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MONETARY FEEDBACK RULES AND EQUILIBRIUM DETERMINACY IN PURE EXCHANGE OVERLAPPING GENERATIONS MODELS

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This paper considers a pure exchange overlapping generations model in which the money-growth rate is endogenous and follows a feedback rule. Different specifications for the monetary policy rule are analyzed, namely a so-called current, forward, or backward-looking feedback rule, depending on whether the monetary authority uses the actual, expected, or last observed values of the inflation rate to set the monetary policy. We study how the responsiveness of the policy rule with respect to inflation affects the determinacy of the monetary equilibrium. A policy rule is called aggressive (moderate) if it responds strongly (moderately) to inflation deviations from the target. We show how aggressive feedback rules, depending on the considered timing, can reinforce mechanisms that lead to indeterminacy or may lead the inflation rate to fluctuate around the monetary equilibrium at which monetary policy is aggressive. A *leaning against the wind* policy seems to be more desirable from an equilibrium determinacy point of view. On the contrary, a *leaning with the wind* policy could not be the recommended policy for the Central Bank.

Keywords: OLG Model, Monetary Policy Rules, Perfect Foresight Equilibrium, Indeterminacy

1. INTRODUCTION

Equilibrium models are appealing tools for understanding the economic environment and for providing a framework in which to design (possibly) successful economic policies. In this sense, such models are especially useful when they provide unique policy prescriptions for specified policy targets. But very often they exhibit multiple equilibria and, as a consequence, different policy prescriptions.

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Policies targeted to stabilization purposes, such as inflation stabilization policies, often present this kind of drawback.

Many developments in monetary economics have emphasized the link between the degree to which monetary policies respond to endogenous variables, as the inflation rate, and macroeconomic stability. Perhaps, the best known result in this stream of literature is that, if fiscal solvency is preserved under all circumstances, then a policy that aggressively reacts (a so called *active* policy) to inflation by raising the nominal interest rate by more than the increase in inflation stabilizes the real side of the economy by ensuring the uniqueness of the equilibrium [see Benhabib et al. (2002)]. At the same time, a policy that under-reacts (a so called *passive* policy) to inflation by raising the nominal interest rate by less than the observed increase in inflation destabilizes the economy.

Many prominent Central Banks have adopted inflation targeting or inflation forecast targeting as their preferred framework for implementing monetary policy. Following the lead of the Bank of England, inflation forecast targeting was adopted by several European Central Banks in the 1990s. This strategy has been adopted by the Bundesbank since 1975 and is connected with its successful low-inflation monetary policy since then [see Clarida and Gertler (1997), Von Hagen (1999)]. In the early 1990s, the French Central Bank started to announce annual monetary targets, too. Nowadays, the inflation-targeting approach, used by most countries, can be characterized as "flexible inflation forecast targeting." Flexible inflation targeting involves the announcement of an explicit target for inflation together with a sensible stabilization policy [see Bullard (2012)]. A specific targeting rule establishes conditions for the target variables, or the forecasts of the target variables.

A lot of relevant contributions proposed the adoption of feedback policies for the conduct of monetary policy, and different instrument rules with varying degrees of responsiveness have been explored across a number of models [see, e.g., Benhabib et al. (2001a, 2001b, 2002), Taylor (2007) among others]. Various feedback rules as well as their impact in terms of stabilization properties have been analyzed, for example, rules that depend on contemporaneous, past, or future inflation.

Following this line of research, this paper studies a standard pure exchange overlapping generations (OLG) model with two-period-lived households. The model assumes that monetary policy is implemented through an endogenous monetary policy rule that could better fit the current practices of Central Banks [see, e.g., Bernanke and Woodford (1997), Bischi and Marimon (2001), Walsh (2009), Creel and Hubert (2015) for theoretical and empirical studies on inflation-targeting policy] instead of a constant growth rate on nominal balances.

In this paper, we investigate the stabilizing virtues of monetary policy. Specifically, a monetary policy is stabilizing when it reduces the conditions for equilibrium indeterminacy. We show that a policy that aggressively fights inflation does not necessarily lead to the determinacy of the target equilibrium, i.e., indeterminacy may arise. Precisely, a given configuration will be referred to as locally indeterminate if there exists a continuum of distinct equilibrium paths starting from the same initial values for inflation. Indeterminacy of perfect foresight equilibrium

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is a sufficient condition for the existence of sunspot equilibria and stochastic fluctuations based upon extrinsic uncertainty. On the other hand, a steady state is locally determinate when, in each small neighborhood, there exists a unique perfect foresight equilibrium trajectory converging to it.

The issues of determinacy and stability of the monetary equilibrium have been thoroughly conducted under the assumption of a constant money supply or a constant growth rate of the money supply. For example, Matsuyama (1990) and Woodford (1994) show that money growth targeting is prone to local real indeterminacy and Matsuyama (1991) shows that an optimizing model of a monetary economy can produce perfect foresight equilibria in which the price level fluctuates forever. Carlstrom and Fuerst (2003) use a generic money-in-the-utility-function model and prove the importance of money demand timing for the local equilibrium determinacy when the monetary policy rule follows an exogenous money-growth rule. Furthermore, Gale (1973) and Tuinstra (2003) show that an exogenous money supply, in a framework of pure exchange OLG model, leads to the stability of the autarkic steady state and, on the contrary, to instability of the monetary equilibrium.

In the context of OLG models, different approaches have been proposed to overcome the issue of equilibria selection and, in particular, to avoid that the economy settles down to equilibria characterized by a high level of inflation. On one hand, there is an extensive literature in which maximizing agents explicitly form their expectations and coordinate them by learning from past experience [see, e.g., Marcet and Sargent (1989a, 1989b)]. Moreover, the works of Bullard (1994) and Tuinstra (2003), in which agents estimate the inflation rate by a least-squares regression, proved that dynamics may converge to learning equilibria that can be characterized by endogenous fluctuations, and even chaotic learning equilibria may exist. With the same model, Tuinstra and Wagener (2007) show that, by changing the estimation procedure only slightly, the monetary equilibrium becomes globally stable for almost any specification of the fundamentals of the OLG economy. Hence, learning may provide an equilibrium selection device in environments with a multiplicity of equilibria under perfect foresight.

An alternative approach studied the empirical relevance of the multiplicity of steady states running laboratory experiments, showing the consistency of the monetary equilibrium as a long-run stationary state under the hypothesis of adaptive learning [see, e.g., Marimon and Sunder (1993)] and that convergence to the monetary equilibrium occurs much less frequently in a scenario with high money-growth rate [see Heemeijer et al. (2012)]. In fact, depending on the level of money supply growth, either participants were able to prevent inflation from widely fluctuating and stabilize it, or none succeeded in keeping the inflation rate stable during the entire experiment. Hence, these experiments also relate to the overriding feature of the existence of multiple equilibria (or indeterminacy of equilibrium): They try to shed light on to which of these equilibria might be attained.

