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Affine interval exchange transformations with wandering intervals

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Abstract. There exist uniquely ergodic bijective affine interval exchange transformations of [0,1) having wandering intervals and such that the support of the invariant measure is a Cantor set.

1. *Introduction*

Let $I = [a, b) \subset \mathbb{R}$ be a finite left-closed right-open interval. A bijective map $T : [a, b) \to [a, b)$ is said to be an *affine interval exchange transformation* (AIET) if there is a finite sequence $a = y_0 < y_1 < y_2 < \cdots < y_{m-1} < y_m = b$ such that for all $i \in \{0, 1, \ldots, m-1\}$, $T|_{[y_i, y_{i+1})}$ is (continuously) differentiable and its derivative is identically equal to a constant, say $\beta_i > 0$. When T is discontinuous exactly at the points $y_1, y_2, \ldots, y_{m-1}$, we shall refer to

$$(y, \tilde{\gamma}) = ((y_0, y_1, \ldots, y_m), (\gamma_1, \gamma_2, \ldots, \gamma_m)),$$

where $\gamma_i = \log \beta_i$, as the *pair of vectors associated to* T.

We say that $E : I \to I$ is an *interval exchange transformation* (IET) with associated vector $x = (x_0, x_1, \ldots, x_m)$ if E is an AIET with associated pair of vectors $(x, \tilde{\gamma})$, such that $\tilde{\gamma}$ is the zero vector. Let $I_1 = [c, d)$ be a proper subinterval of I. We say that the IET E is *renormalizable (on* I_1*)* if there exists an orientation-preserving affine map $L : \mathbb{R} \to \mathbb{R}$ such that $L(I_1) = I$ and $L \circ E_1 \circ L^{-1} = E$, where $E_1 : I_1 \to I_1$ denotes the IET induced by E.

Let N be a finite sub-interval of either $\mathbb{R}$ or the circle S^1. Let $f : N \to N$ be piecewise continuous and $J \subset N$ be an interval. We say that J is a wandering interval of the map f if: (i) $J, f(J), \ldots,$ are pairwise disjoint intervals and (ii) the ω-limit set of J is an infinite set.

The results of this paper are stated in next section. We prove the existence of uniquely ergodic (bijective) affine interval exchange transformations of $[0, 1)$ having densely

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distributed wandering intervals and such that the support of the invariant measure is a Cantor set. We study AIETs semi-conjugate to renormalizable IETs. These maps are uniquely ergodic [**Vee**], [**Mas**]. We show that in many cases, the second vector associated to the AIET determines the absence of wandering intervals. In this respect, Levitt showed the existence of non-uniquely ergodic AIETs having wandering intervals [**Lev**].

This work was motivated by Denjoy's theorem: if f is a C^1-diffeomorphism of the circle, such that the logarithm of its derivative is a function of bounded variation, then f has no wandering intervals. As IETs preserve Lebesgue measure, they cannot have wandering intervals. In this way the supposition that a uniquely ergodic AIET should not have wandering intervals seemed to be correct (see [**Lev**] and [**CLR**]). Other works that reinforced this supposition are those of Guckenheimer [**Guc**], Yoccoz [**Yoc**] and Blokh and Lyubich [**BL**]. These results were generalized by the following theorem of de Melo *et al.* [**MMS**]: if N (as above) is compact and $f : N \to N$ is a C^2-map with non-flat critical points then f has no wandering intervals. (For a more detailed exposition of this, we suggest [**MS**, Chapter IV].)

2. *Statement of the results*

Let $I = [a, b) \subset \mathbb{R}$ be a finite left-closed right-open interval. Let $E : I \to I$ be an IET with associated vector $x = (x_0, x_1, \ldots, x_m)$. Denote by $I_1 = [c, d)$ a proper subinterval of I. Suppose that E is renormalizable (on I_1); that is, there exists an orientation-preserving affine map $L : \mathbb{R} \to \mathbb{R}$ such that $L(I_1) = I$ and $L \circ E_1 \circ L^{-1} = E$, where $E_1 : I_1 \to I_1$ denotes the IET induced by E. Given $i = 0, 1, \ldots, m$, let $z_i = L^{-1}(x_i)$. In this way, the sequence of discontinuities of E_1 is $\{z_1, \ldots, z_{m-1}\}$.

Let $\mathcal{A}$ be a $m \times m$ non-negative matrix whose entries are $\mathcal{A}_{ji} = \#\{0 \le k \le N_i : E^k([z_{i-1}, z_i)) \subset [x_{j-1}, x_j)\}$, where N_i is the least possible non-negative integer such that for some $z \in [z_{i-1}, z_i)$ (and therefore for all $z \in [z_{i-1}, z_i)$), $E^{N_i+1}z \in I_1$. We shall refer to $\mathcal{A}$ as the *matrix associated to* (E, I_1).

Let Λ_m be the cone of positive vectors in $\mathbb{R}^m$. It can be seen that the matrix $\mathcal{A}$ associated to (E, I_1) is aperiodic. Hence, by the Perron–Frobenius theorem [**Gan**], $\mathcal{A}$ possesses exactly one probability right eigenvector $\alpha \in \Lambda_m$; moreover, the eigenvalue μ corresponding to α is simple, real and greater than 1 and, in addition, all other eigenvalues of $\mathcal{A}$ have absolute value less than μ. It was proved by Veech [**Vee**] and Masur [**Mas**] that E (as renormalizable) is uniquely ergodic (the only E-invariant measure is that of Lebesgue). Moreover, following Rauzy [**Rau**], we conclude that

$$\alpha = (x_1 - x_0, x_2 - x_1, \ldots, x_m - x_{m-1}).$$

We say that an AIET $T : [0, 1) \to [0, 1)$ is *semi-conjugate* (respectively *conjugate*) to an IET E if there exists a non-decreasing surjective (respectively bijective) continuous map $h : [0, 1) \to [0, 1)$ such that $E \circ h = h \circ T$.

The first result of this paper is the following.

THEOREM 2.1. *Let $E : [0, 1) \to [0, 1)$ be a renormalizable IET with exactly $m - 1$ discontinuities and associated $m \times m$ matrix $\mathcal{A}$. Let $\mathcal{A}^*$ be the adjoint operator of $\mathcal{A}$. Let $\mathcal{G}^s$ be the $\mathcal{A}^*$-invariant stable subspace of $\mathbb{R}^m$ and let $\mathcal{G}^*$ be the $(m-1)$-dimensional*

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$\mathcal{A}^*$-invariant subspace of $\mathbb{R}^m$ such that the spectral radius of $\mathcal{A}^*|_{\mathcal{G}^*}$ is less than the spectral radius of $\mathcal{A}^*$.

(a) *Let $\tilde{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathbb{R}^m$. There exists an AIET $T : [0, 1) \to [0, 1)$ semi-conjugate to E and with associated pair of vectors $(y, \tilde{\gamma})$ if and only if $\tilde{\gamma} \in \mathcal{G}^*$.*

(b) *Let $\tilde{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathcal{G}^s$. Let $T : [0, 1) \to [0, 1)$ be an AIET semi-conjugate to E and with associated pair of vectors $(y, \tilde{\gamma})$. Then T is conjugate to E.*

To state our second result, we shall restrict to a particular IET. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \Lambda_4$ be the probability (i.e. each $\alpha_i > 0$ and $|\alpha| = \alpha_1+\alpha_2+\alpha_3+\alpha_4 = 1$) Perron–Frobenius right eigenvector of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 2 & 2 \end{pmatrix}.$$

We consider Rauzy's IET $R : [0, 1) \to [0, 1)$ [**Rau**] which is determined by the following conditions: R has associated vector $x = (x_0, x_1, x_2, x_3, x_4)$, where

$$x_0 = 0, \quad x_1 = \alpha_1, \quad x_2 = \alpha_1 + \alpha_2, \quad x_3 = \alpha_1 + \alpha_2 + \alpha_3, \quad x_4 = 1,$$

and $R(0) = R(x_0) > R(x_2) > R(x_3) > R(x_1) = 0$.

