Hasegawa (Hasegawa 1997) and Hyland, was that when the tensor is the categorical product, the trace axioms are equivalent to standard axioms for a fixpoint operator. Dually, with the tensor product being coproduct, the trace axioms are equivalent to standard axioms for the iteration (dagger) operation in iteration theories (Haghverdi 2000a). On the other hand, the work of Peter Hines (Hines 1997; Hines 1999) is a substantial contribution to the categorical axiomatisation of particle-style GoI, and to

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developing connections between it and the theory of inverse semigroups (Lawson 1998), and also with other models of computation (Hines 2000).

Another contribution of Abramsky (1996) was to give a range of examples of traced monoidal categories, including *resumptions* (a stateful version of particle-style GoI), and *stochastic relations* (a probabilistic version).

What remained to be clarified following Abramsky (1996) was how to extend the axiomatics to incorporate the exponentials of Linear Logic. Traced symmetric monoidal categories and the $\mathcal{G}$ construction only account for the GoI interpretation of the *multiplicative* connectives. The technical subtleties of GoI arise in the interpretation of the exponentials. The first main contribution of the present paper is to answer this question in terms of what we call a *Geometry of Interaction situation*, which is a traced symmetric monoidal category equipped with certain additional structure that allows the exponentials to be interpreted in exactly the GoI fashion. This additional structure is fairly simple: the main ingredient is a strong monoidal endofunctor on the category, which will be given by the countable co-power in particle-style models, and by countable power in the wave-style models. This, together with given retractions that allow the arbitrary splitting of an infinite 'address space', enable the characteristic GoI interpretation of contraction by copying to be expressed.

We also show that the various models described in Abramsky (1996) extend to GoI Situations, and hence that the full GoI construction and the corresponding interpretation of Multiplicative-Exponential Linear Logic can be carried out for each of them.

1.2. *Models of what?*

As regards the second question, we offer a robust notion, of independent interest, that captures those equational properties that the Geometry of Interaction *does* satisfy. This is the notion of *linear combinatory algebra*, which stands in the same relation to the Hilbert-style axiomatisation of the !,$\multimap$ fragment of Linear Logic as standard combinatory algebras do to the implicational fragment of intuitionistic (or minimal) logic. The multiplicative part of linear combinatory algebras in fact coincides with the well-known notion of BCI-algebra (Hindley 1997; Troelstra 1992). The main novelty is that, in addition to the binary operation of application, which corresponds to the inference rule of *modus ponens*, there is also a unary operator (written ! by abuse of notation) corresponding to the modal inference rule of necessitation

$$\frac{A}{\Box A}$$

or in Linear Logic terms, seeing ! as an S4 necessitation modality,

$$\frac{A}{!A}.$$

See Avron (1988) and Troelstra (1992) for a Hilbert-style axiomatisation of Linear Logic containing this rule.

Aside: The same idea could be applied to the implicational fragment of Intuitionistic S4 to yield a combinatory equivalent; this might be interesting, for example, as a basis for

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realizability models of modal set theory, or as a combinatory equivalent of a version of Moggi's computational λ-calculus (Moggi 1989).

Conceptually, combinatory algebras are algebras of *closed terms*; they are 'intensional' in that they satisfy weak equational properties, while still being expressive: for example, the partial recursive functions can be interpreted in any (standard) combinatory algebra. Combinatory algebras are also of interest because of their role in realizability semantics, which can be seen as mediating between intensional notions of computation, and extensional mathematical universes.

As regards the Geometry of Interaction, linear combinatory algebras appear to be the right notion, capturing as they do the fact that GoI does model λ-calculus well in computational terms, while not giving rise to a λ-model, or even a λ-algebra (Barendregt 1984). One basic point is that, corresponding to the interpretation of intuitionistic logic in Linear Logic, every linear combinatory algebra gives rise to a (standard) combinatory algebra – or, equivalently, Combinatory Logic can be interpreted in Linear Combinatory Logic.

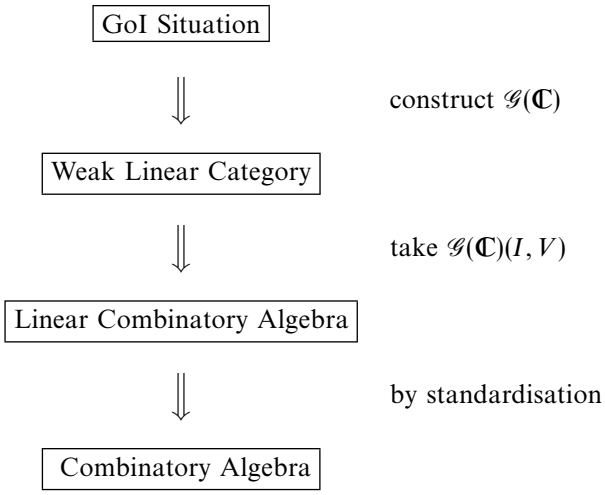
The main result of the present paper is that *every GoI Situation gives rise to a linear combinatory algebra*. We actually proceed *via* an intermediate step: we show that every GoI Situation gives rise to a *weak linear category*; and that every weak linear category gives rise to a linear combinatory algebra. Weak linear categories are simply linear categories with various equational properties subtracted. They are introduced for reasons of technical convenience, rather than as interesting objects in their own right. A consequence of this result is that our range of examples of GoI Situations all give rise to linear combinatory algebras.

This paper provides the general framework and technical background to a series of later developments based on these techniques. For example, the first author and John Longley have shown that one of our examples gives rise *via* realizability to a fully abstract and universal model of PCF (Abramsky and Longley 2000); with Marina Lenisa, he has shown that realizability over an LCA of partial involutions gives rise to a model of System $\mathscr{F}$ which is fully complete for ML types (Abramsky and Lenisa 2000); and he has also shown that a 'wave-style' LCA based on continuous functions gives rise *via* realizability to the Scott model of PCF, while the sub-LCA of sequential functions gives rise to the fully abstract model (Abramsky and Longley 2000). Also, with Paul-André Mellies he has shown that a concurrent games model, based on ideas closely related to wave-style GoI, gives rise to a fully complete model of Multiplicative-Additive Linear Logic (Abramsky and Melliès 1999). The second and third authors have studied axiomatics of categories for 'particle-style' GoI, connecting the framework below to Arbib–Manes' partially additive semantics (Manes and Arbib 1986), as well as to the work of Danos and Regnier mentioned earlier. The details are in the second author's Ph. D. thesis (Haghverdi 2000a) and in Haghverdi (2000b). In Haghverdi (2001) there is a full completeness theorem for the multiplicative fragment (MLL + MIX) for such categories, based on Loader–Hyland–Tan techniques (Loader 1994; Tan 1997). The common framework provided by the present paper allows these and other such results to be understood in a coherent fashion.

The structure of the rest of this paper is as follows. In Section 2, we present some

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background on traced monoidal categories and related notions. In Section 3, we present weak linear categories and linear combinatory algebras, and in Section 4 we prove our main result, which can be seen as the general form of the GoI construction:



In Section 5, we show that the examples described in Abramsky (1996) all extend to GoI Situations, and hence give rise to Linear Combinatory Algebras *via* the GoI construction. Section 6 concludes.

The main ideas of the present paper were introduced by the first author in lectures given in Siena and Edinburgh in April 1997 (Abramsky 1997). The detailed elaboration of the results, some of which first appeared in the second author's thesis (Haghverdi 2000a), is the joint work of the three authors. The authors visited BRICS in May 1997, where some of this material was developed. We thank Glynn Winskel and Prakash Panangaden for their encouragement and hospitality.

2. Traced monoidal categories

Traced monoidal categories were introduced in Joyal *et al.* (1996) and provide a convenient framework for discussing iteration and parametrised feedback in computation. They are general enough to include many previous categorical models of iteration (Bloom and Ésik 1993; Manes and Arbib 1986; Haghverdi 2000a) as well as newer notions arising from linear logic (Abramsky 1996).

In this section we recall the definition of traced symmetric monoidal categories and state some properties of the trace. We also give a normal form theorem for arrows in traced symmetric monoidal categories.

We assume familiarity with the notions of *monoidal category*, *strict monoidal category* and *(strict) monoidal functor* (Mac Lane 1998). A *symmetric* monoidal category is a monoidal category equipped with a natural isomorphism σ with components $\sigma_{A,B} : A \otimes B \longrightarrow B \otimes A$ such that standard coherence diagrams (Mac Lane 1998) commute.

For readability, and without loss of generality, we consider strict monoidal categories.

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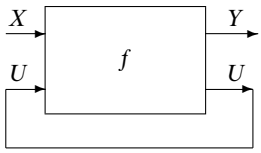


Fig. 1. The trace $\mathtt{Tr}^U_{X,Y}(f)$.

It is well known that every monoidal category is equivalent to a strict one (Mac Lane 1998).

A monoidal functor $(F, \varphi, \varphi_I) : \mathbb{C} \longrightarrow \mathbb{D}$ between symmetric monoidal categories is *symmetric* if $\varphi_{B,A}\sigma_{FA,FB} = F\sigma_{A,B}\varphi_{A,B}$.

Definition 2.1. A *traced symmetric monoidal category* is a symmetric monoidal category $(\mathbb{C}, \otimes, I, \sigma)$ with a family of functions $\mathtt{Tr}^U_{X,Y} : \mathbb{C}(X \otimes U, Y \otimes U) \longrightarrow \mathbb{C}(X, Y)$ pictured in Figure 1, called a *trace*, subject to the following conditions:

— **Natural** in X: $\mathtt{Tr}^U_{X,Y}(f)g = \mathtt{Tr}^U_{X',Y}(f(g \otimes 1_U))$ where $f : X \otimes U \longrightarrow Y \otimes U$, $g : X' \longrightarrow X$,

— **Natural** in Y: $g\mathtt{Tr}^U_{X,Y}(f) = \mathtt{Tr}^U_{X,Y'}((g \otimes 1_U)f)$ where $f : X \otimes U \longrightarrow Y \otimes U$, $g : Y \longrightarrow Y'$,

— **Dinatural** in U: $\mathtt{Tr}^U_{X,Y}((1_Y \otimes g)f) = \mathtt{Tr}^{U'}_{X,Y}(f(1_X \otimes g))$ where $f : X \otimes U \longrightarrow Y \otimes U'$, $g : U' \longrightarrow U$,

— **Vanishing (I,II)**, $\mathtt{Tr}^I_{X,Y}(f) = f$ and $\mathtt{Tr}^{U \otimes V}_{X,Y}(g) = \mathtt{Tr}^U_{X,Y}(\mathtt{Tr}^V_{X \otimes U, Y \otimes U}(g))$ for $f : X \otimes I \longrightarrow Y \otimes I$ and $g : X \otimes U \otimes V \longrightarrow Y \otimes U \otimes V$.

— **Superposing**:

$$g \otimes \mathtt{Tr}^U_{X,Y}(f) = \mathtt{Tr}^U_{W \otimes X, Z \otimes Y}(g \otimes f)$$

for $f : X \otimes U \longrightarrow Y \otimes U$ and $g : W \longrightarrow Z$.

— **Yanking**: $\mathtt{Tr}^U_{U,U}(\sigma_{U,U}) = 1_U$.

We think of $\mathtt{Tr}^U_{X,Y}(f)$ as 'feedback along U', as in Figure 1. The axiomatisation of traces given here differs from those given in Abramsky (1996) and Joyal *et al.* (1996); however, the two versions are equivalent (Haghverdi 2000a).

A monoidal functor $(F, \varphi, \varphi_I) : \mathbb{C} \longrightarrow \mathbb{D}$ between traced symmetric monoidal categories is *traced* if it is symmetric and satisfies

$$\mathtt{Tr}^{FU}_{FA,FB}(\varphi^{-1}_{B,U}(Ff)\varphi_{A,U}) \quad = \quad F(\mathtt{Tr}^U_{A,B}(f)),$$

where $A \otimes U \xrightarrow{f} B \otimes U$ and $FA \otimes FU \xrightarrow{\varphi_{A,U}} F(A \otimes U) \xrightarrow{Ff} F(B \otimes U) \xrightarrow{\varphi^{-1}_{B,U}} FB \otimes FU$.

A monoidal natural transformation m from (F, φ, φ_I) to (G, ψ, ψ_I) is a natural transformation $m : F \Longrightarrow G$ such that $m_{A \otimes B}\varphi_{A,B} = \psi_{A,B}(m_A \otimes m_B)$, and $m_I\varphi_I = \psi_I$. A monoidal *pointwise* natural transformation is a family of maps $m_A : FA \longrightarrow GA$ such that the naturality diagram commutes for morphisms of the form $f : I \longrightarrow A$. That is, we have $Gfm_I = m_A Ff$ for $f : I \longrightarrow A$, A an object in $\mathbb{C}$.

Notation: We will use the following graphical notation: an arrow $U_1 \otimes \cdots \otimes U_m \xrightarrow{f} V_1 \otimes \cdots \otimes V_n$ is represented as a box as in Figure 2. We sometimes emphasise the I/O

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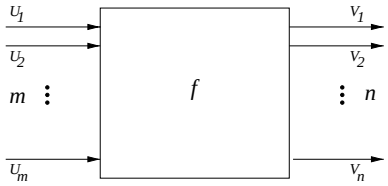


Fig. 2. Graphical representation of f.

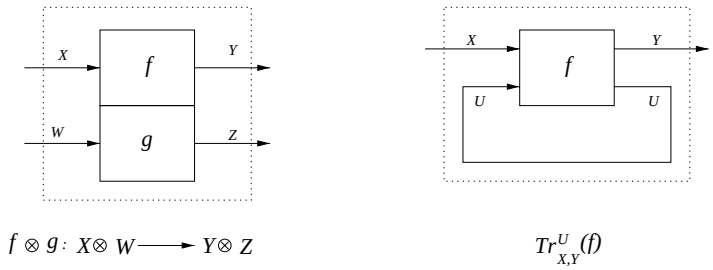


Fig. 3. I/O interfaces.

interface of a tensor or a trace using a dotted box as in Figure 3. We omit the labels on the lines when they are obvious. Graphs for the trace axioms are given in Appendix I.

Example 2.2.

1. The category **Rel** is traced. Let $R : X \times U \longrightarrow Y \times U$ be a morphism in **Rel**. Then $\mathtt{Tr}^U_{X,Y}(R) : X \longrightarrow Y$ is defined by $\mathtt{Tr}^U_{X,Y}(R)(x, y) = \exists u.R(x, u, y, u)$.

2. The category **FDVec** is traced. Given a linear transformation $f : V \otimes U \longrightarrow W \otimes U$ where U, V, W are vector spaces with bases $\{u_i\}, \{v_j\}, \{w_k\}$. $\mathtt{Tr}^U_{V,W}(f) : V \longrightarrow W$ is given by

$$\mathtt{Tr}^U_{V,W}(f)(v_i) = \sum_{j,k} a_{ij}^{kj} w_k \qquad \text{where } f(v_i \otimes u_j) = \sum_{k,m} a_{ij}^{km} w_k \otimes u_m.$$

   This reduces to the usual trace of $f : U \longrightarrow U$ when V and W are one dimensional.