The present paper is in line with the latter issue. We consider an OLG model in which the money-growth rate is endogenous and follows a feedback rule. In so doing, the Central Bank is enabled to choose goals for inflation that may differ

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from those of other countries and allows some response to output fluctuations as well; then, a target for the growth rate of a monetary aggregate provides a nominal anchor that is fairly easily understood by the public and is easily communicated to the public. Here, we study how the monetary policy affects the determinacy of the monetary steady state under perfect foresight. Indeterminacy arises when the monetary steady state is a locally asymptotically stable fixed point of the perfect foresight dynamics. Conversely, if the monetary steady state is a repelling fixed point of the perfect foresight dynamics, then indeterminacy is ruled out. Hence, there exists a close link between indeterminacy of the monetary steady state and the susceptibility of the economy to extrinsic uncertainty, underlining the importance of characterizing the conditions under which determinacy/indeterminacy occurs. Moreover, the local determinacy of the monetary equilibrium is not always sufficient to rule out indeterminacy in a global perspective. Indeed, global indeterminacy may arise whenever there exist perfect foresight paths converging to different equilibrium dynamics. That is, different choices of the variable might imply different long-run behaviors and the economy is path-dependent.

We identify the equilibrium conditions under different timings for the monetary feedback rule and we prove that, by making the monetary policy react aggressively to inflation deviation from its target, the set of the economies for which the monetary equilibrium is indeterminate increases. The adoption of an endogenous monetary policy may lead to the occurrence of multiple monetary equilibria as well and hence, restricting the evaluation of the monetary feedback rules to their consequences for local stability can be misleading. In our baseline analysis, we assume that the inflation rate is a function of the contemporaneous rate of inflation. But, any practical inflation measure to which the policy authority responds must involve some average of past inflation rates or forecasts of expected future rates of inflation, or both. Thus, we extend the baseline scenario to allow for feedback rules in which the inflation rate depends on past or expected future rates of inflation. We find that an aggressive policy may appear to be stabilizing because it locally ensures the uniqueness of the target equilibrium, when in fact it is destabilizing because it gives rise to equilibria in which the economy may converge to a cycle. A general pattern that arises is that if the monetary authority follows an aggressive stance, then a forward-looking component in the monetary policy rule makes indeterminacy more likely, whereas with a backward-looking rule the set of economies for which indeterminacy occurs may be reduced. In conclusion, a *leaning against the wind* policy (i.e., tightening money market conditions in response to higher inflation) seems to be more desirable from an equilibrium determinacy point of view. On the contrary, a *leaning with the wind* policy could not be a beneficial policy for the Central Bank.

The paper is organized as follows: Section 2 presents the OLG model and discusses the conditions for the existence of the steady states; in Section 3, we provide analytical conditions for local determinacy of the monetary equilibrium across the three different policy configurations for the linear money-growth rule; Section 4 concludes. All the proofs are located in the appendix.

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2. THE OLG MODEL WITH MONETARY FEEDBACK RULE

We consider a simple pure exchange OLG economy augmented by a description of a monetary policy. Time is discrete, $t = 0, 1, 2, \ldots$ and in each t a new generation of two-period-lived households (agents) is born, called generation t. Agents are referred to as young in their first period of life and in their second period of life as old. In period 1, there exists also a generation of one-period-lived households, which is called generation 0 and whose members are considered old. All households belonging to the same generation have identical preferences and endowments. In each period, there exists a single homogeneous consumption good that can neither be produced nor stored. Agents are endowed with $w_1 > 0$ units of the consumption good in the first period of their lives and with $w_2 \geq 0$ units of the good in the second period. In addition to the consumption good, there exists fiat money which is the only mean of value store in the economy. Newly created money is provided by the Central Bank in the form of lump-sum transfers to old agents and all agents of the same generation receive the same transfer.

Let p_t be the price of one unit of period t consumption good in terms of money and p^e_{t+1} the expected price for period $t+1$. Households of generation 0 exchange all the money they own for the consumption good and consume it with the units of good they are endowed with. Finally, agents are assumed to have perfect foresight and all the future prices are known with certainty, i.e., $p^e_{t+1} = p_{t+1}$.

Individuals belonging to generation $t \geq 1$ have a strictly monotone, strictly quasi-concave utility function $U(c_1, c_2)$ defined over young-aged consumption c_1 and old-aged consumption c_2, respectively. Thus, households of generation $t \geq 1$ solve the following problem:

$$\max_{c_1, c_2} U(c_1, c_2)$$

s.t

$$p_t c_1 + p^e_{t+1} c_2 \leq p_t w_1 + p^e_{t+1} w_2.$$

The maximization problem results in a savings function of the form

$$S\left(\frac{p^e_{t+1}}{p_t}\right) = w_1 - c_1\left(\frac{p^e_{t+1}}{p_t}\right). \tag{1}$$

We assume the only means of savings is money and, accordingly, the demand for money in period t reads as

$$\frac{M^d_t}{p_t} = S\left(\frac{p^e_{t+1}}{p_t}\right). \tag{2}$$

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If we denote with π_{t+1} the gross rate of inflation between t and $t+1$, and assume perfect foresight about the future price, the money demand reads as

$$\frac{M_t^d}{p_t} = S\left(\frac{p_{t+1}}{p_t}\right) = S(\pi_{t+1}). \tag{3}$$

The government has a series of exogenous expenditures in each period financed by seigniorage (which corresponds to a tax, or subsidy, on old agents). We denote by M_t^s the aggregate nominal money supply in period t and by $\theta(\pi_{t+\tau})$ the gross rate of money growth. This results in the following rule for money supply:

$$M_t^s = \theta(\pi_{t+\tau}) M_{t-1}^s, \qquad \tau \in \{-1, 0, 1\}. \tag{4}$$

Thus, the money stock grows with an endogenous rate $\theta(\pi_{t+\tau})$ which depends on the equilibrium rate of inflation for time $t+\tau$. Depending on the value of τ, the policy rule designs the money growth for period t as a function of the past, actual, or expected inflation. Namely, if $\tau = 1$, the policy rule is said to be forward-looking, if $\tau = 0$, the policy is current-looking and, finally, if $\tau = -1$, we refer to a backward-looking policy rule.

The objective of the monetary authority is that the gross rate of inflation converges asymptotically to an inflation target π^*. Then, a successful policy is one for which

$$\lim_{t \to +\infty} \pi_t = \pi^*.$$

If we also want to consider rules with an explicit inflation target, the function $\theta(\pi_{t+\tau})$ must be structured in such a way the target rate of inflation is a fixed point of $\theta(\pi_{t+\tau})$, i.e.,

$$\theta(\pi^*) = \pi^*.$$

We refer to monetary policy rule as *aggressive* at the inflation steady state if $|\theta'(\pi^*)| > 1$ and as *moderate* if $|\theta'(\pi^*)| < 1$, that is the monetary authority lowers the growth rate of money by more (less) than one-for-one in response to an increase in the inflation rate. Moreover, we distinguish the case in which $\theta'(\pi^*) > 0$ (positive monetary feedback rule or *leaning with the wind* regime) from the case $\theta'(\pi^*) < 0$ (negative monetary feedback rule or *leaning against the wind* regime[1]).

Equilibrium on the money market is then given by

$$S\left(\frac{p_{t+1}}{p_t}\right) p_t = \theta(\pi_{t+\tau}) S\left(\frac{p_t}{p_{t-1}}\right) p_{t-1}, \qquad \tau \in \{-1, 0, 1\} \tag{5}$$

or in terms of gross inflation rate

$$\pi_t S(\pi_{t+1}) = \theta(\pi_{t+\tau}) S(\pi_t), \qquad \tau \in \{-1, 0, 1\}. \tag{6}$$

Equation (6) implicitly defines the model that will be analyzed in the present work, under different timing for monetary policy. From the last expression, it can be seen

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that this system can have two different types of steady states equilibria, that is the monetary steady states such that $\pi = \theta(\pi)$ and, if it exists, the autarkic steady state π^a such that $S(\pi^a) = 0$.