The map R is renormalizable on the interval $[0, \alpha_1)$ and A is precisely the matrix associated to $(R, [0, \alpha_1))$ [**Rau**]. It is numerically seen that $\alpha_1 < \alpha_4$ and so R is also induced on the interval $R([0, \alpha_1))$ and A is also the matrix associated to $(R, R([0, \alpha_1)))$.

The characteristic polynomial of A is the reciprocal polynomial

$$x^4 - 7x^3 + 13x^2 - 7x + 1$$

with four real roots $\rho^{-1}, \lambda^{-1}, \lambda, \rho$ satisfying

$$0 < \rho^{-1} < \lambda^{-1} < 1 < \lambda < \rho.$$

Let A^* be the adjoint operator of A and let G^* (respectively G^s) be the vector 3-space (respectively vector 2-space) A^*-invariant generated by the eigenvectors corresponding to the eigenvalues $\rho^{-1}, \lambda^{-1}, \lambda$ (respectively ρ^{-1}, λ^{-1}).

THEOREM 2.2. *One of the connected components, say C, of $G^* \setminus G^s$ is such that, for all $\tilde{\gamma} = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \in C$, there exists a uniquely ergodic AIET $T : [0, 1) \to [0, 1)$ with associated pair of vectors $(y, \hat{\gamma})$ of the form $y = (y_0, y_1, y_2, y_3, y_4)$ and $\hat{\gamma} = (\delta, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$, satisfying the following*

(a) *$[y_0, y_1) = [0, y_1)$ is a wandering interval of T and, for all $n \in \mathbb{Z}$, $T^n([0, y_1))$ is an interval,*

(b) *If $\theta = \cup_{n\in\mathbb{Z}} T^n([0, y_1))$, there exists a continuous bijective map*

$$h : [0, 1) \setminus \theta \to [0, 1)$$

such that, for all $y \in [0, 1) \setminus \theta$, $R \circ h(y) = h \circ T(y)$.

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3. *Proof of Theorem 2.1(a)*

We shall assume the notation of Theorem 2.1, the proof of which is shown in Lemmas 3.3, 3.4 and 3.6 below.

A proof of the following easy lemma can be found, for instance, in [**Gu2**, Proof of Proposition 1].

LEMMA 3.1. *Let $E : [0, 1) \to [0, 1)$ be an IET with every orbit dense.*

(a) *Given $n \in \mathbb{N}$, there exists $\delta > 0$ such that if $J \subset [0, 1)$ is an interval of length less than δ, then for all $x \in [0, 1)$,*

$$\#(\{x, Ex, \ldots, E^n x\} \cap J) \leq 1.$$

(b) *Given an interval $J \subset [0, 1)$, there exists $K = K(J) \in \mathbb{N}$ such that for all $x \in [0, 1)$,*

$$\#(\{x, Ex, \ldots, E^K x\} \cap J) \geq 1.$$

LEMMA 3.2. *Let $E : [0, 1) \to [0, 1)$ be a uniquely ergodic IET with every orbit dense. Let λ be the Lebesgue measure on $[0, 1)$ and let J be a subinterval of $[0, 1)$. Given $\epsilon > 0$, there exists $N \in \mathbb{N}^*$ such that for all $x \in [0, 1)$ and for all $n > N$,*

$$|\tau(x, J, n) - \lambda(J)| < \epsilon,$$

where

$$\tau(x, J, n) = \frac{\#\{j : E^j x \in J, 0 \leq j \leq n\}}{n + 1}.$$

Proof. Let $\tilde{E} : [0, 1] \to [0, 1]$ be a uniquely ergodic extension of E. Denote the Lebesgue measure on $[0, 1]$ by λ. Assume, by contradiction, that there exists $\epsilon > 0$ such that for each $N \in \mathbb{N}^*$ we can find $n = n(N) > N$ and a point $x_n \in E$ such that

$$|\tau(x_n, J, n) - \lambda(J)| > \epsilon.$$

It follows from Lemma 3.1 that the sequence of (Borel) probabilities

$$\mu_n = \frac{1}{n + 1} \sum_{j=0}^{n} \delta_{E^j(x_n)},$$

where δ_x is the Dirac probability on $[0, 1]$ concentrated at x, has a subsequence which weakly* converges to a probability μ invariant by $\tilde{E}$ and such that

$$|\mu(J) - \lambda(J)| \geq \epsilon.$$

This is the required contradiction because E is uniquely ergodic, μ (restricted to $[0, 1)$) is invariant by E and therefore μ should be equal to λ. □

LEMMA 3.3. *Let T be a uniquely ergodic AIET such that all of its orbits are infinite sets. Let μ be the T-invariant measure and let $(y, \tilde{\gamma})$ be the pair of vectors associated to T, where $y = (y_0, y_1, \ldots, y_m)$ and $\tilde{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_m)$. Then $(\mu_1, \mu_2, \ldots, \mu_m)$ is orthogonal to $\tilde{\gamma}$, where $\mu_i = \mu([y_{i-1}, y_i))$. Here, $\mathbb{R}^m$ is endowed with the usual inner product.*

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Proof. Supposing, by contradiction, that the lemma is false, we shall study only the case in which for all i, $0 < \mu_i$ and

$$0 < \beta_1^{\mu_1} \cdot \beta_2^{\mu_2} \cdots \beta_m^{\mu_m} = c < 1,$$

where $\beta_i = \exp(\gamma_i)$. Let $\epsilon > 0$ and $\delta > 0$ be so small so that $\epsilon < \min\{\mu_1/4, \mu_2/4, \ldots, \mu_m/4\}$ and

$$\beta_1^{\mu_1+\sigma_1\cdot\epsilon} \cdot \beta_2^{\mu_2+\sigma_2\cdot\epsilon} \cdots \beta_m^{\mu_m+\sigma_m\cdot\epsilon} < c + \delta < 1,$$

for all possible choices of $\sigma_i \in \{-1, 1\}$.

By Lemma 3.2 we may choose $N > 0$ such that for all $n > N$, $i = 1, 2, \ldots, m$, and $x \in [0, 1)$,

$$|\tau_i(x, n) - \mu_i| < \epsilon,$$

where $\tau_i(x, n) = \tau(x, [y_{i-1}, y_i), n)$.

Take a subinterval $I \subset [0, 1)$ such that $\mu(I) > 0$ is so small so that if $x \in I$ and $T^j x \in I$, for some $j \geq 1$, then $j \geq N$. Suppose also that the endpoints of I are not endpoints of wandering intervals of T.

Under these circumstances, as T is a bijection, the map $\tilde{T} : I \to I$ induced by T should satisfy $\tilde{T}(I) = I$. Nevertheless, this is not the case because, as we shall see, $\tilde{T}$ is a contraction: this will be the required contradiction. In fact, given $x \in I$, there exists $j = j(x) \geq N$ such that

$$\{Tx, T^2x, \ldots, T^{j-1}x\} \cap I = \emptyset$$

but $T^j x \in I$. Under these conditions, there exists a finite subset S of I such that, if $x \in I \setminus S$ and $j = j(x)$, T^j is (left and right) differentiable at x and

$$((T^j)'(x))^{1/j} = \beta_1^{\tau_1(x,j)} \cdot \beta_2^{\tau_2(x,j)} \cdots \beta_m^{\tau_m(x,j)} \leq c + \delta < 1$$

which implies that

$$(T^j)'(x) \leq (c + \delta)^j < 1$$

and so $\tilde{T}$ is a contraction. □

Let μ be the spectral radius of $\mathcal{A}$. Denote by $\mathcal{F}$ (respectively $\mathcal{F}^*$) the Perron–Frobenius eigenspace associated to the eigenvalue μ of $\mathcal{A}$ (respectively of $\mathcal{A}^*$). Let

$$\mathbb{R}^m = \mathcal{F} + \mathcal{G} \quad \text{and} \quad \mathbb{R}^m = \mathcal{F}^* + \mathcal{G}^*$$

be the Jordan direct sum decompositions invariant by $\mathcal{A}$ and $\mathcal{A}^*$, respectively. In this way, the spectral radius ρ of $\mathcal{A}|_{\mathcal{G}}$ and $\mathcal{A}^*|_{\mathcal{G}^*}$ is the same and $0 < \rho < \mu$.