3. Note that both **Rel** and **FDvec** are compact closed categories. More generally, every compact closed category is canonically traced as follows (Joyal *et al.* 1996): given $f : A \otimes U \longrightarrow B \otimes U$ in a compact closed category $\mathbb{C}$,
   $\mathtt{Tr}^U_{A,B}(f) =$

$$A \xrightarrow{1 \otimes \eta_U} A \otimes U \otimes U^* \xrightarrow{f \otimes 1_{U^*}} B \otimes U \otimes U^* \xrightarrow{1 \otimes \sigma} B \otimes U^* \otimes U \xrightarrow{1_U \otimes \epsilon_U} B$$

Remark 2.3. Following Joyal *et al.* (1996), we will mainly use geometric proofs. Two-dimensional reasoning is valid on the progressive parts of the diagrams because of the results of Joyal and Street (1991) for symmetric monoidal categories. The reasoning in parts involving trace is deduced from the axioms of trace. With these provisos, geometric reasoning is completely rigorous. Indeed, as remarked in Joyal *et al.* (1996, page 450),

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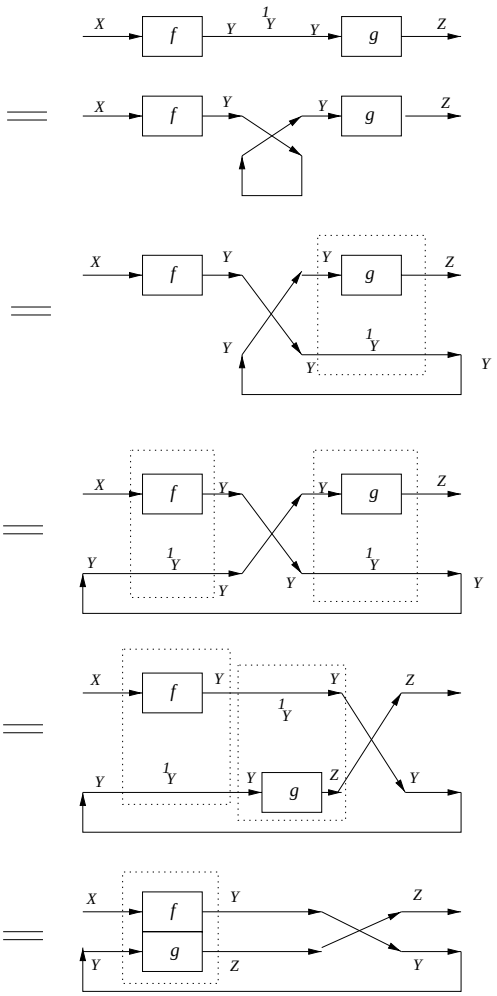


Fig. 4. Graphical proof of Proposition 2.4.

'Algebraic proofs can be constructed from the geometric ones, but algebraic proofs seem only to obfuscate the intuition'. Related geometric calculi for diagrammatic reasoning in monoidal and traced monoidal categories are also developed in the works of Blute, Cockett and Seely (see Blute *et al.* (2000), and the references therein), and arise from their studies of linearly distributive categories and proof-nets.

As an example, we will give algebraic and geometric proofs for the next proposition.

Proposition 2.4 (Generalised Yanking). Let $\mathbb{C}$ be a traced symmetric monoidal category. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be given. Then

$$gf = \mathrm{Tr}^{Y}_{X,Z}(\sigma_{Y,Z}(f \otimes g)).$$

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Proof.

$$\begin{aligned} gf &= g1_Y f && (1)\\ &= g\mathtt{Tr}^Y_{Y,Y}(\sigma_{Y,Y})f && (2)\\ &= \mathtt{Tr}^Y_{Y,Z}((g\otimes 1_Y)\sigma_{Y,Y})f && (3)\\ &= \mathtt{Tr}^Y_{X,Z}((g\otimes 1_Y)\sigma_{Y,Y}(f\otimes 1_Y)) && (4)\\ &= \mathtt{Tr}^Y_{X,Z}(\sigma_{Y,Z}(1_Y\otimes g)(f\otimes 1_Y)) && (5)\\ &= \mathtt{Tr}^Y_{X,Z}(\sigma_{Y,Z}(f\otimes g)). && (6) \end{aligned}$$

Equations 1,2,3,4,5, and 6, respectively, correspond to the diagrams in Figure 4. For clarity, we draw 1_Y as a straight line, omitting the box. □

In particular, one can give an equivalent axiomatisation of traced symmetric monoidal categories with Yanking replaced by Generalised Yanking (Haghverdi 2000a) (*cf.*, also, Blute *et al.* (2000)).

We now give a normal form theorem for arrows in a traced symmetric monoidal category.

Theorem 2.5. Let $\mathbb{C}$ be a traced symmetric monoidal category and T be a set of arrows in $\mathbb{C}$.Then, any expression E built from arrows in T using the tensor product, composition, and trace can be represented as $Tr(\pi F\tau)$ where F consists of a tensor product of arrows in T and π and τ are constructed from symmetry and identity maps using composition and tensor (that is, π and τ are permutations).

Proof. We use induction on the structure of the expression E.

— *Basis step :* Let $E = f : X \longrightarrow Y$ in T. Then $f = \mathtt{Tr}^I_{X,Y}(f)$.

— Let $E = E_1 \otimes E_2$ with $E_1 : X \longrightarrow Y$ and $E_2 : X' \longrightarrow Y'$. By the induction hypothesis, $E_1 = \mathtt{Tr}^U_{X,Y}(\pi_1 F_1 \tau_1)$ and $E_2 = \mathtt{Tr}^{U'}_{X',Y'}(\pi_2 F_2 \tau_2)$. Hence

$$\begin{aligned} E_1\otimes E_2 &= \mathtt{Tr}^U_{X,Y}(\pi_1F_1\tau_1)\otimes \mathtt{Tr}^{U'}_{X',Y'}(\pi_2F_2\tau_2)\\ &= \mathtt{Tr}^{U'}_{X\otimes X',Y\otimes Y'}(\mathtt{Tr}^U_{X,Y}(\pi_1F_1\tau_1)\otimes(\pi_2F_2\tau_2)) \text{ using Superposing,}\\ &= \mathtt{Tr}^{U'}_{X\otimes X',Y\otimes Y'}(\mathtt{Tr}^U_{X\otimes X'\otimes U',Y\otimes Y'\otimes U'}((1_Y\otimes\sigma_{U,Y'\otimes U'})(\pi_1F_1\tau_1\otimes\pi_2F_2\tau_2)\\ &\quad (1_X\otimes\sigma_{X'\otimes U',U}))) \quad \text{using Superposing,}\\ &= \mathtt{Tr}^{U'\otimes U}_{X\otimes X',Y\otimes Y'}((1_Y\otimes\sigma_{U,Y'\otimes U'})(\pi_1F_1\tau_1\otimes\pi_2F_2\tau_2)(1_X\otimes\sigma_{X'\otimes U',U}))\\ &\qquad\qquad \text{using Vanishing II,}\\ &= \mathtt{Tr}^{U'\otimes U}_{X\otimes X',Y\otimes Y'}(\pi F\tau) \end{aligned}$$

where $\pi = (1_Y \otimes \sigma_{U,Y'\otimes U'})(\pi_1 \otimes \pi_2)$, $F = F_1 \otimes F_2$ and $\tau = (\tau_1 \otimes \tau_2)(1_X \otimes \sigma_{X'\otimes U',U})$.

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— Let $E = E_2E_1 : X \longrightarrow Z$ where $E_1 : X \longrightarrow Y$ and $E_2 : Y \longrightarrow Z$.

$$\begin{aligned} E = E_2E_1 &= \mathrm{Tr}^Y_{X,Z}(\sigma_{Y,Z}(E_1 \otimes E_2)) \text{ by Proposition 2.4,} \\ &= \mathrm{Tr}^Y_{X,Z}(\sigma_{Y,Z}\mathrm{Tr}^U_{X\otimes Y, Y\otimes Z}(\pi F\tau)) \text{ using previous part,} \\ &= \mathrm{Tr}^Y_{X,Z}(\mathrm{Tr}^U_{X\otimes Y, Z\otimes Y}((\sigma_{Y,Z} \otimes 1_U)(\pi F\tau))) \text{ using Naturality,} \\ &= \mathrm{Tr}^{Y\otimes U}_{X,Z}(\pi' F\tau) \text{ using Vanishing II.} \end{aligned}$$

where $\pi' = (\sigma_{Y,Z} \otimes 1_U)\pi$.

— Let $E = \mathrm{Tr}^U_{X,Y}(E_1)$ for $E_1 : X \otimes U \longrightarrow Y \otimes U$.

$$\begin{aligned} E &= \mathrm{Tr}^U_{X,Y}(E_1) \\ &= \mathrm{Tr}^U_{X,Y}(\mathrm{Tr}^{U'}_{X\otimes U, Y\otimes U}(\pi F\tau)) \text{ by the induction hypothesis} \\ &= \mathrm{Tr}^{U\otimes U'}_{X,Y}(\pi F\tau) \text{ using Vanishing II.} \end{aligned}$$

□

The following construction on traced monoidal categories (Joyal *et al.* 1996) isolates the key properties of Girard's GoI. Composition – given by trace – corresponds in this setting to Girard's *execution formula* (Girard 1989).

Definition 2.6. (The *Geometry of Interaction* construction) Given a traced monoidal category $\mathbb{C}$, we define a new category, $\mathcal{G}(\mathbb{C})$, as follows:

— **Objects**: Pairs of objects from $\mathbb{C}$, for example, (A^+, A^-) where A^+ and A^- are objects of $\mathbb{C}$.

— **Arrows**: An arrow $f : (A^+, A^-) \longrightarrow (B^+, B^-)$ in $\mathcal{G}(\mathbb{C})$ is $f : A^+ \otimes B^- \longrightarrow A^- \otimes B^+$ in $\mathbb{C}$.

— **Identity**: $1_{(A^+,A^-)} = \sigma_{A^+,A^-}$.

— **Composition**: Composition is given by symmetric feedback. Given

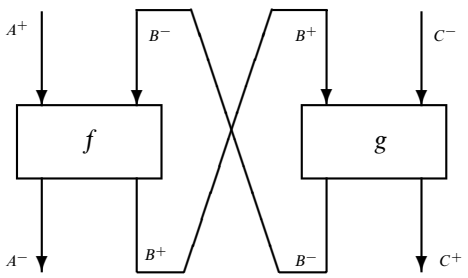
$$f : (A^+, A^-) \longrightarrow (B^+, B^-) \quad \text{and} \quad g : (B^+, B^-) \longrightarrow (C^+, C^-),$$

$$gf : (A^+, A^-) \longrightarrow (C^+, C^-)$$

is given by

$$gf = \mathrm{Tr}^{B^-\otimes B^+}_{A^+\otimes C^-, A^-\otimes C^+}(\beta(f \otimes g)\alpha)$$

where $\alpha = (1_{A^+} \otimes 1_{B^-} \otimes \sigma_{C^-,B^+})(1_{A^+} \otimes \sigma_{C^-,B^-} \otimes 1_{B^+})$ and $\beta = (1_{A^-} \otimes 1_{C^+} \otimes \sigma_{B^+,B^-})(1_{A^-} \otimes \sigma_{B^+,C^+} \otimes 1_{B^-})(1_{A^-} \otimes 1_{B^+} \otimes \sigma_{B^-,C^+})$. An informal picture displaying gf is given below. For the precise graphical representation of gf, see Figure 22 in Appendix I.



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— **Tensor**: $(A^+, A^-) \otimes (B^+, B^-) = (A^+ \otimes B^+, A^- \otimes B^-)$ and for $f : (A^+, A^-) \longrightarrow (B^+, B^-)$ and $g : (C^+, C^-) \longrightarrow (D^+, D^-)$,

$$f \otimes g = (1_{A^-} \otimes \sigma_{B^+,C^-} \otimes 1_{D^+})(f \otimes g)(1_{A^+} \otimes \sigma_{C^+,B^-} \otimes 1_{D^-}).$$

— **Unit**: (I, I).

Remark 2.7. We have given a specific definition for α and β above, but any other permutations $\alpha : A^+ \otimes C^- \otimes B^- \otimes B^+ \stackrel{\cong}{\longrightarrow} A^+ \otimes B^- \otimes B^+ \otimes C^-$ and $\beta : A^- \otimes B^+ \otimes B^- \otimes C^+ \stackrel{\cong}{\longrightarrow} A^- \otimes C^+ \otimes B^- \otimes B^+$ will yield the same result for gf, due to coherence.

Proposition 2.8. Let $\mathbb{C}$ be a traced symmetric monoidal category. Then $\mathcal{G}(\mathbb{C})$ defined as in Definition 2.6 is a compact closed category. Moreover, $F : \mathbb{C} \longrightarrow \mathcal{G}(\mathbb{C})$ with $F(A) = (A, I)$ and $F(f) = f$ is a full and faithful embedding.

Proof. (Sketch) For any two objects (A^+, A^-) and (B^+, B^-) in $\mathcal{G}(\mathbb{C})$, we define $\sigma_{(A^+,A^-),(B^+,B^-)} =_{def}$

$$(1_{A^-} \otimes \sigma_{B^+,B^-} \otimes 1_{A^+})(\sigma_{B^+,A^-} \otimes \sigma_{A^+,B^-})(1_{B^+} \otimes \sigma_{A^+,A^-} \otimes 1_{B^-})(\sigma_{A^+,B^+} \otimes \sigma_{B^-,A^-}).$$

The dual of (A^+, A^-) is given by $(A^+, A^-)^* = (A^-, A^+)$ where the unit $\eta : (I, I) \longrightarrow (A^+, A^-) \otimes (A^+, A^-)^* =_{def} \sigma_{A^-,A^+}$ and counit $\epsilon : (A^+, A^-)^* \otimes (A^+, A^-) \longrightarrow (I, I) =_{def} \sigma_{A^-,A^+}$. The internal homs are given by $(A^+, A^-) \multimap (B^+, B^-) = (B^+ \otimes A^-, B^- \otimes A^+)$. □

It is shown in Joyal *et al.* (1996) that $Int(\mathbb{C})$, which is isomorphic to the above $\mathcal{G}(\mathbb{C})$, is the 'free compact closure' of $\mathbb{C}$, in an appropriate bicategorical sense.

Notation: Let $A \underset{g}{\overset{f}{\rightleftarrows}} B$ be morphisms in a category $\mathbb{C}$. We write $f : A \lhd B : g$ to mean that A is a retract of B, that is, $gf = 1_A$. We write $A \lhd B$ to mean $f : A \lhd B : g$ for some arrows f, g.

Lemma 2.9.

— *Let* $A^+ \cong B^+$ *and* $A^- \cong B^-$ *in* $\mathbb{C}$. *Then* $(A^+, A^-) \cong (B^+, B^-)$ *in* $\mathcal{G}(\mathbb{C})$.

— *If* $f_1 : A^+ \lhd B^+ : g_1$ *and* $f_2 : A^- \lhd B^- : g_2$ *in* $\mathbb{C}$, *then* $\sigma_{B^+,A^-}(f_1 \otimes g_2) : (A^+, A^-) \lhd (B^+, B^-) : \sigma_{A^+,B^-}(g_1 \otimes f_2)$ *in* $\mathcal{G}(\mathbb{C})$.

3. Weak linear categories and linear combinatory algebras

In this section we introduce weak linear categories (WLCs) and linear combinatory algebras (LCAs). We show how to construct such algebras from WLCs with reflexive objects. We also show how to get (standard) combinatory algebras from linear ones.

Definition 3.1. A *weak linear category (WLC)* $(\mathbb{C}, !)$ consists of the following data:

— A symmetric monoidal closed category $\mathbb{C}$;

— A symmetric monoidal functor $! : \mathbb{C} \longrightarrow \mathbb{C}$ (officially, $! = (!, \varphi, \varphi_I)$);

— The following monoidal pointwise natural transformations:

1 $\mathtt{der} : ! \Longrightarrow Id$

2 $\delta : ! \Longrightarrow !!$

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3 $\mathtt{con} : ! \Longrightarrow ! \otimes !$

4 $\mathtt{weak} : ! \Longrightarrow \mathscr{K}_I$. Here $\mathscr{K}_I$ is the constant I functor.