In fact, we state the following

PROPOSITION 1. *The monetary equilibria of the dynamic system expressed by (6) are given as the solutions of the following equation*:

$$\pi = \theta(\pi). \tag{7}$$

The eventual unique autarkic steady state is determined as the solution of

$$S(\pi) = 0. \tag{8}$$

The savings function is an aggregate demand function and therefore, by the Sonnenschein–Debreu–Mantel results [see Debreu (1974), Mantel (1974), Sonnenschein (1973)] it can take on almost any form. Indeed, the work of Sonnenschein (1973) has shown that any continuous function could be an excess demand function; full generality in this regard would allow complicated dynamics under rational expectations, and possibly even chaos, as has been shown by Grandmont (1985). Therefore, the simple OLG model has, as a logical possibility, complicated and chaotic dynamics based on perfect competition, perfect foresight, and exogenous monetary policy. In our paper, this kind of chaotic trajectories will be ruled out by assuming the monotonicity of the savings function. Thus, maintaining this assumption, eventually complicated dynamics will be induced only by the endogenization of the monetary policy rule.

Henceforth [and as in Tuinstra (2003)], we will make the following assumptions about the savings function $S(\cdot)$:

1. S is twice differentiable and $S'(\cdot) < 0$.
2. $S(\pi^*) > 0$.

Assumption 1 states that savings are a decreasing function of the inflation rate, implying that c_1 and c_2 are gross substitutes. Assumption 2 states that we are in the *Samuelson* case where, in the target equilibrium, savings are positive and people transfer income from the present to the future. Assumption 1 and 2 together imply $\pi^a > \pi^*$. For convenience, we introduce the variable

$$\varepsilon_s(\pi) = -\pi \frac{S'(\pi)}{S(\pi)},$$

which is the (negative of the) *inflation elasticity of saving* and plays an important role in the dynamic behavior of the OLG model. Assuming the savings function to be positive $S(\pi) > 0$, under assumptions 1 and 2, we have $\varepsilon_s(\pi) > 0$ for all $\pi > 0$. It also follows that any monetary equilibrium π^m satisfies $\pi^m < \pi^a$.

In general, besides the autarkic steady state, the endogenous money-growth rule may lead to the occurrence of multiple monetary equilibria different from

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the target, as a consequence of the different policy rules. But since money-growth rules estimated from the data tend to take simple forms, we assume a linear rule for analytical tractability and realism. Accordingly, as policy makers would not be inclined to use a monetary policy rule that also has suboptimal steady states, we consider a monetary policy rule which features only one monetary equilibrium.

3. ANALYSIS OF THE MONETARY EQUILIBRIUM DETERMINACY

This section discusses the main results of the paper. In particular, we proceed as follows: we first check the local determinacy of the monetary equilibrium; then, in order to characterize global equilibrium dynamics, we assume a particular (linear) functional form for the monetary policy rule of the type

$$\theta(\pi_{t+\tau}) = \pi^* + \delta\left(\pi_{t+\tau} - \pi^*\right), \qquad \tau \in \{-1, 0, 1\} \tag{9}$$

where the parameter δ tunes the degree of reactivity of the Central Bank in creating new money. We remark that the parameter δ also allows us to distinguish the *leaning with the wind regime* if $\delta > 0$ and the *leaning against the wind regime* if $\delta < 0$.

In order to evaluate how the monetary policy in (9) reacts to inflation deviations and to study its dynamic effects, we parameterize the underlying economy considering a Cobb–Douglas utility function for the maximization problem of the generation born in period t, which reads as

$$U(c_1, c_2) = \ln c_1 + \ln c_2$$

with budget constraint

$$p_t c_1 + p^e_{t+1} c_2 \leq p_t \omega_1 + p^e_{t+1} \omega_2.$$

Hence, the model expressed by equation (6) takes the explicit form

$$\pi_{t+1} = \pi^a + \theta(\pi_{t+\tau})\left(1 - \frac{\pi^a}{\pi_t}\right), \tag{10}$$

where $\pi^a = \frac{w_1}{w_2}$ and the function $\theta(\cdot)$ is defined in (9).

We repeat the analysis for the three different timings of the policy rule. In particular, we analyze the case of $\tau = 0$, $\tau = 1$, and $\tau = -1$ that is the current-looking, forward-looking, and backward-looking cases for the money-growth rule.

Finally, to generate simulations that yield better insights on the model dynamics, we assign values to the parameters. Since OLG models have been widely used to account for the observations of high-inflation episodes [see, e.g., Sargent and Wallace (1987), Marimon and Sunder (1993), Marcet and Nicolini (2003)], we follow this stream of literature. Throughout our numerical analysis, we calibrate the parameters in such a way the model is able to generate episodes of high inflation. During these phases, along with high targets for inflation, we might

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observe a negative real GDP growth as well as a high money-growth rate.[2] In particular, the empirical relevance of this hypothesis is explored with the aid of a calibration that delivers satisfactory results in terms of the ability of the model to mimic high-inflation episodes. We parameterize our model to a developing economy such as the Republic of Zimbabwe, which experienced a hyper-inflation episode in the mid-2000s. First, we fix the annual gross target for inflation rate at $\pi^* = 1.05$, according to the Ministry of Finance and Economic Development of Zimbabwe[3]; second, the autarkic steady state is set to $\pi^a = \frac{w_1}{w_2} = 1.09$, which corresponds to a negative growth rate of -9% per year, which is in line with the average GDP annual growth rate in Zimbabwe during the early 2000s.

3.1. Current-Looking Money Growth Rule

The OLG model with perfect foresight and current-looking money-growth rule reduces to the (implicitly defined) map

$$S(\pi_{t+1}) = \theta(\pi_t)\frac{S(\pi_t)}{\pi_t}. \tag{11}$$

Hence, assuming perfect foresight, equation (11) is a one-dimensional difference equation $\pi_{t+1} = F(\pi_t)$, where the variable π_t denotes the realized rate of inflation between periods $t-1$ and t, which is completely observable at time t. The existence and the stability of the steady states depend on the specific form of function $F(\cdot)$.

PROPOSITION 2. *The unique monetary equilibrium π^* is locally determinate if $\theta'(\pi^*) < 1$ or $\theta'(\pi^*) > 2\varepsilon_s(\pi^*) + 1$, whereas it is indeterminate if $1 < \theta'(\pi^*) < 2\varepsilon_s(\pi^*) + 1$.*

A particular case is the one in which a constant money-growth rule is adopted, as, e.g., in Tuinstra (2003). In fact, regardless of the timing, when $\theta(\pi_{t+\tau}) = \theta$ for all $\pi_{t+\tau}$, that is $\theta'(\pi_{t+\tau}) = 0$, the OLG model has an unstable monetary equilibrium $\pi^* = \theta$ and, if it exists, a locally stable autarkic steady state[4] π^a.