LEMMA 3.4. *Under the above conditions, the orthogonal complement $\mathcal{F}^\perp$ to $\mathcal{F}$ in the usual inner product $\langle\,,\rangle$ of $\mathbb{R}^m$, is $\mathcal{G}^*$.*

Proof. As $\tilde{\mathcal{A}} = \mu^{-1}\mathcal{A}^*|_{\mathcal{G}^*}$ has spectral radius less than one, we have that $\tilde{\mathcal{A}}$ is a contraction in an appropriate norm of $\mathcal{G}^*$ (see [**MP**, Proposition 2.10]).

In this way, if $w \in \mathcal{G}^*$ and $v \in \mathcal{F}$ we have that

$$0 = \left\langle \lim_n \tilde{\mathcal{A}}^n w, v \right\rangle = \lim_n \frac{1}{\mu^n} \langle (\mathcal{A}^*)^n w, v \rangle = \lim_n \frac{1}{\mu^n} \langle w, \mathcal{A}^n v \rangle = \langle w, v \rangle.$$

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This implies that $\mathcal{G}^* \subset \mathcal{F}^\perp$. As the dimensions of $\mathcal{G}^*$ and $\mathcal{F}^\perp$ are equal, the lemma is proved. □

LEMMA 3.5. *Let $(x_1, x_2, \ldots, x_m)$ be the vector associated to E. Let $\varphi = \{1, 2, \ldots, m\}$ and let $u : [0, 1) \to \varphi$ be given by $u(x) = i$ iff $x \in [x_{i-1}, x_i)$. Let $\gamma : \varphi \to \mathbb{R}$ be a map such that $\tilde{\gamma} = (\gamma(1), \gamma(2), \ldots, \gamma(m)) \in \mathcal{G}^*$. Then there exists $x \in [0, 1)$ such that*

$$\limsup_k \left(\sum_{j=0}^{k} \gamma(u(E^j x)) \right) \geq 0 \geq \liminf_k \left(\sum_{j=0}^{k} \gamma(u(E^j x)) \right).$$

Proof. Recall that $(x_1 - x_0, x_2 - x_1, \ldots, x_m - x_{m-1}) \in \mathcal{F}$. Given $n \in \mathbb{N}$ and $\delta \in \{-1, 1\}$, let $\Delta_n(\delta)$ be the set of $x \in [0, 1)$ such that, for some $k = k(x) > n$, E^k is continuous at x and

$$\delta \left(\sum_{j=0}^{k} \gamma(u(E^j x)) \right) > 0.$$

Suppose that there exist $x \in [0, 1)$, $\epsilon > 0$, $n \in \mathbb{N}$ and $\delta \in \{-1, 1\}$ such that $J = [x, x+\epsilon)$ is disjoint from $\Delta_n(\delta)$. To fix ideas assume that $\delta = 1$.

Let $(y_0, y_1, \ldots, y_\ell)$ be the vector associated to the IET $E_J : J \to J$ induced by E. Given $i \in \{1, 2, \ldots, \ell\}$, let N_i be the least possible non-negative integer such that for some $y \in [y_{i-1}, y_i)$ (and therefore for all $y \in [y_{i-1}, y_i)$), $E^{N_i+1} y \in J$.

We may take $\epsilon > 0$ so small that $\min\{N_i : 1 \leq i \leq \ell\} > n$. This and the assumption that $J \cap \Delta_n(1) = \emptyset$ imply that, for all $i \in \{1, 2, \ldots, \ell\}$,

$$\Gamma_i = \gamma(u(y_{i-1})) + \gamma(u(Ey_{i-1})) + \cdots + \gamma(u(E^{N_i} y_{i-1})) \leq 0.$$

Hence, by rearranging terms in an appropriate way,

$$\begin{aligned} &\sum_{i=1}^{\ell} (y_i - y_{i-1}) \Gamma_i \\ &= \sum_{i=1}^{\ell} (y_i - y_{i-1}) [\gamma(u(y_{i-1})) + \gamma(u(Ey_{i-1})) + \cdots + \gamma(u(E^{N_i} y_{i-1}))] \\ &= \sum_{i=1}^{m} (x_i - x_{i-1}) \gamma(u(x_{i-1})) = 0 \quad \text{(see Lemma 3.4).} \end{aligned}$$

As every $y_i - y_{i-1} > 0$, we conclude that, for all $i \in \{1, 2, \ldots, \ell\}$, $\Gamma_i = 0$. From this it can easily be seen that the lemma is true. So we may proceed assuming that, for all $n \in \mathbb{N}$ and $\delta \in \{-1, 1\}$, $\Delta_n(\delta)$ is dense. As these sets are also open and as $[0, 1)$ is a Baire space, there exists x belonging to $\Delta_n(1) \cap \Delta_n(-1)$ for all $n \in \mathbb{N}$. This point x satisfies the required inequalities of this lemma. □

LEMMA 3.6. *Let $\varphi = \{1, 2, \ldots, m\}$ and let $\gamma : \varphi \to \mathbb{R}$ be a map such that $\tilde{\gamma} = (\gamma(1), \gamma(2), \ldots, \gamma(m)) \in \mathcal{G}^*$. Then there exists an AIET $T : [0, 1) \to [0, 1)$ semi-conjugate to E and with associated pair of vectors $(y, \tilde{\gamma})$.*

Proof. Let $(x_0, x_1, \ldots, x_m)$ be the vector associated to E. Let x and $u : [0, 1) \to \varphi$ be as in Lemma 3.5. Let $\{a_n\}_{n \geq 1}$ be a sequence of positive integers such that, for all n,

$$\gamma(u(x)) + \gamma(u(Ex)) + \cdots + \gamma(u(E^{a_n} x)) \leq 0.$$

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Denote by $\vartheta_0 = \delta_x$ and, for all integers $k \geq 1$,

$$\vartheta_k = \exp(\gamma(u(x)) + \gamma(u(Ex)) + \cdots + \gamma(u(E^{k-1}x)))\delta_{E^k x},$$

where δ_x is the Dirac probability concentrated at x. Let

$$\nu_n = c_n(\vartheta_0 + \vartheta_1 + \cdots + \vartheta_{a_n+1}),$$

where c_n is the positive constant that makes ν_n a probability measure. By Lemma 3.5:
(1) $\lim_n c_n = 0$;
(2) $\lim_n (c_n[\exp(\gamma(u(x)) + \gamma(u(Ex)) + \cdots + \gamma(u(E^{a_n}x)))]) = 0$;
(3) for all non-negative integers k, $\vartheta_{k+1} = (\exp(\gamma(u(E^k x))))E_*\vartheta_k$.
These three facts imply that if K is a subinterval of $[x_{i-1}, x_i)$, for some $i \in \varphi$, then
(4) $\lim_{n\to\infty} |\nu_n(E(K)) - \exp(\gamma(i)) \cdot \nu_n(K)| = 0$.
It follows from Lemma 3.3 and its proof that there exists a probability ν which is a weak limit of the sequence $\{\nu_n\}$. Therefore,
(5) $\nu(E(K)) = \exp(\gamma(i)) \cdot \nu(K)$.
This and Lemma 3.1 imply that ν is positive on open intervals.

We now define the map $\phi : [0, 1) \to [0, 1)$ by

$$\phi(x) = \nu([0, x)); \quad \phi(0) = \nu(0).$$

Since ν is positive on open sets, ϕ is strictly increasing and so it has at most countably many discontinuities. It is also clear that ϕ is right semi-continuous. So the set

$$G = [0, 1) \setminus \phi([0, 1))$$

is the union of at most a countable many intervals of the type $[a, b)$, which we call *gaps*.

Define the map $h = [0, 1) \to [0, 1)$ by:
(i) $h = \phi^{-1}$ on $[0, 1) - G$;
(ii) $h(x) = \phi^{-1}(b)$, $a \leq x < b$, for each gap $[a, b) \subset G$.
Since ϕ^{-1} is also right semi-continuous and increasing, it follows that h is continuous, non-decreasing and $h([0, 1)) = [0, 1)$.