We note that in contrast to the usual models of linear logic (Seely 1989; Bierman 1995; Troelstra 1992), in the GoI models below the monoidal transformations $\mathtt{der}, \delta, \mathtt{con}, \mathtt{weak}$ exist but are merely pointwise natural (see Appendix III). Fortunately, pointwise naturality suffices for our main construction of linear combinatory algebras (out of WLCs with a reflexive object) in Theorem 3.5. Similarly, for our purposes we do not require $(!, \mathtt{der}, \delta)$ to form a comonad, nor for $(!A, \mathtt{con}_A, \mathtt{weak}_A)$ to form a comonoid.

Definition 3.2. A *reflexive* object in a WLC $(\mathbb{C}, !)$ is an object V in $\mathbb{C}$ with the following retracts:

— $V \multimap V \lhd V$

— $!V \lhd V$

— $I \lhd V$.

Since CCCs are SMCCs, all the usual domain theoretic constructions of reflexive objects in CCCs also yield reflexive objects in the WLC-sense, as follows.

Proposition 3.3. Let $\mathbb{C}$ be a CCC and V be a reflexive object in $\mathbb{C}$, that is, $V^V \lhd V$. Then $(\mathbb{C}, Id)$ is a WLC and V is a reflexive object in the WLC-sense.

Proof. Any CCC is an SMCC. Id is a symmetric monoidal functor from $\mathbb{C}$ to itself, and the following are monoidal natural transformations:

1 $\mathtt{der}_A = 1_A$

2 $\delta_A = 1_A$

3 $\mathtt{con}_A = \langle 1_A, 1_A \rangle$

4 $\mathtt{weak}_A = f : A \longrightarrow T$; the unique map to the terminal object T.

It can be easily shown that $V^V \lhd V$ implies $T \lhd V$. Therefore $V \multimap V = V^V \lhd V$, $!V = Id(V) = V \lhd V$ and $I = T \lhd V$, and hence V is a reflexive object in the WLC-sense. □

Definition 3.4. A *linear combinatory algebra* $(A, \cdot, !)$ consists of the following data:

— An applicative structure $(A, \cdot)$;

— A unary operator $! : A \to A$;

— Distinguished elements $B, C, I, K, W, D, \delta, F$ of A,

satisfying the following identities (we associate $\cdot$ to the left and write xy for $x \cdot y$, $x!y = x \cdot (!(y))$, and so on) for all variables x, y, z ranging over A:

1 $Bxyz = x(yz)$ Composition, Cut

2 $Cxyz = (xz)y$ Exchange

3 $Ix = x$ Identity

4 $Kx!y = x$ Weakening

5 $Wx!y = x!y!y$ Contraction

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6 $D!x = x$ Dereliction

7 $\delta!x = !!x$ Comultiplication

8 $F!x!y = !(xy)$ Monoidal Functoriality.

The notion of LCA corresponds to a Hilbert style axiomatisation of the $\{!, \multimap\}$ fragment of linear logic (Abramsky 1997; Avron 1988; Troelstra 1992). The *principal types* of the combinators correspond to the axiom schemes that they name. They can be computed by a Hindley–Milner style algorithm (Hindley 1997) from the above equations:

1 $B : (\beta \multimap \gamma) \multimap (\alpha \multimap \beta) \multimap \alpha \multimap \gamma$
2 $C : (\alpha \multimap \beta \multimap \gamma) \multimap (\beta \multimap \alpha \multimap \gamma)$
3 $I : \alpha \multimap \alpha$
4 $K : \alpha \multimap !\beta \multimap \alpha$
5 $W : (!\alpha \multimap !\alpha \multimap \beta) \multimap !\alpha \multimap \beta$
6 $D : !\alpha \multimap \alpha$
7 $\delta : !\alpha \multimap !!\alpha$
8 $F : !(\alpha \multimap \beta) \multimap !\alpha \multimap !\beta$.

Let $\mathbb{C}$ be a WLC and V be a reflexive object in $\mathbb{C}$ with retracts $r : V \multimap V \lhd V : s$ and $p : !V \lhd V : q$ where (p, q) and (r, s) are the retraction morphisms (see *Notation* before Lemma 2.9.) We define the following operations on the set $\mathbb{C}(I, V)$:

— Given $f, g \in \mathbb{C}(I, V)$, $f \cdot g = ev(sf \otimes g)$.
— Given $f \in \mathbb{C}(I, V)$, $!f = p!f\varphi_I$ where $\varphi_I : I \longrightarrow !I$ and $! = (!, \varphi, \varphi_I)$.

Theorem 3.5. Let $(\mathbb{C}, !)$ be a WLC and V be a reflexive object in $\mathbb{C}$ with retracts $r : V \multimap V \lhd V : s$ and $p : !V \lhd V : q$. Then $(\mathbb{C}(I, V), \cdot, !)$, with $\cdot$ and $!$ defined as above, is a linear combinatory algebra.

A detailed proof is given in Haghverdi (2000a, Chapter 6). This proof only uses monoidal *pointwise* naturality of the transformations $\mathtt{der}, \delta, \mathtt{con}, \mathtt{weak}$. See also Remark 4.3 below.

Definition 3.6. A *standard combinatory algebra* consists of a pair $(A, \cdot_s)$ where A is a non-empty set and $\cdot_s$ is a binary operation on A and B_s, C_s, I_s, K_s, and W_s are distinguished elements of A satisfying the following identities for all x, y, z variables ranging over A:

1 $B_s \cdot_s x \cdot_s y \cdot_s z = x \cdot_s (y \cdot_s z)$
2 $C_s \cdot_s x \cdot_s y \cdot_s z = (x \cdot_s z) \cdot_s y$
3 $I_s \cdot_s x = x$
4 $K_s \cdot_s x \cdot_s y = x$
5 $W_s \cdot_s x \cdot_s y = x \cdot_s y \cdot_s y$.

Note that this is equivalent to the more familiar definition of SK-combinatory algebra; in particular, S_s can be defined from B_s, C_s, I_s and W_s (Barendregt 1984; Hindley 1997). Let $(A, \cdot, !)$ be a linear combinatory algebra. We define a binary operation $\cdot_s$ on A as follows: for $\alpha, \beta \in A$, $\alpha \cdot_s \beta =_{def} \alpha \cdot !\beta$. We define $D' = C(BBI)(BDI)$. Note that

$$D'x!y = xy.$$

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Now consider the following elements of A:

1 $B_s =_{def} C\cdot(B\cdot(B\cdot B\cdot B)\cdot(D'\cdot I))\cdot(C\cdot((B\cdot B)\cdot F)\cdot\delta)$
2 $C_s =_{def} D'\cdot C$
3 $I_s =_{def} D'\cdot I$
4 $K_s =_{def} D'\cdot K$
5 $W_s =_{def} D'\cdot W$.

Theorem 3.7. Let $(A, \cdot, !)$ be a linear combinatory algebra. Then $(A, \cdot_s)$ with $\cdot_s$ and the elements B_s, C_s, I_s, K_s, W_s as defined above is a combinatory algebra.

We remark that in the case of WLCs coming from CCCs (see Proposition 3.3 above) the associated linear combinatory algebra agrees with the (standard) combinatory algebra structure, since

$$x\cdot_s y = x\cdot !y = x.y \ .$$

4. General GoI construction

In this section we will present the general form of the GoI construction, extending the $\mathcal{G}$ construction from Definition 2.6 to encompass the exponentials. We then show our main result: that this construction gives rise to linear combinatory algebras (and thus to standard combinatory algebras, using the previous results).

Definition 4.1. A *GoI Situation* is a triple $(\mathbb{C}, T, U)$ where:

— $\mathbb{C}$ is a traced symmetric monoidal category

— $T : \mathbb{C} \longrightarrow \mathbb{C}$ is a traced symmetric monoidal functor with the following retractions (which are monoidal natural transformations):

1 $e : TT \lhd T : e'$ (Comultiplication)
2 $d : Id \lhd T : d'$ (Dereliction)
3 $c : T \otimes T \lhd T : c'$ (Contraction)
4 $w : \mathcal{K}_I \lhd T : w'$ (Weakening), where $\mathcal{K}_I$ is the constant I functor.

— U is an object of $\mathbb{C}$, called *a reflexive object*, with retractions:

1 $j : U \otimes U \lhd U : k$
2 $I \lhd U$
3 $u : TU \lhd U : v$.

Given a GoI Situation $(\mathbb{C}, T, U)$ with $j : U \otimes U \lhd U : k$ and $u : TU \lhd U : v$, where (j, k) and (u, v) are the associated retract pairs, consider the compact closed category $\mathcal{G}(\mathbb{C})$ from Definition 2.6, with the distinguished objects $I = (I, I)$ and $V = (U, U)$. Note that by definition (since we are in the strict case), $\mathcal{G}(\mathbb{C})(I, V) = \mathbb{C}(U, U)$.

We can define an endofunctor $! : \mathcal{G}(\mathbb{C}) \longrightarrow \mathcal{G}(\mathbb{C})$ as follows: $!(A^+, A^-) = (TA^+, TA^-)$ and given $f : (A^+, A^-) \longrightarrow (B^+, B^-)$,

$$!f \ =_{def} \ TA^+ \otimes TB^- \xrightarrow{\cong} T(A^+ \otimes B^-) \xrightarrow{Tf} T(A^- \otimes B^+) \xrightarrow{\cong} TA^- \otimes TB^+.$$

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Proposition 4.2. Let $(\mathbb{C}, T, U)$ be a GoI Situation satisfying the conditions above. Then:

(i) $(\mathcal{G}(\mathbb{C}), !)$ is a WLC with reflexive object $V = (U, U)$,

(ii) $(\mathcal{G}(\mathbb{C})(I, V), \cdot, \mathbf{!})$ is an LCA, where for any $f, g \in \mathcal{G}(\mathbb{C})(I, V) = \mathbb{C}(U, U)$, $f \cdot g = \mathtt{Tr}^U_{U,U}((1_U \otimes g)(kfj))$, and $\mathbf{!}f = u(Tf)v$.

Proof. Note that $\mathcal{G}(\mathbb{C})$ is a compact closed category and hence it is symmetric monoidal closed, see Proposition 2.8.

It can be easily shown that ! is a symmetric monoidal functor. Next we define the following maps:

— $\mathtt{der}_{(A^+,A^-)} : !(A^+, A^-) \longrightarrow (A^+, A^-) =_{def} \sigma_{A^+,TA^-}(d'_{A^+} \otimes d_{A^-})$ where $d_A : A \lhd TA : d'_A$,

— $\delta_{(A^+,A^-)} : !(A^+, A^-) \longrightarrow !!(A^+, A^-) =_{def} \sigma_{T^2A^+,TA^-}(e'_{A^+} \otimes e_{A^-})$ where $e_A : T^2A \lhd TA : e'_A$,

— $\mathtt{con}_{(A^+,A^-)} : !(A^+, A^-) \longrightarrow !(A^+, A^-) \otimes !(A^+, A^-) =_{def} \sigma_{TA^+ \otimes TA^+,TA^-}(c'_{A^+} \otimes c_{A^-})$ where $c_A : TA \otimes TA \lhd TA : c'_A$,

— $\mathtt{weak}_{(A^+,A^-)} : !(A^+, A^-) \longrightarrow (I, I) =_{def} \sigma_{I,TA^-}(w'_{A^+} \otimes w_{A^-})$ where $w_A : I \lhd TA : w'_A$.

A direct calculation shows (see Appendix II) that $\mathtt{der}, \delta, \mathtt{con}, \mathtt{weak}$ form monoidal pointwise natural transformations. With the above definitions it is easily seen that $(\mathcal{G}(\mathbb{C}), !)$ is a WLC. Also, it follows (see Lemma 2.9) that $V = (U, U)$ is a reflexive object in $\mathcal{G}(\mathbb{C})$, and hence we have that $(\mathcal{G}(\mathbb{C})(I, V), \cdot, \mathbf{!})$ is an LCA using Theorem 3.5. □

Remark 4.3. Let us consider the necessity of pointwise naturality. As shown in Appendix II, the family of maps $\mathtt{der}, \delta, \mathtt{con}, \mathtt{weak}$ are monoidal *pointwise* natural transformations and not natural in general. These maps will be natural transformations iff the underlying monoidal natural transformations used in their definitions are monoidal natural isomorphisms. For example `der` will be a natural transformation iff (d_A, d'_A) is an isomorphism with $d_A^{-1} = d'_A$ (a proof of this is sketched at the end of Appendix II). This will, in general, produce undesirable effects: for example, asking for (d, d') to form an isomorphism means that every object in $\mathbb{C}$ must be a fixed point of T. Similarly, requiring (w, w') to be an isomorphism means TA isomorphic to I for all A, and together this means that $\mathbb{C}$ must be a trivial one object category. The isomorphisms (c, c') and (e, e') are on the other hand benign. For example, in the case of **PInj** below we have chosen (c, c') and (e, e') that are isomorphisms and hence δ and `con` are actually natural transformations.

The diagrammatic definitions of the combinators show vividly the geometric, information flow perspective on computation yielded by GoI, and we shall give them all explicitly. The reductions of the combinators to normal forms is included in Appendix III.

Notation: We introduce in Figure 5 an explicit graphical notation for the data in a GoI Situation.

The combinators are defined as follows:

1 $I =_{def} \alpha\gamma\beta$, where

(a) $\alpha = j$

(b) $\beta = k$

(c) $\gamma = \sigma_{U,U}$. See Figure 6.

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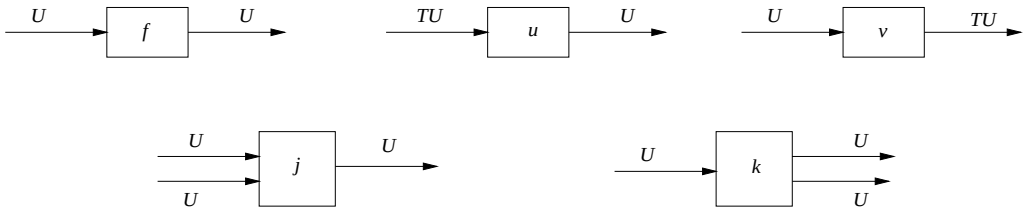


Fig. 5. Graphical representations.

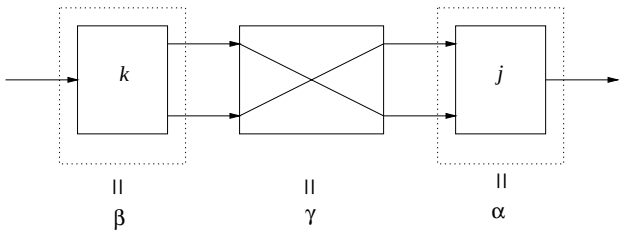


Fig. 6. Identity combinator I.

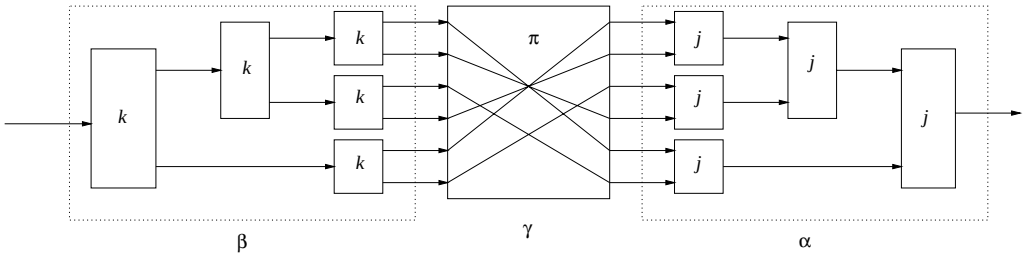


Fig. 7. Composition combinator B.