Proposition 2 shows that the monetary equilibrium may lose stability and periodic cycles emerge as the feedback parameter becomes larger than a certain threshold. A simple economic intuition for indeterminacy of the monetary equilibrium and for the occurrence of cycles may be the following[5]: Let $\{\pi^L, \pi^H\}$ be a cycle with $\pi^L < \pi^* < \pi^H$. Suppose that agents expect $\pi_{t+1} = \pi^H > \pi^*$. Then, savings $S(\pi_{t+1})$ drop below their steady-state value S^*. Recall the money market equilibrium

$$S(\pi_{t+1}) = \frac{\theta(\pi_t)}{\pi_t}S(\pi_t) \equiv v(\pi_t).$$

Straightforward calculus shows that $v'(\pi_t) < 0$ always under a *leaning against the wind* regime, while $v'(\pi_t) > 0$ under a *leaning with the wind* regime if $\theta'(\pi^*)$ is sufficiently positive. This implies that, under a *leaning against the wind* regime, if $\pi_{t+1} > \pi^*$, the money market equilibrium requires also π_t to increase above

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π^*: Thus, no cycle can occur. Otherwise, in the *leaning with the wind* regime, the money market equilibrium implies that for $\pi_{t+1} = \pi^H > \pi^*$, we must have $\pi_t = \pi^L < \pi^*$, hence a cycle. Although a drop in current inflation π_t increases the value of initial savings $S(\pi_t)$, the Central Bank lowers money growth $\theta(\pi_t)$ so aggressively that the induced fall in $\frac{\theta(\pi_t)}{\pi_t}$ more than compensates the raise in $S(\pi_t)$.

In order to study global equilibrium dynamics, we focus on the model expressed by equation (10) which, in the case of current-looking policy rule, specifies to

$$\pi_{t+1} = f(\pi_t) = \pi^a - \left[\pi^* + \delta\left(\pi_t - \pi^*\right)\right]\left(\frac{\pi^a}{\pi_t} - 1\right). \tag{12}$$

It is straightforward to see that the map always owns the target steady state π^* and the autarkic equilibrium $\pi^a = \frac{w_1}{w_2}$. The Samuelson case, $S(\pi^*) > 0$, implies $\pi^* < \pi^a$.

The following proposition provides analytical conditions for local determinacy of the monetary equilibrium[6] π^*.

COROLLARY 3. *The monetary equilibrium π^* is locally determinate if $\delta < 1$ or $\delta > \bar{\delta}$, where $\bar{\delta} \equiv \frac{\pi^a + \pi^*}{\pi^a - \pi^*}$.*

The general finding is that an aggressive rule may lead to indeterminacy of the monetary steady state: Indeed, when the monetary equilibrium is indeterminate, there exists a continuum of initial conditions π_0 such that $\pi_t \to \pi^*$, i.e., there is an infinite number of trajectories that settle down to the target steady state (see Figure 1a). An aggressive monetary policy does not necessarily bring about the determinacy of the monetary equilibrium: Hence, restricting the evaluation of the monetary policy rule to its consequence for local stability can be misleading. Indeed, when the policy responds aggressively to inflation deviation from its target, global indeterminacy in the form of equilibrium cycles occurs (see Figure 1b) in regions where, by local analysis, one concludes in favor of a unique monetary equilibrium.

Figure 1c presents the bifurcation diagram of the map f with respect to δ: When $\delta = \bar{\delta}$ the monetary equilibrium undergoes a period doubling bifurcation and, immediately after this value, a two-cycle arises. As δ grows further, a period-doubling route to chaos takes place.[7] The present model also features cases where paths starting off the target settle down to chaotic trajectories. In fact, when $\delta = 75.5$, an orbit of period three exist (see the red arrow in Figure 1c) whose values are $\pi_1 = 1.081365$, $\pi_2 = 1.062705$, and $\pi_3 = 1.038394$. Therefore, according to the Li–Yorke theorem (1975), the period three orbit implies chaos. Furthermore, since an orbit of period three exists, then according to the Sharkovsky's theorem (1995), there are periodic orbits of period n for every integer n.

Summing up, there exist cases where paths starting off the target equilibrium converge to equilibrium cycles (or chaos): Aggressive money-growth rules may open the door to equilibrium cycles of any periodicity and even chaos. Interestingly, such equilibrium dynamics exist precisely when the target equilibrium is unique

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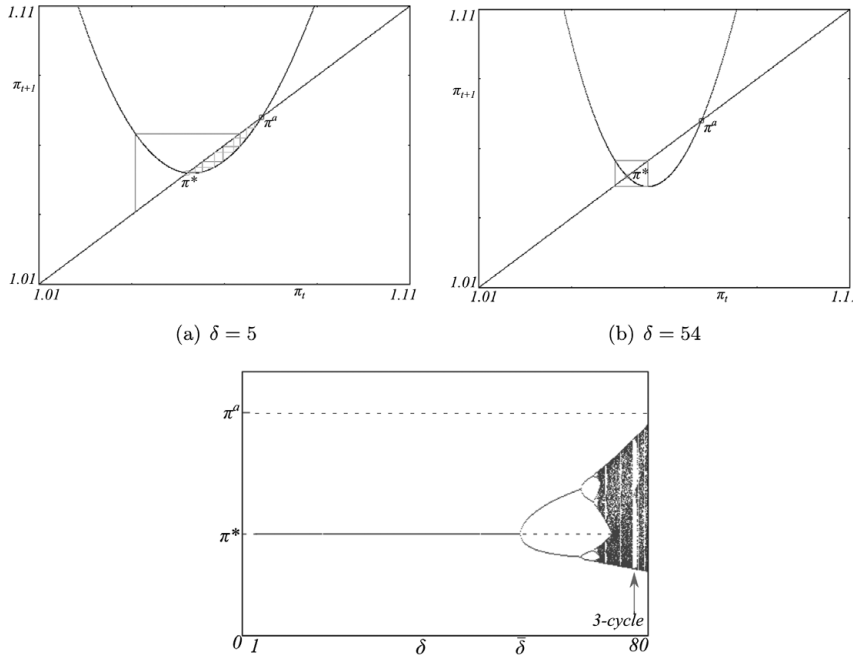


FIGURE 1. Indeterminacy of the perfect foresight dynamics. (a) If one limits the analysis to equilibria in which π_t remains forever in a small neighborhood of π^*, then the only perfect foresight equilibrium is the monetary steady state π^* itself. However, if one allows equilibria in which π_t can take values in a larger neighborhood of π^*, then a large number of equilibrium trajectories become possible: In particular, any inflation path starting between π^* and π^a represent a perfect foresight equilibrium. (b) As the policy responds aggressively to inflation deviation from its target, global indeterminacy in the form of equilibrium cycles occurs in regions where, by local analysis, one concludes in favor of a unique monetary equilibrium. (c) Bifurcation diagram on varying the reaction parameter δ. Dashed line refers to a determinate steady state. Parameter values are as follows: $\pi^* = 1.05$, $\pi^a = 1.09$, and $\bar{\delta} = 53.5$. We show the bifurcation diagram only for positive values of δ, since the monetary equilibrium is always locally determinate for $\delta < 0$.

from a local point of view, that is, when the inflation target is the only equilibrium level of inflation within a sufficiently small neighborhood around the target itself. We also would like to note that the value obtained for $\bar{\delta} = 53.5$ obtained with our calibration is in line with the average money-growth rate observed for the Republic of Zimbabwe in the early 2000s.