Let us denote by $|J|$ the length of the interval J. If $J = [a, b)$ is a gap, $p = h(J)$ is the discontinuity which generates it and i is such that $p \in [x_{i-1}, x_i)$ then, by (5), we have that:
(6) $J' = h^{-1}(Eh(J))$ is also a gap and $|J'| = \exp(\gamma(i)) \cdot |J|$
i.e. $E(p)$ is also a discontinuity for ϕ and the length of the gap generated by $E(p)$ is $\exp(\gamma(i)) \cdot \nu(\{p\}) = \exp(\gamma(i)) \cdot |J|$.

Let us now define the affine map $T : [0, 1) \to [0, 1)$. Initially, define $(y_0, y_1, \ldots, y_m)$ in the following way:

$$\begin{cases} y_0 = 0 \\ y_i = \phi(x_i) = \nu([0, x_i)), \quad i = 1, 2, \ldots, m-1 \\ y_m = 1. \end{cases}$$

Since ν is positive on open intervals we have that $0 = y_0 < y_1 < \cdots < y_m = 1$. On each subinterval $[y_{i-1}, y_i)$ define T in the following way:
(i) on $[y_{i-1}, y_i) \setminus G$, h is one-to-one, so define $T(x) = h^{-1}Eh(x)$;

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(ii) on G, if J is a gap and J' is the corresponding image gap given by statement (6), then define $T|_J : J \to J'$ linearly and orientation preserving.

Since (6) also says that each gap is the image of another gap, we have that T is one-to-one and onto.

We can now finish the first part of this proposition.

(7) The map $T : [0, 1) \to [0, 1)$ defined above is an AIET semi-conjugate to E.

Given $[a, b) \subset [y_{i-1}, y_i)$, then $T([a, b))$ is an interval by definition. If a and b do not belong to a gap, in view of (5), we have that

$$T(b) - T(a) = \exp(\gamma(i)) \cdot (b - a).$$

So T is affine on $[a, b)$ with slope $\exp(\gamma(i))$. The same conclusion holds if $[a, b)$ is a gap, as can be seen from the definition of T. As for the semi-conjugation, if $x \in [0, 1)$ does not belong to a gap then (from the definition) we have that $Eh(x) = hT(x)$. If x does belong to a gap J, let $p = h(J)$ and let J' be the image gap given by (6). Then $T(p) \in J'$ and $h(J') = E(p)$, so $Eh(x) = E(p) = hT(x)$. □

4. *Substitutions and the Proof of Theorem 2.1(b)*

Lemma 4.4 takes care of Theorem 2.1(b).

Let $m \geq 2$ be an integer and let $W(\varphi)$ be the set of words on the alphabet $\varphi = \{1, 2, \ldots, m\}$ (i.e. the free associative monoid generated by φ). We shall assume that the empty word does not belong to $W(\varphi)$. Any map $\pi : \varphi \to W(\varphi)$ can be extended to a map $\pi : W(\varphi) \to W(\varphi)$, called a *substitution*, defined by recurrence as follows:

$$\begin{aligned} \pi(Vx) &= \pi(V)\pi(x) \\ \pi(xV) &= \pi(x)\pi(V), \end{aligned}$$

where $V \in W(\varphi)$ and $x \in \varphi$.

By induction on n it is easy to see the following.

LEMMA 4.1. *Given $V_1, V_2, \ldots, V_k \in W(\varphi)$ and $n \in \mathbb{N}$, the following equation is satisfied:*

$$\pi^n(V_1 V_2 \cdots V_k) = \pi^n(V_1)\pi^n(V_2) \cdots \pi^n(V_k).$$

In this section we shall suppose that $\pi : W(\varphi) \to W(\varphi)$ is a substitution such that, for all $i \in \varphi$, $\pi(i)$ has the form

$$\pi(i) = B_{i1} B_{i2} \cdots B_{iN_i}.$$

Observe that the substitution π induces the map $\pi^* : \varphi^{\mathbb{N}} \to \varphi^{\mathbb{N}}$ which takes the infinite sequence

$$v = v_0 v_1 v_2 \cdots v_n \cdots$$

to the sequence

$$\pi^*(v) = B_0 B_1 B_2 \cdots B_n \cdots,$$

where B_n denotes the block $\pi(i) = B_{i1} B_{i2} \cdots B_{iN_i}$, if $v_n = i$.

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The *length* is the function $\ell : W(\varphi) \to \mathbb{N}$ which takes the word V to the non-negative integer defined by recurrence:

$$\begin{aligned} \ell(x) &= 1 \quad \text{for all } x \in \varphi \\ \ell(xV) &= \ell(Vx) = \ell(V) + 1 \quad \text{for all } x \in \varphi \text{ and } V \in W(\varphi). \end{aligned}$$

Remark 4.2. Let π be as above and suppose that, for all $i \in \varphi$, $\ell(\pi(i)) \geq 2$. If there is $i_0 \in \varphi$ such that, for all $i \in \varphi$, $B_{i1} = i_0$ then the transformation π^*, obviously contracting for the natural metric on $\varphi^{\mathbb{N}}$, has a unique fixed point which is precisely the infinite sequence $u = \lim_{n\to\infty} \pi^n(i_0)$. The reader may easily find other weaker sufficient conditions that guarantee the existence of the unique fixed point.

Let $j \in \varphi$ and $V = x_1 x_2 \cdots x_s$ be an element of $W(\varphi)$ (where $x_k \in \varphi$ for $k = 1, 2, \ldots, s$). Let

$$r_j(V) = \text{card}\{k : x_k = j\};$$

that is, the number of times that the letter j appears in the word V. Let us denote by $r(V)$ the column vector

$$r(V) = \begin{pmatrix} r_1(V) \\ r_2(V) \\ \vdots \\ r_m(V) \end{pmatrix}.$$

Let $\mathcal{A}$ be a $m \times m$ non-negative matrix whose entries are $\mathcal{A}_{ji} = r_j(\pi(i)) = \#\{0 \leq k \leq N_i : B_{ik} = j\}$. We shall refer to $\mathcal{A}$ as the *matrix associated to* π.

As $\ell(V) = \sum_{j\in\varphi} r_j(V)$ and $\pi(V) = \pi(x_1)\pi(x_2)\cdots\pi(x_s)$, the following matrix equality can be deduced:

$$r(\pi(V)) = \mathcal{A} \cdot r(V), \tag{1}$$

and therefore, for $i \in \varphi$,

$$r(\pi^n(i)) = \mathcal{A}^n \cdot r(i). \tag{2}$$

In this way, for $i \in \varphi$ and $n \geq 0$ an integer,

$$r(\pi^n(i)) \text{ is the } i\text{th column of the matrix } \mathcal{A}^n,$$

where $\pi^0(i) = i$ and $\mathcal{A}^0$ denotes the identity matrix.

Any function $\gamma : \varphi \to \mathbb{R}$ may be extended, by recurrence, to a map $\gamma : W(\varphi) \to \mathbb{R}$ as follows:

$$\gamma(Vx) = \gamma(xV) = \gamma(V) + \gamma(x), \tag{3}$$

where $V \in W(\varphi)$ and $x \in \varphi$.