2 $B =_{def} \alpha\gamma\beta$, where
   (a) $\alpha = j(j \otimes 1_U)(j \otimes j \otimes j)$
   (b) $\beta = (k \otimes k \otimes k)(k \otimes 1_U)k$
   (c) $\gamma =$ the permutation π in Figure 7.

3 $C =_{def} \alpha\gamma\beta$, where
   (a) $\alpha = j(j \otimes j)(j \otimes 1_U \otimes j \otimes 1_U)$
   (b) $\beta = (k \otimes 1_U \otimes k \otimes 1_U)(k \otimes k)k$
   (c) $\gamma =$ the permutation π in Figure 8.

4 $K =_{def} \alpha\gamma\beta$, where
   (a) $\alpha = j(j \otimes 1)$
   (b) $\beta = (k \otimes 1)k$
   (c) $\gamma = \pi(1_U \otimes f_K \otimes 1_U)$, where $f_K = uw_U w'_U v$ and π is the permutation given in Figure 9.

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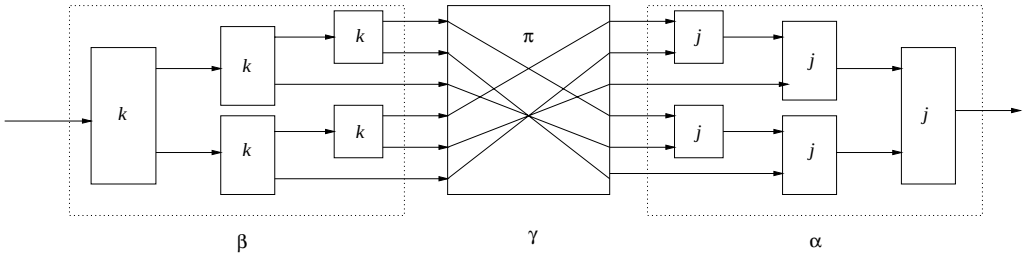


Fig. 8. Exchange combinator C.

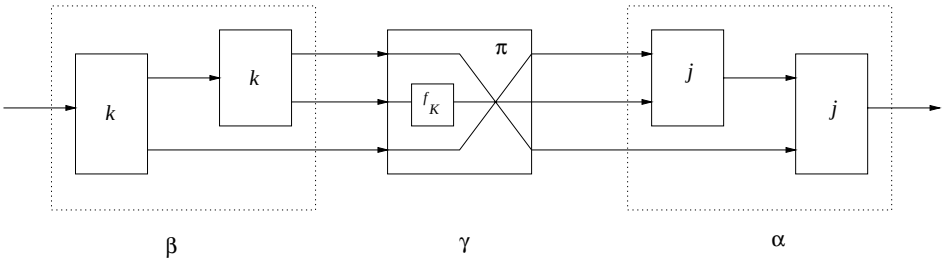


Fig. 9. Weakening combinator K.

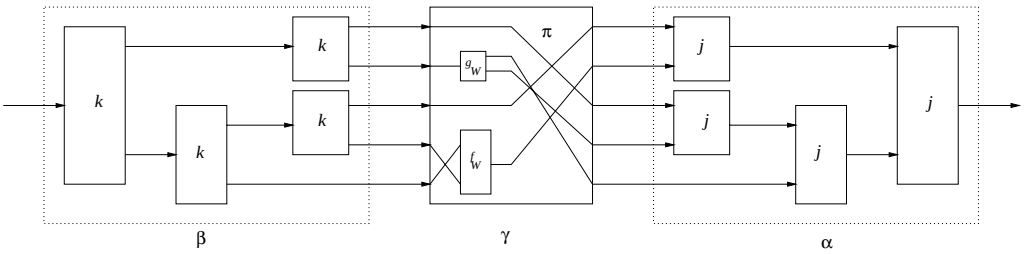


Fig. 10. Contraction combinator W.

5 $W =_{def} \alpha\gamma\beta$, where

(a) $\alpha = j(1_U \otimes j)(j \otimes j \otimes 1_U)$

(b) $\beta = (k \otimes k \otimes 1_U)(1_U \otimes k)k$

(c) $\gamma = \pi(1_U \otimes g_W \otimes 1_U \otimes f_W)(1_U \otimes 1_U \otimes 1_U \otimes \sigma)$, where $g_W = (u \otimes u)c'_U v, f_W = uc_U(v \otimes v)$, and π is the permutation given in Figure 10.

6 $D =_{def} \alpha\gamma\beta$, where

(a) $\alpha = j(j \otimes j)$

(b) $\beta = (k \otimes k)k$

(c) $\gamma = \pi(1_U \otimes g_D \otimes 1_U \otimes f_D)$, where $f_D = ud_U$, $g_D = d'_U v$ and π is the permutation given in Figure 11

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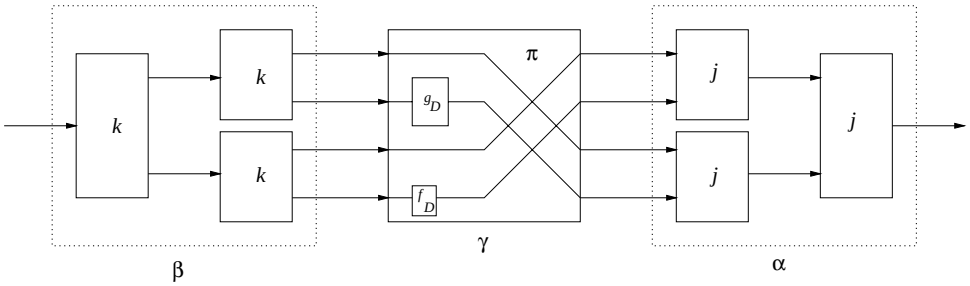


Fig. 11. Dereliction combinator D.

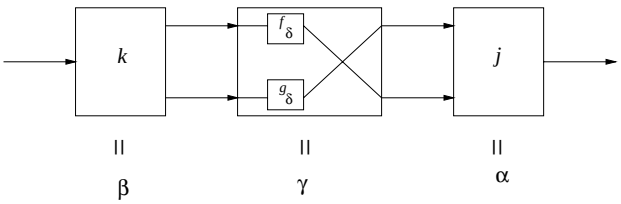


Fig. 12. Comultiplication combinator δ.

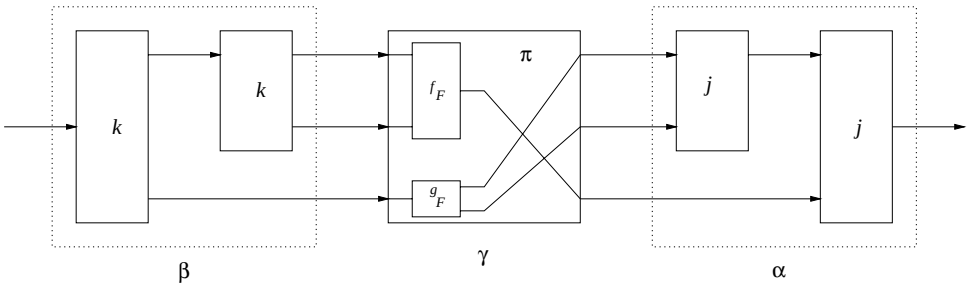


Fig. 13. Functoriality combinator F.

7 $\delta =_{def} \alpha\gamma\beta$, where
   (a) $\alpha = j$
   (b) $\beta = k$
   (c) $\gamma = \sigma_{U,U}(f_\delta \otimes g_\delta)$, where $f_\delta = ue_U T(v)v$ and $g_\delta = uT(u)e'_U v$. See Figure 12.

8 $F =_{def} \alpha\gamma\beta$, where
   (a) $\alpha = j(j \otimes 1_U)$
   (b) $\beta = (k \otimes 1_U)k$
   (c) $\gamma = \pi(f_F \otimes g_F)$, where $f_F = uT(j)\psi_{U,U}(v \otimes v)$, $g_F = (u \otimes u)\psi^{-1}_{U,U}T(k)v$, π is the permutation given in Figure 13, and $T = (T, \psi, \psi_I)$.

5. Examples

In this section we illustrate the construction given in the previous sections in some specific categories: **PInj** (sets and partial injective functions), **Pfn** (sets and partial functions),

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Rel$_+$ (sets and relations), **Res** (the category of Resumptions), **SRel** (measurable spaces and stochastic kernels), and ω-**CPO** (ω-complete partial orders and continuous maps). The role of these categories in GoI and the connection of their trace structure to Girard's *execution formula* has been emphasised in Abramsky (1996), Abramsky (1997) and Haghverdi (2000b). The categories **PInj**, **Pfn**, **Rel**$_+$ and **SRel** are all examples of a special subclass of traced symmetric monoidal categories called *traced unique decomposition categories*, which were studied in Haghverdi (2000a) and Haghverdi (2000b). These categories generalise Manes and Arbib's *partially additive semantics* (Manes and Arbib 1986) and support a convenient computational framework for many of the 'particle-style models' of GoI emphasised in Abramsky (1996). Our uniform treatment of the trace formula in **PInj**, **Pfn**, **Rel**$_+$ and **SRel** is motivated from this viewpoint.

5.1. **PInj**

Consider the category **PInj** of sets and partial injective functions. **PInj** is a traced symmetric monoidal category. The tensor product is given by the disjoint union of sets, where we identify $A \uplus B = \{1\} \times A \cup \{2\} \times B$ (note that this is not a coproduct in **PInj**). There are the obvious injections $\mathtt{inl}^{A,B} : A \to A \uplus B$ and $\mathtt{inr}^{A,B} : B \to A \uplus B$ given as follows (we omit superscripts) $\mathtt{inl}(a) = (1, a)$ and $\mathtt{inr}(b) = (2, b)$. There are also 'quasiprojections' $\rho_1 : A \uplus B \longrightarrow A$ and $\rho_2 : A \uplus B \longrightarrow B$ given by $\rho_1((1, a)) = a$ (where $\rho_1((2, b))$ is undefined) and by $\rho_2((2, b)) = b$ (where $\rho_2((1, a))$ is undefined.)

Given a morphism $f : X \uplus U \longrightarrow Y \uplus U$, we may consider its four 'components' $f_{XY} : X \to Y$, $f_{XU} : X \to U$,$f_{UX} : U \to X$, and $f_{UU} : U \to U$ obtained by pre- and post-composing with injections and quasiprojections: for example, $f_{XY} = X \xrightarrow{\mathtt{inl}} X \uplus U \xrightarrow{f} Y \uplus U \xrightarrow{\rho_1} Y$, and so on (see Figure 14); it may be represented as a matrix $f = \begin{bmatrix} f_{XY} & f_{UY} \\ f_{XU} & F_{UU} \end{bmatrix}$.

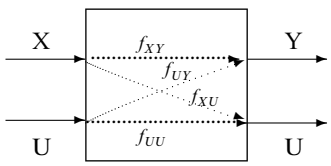


Fig. 14. Components of $f : X \uplus U \to Y \uplus U$.

The trace is given by the formula

$$\mathtt{Tr}^U_{X,Y}(f) = f_{XY} + \sum_{n \in \omega} f_{UY} f^n_{UU} f_{XU} \ ,$$

which we interpret as follows: a family $\{h_i\}_{i \in I} : X \longrightarrow Y$ is said to be *summable* if the h_i's have pairwise disjoint domains and codomains. In that case, we define their sum

$$(\sum_{i \in I} h_i)(x) = \begin{cases} h_j(x), & \text{if } x \in \mathtt{Dom}(h_j) \text{ for some } j \in I\,; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

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In the next paragraph we show that the above trace formula is well defined. But we note that from the 'particle-style' viewpoint, we may picture the trace formula above as follows: consider the trace in Figure 3, superposed on Figure 14. Imagine a token entering the box through the wire labelled X, and either exiting at Y by following f_{XY}, or instead following f_{XU}, cycling some number of times *via* $f_{UU} : U \to U$, and again exiting through Y *via* f_{UY}.

Given $f : X \uplus U \longrightarrow Y \uplus U$, it remains to show that the sum appearing in the trace above, that is, $f_{XY} + \sum_{n\in\omega} f_{UY} f_{UU}^n f_{XU}$, is well defined. That is, the family of functions appearing in the above sum have pairwise disjoint domains and codomains.

(i) Suppose $\mathtt{Dom}(f_{XY}) \cap \mathtt{Dom}(f_{UY} f_{UU}^j f_{XU}) \neq \emptyset$, for some j. If $x \in \mathtt{Dom}(f_{XY}) \cap \mathtt{Dom}(f_{UY} f_{UU}^j f_{XU})$, then $f_{XY}(x) = y$ for some $y \in Y$, and hence $f((x,1)) = (y,1)$. On the other hand, $f_{UY} f_{UU}^j f_{XU}(x) = y'$, for some $y' \in Y$; hence $f_{XU}(x) = u$, for some $u \in U$, so $f((x,1)) = (u,2)$, which gives a contradiction.

(ii) Suppose $\mathtt{Dom}(f_{UY} f_{UU}^j f_{XU}) \cap \mathtt{Dom}(f_{UY} f_{UU}^k f_{XU}) \neq \emptyset$, say $x \in \mathtt{Dom}(f_{UY} f_{UU}^j f_{XU}) \cap \mathtt{Dom}(f_{UY} f_{UU}^k f_{XU})$ for $k < j$. Let $f_{XU}(x) = u$ and $f_{UU}^k(u) = u'$; then $u' \in \mathtt{Dom}(f_{UY}) \cap \mathtt{Dom}(f_{UU}^{j-k})$. Hence $f_{UY}(u') = y$, for some $y \in Y$. Since $j-k > 0$, $f_{UU}(u') = u''$, for some $u'' \in U$, we have $f((u',2)) = (y,1)$ and $f((u',2)) = (u'',2)$, which gives a contradiction.

(iii) To show the above family of functions have disjoint codomains, one shows the inverse maps have disjoint domains. The proof is similar to the above cases and is left to the reader.

Finally, the fact that the above trace formula satisfies the axioms for a trace follows from Haghverdi (2000b, Proposition 2.12).

We will now show that $(\mathbf{PInj}, \mathbb{N} \times -, \mathbb{N})$ is a GoI Situation. However, we first state a lemma that will be used to prove that the functor T is a traced functor, as required by a GoI Situation.

Lemma 5.1. Let $T : \mathbf{PInj} \longrightarrow \mathbf{PInj}$ be an additive functor, that is, a functor preserving the additive structure on the homsets. Then, T is traced.

Proof. Let $f : X \otimes U \longrightarrow Y \otimes U$ be a morphism in **PInj**. Then $\mathtt{Tr}_{X,Y}^U(f) = f_{XY} + \sum_{n\in\omega} f_{UY} f_{UU}^n f_{XU}$. Let $T = (T, \psi, \psi_I)$ and $g = \psi_{Y,U}^{-1} T(f) \psi_{X,U}$. Then,

$$\begin{aligned} g_{XY} &= \rho_1^{TY,TU} \mathtt{g inl}^{TX,TU} \\ &= T(\rho_1^{Y,U}) \psi_{Y,U} g \psi_{X,U}^{-1} T(\mathtt{inl}^{X,U}) \\ &= T(\rho_1^{Y,U}) \psi_{Y,U} \psi_{Y,U}^{-1} T(f) \psi_{X,U} \psi_{X,U}^{-1} T(\mathtt{inl}^{X,U}) \\ &= T(\rho_1^{Y,U} f \mathtt{inl}^{X,U}) \\ &= T(f_{XY}). \end{aligned}$$

Similarly, we have $g_{UY} = T(f_{UY}), g_{XU} = T(f_{XU})$, and $g_{UU} = T(f_{UU})$.

$$\begin{aligned} T(\mathtt{Tr}_{X,Y}^U(f)) &= T(f_{XY} + \textstyle\sum_{n\in\omega} f_{UY} f_{UU}^n f_{XU}) \\ &= T(f_{XY}) + \textstyle\sum_{n\in\omega} T(f_{UY}) T(f_{UU})^n T(f_{XU}), \quad T \text{ is additive} \\ &= g_{XY} + \textstyle\sum_{n\in\omega} g_{UY} g_{UU}^n g_{XU} \\ &= \mathtt{Tr}_{TX,TY}^{TU}(\psi_{Y,U}^{-1} T(f) \psi_{X,U}). \end{aligned}$$

Therefore, T is traced. □

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Proposition 5.2. (**PInj**, $\mathbb{N} \times -, \mathbb{N}$) is a GoI Situation.