3.2. Forward-Looking Money-Growth Rule

Several Central Banks around the world have been using inflation forecasts as an important part of their decision-making on policy issues[8] and a standard argument

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in the literature is that, to avoid real indeterminacy, the Central Bank must respond aggressively to expected inflation [see Bernanke and Woodford (1997), Clarida and Gertler (1997), Carlstrom and Fuerst (2000)].

We then suppose that the monetary policy is forward-looking and, accordingly, the OLG model with perfect foresight and forward-looking money-growth rule reduces to the (implicitly defined) map

$$\frac{S(\pi_{t+1})}{\theta(\pi_{t+1})} = \frac{S(\pi_t)}{\pi_t}, \tag{13}$$

which can be rewritten as

$$G(\pi_t, \pi_{t+1}) = S(\pi_{t+1})\pi_t - \theta(\pi_{t+1})S(\pi_t). \tag{14}$$

Here, the variable π_{t+1} denotes the Central Bank's forecast of the rate of inflation between periods t and $t+1$. Assuming that the Central Bank has perfect foresight, the inflation forecast π_{t+1} coincides with the rate of inflation between t and $t+1$.

By the implicit function theorem, we differentiate (14) as

$$\frac{\partial \pi_{t+1}}{\partial \pi_t} = \frac{\theta(\pi_{t+1})S'(\pi_t) - S(\pi_{t+1})}{S'(\pi_{t+1})\pi_t - \theta'(\pi_{t+1})S(\pi_t)}.$$

The following proposition establishes conditions for local determinacy in the forward-looking case:

PROPOSITION 4. *The monetary steady state π^* is locally determinate if* $-1 - 2\varepsilon_s(\pi^*) < \theta'(\pi^*) < 1$.

We can provide a simple economic intuition for such a forward-looking inflation targeting policy: If the inflation forecast for period $t+1$ is high, an aggressive policy will result in a tight money supply in period $t+1$. Households born in generation t will therefore rationally expect to receive a low transfer in their old age. This reinforces their desire to transfer wealth from period t to period $t+1$ and, therefore, makes indeterminacy more likely to occur. A moderate monetary policy, on the other hand, reduces the desire to transfer wealth from young age to old age. Thus, in this case, fluctuations and indeterminacy of the monetary steady state are less likely to occur.

Then, specifying the rule in equation (9) for the forward-looking case, the map expressing the dynamics of the inflation rate reads as

$$\pi_{t+1} = g(\pi_t) = \frac{[(\delta+1)\pi^* - \pi^a]\pi_t + (1-\delta)\pi^a\pi^*}{(\delta-1)\pi_t + \delta\pi^a}. \tag{15}$$

The following proposition provides analytical conditions for local determinacy of the monetary equilibrium[9] π^*.

COROLLARY 5. *The monetary equilibrium is locally determinate if* $\delta_L < \delta < 1$, *where* $\delta_L = \frac{\pi^a+\pi^*}{\pi^*-\pi^a}$.

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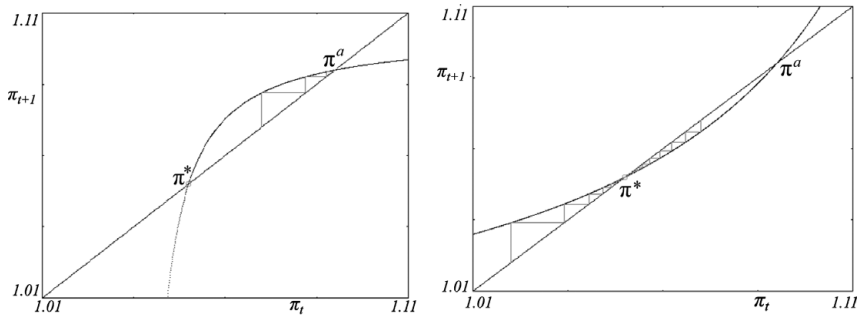


FIGURE 2. Perfect foresight dynamics with forward-looking money-growth rule. Panel (a) reports a configuration in which the monetary steady state is locally determinate under moderate monetary policy in the *leaning against the wind* regime, while panel (b) shows that an aggressive monetary policy in the *leaning with the wind* regime may turn the target equilibrium indeterminate with an infinite number of equilibrium trajectories originating arbitrarily closed to that steady state and converging to it. Parameter values are as follows: $\pi^* = 1.05$ and $\pi^a = 1.09$.

From the analysis of Figure 2, one can observe the indeterminacy problem under an aggressive monetary policy rule. There exist infinitely many trajectories originating close to the target equilibrium that converge to it, determining a close link between indeterminacy of the monetary steady state and the susceptibility of the economy to extrinsic uncertainty. This shows that aggressive forward-looking monetary policy renders the economy more susceptible to indeterminacy than moderate inflation forecast targeting rules.

3.3. Backward-Looking Money-Growth Rule

It is often argued that backward-looking monetary policy rules tend to avoid the possible indeterminacy of equilibrium [see, e.g., Benhabib et al. (2001a, 2003)]. Hence, we consider a backward-looking configuration for the money-growth rule and accordingly, the OLG model reduces to

$$S(\pi_{t+1}) = \theta(\pi_{t-1})\frac{S(\pi_t)}{\pi_t}. \tag{16}$$

In this configuration, the monetary authority employs the last observed price change, π_{t-1}, occurred between time $t-2$ and time $t-1$ as a reference to set the policy.

Since $S(\cdot)$ is assumed to be monotonic, it is possible to retrieve π_{t+1} as

$$\pi_{t+1} = S^{-1}\left[\frac{\theta(\pi_{t-1})S(\pi_t)}{\pi_t}\right].$$

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The map can be reformulated as a bi-dimensional system which takes the form

$$\begin{cases} \pi_{t+1} = h(\pi_t, z_t) \\ z_{t+1} = \pi_t \end{cases},$$

whose Jacobian matrix is given by

$$J = \begin{bmatrix} \dfrac{\partial h}{\partial \pi_t} & \dfrac{\partial h}{\partial z_t} \\ 1 & 0 \end{bmatrix}.$$

The following proposition establishes conditions for local determinacy in the backward-looking case:

PROPOSITION 6. *The monetary equilibrium π^* is locally determinate if the condition $\theta'(\pi^*) < 1$ or $\theta'(\pi^*) > \varepsilon_s(\pi^*)$ holds, whereas it is indeterminate if $1 < \theta'(\pi^*) < \varepsilon_s(\pi^*)$.*

The previous result shows how an *aggressive* monetary policy cannot always lead to a locally determinate monetary equilibrium but globally indeterminacy may also arise. The possible occurrence of a Neimark–Sacker bifurcation suggests that a backward-looking policy rule may be associated also to the existence of endogenous fluctuations in the inflation rate dynamics and global indeterminacy.