LEMMA 4.3. *Let $\mathcal{A}$ and γ be as above. Let $\tilde{\gamma} = (\gamma(1), \gamma(2), \ldots, \gamma(m))$ and denote by $\mathcal{A}^*$ the adjoint operator of $\mathcal{A}$.*

(a) *If $\tilde{\gamma}$ is an eigenvector of $\mathcal{A}^*$ with corresponding eigenvalue λ, then for all $n \geq 0$ and for $x \in \varphi$,*

$$\gamma((\pi^n(x))) = \lambda^n \gamma(x).$$

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(b) *If $\tilde{\gamma}$ belongs to the stable space of $\mathcal{A}^*$, then*

$$\lim_{n\to\infty} \gamma((\pi^n(x))) = 0.$$

Proof. Let us prove (a):

$$\begin{aligned} \gamma((\pi^n(x))) &= \tilde{\gamma} \cdot r(\pi^n(x)) \quad \text{(by definition)} \\ &= \tilde{\gamma} \cdot \mathcal{A}^n(r(x)) \\ &= \lambda^n(\tilde{\gamma} \cdot r(x)) \\ &= \lambda^n(\gamma(x)). \end{aligned}$$

Let us prove (b). Observe that, by [**MP**, Proposition 2.10], $\lim_{n\to\infty} \tilde{\gamma} \cdot \mathcal{A}^n = 0$. Therefore,

$$\begin{aligned} \lim_{n\to\infty} \gamma((\pi^n(x))) &= \lim_{n\to\infty} \tilde{\gamma} \cdot r(\pi^n(x)) \quad \text{(by definition (2))} \\ &= \lim_{n\to\infty} \tilde{\gamma} \cdot \mathcal{A}^n(r(x)) \\ &= 0. \end{aligned}$$

This proves the lemma. □

LEMMA 4.4. *Let $\varphi = \{1, 2, \ldots, m\}$ and let $\gamma : \varphi \to \mathbb{R}$ be a map such that $\tilde{\gamma} = (\gamma(1), \gamma(2), \ldots, \gamma(m)) \in \mathcal{G}^s$. Let $T : [0, 1) \to [0, 1)$ be an AIET semi-conjugate to E and with associated pair of vectors $(y, \tilde{\gamma})$. Then T is conjugate to E.*

Proof. Denote by $h : [0, 1) \to [0, 1)$ a semi-conjugacy between T and E ($E \circ h = h \circ T$). We say that an open non-empty interval $J \subset [0, 1)$ is a gap if $h(J)$ is a one point set. All we have to do is to show that there are no gaps. In this case the semi-conjugacy h will be bijective and so a conjugation. Recall that since E is renormalizable, there is a sequence of IETs $E_n : I_n \to I_n$, induced by E, such that $L^n \circ E_n \circ L^{-n} = E$, where L is an orientation-preserving affine map. Consequently, T induces on $I'_n = h^{-1}(I_n)$, $T_n : I'_n \to I'_n$. The maps T_n also satisfy $hT_n = E_n h$. Since the itineraries of T are the same as those of E (in terms of the semi-conjugacy h) then, by the chain rule, the derivatives of T_n satisfy

$$T'_n = \exp(\gamma(\pi^n(i))), \quad i = 1, 2, \ldots, m.$$

Let $F_n : [0, 1) \to [0, 1)$ denote the renormalization of T_n to $[0, 1)$. Then we also have

$$F'_n = \exp(\gamma(\pi^n(i))), \quad i = 1, 2, \ldots, m.$$

By Lemma 4.3(b) we have that $F'_n \to 1$ as $n \to \infty$ for each $i = 1, 2, \ldots, m$. If we denote by K_n the interval on which F_n induces, then as $|I_1| = (1/\lambda) < 1$, where λ is the Perron–Frobenius eigenvalue, there exists N_0 such that for all $n \geq N_0$,

$$|K_n| \leq \frac{s}{s+1} < 1$$

for some $s > 0$. We can also take N_0 such that

$$(F_n^k)' > \frac{s+1}{s+2}$$

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for $1 \leq k \leq N_0$, where N_0 is the maximum return time of each subinterval of I_1 ($N_0 = \max\{\sum_{i=1}^{m} \mathcal{A}_{ij};\ j = 1, 2, \ldots, m\}$). Now if T has a gap J, then F_{N_0} will have a gap J'. Iterating J' at most N_0 times we will obtain a new gap $J_{N_0} \subset K_{N_0}$. Moreover,

$$|J_{N_0}| > \frac{s+1}{s+2} \cdot |J'|.$$

Renormalizing J_{N_0} we obtain a new gap J'_{N_0} which satisfies

$$\begin{aligned} |J'_{N_0}| &= |J_{N_0}|/|K_{N_0}| \\ &> \frac{s+1}{s+2} \cdot |J'| \cdot \frac{s+1}{s} \\ &= \left(1 + \frac{1}{s(s+2)}\right) \cdot |J'|. \end{aligned}$$

Arguing inductively, we get a sequence $\{J'_n\}_{n \geq N_0}$ of gaps, where J'_n is the gap relative to F_n, such that

$$|J'_n| = \left(1 + \frac{1}{s(s+2)}\right)^{n-N_0+1} \cdot |J'|.$$

So $|J'_n| \to \infty$ as $n \to \infty$ which is a contradiction. □

5. *IETs and substitutions*

An interval exchange transformation is usually defined as follows. Fix an integer $m > 1$, take ψ to be an irreducible permutation $j \to \psi j$, $1 \leq j \leq m$, which is to say $\psi(\{1, 2, \ldots, k\}) = \{1, 2, \ldots, k\}$ implies $k = m$. Given $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \Lambda_m$, one associates $x_0 = x_0(\alpha) = a$, $x_j = x_j(\alpha) = a + \sum_{i=1}^{j} \alpha_i$, with $1 \leq j \leq m$, $|\alpha| = b - a$, and $I_j^\alpha = [x_{j-1}(\alpha), x_j(\alpha)) \subset [a, b) = I^\alpha$. Also define $\alpha_j^\psi = \alpha_{\psi^{-1} j}$, $1 \leq j \leq m$. The (α, ψ) IET, $E = E_{\alpha,\psi}$, is the map of I^α defined by

$$Ex = x + x_{\psi i - 1}(\alpha^\psi) - x_{i-1}(\alpha) \quad (x \in I_i^\alpha).$$

Let $E : I \to I$ as above and assume that $(x_0, x_1, \ldots, x_m)$ is the vector associated to E. Denote by $I_1 = [c, d)$, a proper subinterval of I. Suppose that E is *renormalizable* (on I_1); that is, there exists an orientation-preserving affine map $L : \mathbb{R} \to \mathbb{R}$ such that $L(I_1) = I$ and $L \circ E_1 \circ L^{-1} = E$, where $E_1 : I_1 \to I_1$ denotes the IET induced by E. Let $\varphi = \{1, 2, \ldots, m\}$. Given $i \in \varphi$, let $z_i = L^{-1}(x_i)$. In this way, the sequence of discontinuities of E_1 is $\{z_1, \ldots, z_{m-1}\}$. Proceeding inductively, we may find a decreasing sequence of intervals $\{I_n\}$ and a sequence of IETs $E_n : I_n \to I_n$, induced by E, such that $L^n \circ E_n \circ L^{-n} = E$. Moreover, if $\overline{I_1} \subset I$ then $\cap_n I_n$ is a one point set, say p_0, called the *distinguished point of* (E, I_1), whose itinerary (under the map E) will be described next.

Given $i \in \varphi$, let N_i be the least possible non-negative integer such that for some $z \in [z_{i-1}, z_i)$ (and therefore for all $z \in [z_{i-1}, z_i)$), $E^{N_i+1} z \in I_1$. Let $u : [0, 1) \to \varphi$ be given by $u(x) = i$ iff $x \in [x_{i-1}, x_i)$.

Consider the substitution

$$\begin{aligned} \pi : 1 &\to u(z_0)u(Ez_0)\cdots u(E^{N_1} z_0) \\ \pi : 2 &\to u(z_1)u(Ez_1)\cdots u(E^{N_2} z_1) \\ \vdots \quad &\quad \vdots \quad \vdots \\ \pi : m &\to u(z_{m-1})u(Ez_{m-1})\cdots u(E^{N_m} z_{m-1}) \end{aligned}$$

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which gives the itinerary of the subintervals of I_1, under the action of E, until its return to I_1.

Under these conditions, if the distinguished point $p_0 = \cap_n I_n$ belongs to $[x_{i_0-1}, x_{i_0})$, then its itinerary is given by the infinite sequence

$$u^* = u_0 u_1 \cdots u_k \cdots = \lim_{n\to\infty} \pi^n(i_0).$$

This sequence u^* is the unique fixed point of the map $\pi^* : \varphi^{\mathbb{N}} \to \varphi^{\mathbb{N}}$ that is naturally induced by the map π (see [**Rau**]). In this way, for all integers $k \geq 0$, $E^k p_0 \in [x_{u_k-1}, x_{u_k})$.