Proof. We have previously observed that **PInj** is traced symmetric monoidal category, with the tensor product taken to be the disjoint union of sets with $\varnothing$ as the unit. Clearly, $T = \mathbb{N} \times -$, with $T = (T, \psi, \psi_I)$, is a symmetric monoidal functor where $\psi_{X,Y} : \mathbb{N} \times X \uplus \mathbb{N} \times Y \longrightarrow \mathbb{N} \times (X \uplus Y)$ is given by $(1,(n,x)) \mapsto (n,(1,x))$ and $(2,(n,y)) \mapsto (n,(2,y))$, and it has an inverse defined by $(n,(1,x)) \mapsto (1,(n,x))$ and $(n,(2,y)) \mapsto (2,(n,y))$. Also, $\psi_I : \varnothing \longrightarrow \mathbb{N} \times \varnothing$ given by $\langle \varnothing, 1_\varnothing \rangle$ is clearly an isomorphism.

We show that T is additive. Let $\{f_i\}_{i \in I}$ be a summable family in $\mathbf{PInj}(X, Y)$. Then

$$(1_{\mathbb{N}} \times \textstyle\sum_I f_i)(n,x) = \begin{cases} (n, f_j(x)), & \text{if there exists a } j \in I \text{ such that } x \in \mathtt{Dom}(f_j); \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

But this is exactly the definition of $(\sum_I (1_{\mathbb{N}} \times f_i))(n,x)$ for all $(n,x) \in \mathbb{N} \times X$. Therefore, $T = \mathbb{N} \times -$ is an additive functor and thus, by Lemma 5.1, it is also traced. In other words, given $f : X \uplus U \longrightarrow Y \uplus U$, we have $1_{\mathbb{N}} \times \mathtt{Tr}^U_{X,Y}(f) = \mathtt{Tr}^{\mathbb{N}\times U}_{\mathbb{N}\times X, \mathbb{N}\times Y}(\psi^{-1}(1_{\mathbb{N}} \times f)\psi)$.

We next define the necessary monoidal natural transformations.

— (Comultiplication) $\mathbb{N} \times (\mathbb{N} \times X) \xrightarrow{e_X} \mathbb{N} \times X$ and $\mathbb{N} \times X \xrightarrow{e'_X} \mathbb{N} \times (\mathbb{N} \times X)$.
$\mathbb{N} \times (\mathbb{N} \times X) \xrightarrow{e_X} \mathbb{N} \times X$ is defined by, $e_X(n_1,(n_2,x)) = (\langle n_1, n_2 \rangle, x)$. Given $f : X \longrightarrow Y$, $(1_{\mathbb{N}} \times f)e_X((n_1,(n_2,x)) = (\langle n_1, n_2 \rangle, f(x)) = e_Y(1_{\mathbb{N}} \times (1_{\mathbb{N}} \times f)(n_1,(n_2,x))$ for all $n_1, n_2 \in \mathbb{N}$ and $x \in X$, proving the naturality of e_X. $e'_X(n,x) = (n_1,(n_2,x))$ where $\langle n_1, n_2 \rangle = n$. $e'_X e_X(n_1,(n_2,x)) = e'_X(\langle n_1, n_2 \rangle, x) = (n_1,(n_2,x))$ for all $n_1, n_2 \in \mathbb{N}$ and $x \in X$.

— (Dereliction) $X \xrightarrow{d_X} \mathbb{N} \times X$ and $\mathbb{N} \times X \xrightarrow{d'_X} X$.
$d_X(x) = (n_0, x)$ for a fixed $n_0 \in \mathbb{N}$. Given $f : X \longrightarrow Y$, $(1_{\mathbb{N}} \times f d_X(x) = (n_0, f(x)) = d_Y f(x)$ for any $x \in X$, proving the naturality of d_X.

$$d'_X(n,x) = \begin{cases} x, & \text{if } n = n_0; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

$d'_X d_X(x) = d'_X(n_0, x) = x$ for all $x \in X$.

— (Contraction) $(\mathbb{N} \times X) \uplus (\mathbb{N} \times X) \xrightarrow{c_X} \mathbb{N} \times X$ and $\mathbb{N} \times X \xrightarrow{c'_X} (\mathbb{N} \times X) \uplus (\mathbb{N} \times X)$.

$$c_X = \begin{cases} (1,(n,x)) \mapsto (2n, x) \\ (2,(n,x)) \mapsto (2n+1, x). \end{cases}$$

Given $f : X \longrightarrow Y$, $(1_{\mathbb{N}} \times f)c_X(1,(n,x)) = (2n, f(x)) = c_Y(1_{\mathbb{N}} \times f \uplus 1_{\mathbb{N}} \times f)(1,(n,x))$ for all $n \in \mathbb{N}$ and $x \in X$. Similarly $(1_{\mathbb{N}} \times f)c_X(2,(n,x)) = (2n+1, f(x)) = c_Y(1_{\mathbb{N}} \times f \uplus 1_{\mathbb{N}} \times f)(2,(n,x))$ for all $n \in \mathbb{N}$ and $x \in X$, proving the naturality of c_X.

$$c'_X(n,x) = \begin{cases} (1,(n/2,x)), & \text{if } n \text{ is even;} \\ (2,((n-1)/2,x)), & \text{if } n \text{ is odd.} \end{cases}$$

Finally, $c'_X c_X(1,(n,x)) = c'_X(2n,x) = (1,(n,x))$ and $c'_X c_X(2,(n,x)) = c'_X(2n+1,x) = (2,(n,x))$.

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— (Weakening) $\varnothing \xrightarrow{w_X} \mathbb{N} \times X$ and $\mathbb{N} \times X \xrightarrow{w'_X} \varnothing$.
Let w_X and w'_X both be the empty partial function. Clearly, for any $f : X \longrightarrow Y$, $(1_{\mathbb{N}} \times f)w_X = w'_X 1_{\varnothing}$, proving the naturality of w_X. $w'_X w_X = 1_{\varnothing}$.

Finally, we show that $\mathbb{N}$ is a reflexive object.

— $j : \mathbb{N} \uplus \mathbb{N} \lhd \mathbb{N} : k$ is given by $j : \mathbb{N} \uplus \mathbb{N} \longrightarrow \mathbb{N}, j(1,n) = 2n, j(2,n) = 2n+1$ and $k : \mathbb{N} \longrightarrow \mathbb{N} \uplus \mathbb{N}$,

$$k(n) = \begin{cases} (1, n/2), & \text{if } n \text{ even;} \\ (2, (n-1)/2), & \text{if } n \text{ odd .} \end{cases}$$

Clearly, $kj = 1_{\mathbb{N} \uplus \mathbb{N}}$.

— $\varnothing \lhd \mathbb{N}$ using the empty partial function as the retract morphisms.

— $u : \mathbb{N} \times \mathbb{N} \lhd \mathbb{N} : v$ is defined as $u(m,n) = \langle m, n \rangle = \frac{(m+n+1)(m+n)}{2} + n$ (Cantor surjective pairing) and v as its inverse, $v(n) = (n_1, n_2)$ with $\langle n_1, n_2 \rangle = n$. Clearly, $vu = 1_{\mathbb{N} \times \mathbb{N}}$.

□

We can generalise the above example to the case of partial functions in the next section.

5.2. **Pfn**

Consider the category **Pfn** of sets and partial functions. **Pfn** is a traced symmetric monoidal category with disjoint union (= coproduct) as the tensor product, $I = \varnothing$. Given $f : X \oplus U \longrightarrow Y \oplus U$, (we use $\oplus$ for coproduct),

$$\mathtt{Tr}^U_{X,Y}(f) = f_{XY} + \sum_{n \in \omega} f_{UY} f^n_{UU} f_{XU}.$$

Here, a family $\{h_i\}_{i \in I} : X \longrightarrow Y$ is said to be summable if the h_i's have pairwise disjoint domains. In that case

$$(\sum_{i \in I} h_i)(x) = \begin{cases} h_j(x), & \text{if } x \in \mathtt{Dom}(h_j) \text{ for some } j \in I\,; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

The component maps are as in the previous example, with $\mathtt{inl}, \mathtt{inr}, \rho_i$ as before. Clearly, $\mathbb{N} \times -$ is an additive and hence a traced functor. Note that the morphisms used in the previous example are partial injective functions, hence partial functions. In view of this we have the following proposition.

Proposition 5.3. (**Pfn**, $\mathbb{N} \times -, \mathbb{N}$) is a GoI Situation.

5.3. $\mathbf{Rel}_+$

The category $\mathbf{Rel}_+$ of sets and binary relations is a traced symmetric monoidal category. Tensor product is again taken to be disjoint union (which will in fact be a biproduct). To keep our discussion in the framework of the previous two examples, we shall declare *any* family of relations $\{R_i\}_{i \in I} \in \mathbf{Rel}_+(X, Y)$ to be summable, with $\sum_{i \in I} R_i = \cup_{i \in I} R_i$.

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As in the previous examples, we obtain a trace as follows (*cf.* Haghverdi (2000b) and Abramsky (1996)): given $R : X \oplus U \longrightarrow Y \oplus U$, define

$$\begin{aligned} \mathtt{Tr}^U_{X,Y}(R) &= R_{XY} + \sum_{n\in\omega} R_{UY} R^n_{UU} R_{XU} \\ &= R_{XY} \cup R_{UY} R^*_{UU} R_{XU}. \end{aligned}$$

Here, the components of R in the trace are as before, where now ρ_i and $\mathtt{inl}, \mathtt{inr}$ become the projections and injections of the biproduct, and R^* denotes the reflexive, transitive closure of R.

As we have seen above, $\mathbb{N}\times -$ is an additive and hence a traced functor (by the analogue of Lemma 5.1). The morphisms used in the case of **PInj** are partial injective functions and hence, in particular, they are relations. Thus we have the following proposition.

Proposition 5.4. $(\mathbf{Rel}_+, \mathbb{N} \times -, \mathbb{N})$ is a GoI Situation.

5.4. *Resumptions*

5.4.1. *A brief history of resumptions.* Our starting point is the classical automata-theoretic notion of *transducers*, that is, structures (Q, X, Y, q_0, δ) where Q is a set of states, $q_0 \in Q$ the initial state, X the set of inputs, Y the set of outputs, and $\delta : Q \times X \rightharpoonup Y \times Q$ is the transition function (here a partial function). If we supply a sequence of inputs $x_0, \ldots, x_k$ to such a transducer, we obtain the orbit

$$q_0 \xrightarrow{x_0} y_0, q_1 \xrightarrow{x_1} y_1, q_2 \xrightarrow{x_2} \cdots \xrightarrow{x_k} y_k, q_{k+1}$$

where $\delta(q_i, x_i) = y_i, q_{i+1}$, $0 \leqslant i \leqslant k$. This generalises to non-deterministic transducers with transition function $\delta : Q \times X \longrightarrow \mathscr{P}(Y \times Q)$ in an evident fashion (where for any set X, $\mathscr{P}(X) = \{U \mid U \subseteq X\}$ denotes the power set of X).

A key idea introduced in Milner (1975) is to give a denotational semantics for concurrent programs as *processes*, which were taken to be extensional versions of transducers. There are two ingredients to this idea:

(i) Instead of modelling programs by functions or relations, we model them by entities with more complex behaviours, taking account of the possible interactions between a program and its environment during the course of a computation.
(ii) Instead of modelling concurrent programs by automata, with all the intensionality this entails, we look for a more extensional description of the *behaviours* of transducers.

To obtain this extensional view of transducers, consider the recursive definition

$$R = X \rightharpoonup (Y \times R).$$

This defines a mathematical space of 'resumptions' in which the states of transducers are 'unfolded' into their observable behaviours. Milner solved equations such as this over a category of domains in Milner (1975), but in fact it can be solved in a canonical fashion over **Set** – in modern terminology, the functor $T_{X,Y} : \mathbf{Set} \longrightarrow \mathbf{Set}$ defined by $T_{X,Y}(S) = X \rightharpoonup Y \times S$ has a final coalgebra $R \xrightarrow{\cong} T_{X,Y}(R)$. Indeed, Milner defined a notion $\sim$ of behavioural equivalence between transducers, and for any transducer

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(Q, X, Y, q_0, δ) a map $h_\delta : Q \longrightarrow R$, which is, in fact, the final coalgebra homomorphism from the coalgebra

$$\hat{\delta} : Q \longrightarrow T_{X,Y}(Q)$$

to R (where $\hat{\delta}$ is the exponential transpose of δ), and proved that

$$(Q, X, Y, q_0, \delta) \sim (Q', X, Y, q_0', \delta') \iff h_\delta(q_0) = h_{\delta'}(q_0').$$

From a modern perspective, we can also make light of a technical problem that figured prominently in Milner (1975), namely how to model non-determinism. Historically this inspired Plotkin's work on powerdomains (Plotkin 1976), but, for the specific application at hand, the equation

$$R = X \longrightarrow \mathscr{P}(Y \times R)$$

has a final coalgebra in the category of classes in Peter Aczel's non-well-founded set theory (Aczel 1988), and if we are content to bound the cardinality of subsets by an inaccessible cardinable κ, then the equation

$$R = X \longrightarrow \mathscr{P}^{<\kappa}(Y \times R)$$

has a final coalgebra in **Set** (Barr 1993). Moreover, the equivalence induced by this model coincides with strong bisimulation (Aczel 1988).

5.4.2. *The category* **Res**. The category **Res** of resumptions (we will for simplicity confine ourselves to deterministic resumptions) has as objects sets, and as morphisms

$$\mathbf{Res}(X, Y) = X \rightharpoonup (Y \times \mathbf{Res}(X, Y)),$$

that is, the space of resumptions parameterised by the sets of 'inputs' X and 'outputs' Y. The composition of resumptions $f \in \mathbf{Res}(X, Y)$ and $g \in \mathbf{Res}(Y, Z)$ is defined (coinductively (Aczel 1988)) by

$$f ; g(x) = \begin{cases} (z, f' ; g') & f(x) = (y, f'),\ g(y) = (z, g') \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The identity resumption $\mathtt{id}_X \in \mathbf{Res}(X, X)$ is defined by

$$\mathtt{id}_X(x) = (x, \mathtt{id}_X).$$

We can picture this composition as sequential (or 'series') composition of transducers.

We can define a monoidal structure on **Res** by

$$X \otimes Y = X + Y \quad \text{(disjoint union of sets)}$$

and if $f \in \mathbf{Res}(X, Y)$, $g \in \mathbf{Res}(X', Y')$, $f \otimes g \in \mathbf{Res}(X \otimes X', Y \otimes Y')$ is defined by

$$f \otimes g(\mathtt{inl}(x)) = \begin{cases} (\mathtt{inl}(y), f' \otimes g), & f(x) = (y, f') \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$f \otimes g(\mathtt{inr}(x')) = \begin{cases} (\mathtt{inr}(y'), f \otimes g'), & g(x') = (y', g') \\ \text{undefined} & \text{otherwise.} \end{cases}$$

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This is (asynchronous) parallel composition of transducers: at each stage, we respond to an input on the X 'wire' according to f, with output appearing on the Y wire, and to an input on the X' wire according to g, with output appearing on the Y' wire.