The intuition for the possible occurrence of cycles can be the following: Assume that the last observed price change is above its steady-state value leading savings $S(\pi_{t+1})$ to drop below their steady-state value $S(\pi^*)$. Under an *aggressive* monetary policy, from the money market equilibrium in (16), we have that for $\pi_{t+1} > \pi^*$, we must have $\pi_t < \pi^*$, and thus cycles and fluctuations may arise; under a *moderate* monetary policy, if $\pi_{t+1} > \pi^*$, also π_t will increase above π^* and no cycle occurs. In particular, note that the equilibrium dynamics in (16) is an implicitly defined second-order difference equation. Hence, if all eigenvalues of the linearization of (16) are in absolute value greater than 1, then the monetary steady state is determinate. If the Central Bank starts using the backward-looking policy rule in period t, then the monetary steady state will not be reached unless the past inflation rate coincides with the steady-state value π^*. If one eigenvalue of the linearization of (16) is in absolute value smaller than 1, whereas the other one is larger, then the monetary steady state is determinate and stable. This means that starting from any past value π_{t-1} close to π^*, there exists a unique equilibrium that remains close to π^* and converges to the monetary steady state. Finally, if both eigenvalues of the linearization of (16) are in absolute value smaller than 1 then, from any initial value π_{t-1} close to π^*, there exist infinitely many trajectories that converge to the monetary steady state π^*. The monetary steady state is therefore indeterminate.

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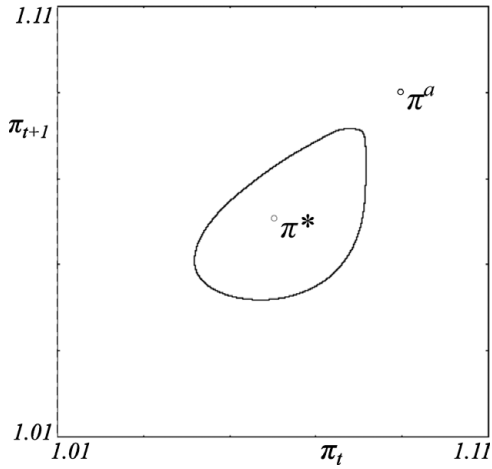


FIGURE 3. Perfect foresight dynamics with backward-looking money-growth rule. If with an aggressive monetary policy, the target equilibrium π^* is locally the unique monetary equilibrium, then there exist equilibrium trajectories originating arbitrarily close to the monetary steady state that converge either to a closed invariant curve or to the autarkic equilibrium. Parameter values are as follows: $\pi^* = 1.05$, $\pi^a = 1.09$, and $\delta = 18.5$.

Finally, according to the money-growth rule specified in equation (9), the map under scrutiny takes the following form:

$$\begin{cases} z_{t+1} = \pi_t \\ \pi_{t+1} = \pi^a - [\pi^* + \delta(z_t - \pi^*)]\left(\dfrac{\pi^a}{\pi_t} - 1\right) \end{cases}. \tag{17}$$

The determinacy condition for the monetary equilibrium is provided in the following

COROLLARY 7. *The monetary equilibrium is determinate if* $\delta < 1$. *Furthermore, if* $\delta > \tilde{\delta}$, *trajectories originating in a neighborhood of* π^* *may settle down to an attracting closed invariant curve or to the autarkic steady state, where* $\tilde{\delta} = \frac{\pi^*}{\pi^a - \pi^*}$.

Figure 3 illustrates the existence of a closed invariant curve surrounding the unstable monetary equilibrium, which is the unique perfect foresight equilibrium from a local viewpoint. The closed invariant curve appears via a supercritical Neimark–Sacker bifurcation as δ crosses the bifurcation value $\tilde{\delta}$. Formally, a Neimark–Sacker bifurcation occurs when the fixed point changes stability when a pair of complex eigenvalues exit the unit circle. In a nonlinear discrete map, endogenous fluctuations naturally appear when a fixed point is destabilized through a supercritical Neimark–Sacker bifurcation: A stable focus becomes unstable and

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an attracting closed curve appears around it, becoming wider and wider when the parameters move away from the bifurcation value.

Interestingly along the closed invariant curve, the inflation rate exhibits endogenous fluctuations around the monetary equilibrium at which the monetary policy is aggressive. Such a behavior is not inconsistent with the observation that the inflation rate fluctuates for long[10] in a region in which the policy is aggressive, as it has been argued for Zimbabwe during the early 2000s (see OECD African Economic Outlook[11]).

The main result that arises from this example is that, with an aggressive monetary policy, there exists an infinite number of equilibrium trajectories originating arbitrarily close to the monetary steady state that may converge to the autarkic steady state or to a closed invariant curve. Indeed, as already stated in the previous cases, if one restricts the analysis to an arbitrarily small neighborhood of the monetary steady state, then the unique perfect foresight equilibrium is the monetary equilibrium itself. Thus, the picture that arises from a local analysis might wrongly lead to the conclusion that an aggressive monetary policy is stabilizing when in fact it is not.

4. CONCLUDING REMARKS

In recent years, many prominent Central Banks have adopted inflation based rules as their preferred framework for the conduct of monetary policy. Standard analysis (meaning, local) would recommend to adopt a policy rule that eliminates the possibility of (local) sunspots around the monetary equilibrium. In other words, it is required that such an equilibrium is dynamically unstable. Since the analysis is based on a linearized model, this guarantees that any path of inflation starting slightly off that steady state would generate a nonstationary explosive or implosive inflationary path. The problem is that this logic is faulty if the economy possesses more steady states, as Obstfeld and Rogoff (1984, 1986) and Benhabib et al. (2001b) shown. If the other steady state (in this case, the autarkic steady state) is dynamically stable (or, locally indeterminate), we might have an equilibrium trajectory that starts off the dynamically unstable monetary equilibrium and move toward the nonmonetary one. If instead both steady states are dynamically unstable, we might have the occurrence of stable cycles, around the monetary equilibrium.

The purpose of the present work was to study how monetary policy rules affect the determinacy of the monetary equilibrium in a standard OLG model with perfect foresight and under which conditions the dynamics of the model is affected by the existence of a nonmonetary steady state. We have considered different timings for the policy rule, namely we analyzed rules that respond to current, past, or expected level of inflation. We show analytical conditions under which indeterminacy arises and a number of examples confirming that the different feedback rules adopted may lead to different outcomes in terms of determinacy of the monetary steady state. Our main findings are that an aggressive monetary policy rule reinforces mechanism that lead to indeterminacy of the monetary steady state, especially

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when the policy is designed as forward-looking, whereas a backward-looking policy may render determinacy more likely. Furthermore, a *leaning against the wind* policy seems to be more desirable from an equilibrium determinacy point of view. On the contrary, a *leaning with the wind* policy could not be a recommended policy for the Central Bank. We also confirm that restricting the evaluation of the monetary policy rules to their local stability properties can be misleading: An active monetary policy stance may appear to be stabilizing because it ensures the uniqueness of the equilibrium locally, when in fact it is destabilizing, giving rise to endogenous fluctuations as well.

NOTES

1. The work of Clarida and Gertler (1997) reports empirical evidence about the strategy of *leaning against the wind* pursued by the Bundesbank during the 80's and 90's in the sense that it tightens monetary policy if inflation (and output) are rising.

2. Notice that this framework has been used also in Bullard (1994), Marimon and Sunder (1995), and Schönhofer (2001) without an explicit assumption on the sign of real economic growth.

3. www.zimtreasury.gov.zw

4. The proof of this result can be easily derived from the results in appendix and can also be found in Tuinstra (2003).

5. We thank an anonymous referee for helpful suggestions.

6. The local determinacy conditions for the autarkic steady state π^a are provided in the appendix since we are mainly interested in the monetary equilibrium determinacy properties.

7. It is noteworthy that as the parameter δ is further increased, the cycle of period two undergoes a period doubling bifurcation which starts a Feigenbaum cascade of period doubling bifurcations [see Mira et al. (1998)] leading to the chaotic behavior. At the accumulation point, δ_∞ (that is the Feigenbaum constant) of the sequence of the period doubling bifurcation values infinitely many cycles exist (all of even period).