If I_1 is of the form $[c, 1)$ and $u^* = \lim_{n\to\infty} \pi^n(1) = u_0 u_1 \cdots u_k \cdots$, where π is now the substitution associated to (E, I_1), then $p_0 = \lim_{t\to 1^-} E(t)$ will be called the *distinguished point associated* to (E, I_1). The itinerary of p_0 is given by the infinite sequence

$$u_1 u_2 \cdots u_k \cdots,$$

where $p_0 \in I_{u_1}$. Therefore, for all integers $k \geq 1$, $E^{k-1} p_0 \in [x_{u_k-1}, x_{u_k})$.

Remark 5.1. The matrices associated to E and π (as above) are the same.

6. *Rauzy's map*

We consider Rauzy's map. Let $R : [0, 1) \to [0, 1)$ be the IET for which $m = 4$, the permutation $\psi : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ is (142) and $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \Lambda_4$ is the Perron–Frobenius normalized (i.e. $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$) right eigenvector of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 2 & 2 \end{pmatrix}.$$

It has been observed that $\alpha_1 < \alpha_4$ and that R is renormalizable on the interval $R(I_1^\alpha) \subset I_4^\alpha$. In this case the associated substitution is

$$\begin{aligned} \pi(1) &= 421 \\ \pi(2) &= 4221 \\ \pi(3) &= 433421 \\ \pi(4) &= 43421. \end{aligned}$$

As $R(I_1^\alpha)$ is of the form $[x, 1)$, the distinguished point associated to $(R, R(I_1^\alpha))$ is $\lim_{t\to 1^-} R(t) = R(x_2(\alpha))$ and therefore if

$$\begin{aligned} u^* &= 4 \cdots \\ u^* &= 43421 \cdots \\ u^* &= 43421\,433421\,4221\,421 \cdots \\ u^* &= u_1 u_2 \cdots u_n \cdots \end{aligned}$$

denotes the fixed point of this substitution, then $v^* = u_2 \cdots u_n \cdots$ (which starts at $u_2 = 3$) gives the itinerary of the positive orbit of $R(x_2(\alpha))$.

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The matrix A has four real eigenvalues whose approximate values are

$$4.39026, \quad 1.83785, \quad 0.544113, \quad 0.227777.$$

Denote by λ the eigenvalue that is approximated by 1.83785 and let $\varphi = \{1, 2, 3, 4\}$ and

$$\gamma : \varphi \to \mathbb{R}$$

be such that

$$\tilde{\gamma} = (\gamma(1), \gamma(2), \gamma(3), \gamma(4))$$

is a left eigenvector, having its last entry negative, corresponding to the eigenvalue λ (i.e. $\tilde{\gamma}A = \lambda\tilde{\gamma}$). Up to multiplication by a positive real number, an approximate value for $\tilde{\gamma}$ is:

$$(0.24624, \ 0.540134, \ -0.732246, \ -0.333821).$$

We may extend γ, by recurrence, to a map $\gamma: W(\varphi) \to \mathbb{R}$ as in equation (3) of §4. Let $n, k \in \mathbb{N}$, with $n \geq 1$. Define $\psi_n : \{V \in W(\varphi) : \ell(V) \geq n\} \to W(\varphi)$ by

$$\psi_n(v_1 v_2 \cdots v_n \cdots v_{n+k}) = (v_1 v_2 \cdots v_n).$$

Let $\Delta : W(\varphi) \to (0, \infty)$ be defined by

$$\Delta(V) = \exp(\gamma(\psi_1(V))) + \exp(\gamma(\psi_2(V))) + \cdots + \exp(\gamma(\psi_{\ell(V)}(V))).$$

LEMMA 6.1. *Given $V_1, V_2, \ldots, V_k \in W(\varphi)$, the following equation is satisfied:*

$$\begin{aligned} \Delta(V_1 V_2 \cdots V_k) &= \Delta(V_1) + \exp(\gamma(V_1)) \cdot \Delta(V_2) \\ &\quad + \exp(\gamma(V_1 V_2)) \cdot \Delta(V_3) + \cdots + \exp(\gamma(V_1 V_2 \cdots V_{k-1})) \cdot \Delta(V_k). \end{aligned}$$

Proof. We will prove the lemma for $k = 2$ only. Let $V, W \in W(\varphi)$ and let $n = \ell(V)$ and $m = \ell(W)$. Then

$$\begin{aligned} \Delta(VW) &= \exp(\gamma(\psi_1(VW))) + \cdots + \exp(\gamma(\psi_n(VW))) \\ &\quad + \exp(\gamma(\psi_{n+1}(VW))) + \cdots + \exp(\gamma(\psi_{n+m}(VW))) \\ &= \exp(\gamma(\psi_1(V))) + \cdots + \exp(\gamma(\psi_n(V))) \\ &\quad + \exp(\gamma(\psi_n(V)) + \gamma(\psi_1(W))) + \cdots + \exp(\gamma(\psi_n(V)) + \gamma(\psi_m(W))) \\ &= \Delta(V) + \exp(\gamma(V))\dot{\Delta}(W). \end{aligned}$$

This finishes the proof. □

LEMMA 6.2. *Let $(a, b, -c, -d) = (0.25, 0.55, -0.73, -0, 33)$. Given $n \in \mathbb{N}$, take $C = C(n) > 0$ such that*

$$\Delta(\pi^n(i)) \leq C, \quad i = 1, 2, 3, 4.$$

If $n \geq 1$ then, for all $k \in \mathbb{N}$, the following are satisfied:
(a) $\Delta(\pi^{n+k}(4)) \leq C[1+6\exp(-\lambda^n d/2)+6^2\exp(-\lambda^{n+1}d/2)+\cdots+6^k\exp(-\lambda^{n+k+1}d/2)]$;
(b) $\Delta(\pi^{n+k}(4)) \leq 6^k C$;
(c) $\Delta(\pi^{n+k}(3)) \leq 6^k C$;

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(d) $\Delta(\pi^{n+k}(2)) \leq 6^k C \cdot \exp(\lambda^n(\lambda^{k-1} + \lambda^{k-2} + \cdots + \lambda + 1)(2b - d))$;
(e) $\Delta(\pi^{n+k}(1)) \leq 6^k C \cdot \exp(\lambda^n(\lambda^{k-1} + \lambda^{k-2} + \cdots + \lambda + 1)(\frac{3}{2}b - d))$.

Proof. We use induction on k. For $k = 1$ we have the following.

(a) By Lemma 6.1, Lemma 4.3(a), and because $\Delta(\pi^n(i)) \leq C,\ i = 1, 2, 3, 4$, we have that

$$\begin{aligned}
\Delta(\pi^{n+1}(4)) &= \Delta(\pi^n(4)\pi^n(3)\pi^n(4)\pi^n(2)\pi^n(1)) \\
&= \Delta(\pi^n(4)) \\
&\quad +\Delta(\pi^n(3))\exp(\gamma(\pi^n(4))) \\
&\quad +\Delta(\pi^n(4))\exp(\gamma(\pi^n(4)\pi^n(3))) \\
&\quad +\Delta(\pi^n(2))\exp(\gamma(\pi^n(4)\pi^n(3)\pi^n(4))) \\
&\quad +\Delta(\pi^n(1))\exp(\gamma(\pi^n(4)\pi^n(3)\pi^n(4)\pi^n(2))) \\
&\leq C + C \cdot \exp(\lambda^n(\gamma(4))) \\
&\quad +C \cdot \exp(\lambda^n(\gamma(3) + \gamma(4))) \\
&\quad +C \cdot \exp(\lambda^n(\gamma(3) + 2\gamma(4))) \\
&\quad +C \cdot \exp(\lambda^n(\gamma(2) + \gamma(3) + 2\gamma(4))) \\
&\leq C + C \cdot \exp(\lambda^n(-d)) \\
&\quad +C \cdot \exp(\lambda^n(-c - d)) \\
&\quad +C \cdot \exp(\lambda^n(-c - 2d)) \\
&\quad +C \cdot \exp(\lambda^n(b - c - 2d)) \\
&\leq C \cdot \left[1 + 6\exp\left(\lambda^n\left(\frac{-d}{2}\right)\right)\right].
\end{aligned}$$

The last inequality comes from the fact that $b - c - 2d < -d/2$.