Let us prove that composition in **Res** is associative. Given $f \in \mathbf{Res}(X, Y), g \in \mathbf{Res}(Y, Z), h \in \mathbf{Res}(Z, W)$, we wish to prove that $(f;g);h = f;(g;h)$. Define

$$R_3 = \mathbf{Res}(X, Y) \times \mathbf{Res}(Y, Z) \times \mathbf{Res}(Z, W)$$

and

$$T_{X,Y}(A) = X \rightharpoonup Y \times A.$$

We define a $T_{X,W}$-coalgebra structure on R_3, $\alpha : R_3 \to T_{X,W}(R_3)$ by

$$\alpha(f,g,h)(x) = \begin{cases} (w,(f',g',h')) & \text{if } f(x) = (y,f'),\ g(y) = (z,g'), \text{and } h(z) = (w,h') \\ \text{undefined} & \text{otherwise.} \end{cases}$$

We define maps $\beta_1, \beta_2 : R_3 \to \mathbf{Res}(X, W)$ by $\beta_1(f,g,h) = (f;g);h$ and $\beta_2(f,g,h) = f;(g;h)$. It suffices to show that β_1 andβ_2 are $T_{X,W}$-coalgebra homomorphisms: since then, by the final coalgebra property of $\mathbf{Res}(X, W)$, they are equal.

This amounts to showing that

$$\beta_1(f,g,h)(x) = \begin{cases} (w,\beta_1(f',g',h')) & \text{if } \alpha(f,g,h) = (w,(f',g',h')) \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$\beta_2(f,g,h)(x) = \begin{cases} (w,\beta_2(f',g',h')) & \text{if } \alpha(f,g,h) = (w,(f',g',h')) \\ \text{undefined} & \text{otherwise,} \end{cases}$$

which is immediate from the definition.

The remaining definitions to make this into a symmetric monoidal structure on **Res** are straightforward, and left to the reader. Note that the associativity and symmetry isomorphisms, like the identities, have just one state; they are 'history-free'.

Finally, there is a feedback operator: for each X, Y, U a function

$$\mathtt{Tr}^U_{X,Y} : \mathbf{Res}(X \otimes U, Y \otimes U) \longrightarrow \mathbf{Res}(X, Y)$$

defined by

$$\mathtt{Tr}^U_{X,Y}(f)(x) = \begin{cases} (y,f'), & f(x) = (y,f') \vee \\ & \exists k.\, f(x) = (u_0,f_0), \\ & f_0(u_0) = (u_1,f_1), \\ & \vdots \\ & f_k(u_k) = (y,f') \\ \text{undefined} & \text{otherwise.} \end{cases}$$

One has the same 'particle-style' imagery as discussed earlier for **PInj**: imagine a token entering at the X wire, circulating k times around the feedback loop at the U wire, and exiting at Y.

It can be verified that this satisfies the axioms for a trace: let us look at the cases of

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Yanking and Vanishing II. Note that **Pfn** can be embedded in **Res** by an identity-on-objects embedding $r : \mathbf{Pfn} \to \mathbf{Res}$: if $f : X \rightharpoonup Y$, then $r(f) \in \mathbf{Res}(X, Y)$ is defined by

$$r(f)(x) = \begin{cases} (f(x), r(f)), & f(x) \text{ defined} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Thus $r(f)$ is the 'history-free extension in time' of f. Moreover, observe that r is a traced monoidal functor. It immediately follows that Yanking is valid in **Res**, since the symmetry morphism $\sigma_{U,U}$ in **Res** is the image under σ of the symmetry in **Pfn**.

To prove Vanishing II, if $g : X \otimes U \otimes V \to Y \otimes U \otimes V$, let $T_1(g) = \mathtt{Tr}^{U \otimes V}_{X,Y}(g)$ and let $T_2(g) = \mathtt{Tr}^{U}_{X,Y}(\mathtt{Tr}^{V}_{X \otimes U, Y \otimes U}(g))$. Unpacking the definitions, we get:

$$T_1(g)(x) = \begin{cases} (y, T_1(g')), & \text{if } g(x) = (y, g') \vee \\ & \exists k \geqslant 0\, g(x) = (w_0, g_0), \\ & g_0(w_0) = (w_1, g_1), \\ & \vdots \\ & g_k(w_k) = (y, g'), w_i \in U \uplus V, 0 \leqslant i \leqslant k \\ \text{undefined} & \text{otherwise.} \end{cases}$$

$$T_2(g)(x) = \begin{cases} (y, T_2(g')), & \text{if } g(x) = (y, g') \vee \\ & \exists k \geqslant 0\, g(x) = (w_0, g_0), \\ & g_0(w_0) = (w_1, g_1), \\ & \vdots \\ & g_k(w_k) = (y, g'), w_i \in U \uplus V, 0 \leqslant i \leqslant k \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The fact that $T_1(g) = T_2(g)$ then follows by standard co-inductive reasoning.

The functor $\mathbb{N} \times -$ can also be extended from **Pfn** to **Res**:

$$\mathbb{N} \times f(n, x) = \begin{cases} ((n, y), \mathbb{N} \times f'), & f(x) = (y, f') \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Using these extensions, the GoI Situation structure given for **PInj** and **Pfn** can be transferred to **Res**.

5.4.3. *From resumptions to strategies.* To interpret the category $\mathscr{G}(\mathbf{Res})$, think of an object (X^+, X^-) as a rudimentary two-person game, in which X^+ is the set of moves for Player, and X^- the set of moves for Opponent. A resumption $f : X^- \longrightarrow X^+$ is then a *strategy* for Player. Note that we can represent such a strategy by its set of *plays*:

$$P(f) = \{x_1 y_1 \cdots x_k y_k \mid f(x_1) = (y_1, f_1), \ldots, f_{k-1}(x_k) = (y_k, f_k)\}.$$

One can then show that composition in $\mathscr{G}(\mathbf{Res})$ is given by 'parallel composition plus hiding' (Abramsky 1994; Abramsky and Jagadeesan 1994b):

$$P(f;g) = \{s \upharpoonright X, Z \mid s \in P(f) \| P(g)\}$$

$$S \| T = \{s \in \mathscr{L}(X, Y, Z) \mid s \upharpoonright X, Y \in S \wedge s \upharpoonright Y, Z \in T\}$$

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where $X = X^+ + X^-$, $Y = Y^+ + Y^-$, $Z = Z^+ + Z^-$, and

$$\mathscr{L}(S_1, S_2, S_3) = \{s \in (S_1 + S_2 + S_3)^* \mid s_i \in S_j \wedge s_{i+1} \in S_k \implies |j - k| \leqslant 1\}.$$

The identities are the 'copycat' strategies as in Abramsky and Jagadeesan (1994a). We can then obtain the simple category of games by applying a specification structure in the sense of Abramsky *et al.* (1996) to $\mathscr{G}$(**Res**), in which the properties over (X^+, X^-) are the prefix-closed subsets of $(X^-X^+)^*$, that is, the 'safety properties', which in this context are the game trees.

5.5. **SRel**

We now consider a probabilistic version of **Pfn** or **Rel** as a GoI Situation. Consider the category **SRel** of *stochastic relations*, with measurable spaces $(X, \mathscr{F}_X)$ as objects and stochastic kernels as arrows. An arrow $f : (X, \mathscr{F}_X) \longrightarrow (Y, \mathscr{F}_Y)$ is a map $f : X \times \mathscr{F}_Y \longrightarrow [0,1]$ such that $f(\cdot, B) : X \longrightarrow [0,1]$ is a bounded measurable function for fixed $B \in \mathscr{F}_Y$ and $f(x, \cdot) : \mathscr{F}_Y \longrightarrow [0,1]$ is a subprobability measure (that is, σ-additive, set function, $f(x, \varnothing) = 0$ and $f(x, Y) \leqslant 1$). The identity morphism $1_X : (X, \mathscr{F}_X) \longrightarrow (X, \mathscr{F}_X)$ is $1_X : X \times \mathscr{F}_X \longrightarrow [0,1]$, and is defined by

$$1_X(x, A) = \delta(x, A) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$$

For A fixed, $\delta(x, A)$ is the characteristic function of A, and for x fixed, $\delta(x, A)$ is the Dirac distribution. Finally, composition is defined as follows: given $f : (X, \mathscr{F}_X) \longrightarrow (Y, \mathscr{F}_Y)$ and $g : (Y, \mathscr{F}_Y) \longrightarrow (Z, \mathscr{F}_Z)$, we have $gf : (X, \mathscr{F}_X) \longrightarrow (Z, \mathscr{F}_Z)$ is given by

$$gf(x, C) = \int_Y g(y, C)\, f(x, dy),$$

where we are using $f(x, \cdot)$ as the measure for the integration and the function being integrated is the measurable function $g(\cdot, C)$.

This category is based on the work of Giry (Giry 1981) and Lawvere (Lawvere 1963). In fact **SRel** is the Kleisli category of a certain functor Π over the category **Mes** of measurable spaces and measurable functions. The category **SRel** was defined by Panangaden (Panangaden 1997) with slight modifications on Giry's work. This category first appeared in Abramsky (1996). More details on **SRel** and the measure theory background can be found in Panangaden (1997).

We will sometimes refer to objects in **SRel** by their underlying sets, so we write X for $(X, \mathscr{F}_X)$.

Proposition 5.5. The category **SRel** has countable coproducts.

Proof. Given a family $\{(X_i, \mathscr{F}_{X_i})\}_{i \in I}$ of objects in **SRel**, the coproduct $(X, \mathscr{F}_X)$ is defined as follows. The set X is the disjoint union of the X_i. The σ-field on X is generated by the measurable sets of each component. Thus, a measurable set in $\mathscr{F}_X$ is of the form $\biguplus_{i \in I} A_i$, where $A_i \in \mathscr{F}_{X_i}$ for all $i \in I$. The injections $in_i : X_i \longrightarrow X$ are $in_i(x, \biguplus_{k \in I} A_k) = \delta(x, A_i)$. Given an object $(Y, \mathscr{F}_Y)$ and a family $f_j : X_j \longrightarrow Y$ with $j \in I$,

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the mediating morphism $f : X \longrightarrow Y$ is defined by $f((x,i),B) = f_i(x,B)$. For further details see Abramsky *et al.* (2001) and Haghverdi (2000a). □

SRel is a symmetric monoidal category with coproduct as the tensor product and $I = (\emptyset, \mathscr{F}_\emptyset)$. As in **PInj**, **Pfn**, and $\mathbf{Rel}_+$, we can define a trace by a general summation formula: given $f : (X \uplus U, \mathscr{F}_{X\uplus U}) \longrightarrow (Y \uplus U, \mathscr{F}_{Y\uplus U})$,

$$\mathtt{Tr}^U_{X,Y}(f) = f_{XY} + \sum_{n\in\omega} f_{UY} f^n_{UU} f_{XU}.$$

Here we define $\{f_i\}_{i\in I}$ in $\mathbf{SRel}(X,Y)$ to be summable if

$$\textstyle\sum_i f_i(x,Y) \leqslant 1 \text{ for all } x \in X.$$

In that case, $(\sum_i f_i)(x,B) = \sum_i f_i(x,B)$, where the latter is the usual sum of real numbers. The component maps in the trace formula are as before: $\mathtt{inl} = in_1, \mathtt{inr} = in_2, \rho_1 : Y \uplus U \longrightarrow Y$ defined by $\rho_1((1,y),B) = \delta(y,B)$ and $\rho_1((2,u),B) = 0$; and dually, $\rho_2 : Y \uplus U \longrightarrow U$, given by $\rho_2((2,u),C) = \delta(u,C)$ and $\rho_2((1,y),C) = 0$.

We define $T : \mathbf{SRel} \longrightarrow \mathbf{SRel}$ as $T(X, \mathscr{F}_X) = (\mathbb{N}\times X, \mathscr{F}_{\mathbb{N}\times X})$ where $\mathscr{F}_{\mathbb{N}\times X}$ is the σ-field on $X \uplus X \uplus X \cdots$ (ω copies). For a given $f : (X, \mathscr{F}_X) \longrightarrow (Y, \mathscr{F}_Y)$, $Tf((n,x), \biguplus_{i\in\omega} B_i) = f(x, B_n)$. Let $U = \mathbb{N}^\infty$ with the Baire metric, which induces a topology on $\mathbb{N}^\infty$ corresponding to the product topology obtained from $\mathbb{N}$ with discrete topology. The following proposition is due to P. Panangaden and the second author.

Proposition 5.6. $(\mathbf{SRel}, T, \mathbb{N}^\infty)$ is a GoI Situation.

Proof. We show that T as defined above is an additive and hence a traced symmetric monoidal functor (by the analogue of Lemma 5.1). First, observe that $T = (T, \psi, \psi_I)$ is a monoidal functor with

$$\psi_{X,Y} : ((\mathbb{N}\times X) \uplus (\mathbb{N}\times Y), \mathscr{F}_{(\mathbb{N}\times X)\uplus(\mathbb{N}\times Y)}) \longrightarrow (\mathbb{N}\times(X\uplus Y), \mathscr{F}_{\mathbb{N}\times(X\uplus Y)})$$

given by

$$\psi_{X,Y}((1,(n,x)), \biguplus_{i\in\omega}(A_i \uplus B_i)) = \delta(x, A_n)$$

$$\psi_{X,Y}((2,(n,y)), \biguplus_{i\in\omega}(A_i \uplus B_i) = \delta(y, B_n).$$

Also,

$$\psi^{-1}_{X,Y}((n,(1,x)), (\biguplus_{i\in\omega} A_i) \uplus (\biguplus_{i\in\omega} B_i)) = \delta(x, A_n)$$

$$\psi^{-1}_{X,Y}((n,(2,y)), (\biguplus_{i\in\omega} A_i) \uplus (\biguplus_{i\in\omega} B_i)) = \delta(y, B_n).$$

Finally, $\psi_I = 1_I$.

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Given a summable family $\{f_i\}_{i\in I} \in \mathbf{SRel}(X, Y)$, we have $\sum_I f_i(x, Y) \leqslant 1$ since $\{f_i\}$ is summable, and hence $\{Tf_i\}$ is summable.

$$\begin{aligned}(\textstyle\sum_I Tf_i)((n,x), \biguplus_{i\in\omega} B_i) &= \textstyle\sum_I Tf_i((n,x), \biguplus_{i\in\omega} B_i)\\ &= \textstyle\sum_I f_i(x, B_n)\\ &= (\textstyle\sum_I f_i)(x, B_n)\\ &= T(\textstyle\sum_I f_i)((n,x), \biguplus_{i\in\omega} B_i).\end{aligned}$$

Hence, T is an additive and, by the analogue of Lemma 5.1, a traced functor.

Next we consider the monoidal natural transformations.

— (Comultiplication): $e_X : \mathbb{N}\times(\mathbb{N}\times X) \lhd \mathbb{N}\times X : e'_X$
$e_X((n_1,(n_2,x)), \biguplus_{i\in\omega} A_i) = \delta(x, A_{\langle n_1,n_2\rangle})$ and $e'_X((n,x), \biguplus_i(\biguplus_j A_{ij})) = \delta(x, A_{n_1n_2})$ where $n = \langle n_1, n_2\rangle$.

— (Dereliction) $d_X : X \lhd \mathbb{N}\times X : d'_X$
$d_X(x, \biguplus_i A_i) = \delta(x, A_{n_0})$ for a fixed $n_0 \in \mathbb{N}$ and $d'_X((n,x), A) = \delta(x, A)$, if $n = n_0$ and 0, otherwise.