8. Countries with explicit inflation targets (for example, Canada, New Zealand, and the United Kingdom) all base their policy actions on inflation forecasts.

9. As before, the local determinacy conditions for the autarkic steady state π^a are provided in the appendix since we are mainly interested in the monetary equilibrium determinacy properties.

10. This behavior has also been observed for the US monetary policy since the Volcker era [see Clarida and Gertler (1997), Benhabib et al. (2001b)].

11. www.oecd.org/countries/zimbabwe/

12. We recall that a periodic orbit of period k for a map $x_{i+1} = g(x_i)$, $x \in \mathbb{R}^n$, $n \geq 1$ is a set of k distinct points $\{p_j = g^j(p_0) | j = 0, \ldots, k-1\}$ with $g^k(p_0) = p_0$ (Guckenheimer and Holmes, 2013). Here g^k represents the composition of g with itself k times. The smallest positive value of k for which this equality holds is the period of the orbit.

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APPENDIX

PROOF OF PROPOSITION 1

The steady-state condition implies $S(\pi)\,[\pi - \theta(\pi)] = 0$. From the previous expression, it can be seen that the steady states of the map are such that $S(\pi) = 0$, [if exist, the solution of this equation is called *autarkic* steady state ($\pi = \pi^a$)] while the other (*monetary*) equilibria are found as solutions of $\pi = \theta(\pi)$, depending on the properties of function $\theta(\pi)$. Since we have made the assumption $\theta(\pi^*) = \pi^*$, then the map always features the target equilibrium $\pi = \pi^*$. ■

PROOF OF PROPOSITION 2

Linearizing equation (11) around the monetary steady state $\pi = \pi^*$, we get

$$S'(\pi^*)\,d\pi_{t+1} = \frac{\left[\theta'(\pi^*)S(\pi^*) + \theta(\pi^*)S'(\pi^*)\right]\pi^* - \theta(\pi^*)S(\pi^*)}{(\pi^*)^2} d\pi_t,$$

which can be rewritten as

$$d\pi_{t+1} = \left[-\frac{\theta'(\pi^*)}{\varepsilon_s(\pi^*)} + 1 + \frac{1}{\varepsilon_s(\pi^*)}\right] d\pi_t$$

since in equilibrium $\theta(\pi^*) = \pi^*$. The monetary steady state is locally determinate if

$$\left|-\frac{\theta'(\pi^*)}{\varepsilon_s(\pi^*)} + 1 + \frac{1}{\varepsilon_s(\pi^*)}\right| > 1. \tag{A.1}$$

Since $\varepsilon_s(\pi^*) > 0$, it is straightforward to see that the target steady state is locally determinate if $\theta'(\pi^*) < 1$. Furthermore, for local determinacy, we have to check when

$$-\frac{\theta'(\pi^*)}{\varepsilon_s(\pi^*)} + 1 + \frac{1}{\varepsilon_s(\pi^*)} < -1,$$

i.e., when $-\theta'(\pi^*) + 2\varepsilon_s(\pi^*) + 1 < 0$ which is satisfied for $\theta'(\pi^*) > 2\varepsilon_s(\pi^*) + 1$.

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In summary, the target steady state is locally determinate if $\theta'(\pi^*) < 1$ or if $\theta'(\pi^*) > 2\varepsilon_s(\pi^*) + 1$.

For sake of completeness, we report the results about the determinacy of the autarkic steady state. The autarkic steady state π^a is locally determinate if the condition $\theta'(\pi^*) > 1$ holds, whereas it is indeterminate if $\theta'(\pi^*) < 1$. In fact, we have that $S(\pi^a) = 0$ and so $\frac{1}{\varepsilon_s(\pi^a)} = 0$. Therefore, linearizing equation (11) around π^a, the determinacy of the autarkic equilibrium requires

$$\left|\frac{\theta(\pi^a)}{\pi^a}\right| > 1,$$

which is satisfied if and only if the condition $\theta'(\pi^*) > 1$ holds. Otherwise there would be an intersection point between the function $\theta(\pi^a)$ and the 45° line below the saving function, implying $\theta(\pi^a) < \pi^a$ and thus indeterminacy for the autarkic equilibrium. ■

PROOF OF COROLLARY 3

The determinacy condition for the monetary equilibrium π^* reads as

$$\left|f'(\pi^*)\right| = \left|\frac{\pi^a\pi^* - \pi^*\pi^a\delta + (\pi^*)^2\delta}{(\pi^*)^2}\right| > 1.$$

First, we check when $f'(\pi^*) > 1$, i.e., when $\frac{(\pi^a-\pi^*)(1-\delta)}{\pi^*} > 0$. Since we are assuming $\pi^a > \pi^*$, the condition is satisfied when $\delta < 1$. Then, we check when $f'(\pi^*) < -1$, i.e., when $\frac{\delta\pi^*(\pi^*-\pi^a)+\pi^*(\pi^*+\pi^a)}{(\pi^*)^2} < 0$ which is satisfied for $\delta > \frac{\pi^a+\pi^*}{\pi^a-\pi^*}$ if $\delta > 0$ while, when $\delta < 0$, the monetary equilibrium is always determinate. As $\frac{\pi^a+\pi^*}{\pi^a-\pi^*} > 1$, the target equilibrium is locally determinate for $\delta < 1$ or $\delta > \frac{\pi^a+\pi^*}{\pi^a-\pi^*}$.

When $f'(\pi^*) = -1$, occurring at $\delta = \bar{\delta} = \frac{\pi^a+\pi^*}{\pi^a-\pi^*}$, the monetary equilibrium loses stability via period doubling bifurcation and, immediately after this values, a stable orbit of period two arises around the unstable equilibrium.[12]

For sake of completeness, we report the results about the determinacy of the autarkic steady state π^a. The steady state π^a is determinate if $\delta < \hat{\delta}$ or $\delta > 1$, where $\hat{\delta} = \frac{\pi^a+\pi^*}{\pi^*-\pi^a}$.

In fact, the determinacy condition requires that

$$\left|f'(\pi^a)\right| = \left|\frac{\pi^*\pi^a - \pi^*\pi^a\delta + (\pi^a)^2\delta}{(\pi^a)^2}\right| > 1.$$

First, we check when $f'(\pi^a) > 1$, i.e., when $\frac{[\pi^a(\pi^*-\pi^a)(1-\delta)]}{(\pi^a)^2} > 0$. Since we are assuming $\pi^a > \pi^*$, the condition is satisfied when $\delta < 1$.