(b) Since each exponential in the penultimate inequality above is less then one, we also have

$$\Delta(\pi^{n+1}(4)) \leq 6C.$$

(c) As in (a), we have that

$$\begin{aligned}
\Delta(\pi^{n+1}(3)) &= \Delta(\pi^n(4)\pi^n(3)\pi^n(3)\pi^n(4)\pi^n(2)\pi^n(1)) \\
&\leq C + C \cdot \exp(\lambda^n(-d)) \\
&\quad +C \cdot \exp(\lambda^n(-c - d)) \\
&\quad +C \cdot \exp(\lambda^n(-2c - d)) \\
&\quad +C \cdot \exp(\lambda^n(-2c - 2d)) \\
&\quad +C \cdot \exp(\lambda^n(b - 2c - 2d)) \\
&\leq C \cdot (1 + \exp(\lambda^n(-d))) \\
&\leq 6C.
\end{aligned}$$

(d) As before,

$$\Delta(\pi^{n+1}(2)) = \Delta(\pi^n(4)\pi^n(2)\pi^n(2)\pi^n(1))$$

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$$\begin{aligned} &\leq C + C \cdot \exp(\lambda^n(-d)) \\ &\quad + C \cdot \exp(\lambda^n(b-d)) \\ &\quad + C \cdot \exp(\lambda^n(2b-d)) \\ &\leq 6C \cdot \exp(\lambda^n(2b-d)). \end{aligned}$$

(e) Again,

$$\begin{aligned} \Delta(\pi^{n+1}(1)) &= \Delta(\pi^n(4)\pi^n(2)\pi^n(1)) \\ &\leq C + C \cdot \exp(\lambda^n(-d)) \\ &\quad + C \cdot \exp(\lambda^n(b-d)) \\ &\leq 6C \cdot \exp(\lambda^n(\tfrac{3}{2}b-d)) \end{aligned}$$

as $b - d \leq \frac{3}{2}b - d$.

Suppose now that the equations are valid for k and we shall proceed to prove that they are valid for $k+1$.

(a′)

$$\begin{aligned} &\Delta(\pi^{n+k+1}(4)) \\ &= \Delta(\pi^{n+k}(4)\pi^{n+k}(3)\pi^{n+k}(4)\pi^{n+k}(2)\pi^{n+k}(1)) \\ &= \Delta(\pi^{n+k}(4)) \\ &\quad + \Delta(\pi^{n+k}(3))\exp(\gamma(\pi^{n+k}(4))) \\ &\quad + \Delta(\pi^{n+k}(4))\exp(\gamma(\pi^{n+k}(4)\pi^{n+k}(3))) \\ &\quad + \Delta(\pi^{n+k}(2))\exp(\gamma(\pi^{n+k}(4)\pi^{n+k}(3)\pi^{n+k}(4))) \\ &\quad + \Delta(\pi^{n+k}(1))\exp(\gamma(\pi^{n+k}(4)\pi^{n+k}(3)\pi^{n+k}(4)\pi^{n+k}(2))) \\ &\leq C \cdot \left[1 + 6\exp\left(\lambda^n\left(\frac{-d}{2}\right)\right) + 6^2\exp\left(\lambda^{n+1}\left(\frac{-d}{2}\right)\right) + \cdots \right. \\ &\quad \left. + 6^k\exp\left(\lambda^{n+k-1}\left(\frac{-d}{2}\right)\right)\right] \\ &\quad + 6^k C \cdot \exp(\lambda^{n+k}(-d)) \\ &\quad + 6^k C \cdot \exp(\lambda^{n+k}(-c-d)) \\ &\quad + 6^k C \cdot \exp(\lambda^n(\lambda^{k-1}+\lambda^{k-2}+\cdots+\lambda+1)(2b-d)) \cdot \exp(\lambda^{n+k}(-c-2d)) \\ &\quad + 6^k C \cdot \exp(\lambda^n(\lambda^{k-1}+\lambda^{k-2}+\cdots+\lambda+1)(\tfrac{3}{2}b-d)) \cdot \exp(\lambda^{n+k}(b-c-2d)) \\ &\leq C \cdot \left[1 + 6\exp\left(\lambda^n\left(\frac{-d}{2}\right)\right) + 6^2\exp\left(\lambda^{n+1}\left(\frac{-d}{2}\right)\right) + \cdots \right. \\ &\quad \left. + 6^k\exp\left(\lambda^{n+k-1}\left(\frac{-d}{2}\right)\right) + 6^{k+1}\exp\left(\lambda^{n+k}\left(\frac{-d}{2}\right)\right)\right]. \end{aligned}$$

The last inequality comes from the fact that $\frac{3}{2}b - d < (\lambda - 1)(-b + c - \frac{3}{2}d)$.

(b′) By this same fact we have that

$$\begin{aligned} &\Delta(\pi^{n+k+1}(4)) \\ &= \Delta(\pi^{n+k}(4)\pi^{n+k}(3)\pi^{n+k}(4)\pi^{n+k}(2)\pi^{n+k}(1)) \end{aligned}$$

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$$\begin{aligned}
&\leq 6^k C \\
&\quad +6^k C \cdot \exp(\lambda^{n+k}(-d)) \\
&\quad +6^k C \cdot \exp(\lambda^{n+k}(-c-d)) \\
&\quad +6^k C \cdot \exp(\lambda^n(\lambda^{k-1}+\lambda^{k-2}+\cdots+\lambda+1)(2b-d)) \cdot \exp(\lambda^{n+k}(-c-2d)) \\
&\quad +6^k C \cdot \exp(\lambda^n(\lambda^{k-1}+\lambda^{k-2}+\cdots+\lambda+1)(\tfrac{3}{2}b-d)) \cdot \exp(\lambda^{n+k}(b-c-2d)) \\
&\leq 6^{k+1} C.
\end{aligned}$$

(c′) As above,

$$\begin{aligned}
&\Delta(\pi^{n+k+1}(3)) \\
&= \Delta(\pi^{n+k}(4)\pi^{n+k}(3)\pi^{n+k}(3)\pi^{n+k}(4)\pi^{n+k}(2)\pi^{n+k}(1)) \\
&\leq 6^k C \\
&\quad +6^k C \cdot \exp(\lambda^{n+k}(-d)) \\
&\quad +6^k C \cdot \exp(\lambda^{n+k}(-c-d)) \\
&\quad +6^k C \cdot \exp(\lambda^{n+k}(-2c-d)) \\
&\quad +6^k C \cdot \exp(\lambda^n(\lambda^{k-1}+\lambda^{k-2}+\cdots+\lambda+1)(2b-d)) \cdot \exp(\lambda^{n+k}(-2c-2d)) \\
&\quad +6^k C \cdot \exp(\lambda^n(\lambda^{k-1}+\lambda^{k-2}+\cdots+\lambda+1)(\tfrac{3}{2}b-d)) \cdot \exp(\lambda^{n+k}(b-2c-2d)) \\
&\leq 6^{k+1} C.
\end{aligned}$$

(d′) In the same way,

$$\begin{aligned}
&\Delta(\pi^{n+k+1}(2)) \\
&= \Delta(\pi^{n+k}(4)\pi^{n+k}(2)\pi^{n+k}(2)\pi^{n+k}(1)) \\
&\leq 6^k C \\
&\quad +6^k C \cdot \exp(\lambda^n(\lambda^{k-1}+\lambda^{k-2}+\cdots+\lambda+1)(2b-d)) \cdot \exp(\lambda^{n+k}(-d)) \\
&\quad +6^k C \cdot \exp(\lambda^n(\lambda^{k-1}+\lambda^{k-2}+\cdots+\lambda+1)(2b-d)) \cdot \exp(\lambda^{n+k}(b-d)) \\
&\quad +6^k C \cdot \exp(\lambda^n(\lambda^{k-1}+\lambda^{k-2}+\cdots+\lambda+1)(\tfrac{3}{2}b-d)) \cdot \exp(\lambda^{n+k}(2b-d)) \\
&\leq 6^{k+1} C \cdot \exp(\lambda^n(\lambda^k+\lambda^{k-1}+\cdots+\lambda+1)(2b-d)).
\end{aligned}$$