— (Contraction) $c_X : \mathbb{N}\times X \uplus \mathbb{N}\times X \lhd \mathbb{N}\times X : c'_X$
$c_X((1,(n,x)), \biguplus_{i\in\omega} A_i) = \delta(x, A_{2n})$,
$c_X((2,(n,x)), \biguplus_i A_i) = \delta(x, A_{2n+1})$ and

$$c'_X((n,x), (\biguplus_i A_i) \uplus (\biguplus_i B_i)) = \begin{cases}\delta(x, A_{n/2}), & \text{if } n \text{ is even;}\\ \delta(x, B_{(n-1)/2}), & \text{if } n \text{ is odd.}\end{cases}$$

— (Weakening) $w_X : \varnothing \lhd \mathbb{N}\times X : w'_X$. We set w_X to be the empty function and $w'_X((n,x), \varnothing) = 0$. Clearly, $w'_X w_X = 1_{(\varnothing, \mathscr{F}_\varnothing)}$.

We will prove naturality and the retract property for comultiplication. The proofs of naturality and the retract property for d_X, c_X and w_X are similar and can be found in Abramsky *et al.* (2001). Let $f : X \longrightarrow Y$. Then:

$$\begin{aligned}(Tf)e_X((n_1,(n_2,x)), \biguplus_{i\in\omega} B_i) &= \int_{\mathbb{N}\times X} Tf((n',x'), \biguplus_i B_i) e_X((n_1,(n_2,x)), d(n',x'))\\ &= \int_{\mathbb{N}\times X} Tf((n',x'), \biguplus_i B_i)\delta(x, d(\langle n_1,n_2\rangle, x'))\\ &= Tf((\langle n_1,n_2\rangle, x), \biguplus_i B_i)\\ &= f(x, B_{\langle n_1,n_2\rangle}).\end{aligned}$$

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$$\begin{aligned} e_Y(T(Tf))((n_1,(n_2,x)),\biguplus_i B_i) &= \int_{\mathbb{N}\times(\mathbb{N}\times Y)} e_Y((n_1',(n_2',y)),\biguplus_i B_i)T(Tf) \\ &\qquad ((n_1,(n_2,x)),d(n_1',(n_2',y))) \\ &= \int_{\mathbb{N}\times(\mathbb{N}\times Y)} e_Y((n_1',(n_2',y)),\biguplus_i B_i)f(x,d(n_1,(n_2,y))) \\ &= \int_{\mathbb{N}\times(\mathbb{N}\times Y)} \delta(y,B_{\langle n_1',n_2'\rangle})f(x,d(n_1,(n_2,y))) \\ &= f(x,B_{\langle n_1,n_2\rangle}). \end{aligned}$$

Therefore, $(Tf)e_X = e_Y(T(Tf))$ for all $f : X \longrightarrow Y$.

$$\begin{aligned} e_X'e_X((n_1,(n_2,x)),\biguplus_{ij}A_{ij}) &= \int_{\mathbb{N}\times X} e_X'((n,x'),\biguplus_{ij}A_{ij})e_X((n_1,(n_2,x)),d(n,x')) \\ &= \int_{\mathbb{N}\times X} e_X'((n,x'),\biguplus_{ij}A_{ij})\delta(x,d(\langle n_1,n_2\rangle,x')) \\ &= e_X'((\langle n_1,n_2\rangle,x),\biguplus_{ij}A_{ij}) \\ &= \delta(x,A_{n_1n_2}) \\ &= 1_{\mathbb{N}\times(\mathbb{N}\times X)}((n_1,(n_2,x)),\biguplus_{ij}A_{ij}). \end{aligned}$$

Finally, we show that $\mathbb{N}^\infty$ is a reflexive object. We will denote the objects in **SRel** by their first components in order not to overload the notation.

— $j : \mathbb{N}^\infty \uplus \mathbb{N}^\infty \lhd \mathbb{N}^\infty : k$
Consider the maps $j : \mathbb{N}^\infty \uplus \mathbb{N}^\infty \longrightarrow \mathbb{N}^\infty$ defined by $j((1,\vec{n}),A) = \delta(0.\vec{n},A)$, $j((2,\vec{n}),A) = \delta(1.\vec{n},A)$ and $k : \mathbb{N}^\infty \longrightarrow \mathbb{N}^\infty \uplus \mathbb{N}^\infty$ defined by

$$k(\vec{n},A_1 \uplus A_2) = \begin{cases} \delta(\vec{n}',A_1), & \text{if } \vec{n} = 0.\vec{n}'; \\ \delta(\vec{n}',A_2), & \text{if } \vec{n} = 1.\vec{n}'; \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} kj((1,\vec{n}),A_1 \uplus A_2) &= \int_{\mathbb{N}^\infty} k(\vec{n}',A_1 \uplus A_2)j((1,\vec{n}),d\vec{n}') \\ &= \int_{\mathbb{N}^\infty} k(\vec{n}',A_1 \uplus A_2)\delta(0.\vec{n},d\vec{n}') \\ &= k(0.\vec{n},A_1 \uplus A_2) \\ &= \delta(\vec{n},A_1) \\ &= 1_{\mathbb{N}^\infty\uplus\mathbb{N}^\infty}((1,\vec{n}),A_1 \uplus A_2). \end{aligned}$$

Similarly, $kj((2,\vec{n}),A_1 \uplus A_2) = \delta(\vec{n},A_2) = 1_{\mathbb{N}^\infty\uplus\mathbb{N}^\infty}((2,\vec{n}),A_1 \uplus A_2)$.

— $\varnothing \lhd \mathbb{N}^\infty$, as $(\varnothing,\mathscr{F}_\varnothing)$ is the zero object.

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— $u : \mathbb{N} \times \mathbb{N}^\infty \longrightarrow \mathbb{N}^\infty : v$.
u is defined by $u((i,\vec{n}),A) = \delta(i.\vec{n},A)$ and $v : \mathbb{N}^\infty \longrightarrow \mathbb{N} \times \mathbb{N}^\infty$ by $v(i.\vec{n}, \biguplus_{i\in\omega} A_i) = \delta(\vec{n}, A_i)$.

$$\begin{aligned} vu((i,\vec{n}),\biguplus_{i\in\omega}A_i) &= \int_{\mathbb{N}^\infty} v(\vec{n}',\biguplus_{i\in\omega}A_i)u((i,\vec{n}),d\vec{n}') \\ &= \int_{\mathbb{N}^\infty} v(\vec{n}',\biguplus_{i\in\omega}A_i)\delta(i.\vec{n},d\vec{n}') \\ &= v(i.\vec{n},\biguplus_{i\in\omega}A_i) \\ &= \delta(\vec{n},A_i) \\ &= 1_{\mathbb{N}\times\mathbb{N}^\infty}((i,\vec{n}),\biguplus_{i\in\omega}A_i). \end{aligned}$$ □

5.6. *ω-CPO*

Consider the category ω-**CPO** of ω-complete partial orders with bottom element and continuous maps as morphisms. ω-**CPO** is a traced symmetric monoidal category with product as the tensor, $I = \{\perp\}$ and given $f : X \times U \longrightarrow Y \times U$, $\mathtt{Tr}^U_{X,Y}(f) = f_1\langle 1_X, \mathsf{Y}(f_2)\rangle$ where $f_1 = pr_1 f$ and $f_2 = pr_2 f$ and $\mathsf{Y} = \mathsf{Y}_{X,U} : \omega\text{-}\mathbf{CPO}(X \times U, U) \to \omega\text{-}\mathbf{CPO}(C, U)$ is the (parametrised) Tarski least fixed-point operator: given $k : X \times U \longrightarrow U$, we have $\mathsf{Y}(k)(x) = \mu u.k(x,u) = u$.

Proposition 5.7. (ω-**CPO**, $(-)^{\mathbb{N}}, A^{\mathbb{N}}$) is a GoI Situation where A is any object in ω-**CPO** (and the associated LCA will be non-trivial if A is not $\{\perp\}$).

Proof. Note that the functor $T = (-)^{\mathbb{N}}$ determines a symmetric monoidal functor (T, ψ, ψ_I), where we define the isomorphisms $\psi_{A,B}(f,g) = \langle f, g\rangle$, with $\langle f, g\rangle(n) = (f(n), g(n))$ for $n \in \mathbb{N}, f \in A^{\mathbb{N}}, g \in B^{\mathbb{N}}$ and $\psi_I(\perp) = f_\perp$, where $f_\perp$ is the constant-$\perp$ sequence. In order to show that T is traced, it suffices to show that, given $f : X \times U \longrightarrow U$, $T(\mathsf{Y}(f)) = \mathsf{Y}'((Tf)\psi)$, where $\mathsf{Y}' = \mathsf{Y}_{TX,TU}$. Note that, for $g \in TX$,

$$\begin{aligned} T(\mathsf{Y}(f))(g)(n) &= \mathsf{Y}(f)(g(n)), \quad \text{by definition of } T \\ &= \mu u.f(g(n),u) = u, \quad \text{by definition of } \mathsf{Y}. \end{aligned}$$

On the other hand, $\mathsf{Y}'((Tf)\psi)(g) = \mu h.Tf\psi(g,h) = h$, therefore $f(g(n),h(n)) = h(n)$ for all $n \in \mathbb{N}$ and h is the least element in TU with this property. However, $f(g(n),u) = u$, thus $h(n) \leqslant \mathscr{K}_u(n) = u$ where $\mathscr{K}_u$ is the constant-u function in TU. Also, $u \in U$ is the least element with $f(g(n),u) = u$ and therefore $u \leqslant h(n)$. Hence $h(n) = u$, and $T(\mathsf{Y}(f))g = \mathsf{Y}'(Tf)\psi g$ for all $g \in TX$, which implies $T(\mathsf{Y}(f)) = \mathsf{Y}'(Tf\psi)$.

Now, given $f : X \times U \longrightarrow Y \times U$,

$$\begin{aligned} T(\mathtt{Tr}(f)) &= T(f_1\langle 1_X, \mathsf{Y}(f_2)\rangle) \\ &= T(f_1)T(\langle 1_X, \mathsf{Y}(f_2)\rangle) \\ &= T(f_1)\psi\langle 1_{TX}, \mathsf{Y}'(Tf_2\psi)\rangle \\ &= \mathtt{Tr}(\psi^{-1}Tf\psi). \end{aligned}$$

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Hence T is traced. We next discuss the necessary retractions:

1 Define $e_A : (A^{\mathbb{N}})^{\mathbb{N}} \longrightarrow A^{\mathbb{N}}$ by $e_A(f) = g$ for $f \in (A^{\mathbb{N}})^{\mathbb{N}}$ and $g(n) = f(n_1, n_2)$ where $n = \langle n_1, n_2 \rangle$. And $e'(A) : A^{\mathbb{N}} \longrightarrow (A^{\mathbb{N}})^{\mathbb{N}}$ by $e'_A(h) = k$ where $k(n_1, n_2) = h(\langle n_1, n_2 \rangle)$. Clearly, $e'_A e_A = 1$ for all A and e, e' are natural.

2 Define $d_A : A \longrightarrow A^{\mathbb{N}}$ by $d_A(a) = f$ where $f(n) = a$ for all $n \in \mathbb{N}$. And $d'_A : A^{\mathbb{N}} \longrightarrow A$ by $d'_A(f) = f(0)$. Clearly, d, d' are natural and (d, d') forms a retract pair.

3 Define $c_A : A^{\mathbb{N}} \times A^{\mathbb{N}} \longrightarrow A^{\mathbb{N}}$ by $c_A(f, g) = h$ where

$$h(n) = \begin{cases} f(n/2), & \text{if } n \text{ is even;} \\ g((n-1)/2), & \text{if } n \text{ is odd.} \end{cases}$$

And $c'_A : A^{\mathbb{N}} \longrightarrow A^{\mathbb{N}} \times A^{\mathbb{N}}$ by $c'_A(h) = (f, g)$ where $f(n) = h(2n)$ and $g(n) = h(2n + 1)$. Clearly, c, c' are natural and (c, c') is a retract pair.

4 Define $w_A : I \longrightarrow A^{\mathbb{N}}$ by $w_A(\perp) = f$ where $f(n) = \perp$ for all $n \in \mathbb{N}$ and $w'_A : A^{\mathbb{N}} \longrightarrow I$ by $w'_A(f) = \perp$ for all $f \in A^{\mathbb{N}}$. Clearly, w, w' are natural and (w, w') is a retract pair.

Having found the functor T, it follows automatically that $U = TA$ is a reflexive object for any object A. In fact:

(i) $U \times U \lhd U$ follows from (3) above,

(ii) $TU \lhd U$ from (1) above,

and, finally,

(iii) $I \lhd U$ from (4) above.

Hence $U = TA = A^{\mathbb{N}}$ is a reflexive object, where A is any object in ω-**CPO**.

Finally, observe that if A is $I = \{\perp\}$, then the LCA is trivial since ω-**CPO**$(U, U) = \omega$-**CPO**$(TI, TI) = \{id_{TI}\}$. □

Remark 5.8. GoI Situations (and their associated LCA's) capture the essence of Girard's GoI interpretation: of the examples above (*cf.* Abramsky (1996)), $\mathcal{G}$(**PInj**) is essentially the original GoI construction in Girard (1989), while $\mathcal{G}(\omega$-**CPO**) is the 'wave-style' model in Abramsky and Jagadeesan (1994b). In particular, the fact that our semantics coincides with Girard-Danos–Regnier in the case of partial injective maps has already been observed both in the papers Abramsky and Jagadeesan (1994a) and Abramsky *et al.* (2000) on history-free game semantics, and at greater length in Patrick Baillot's Thèse du Troisième Cycle (Baillot 1995)

Moreover, Girard's original operator-theoretic models (in the category of Hilbert spaces **Hilb**$_2$), as well as Danos and Regnier's *small model* are also captured by particle-style GoI Situations with some additional structure (Haghverdi 2000b, Section 6).

6. Conclusion and future work

As we have seen, the abstract framework introduced above captures Girard's Geometry of Interaction interpretation for the full multiplicative-exponential fragment of linear logic. Moreover, from each such GoI Situation we obtain a (standard) combinatory algebra. This suggests several intriguing directions for future work.

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1 It would be interesting to compare different LCA's: for example, to analyse and classify the fine structure of those combinatory algebras (and their subalgebras) arising from our construction. One candidate for such a classification are realizability toposes generated by partial combinatory algebras (and their associated full subcategories of assemblies and modest sets). These structures have recently enjoyed considerable interest in providing new models of intuitionistic type theories and also fine-grained detail on models of computation, especially computability at higher types (Longley 1995; Longley 1998). For example, as mentioned in the *Introduction*, recent work of Abramsky and Longley (Abramsky and Longley 2000) presents a combinatory algebra $\mathscr{A}$ of history-free strategies arising from our LCA construction on **Pfn** above. $\mathscr{A}$ realizes the finite type structure of the strongly stable (= sequentially realizable) functionals, while a sub-PCA of $\mathscr{A}$ gives rise to the PCF-sequential ones. Similarly, Abramsky and Lenisa (2000) explores the subalgebra of **Pfn** of partial involutions. However, many other possibilities remain, for example subalgebras of **SRel**.
2 While the GoI interpretation for the multiplicative-exponential fragment is under control, the extension of our abstract treatment of GoI to cover the additives is open, at least for particle-style semantics. Such a treatment would need to account for Girard's work in Girard (1995) (for wave-style models, an adequate treatment of the additives is described in Abramsky and Jagadeesan (1994b)).
3 Finally, related to (1) above, it is known how to interpret Girard's system $\mathscr{F}$ over a combinatory algebra. For example, if one works in $\mathscr{G}$(**PInj**), this is essentially given in Abramsky and Jagadeesan (1994b). A detailed study of the interpretation of system $\mathscr{F}$ in our other GoI models would be useful (*cf.* Girard (1989)). This is especially so for the question of finding fully complete models (arising from a GoI Situation) both for system $\mathscr{F}$ as well as for fragments of Linear Logic (starting with MLL + MIX). Such problems are currently being actively pursued by the authors.