Then, we check when $f'(\pi^a) < -1$, i.e., when $\frac{\delta\pi^a(\pi^a-\pi^*)+\pi^a(\pi^a+\pi^*)}{(\pi^a)^2} < 0$ which is always satisfied when $\delta < 0$ and, when $\delta > 0$, the condition is satisfied for $\delta < \frac{\pi^a+\pi^*}{\pi^*-\pi^a}$. As $\frac{\pi^a+\pi^*}{\pi^*-\pi^a} < 0$, the autarkic steady state is locally determinate for $\delta < \frac{\pi^a+\pi^*}{\pi^*-\pi^a}$ or $\delta > 1$. ■

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PROOF OF PROPOSITION 4

The derivative of (14) evaluated at the monetary equilibrium π^* reduces to

$$\left(\frac{\partial \pi_{t+1}}{\partial \pi_t}\right)_{\pi=\pi^*} = \frac{\pi^* S'(\pi^*) - S(\pi^*)}{S'(\pi^*)\pi^* - \theta'(\pi^*)S(\pi^*)} = \frac{1+\varepsilon_s(\pi^*)}{\theta'(\pi^*)+\varepsilon_s(\pi^*)}.$$

The condition for local determinacy requires that

$$\left|\frac{1+\varepsilon_s(\pi^*)}{\theta'(\pi^*)+\varepsilon_s(\pi^*)}\right| > 1.$$

The solution reads as $-1-2\varepsilon_s(\pi^*) < \theta'(\pi^*) < 1$ which proves the result and concludes the proof.

Moreover, the autarkic steady state π^a is locally determinate if $\theta(\pi^a) < \pi^a$. In fact, the determinacy condition reads as

$$\left|\frac{\theta(\pi^a)}{\pi^a}\right| > 1$$

that is when $\theta(\pi^a) > \pi^a$. Such a condition is only satisfied when $\theta'(\pi_t) > 0$ (*leaning with the wind regime*). Therefore, the autarkic equilibrium is always indeterminate in the *leaning against the wind* regime. ■

PROOF OF COROLLARY 5

The determinacy condition for the monetary equilibrium π^* reads as

$$\left|f'(\pi^*)\right| = \left|\frac{\pi^a}{\pi^* + \delta(\pi^a - \pi^*)}\right| > 1.$$

Straightforward computations show that the condition for determinacy is satisfied when $\frac{\pi^a+\pi^*}{\pi^*-\pi^a} < \delta < 1$.

For sake of completeness, we report the results about the determinacy of the autarkic steady state π^a. The steady state π^a is determinate if $\delta < \frac{\pi^a+\pi^*}{\pi^*-\pi^a}$ or $\delta > 1$.

In fact, the determinacy condition requires that

$$\left|f'(\pi^a)\right| = \left|\frac{\delta(\pi^a - \pi^*) + \pi^*}{\pi^a}\right| > 1.$$

Simple algebra leads to $\delta < \frac{\pi^a+\pi^*}{\pi^*-\pi^a}$ or $\delta > 1$, providing the determinacy condition for the autarkic steady state. ■

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PROOF OF PROPOSITION 6

The Jacobian matrix evaluated at the target steady state $\pi = \pi^*$ is

$$J = \left[\begin{array}{cc} 1 + \dfrac{1}{\varepsilon_s(\pi^*)} & -\dfrac{\theta'(\pi^*)}{\varepsilon_s(\pi^*)} \\ 1 & 0 \end{array} \right]_{\pi=\pi^*}.$$

In order to determine the location of the eigenvalues, using the triangle of stability [see Medio and Lines (2001)], we compute the trace and the determinant of matrix J:

$$\text{tr } J = 1 + \frac{1}{\varepsilon_s(\pi^*)}$$

and

$$\det J = \frac{\theta'(\pi^*)}{\varepsilon_s(\pi^*)}.$$

The triangle of stability is given by

$$\left\{ \begin{array}{c} 1 - \text{tr } J + \det J = 1 - \left[1 + \dfrac{1}{\varepsilon_s(\pi^*)}\right] + \dfrac{\theta'(\pi^*)}{\varepsilon_s(\pi^*)} > 0 \\ 1 + \text{tr } J + \det J = 1 + \left[1 + \dfrac{1}{\varepsilon_s(\pi^*)}\right] + \dfrac{\theta'(\pi^*)}{\varepsilon_s(\pi^*)} > 0 \\ \det J = \dfrac{\theta'(\pi^*)}{\varepsilon_s(\pi^*)} < 1 \end{array} \right. .$$

The first condition is satisfied when $\theta'(\pi^*) > 1$, the second when $\theta'(\pi^*) > -1 - 2\varepsilon_s(\pi^*)$, and the third when $\theta'(\pi^*) < \varepsilon_s(\pi^*)$.

From the previous conditions, it follows that the monetary equilibrium π^* is locally determinate if $\theta'(\pi^*) < 1$ or $\theta'(\pi^*) > \varepsilon_s(\pi^*)$ holds, otherwise it is indeterminate.

For sake of completeness, we also provide the determinacy condition for the autarkic steady state. π^a is locally determinate if $\theta(\pi^a) < \pi^a$. Indeed, evaluating matrix J at $\pi = \pi^a$, we get

$$J = \left[\begin{array}{cc} \dfrac{\theta(\pi^a)}{\pi^a} & 0 \\ 1 & 0 \end{array} \right]_{\pi=\pi^a}$$

with eigenvalues 0 and $\frac{\theta(\pi^a)}{\pi^a}$. The local determinacy condition of the autarkic steady state π^a immediately follows from:

$$\left| \frac{\theta(\pi^a)}{\pi^a} \right| > 1.$$

■

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PROOF OF COROLLARY 7

The Jacobian matrix evaluated at the monetary steady state π^* is

$$J = \left[\begin{array}{cc} \dfrac{\pi^a}{\pi^*} & \dfrac{\delta(\pi^* - \pi^a)}{\pi^*} \\ 1 & 0 \end{array} \right]_{\pi=\pi^*}.$$

The location of the eigenvalues of the Jacobian matrix J evaluated at π^* is obtained using the triangle of stability which considers the trace and the determinant of J:

$$\text{tr}\,(J) = \frac{\pi^a}{\pi^*}$$

and

$$\det(J) = \frac{\delta(\pi^a - \pi^*)}{\pi^*}.$$

The triangle of stability is given by

$$\left\{ \begin{array}{c} 1 - \text{tr}\, J + \det J = \dfrac{(\delta - 1)(\pi^a - \pi^*)}{\pi^*} > 0 \\ 1 + \text{tr}\, J + \det J = \dfrac{(\delta + 1)\pi^a - (\delta - 1)\pi^*}{\pi^*} > 0 \\ \det J = \dfrac{\delta(\pi^a - \pi^*)}{\pi^*} < 1 \end{array} \right. .$$

The first condition is satisfied for $\delta > 1$. The second condition is always satisfied if $\delta > 0$, whereas if $\delta < 0$ then $\delta > \frac{\pi^* + \pi^a}{\pi^* - \pi^a}$, and finally the third for $\delta < \frac{\pi^*}{\pi^a - \pi^*}$. The determinacy condition easily follows.

As before, we provide also the determinacy condition for the autarkic steady state. The steady state π^a is determinate if $\delta > 1$.

In fact, the Jacobian matrix evaluated at the autarkic steady state π^a reads as

$$J = \left\{ \begin{array}{cc} \dfrac{1}{\pi^a}\left[\delta(\pi^a - \pi^*) + \pi^*\right] & 0 \\ 1 & 0 \end{array} \right\}_{\pi=\pi^a}.$$

It is straightforward to see that the two eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = \frac{1}{\pi^a}\left[\delta(\pi^a - \pi^*) + \pi^*\right]$. Then, the autarkic steady state π^a is determined if $\delta > 1$, whereas it is indeterminate if $\delta < 1$ provided that $\frac{\pi^a - \pi^*}{\pi^a} > 0$ (which is always satisfied under the Samuelson case). ■

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