Appendix I: The trace axioms

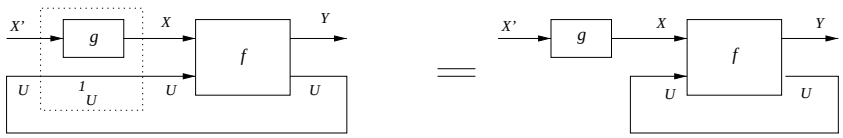


Fig. 15. Naturality in X.

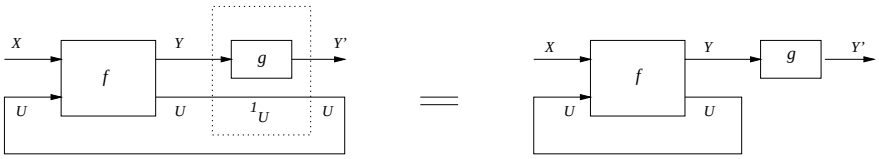


Fig. 16. Naturality in Y.

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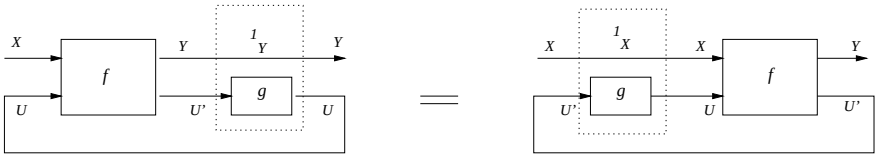


Fig. 17. Dinaturality in U.

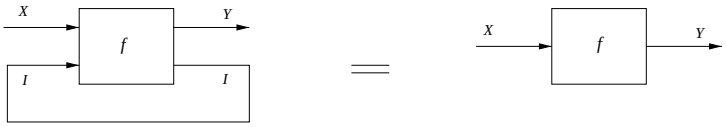


Fig. 18. Vanishing I.

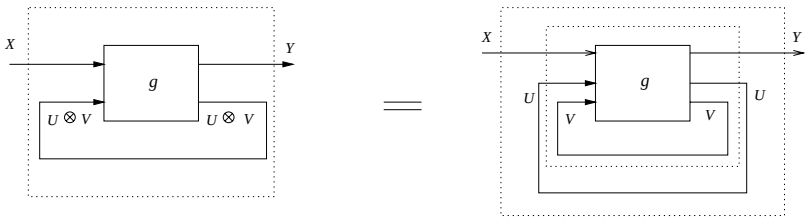


Fig. 19. Vanishing II.

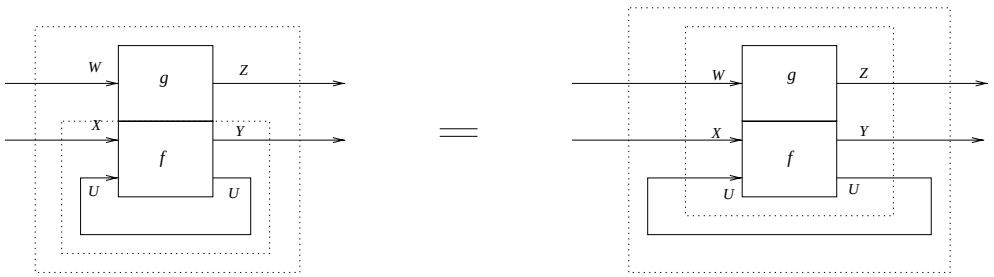


Fig. 20. Superposing.



Fig. 21. Yanking.

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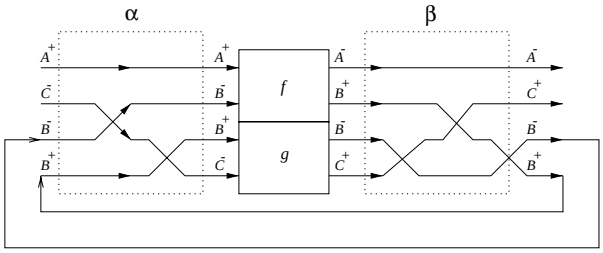


Fig. 22. Composition in $\mathscr{G}(\mathbb{C})$.

Appendix II: Pointwise naturality

Notation: To save space and make the text more readable, we sometimes abbreviate the object $(A^+, A^-) \in \mathscr{G}(\mathbb{C})$ to $\underline{A}$.

We occasionally use the graphical calculus (*cf.* Appendix III), but only to simplify the composition of morphisms in $\mathscr{G}(\mathbb{C})$.

We will prove that δ is a monoidal pointwise natural transformation. For the remaining cases, we refer the reader to an extended version of this paper, available at Abramsky *et al.* (2001).

To prove pointwise naturality, we need to show that

$$!!f\delta_{(I,I)} = \delta_{(B^+,B^-)}!f$$

for all $f : (I,I) \longrightarrow (B^+, B^-)$.

$$\begin{aligned}
!!f\delta_{(I,I)} &= (e_I \otimes 1_{T^2B^+})\psi^{-1}_{TI,TB^+} T\psi^{-1}_{I,B^+} T^2 f T\psi_{I,B^-}\psi_{TI,TB^-}(e'_I \otimes 1_{T^2B^-}),\\
&\quad \text{using graphical calculus}\\
&= (e_I \otimes 1_{T^2B^+})\psi^{-1}_{TI,TB^+} T\psi^{-1}_{I,B^+} e'_{B^+} T f e_{B^-} T\psi_{I,B^-}\psi_{TI,TB^-}(e'_I \otimes 1_{T^2B^-}),\\
&\quad e, e' \text{ are natural transformations and } (e, e') \text{ is a retract pair}\\
&= (e_I \otimes 1_{T^2B^+})(e'_I \otimes e'_{B^+})\psi^{-1}_{I,B^+} T f\psi_{I,B^-}(e_I \otimes e_{B^-})(e'_I \otimes 1_{T^2B^-}),\\
&\quad e, e' \text{ are monoidal}\\
&= (1_{TI} \otimes e'_{B^+})\psi^{-1}_{I,B^+} T f\psi_{I,B^-}(1_{TI} \otimes e_{B^-}),\\
&\quad e_I = \psi_I\psi_I^{-1} T\psi_I^{-1}, e'_I = T\psi_I\psi_I\psi_I^{-1} \text{ (as } e, e' \text{ are monoidal)}\\
&\quad \text{and } T \text{ is a strong monoidal functor}\\
&= \delta_{(B^+,B^-)}!f, \quad \text{using graphical calculus.}
\end{aligned}$$

δ is monoidal as follows:

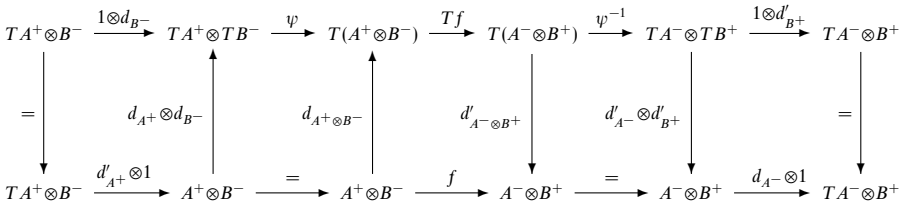
$$\begin{aligned}
!\varphi_{\underline{A},\underline{B}}\varphi_{!\underline{A},!\underline{B}}(\delta_{\underline{A}} \otimes \delta_{\underline{B}}) &= \sigma_{T^2(A^+\otimes B^+),TA^-\otimes TB^-}(T\psi_{A^+,B^+} \otimes e_{A^-} \otimes e_{B^-})\\
&\quad (\psi_{TA^+,TB^+} \otimes \psi^{-1}_{TA^-,TB^-})(e'_{A^+} \otimes e'_{B^+} \otimes T\psi^{-1}_{A^-,B^-})\\
&= \sigma_{T^2(A^+\otimes B^+),TA^-\otimes TB^-}(e'_{A^+\otimes B^+} \otimes \psi^{-1}_{A^-,B^-})\\
&\quad (\psi_{A^+,B^+} \otimes e_{A^-\otimes B^-})\\
&= \delta_{\underline{A}\otimes\underline{B}}\varphi_{\underline{A},\underline{B}}
\end{aligned}$$

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$$\begin{aligned}\delta_{(I,I)}\varphi_{(I,I)} &= \sigma_{T^2I,I}(e'_I \otimes \psi_I^{-1})(\psi_I \otimes e_I)\\ &= \sigma_{T^2I,I}(T\psi_I \otimes \psi_I^{-1})(\psi_I \otimes T\psi_I^{-1})\\ &= !\varphi_{(I,I)}\varphi_{(I,I)}.\end{aligned}$$

Finally, let us discuss pointwise naturality versus ordinary naturality of the transformations of WLC's. We examine the case of `der`, the rest being similar.

Observe that `der` $:! \to Id$ is a natural transformation iff the following diagram commutes:



The three squares in the middle commute because d is a monoidal natural transformation. The right- and left-most squares commute iff $d_{A^+}d'_{A^+} = 1$ and $d_{A^-}d'_{A^-} = 1$ (essentially, by setting $B = I$ and using monoidal identities.) We already have that $d'_{A^+}d_{A^+} = 1$, as (d, d') is a retract pair. Since A^+ is arbitrary, (d, d') must be an isomorphism.

Appendix III: Normal forms for combinators

Recall the equations for the combinators:

$I{\cdot}x = x.$

$B{\cdot}x{\cdot}y{\cdot}z = x{\cdot}(y{\cdot}z).$

$C{\cdot}x{\cdot}y{\cdot}z = x{\cdot}z{\cdot}y.$

$W{\cdot}x{\cdot}!y = x{\cdot}!y{\cdot}!y.$

$D{\cdot}x{\cdot}!y = x{\cdot}y.$

$\delta{\cdot}!x = !!x.$

$F{\cdot}!x{\cdot}!y = !(x{\cdot}y).$

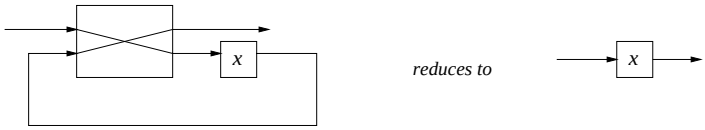


Fig. 23. Identity combinator I.

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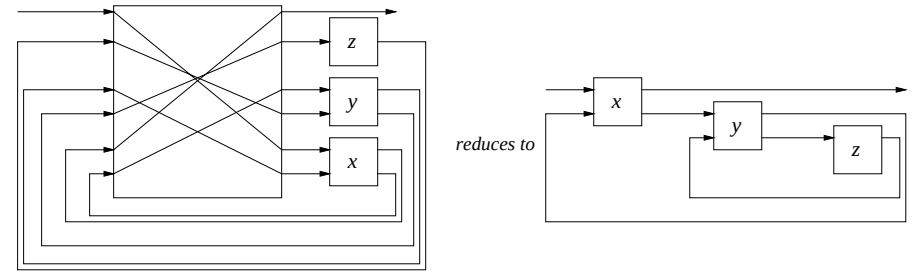


Fig. 24. Composition combinator B.

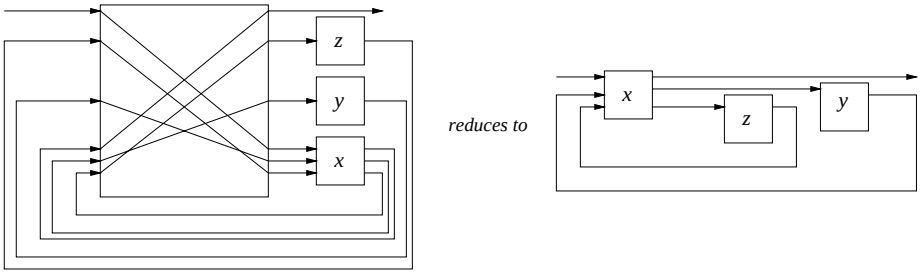


Fig. 25. Exchange combinator C.

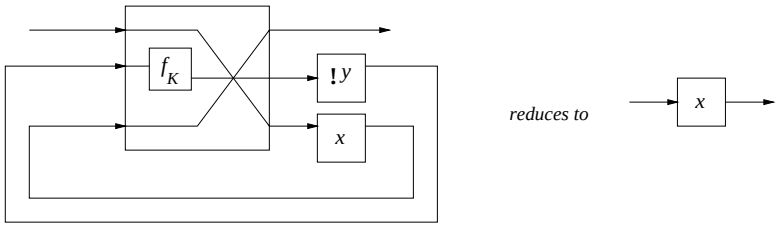


Fig. 26. Weakening combinator K.

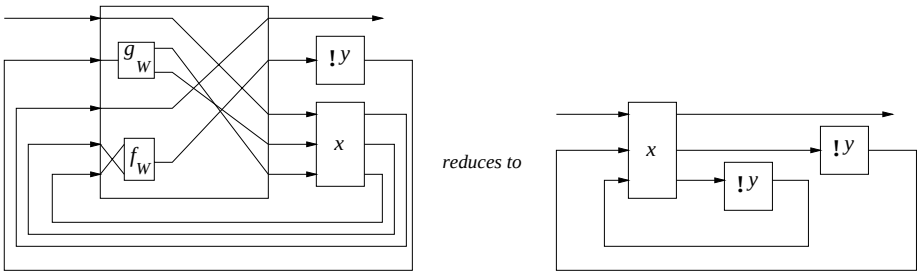


Fig. 27. Contraction combinator W.

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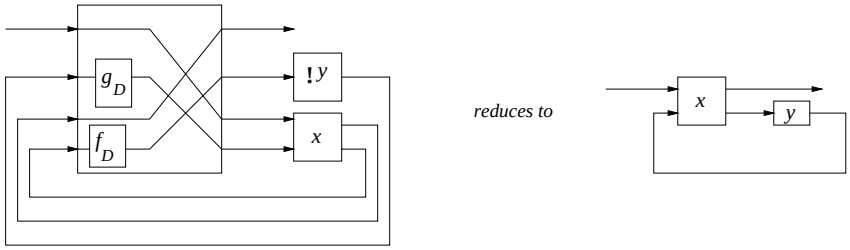


Fig. 28. Dereliction combinator D.

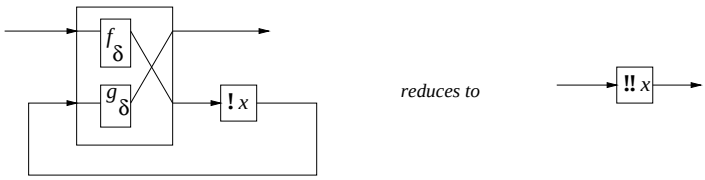


Fig. 29. Comultiplication combinator δ.

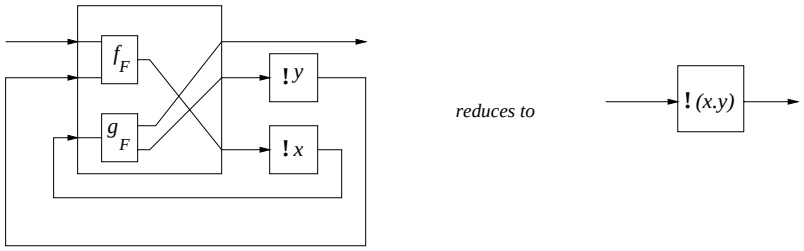


Fig. 30. Functoriality combinator F.

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