(e′) From the fact that $1 < 3(\lambda - 1)$,

$$\begin{aligned}
&\Delta(\pi^{n+k+1}(1)) \\
&= \Delta(\pi^{n+k}(4)\pi^{n+k}(2)\pi^{n+k}(1)) \\
&\leq 6^k C \\
&\quad +6^k C \cdot \exp(\lambda^n(\lambda^{k-1}+\lambda^{k-2}+\cdots+\lambda+1)(2b-d)) \cdot \exp(\lambda^{n+k}(-d)) \\
&\quad +6^k C \cdot \exp(\lambda^n(\lambda^{k-1}+\lambda^{k-2}+\cdots+\lambda+1)(\tfrac{3}{2}b-d)) \cdot \exp(\lambda^{n+k}(b-d)) \\
&\leq 6^{k+1} C \cdot \exp(\lambda^n(\lambda^k+\lambda^{k-1}+\cdots+\lambda+1)(\tfrac{3}{2}b-d)).
\end{aligned}$$

□

PROPOSITION 6.3. *The sum*

$$\lim_{k\to\infty} \Delta(\pi^k(4)) = \exp(\gamma(u_1)) + \exp(\gamma(u_1u_2)) + \cdots + \exp(\gamma(u_1u_2,\ldots u_k)) + \cdots$$

is finite.

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Proof. As $-d/2 < 0$, by using Lemma 6.2, with $n = 1$, we conclude that the $(k+1)$th root of

$$C6^k \exp\left(-\lambda^{2+k} \cdot \frac{d}{2}\right)$$

goes to zero as $k \to \infty$. It follows from the root test that the required sum is finite. □

7. *The inverse of Rauzy's map*

In the following we shall keep the notation of the previous section.

Let us take $m = 4$, let Ψ be the (124) permutation and ξ the Perron–Frobenius normalized eigenvector associated to the matrix

$$B = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Then the inverse of Rauzy's map is the (ξ, Ψ) IET, $F = F_{\xi,\Psi}$. It is clear that this map is minimal, uniquely ergodic and self-induced on the last interval $[\beta_3(\xi), 1)$.

LEMMA 7.1. *The matrices A and B have the same eigenvalues. Moreover,*

(a) (v_1, v_2, v_3, v_4) *is a left eigenvector of A if and only if* (v_2, v_4, v_3, v_1) *is a left eigenvector of B;*

(b) $(v_1, v_2, v_3, v_4)^t$ *is a right eigenvector of A if and only if* $(v_2, v_4, v_3, v_1)^t$ *is a right eigenvector of B.*

Proof. Let

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It may be seen that

$$B = MAM^{-1}.$$

In particular, A and B have the same eigenvalues.

Let $v = (v_1, v_2, v_3, v_4)$ and $u = (v_4, v_1, v_3, v_2)$. We shall only prove that if v is a left eigenvector of A corresponding to the eigenvalue δ then u is a left eigenvector of B corresponding to B. In fact,

$$uB = uMAM^{-1} = vAM^{-1} = \delta vM^{-1} = \delta u. \qquad \square$$

Since ξ is the Perron–Frobenius normalized (right) eigenvector we have, by Lemma 7.1, that

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4) = (\alpha_2, \alpha_4, \alpha_3, \alpha_1).$$

It can be seen numerically that $\alpha_1 < \alpha_2$, so R is renormalizable on the interval $RI_4^\xi \subset I_1^\xi$ and the substitution $1 \to 1124$, $2 \to 12324$, $3 \to 123324$, $4 \to 124$, denoted by Π, gives the itinerary of the subintervals of $RI_4^\xi \subset I_1^\xi$ until its return to RI_4^ξ.

Let $\Pi: W(\varphi) \to W(\varphi)$ be defined by recurrence as follows:

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(a)

$$\begin{aligned}\Pi(1) &= 1124\\ \Pi(2) &= 12324\\ \Pi(3) &= 123324\\ \Pi(4) &= 124;\end{aligned}$$

(b)

$$\begin{aligned}\Pi(Vi) &= \Pi(V)\Pi(i)\\ \Pi(iV) &= \Pi(i)\Pi(V),\end{aligned}$$

where $V \in W(\varphi)$ and $i \in \varphi$. Hence,

$$\begin{aligned}v &= v_1 v_2 \cdots v_n \cdots\\ &= \lim_{n\to\infty} \Pi^n(1)\\ &= 1124\,1124\,12324\,124\cdots\end{aligned}$$

gives the itinerary of the positive orbit of $0 = \beta_0(\xi) = F\beta_3(\xi)$.

As before, denote by $\Pi^*: \varphi^{\mathbb{N}} \to \varphi^{\mathbb{N}}$ the map which takes the sequence

$$w = w_1 w_2 \cdots w_n \cdots$$

to the sequence

$$\Pi^*(w) = B_1 B_2 \cdots B_n \cdots,$$

where B_i denotes the block

$$\begin{aligned}1124 &\quad \text{if } w_i = 1\\ 12324 &\quad \text{if } w_i = 2\\ 123324 &\quad \text{if } w_i = 3\\ 124 &\quad \text{if } w_i = 4.\end{aligned}$$

In this way v is the unique fixed point of Π^*.

In view of Lemma 7.1, if $\tilde{\gamma} = (a_0, b_0, -c_0, -d_0)$ is the left eigenvector of A corresponding to the eigenvalue λ (considered in the previous section), then

$$\tilde{\Gamma} = (\Gamma(1), \Gamma(2), \Gamma(3), \Gamma(4)) = (-b_0, d_0, c_0, -a_0)$$

is a left eigenvector of B also corresponding to λ.

We may extend Γ, by recurrence, to a map $\Gamma : W(\varphi) \to \mathbb{R}$ as in equation (3) of §4.

LEMMA 7.2. *Let* $(-b, d, c, -a) = (-0.54, 0.34, 0.74, -0.24)$. *Given* $n \in \mathbb{N}$, *take* $C = C(n) > 0$ *such that*

$$\Delta(\Pi^n(i)) \le C, \quad i = 1, 2, 3, 4.$$

If $n \ge 1$ *then, for all* $k \in \mathbb{N}$, *the following are satisfied:*

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(a) $\Delta(\Pi^{n+k}(1)) \le C\left[1 + 6\exp\left(-\lambda^n\frac{b}{2}\right) + b^2\exp\left(-\lambda^{n+1}\frac{b}{2}\right) + \cdots + 6^k\exp\left(-\lambda^{n+k-1}\frac{b}{2}\right)\right]$;

(b) $\Delta(\Pi^{n+k}(1)) \le C\cdot 6^k$;

(c) $\Delta(\Pi^{n+k}(2)) \le C\cdot 6^k\exp[\lambda^n((\lambda^{k-1}+\lambda^{k-2}+\cdots+\lambda+1)(-b+d+c)+d)]$;

(d) $\Delta(\Pi^{n+k}(3)) \le C\cdot 6^k\exp[\lambda^n((\lambda^{k-1}+\lambda^{k-2}+\cdots+\lambda+1)(-b+d+c)+\lambda^{k-1}c+d)]$;

(e) $\Delta(\Pi^{n+k}(4)) \le C\cdot 6^k\exp[\lambda^n((\lambda^{k-1}+\lambda^{k-2}+\cdots+\lambda+1)(-b+d+c)-\lambda^k b+d)]$.

Proof. Analogous to Lemma 6.2. □

PROPOSITION 7.3. *The sum*

$$\lim_{k\to\infty}\Delta(\Pi^{n+k}(1)) = \exp(\Gamma(v_1)) + \exp(\Gamma(v_1v_2)) + \cdots + \exp(\Gamma(v_1v_2\ldots v_{n+k})) + \cdots$$

is finite.

Proof. Analogous to Proposition 6.3. □

8. *Proof of Theorem 2.2*

We know from Proposition 6.3 that $\lim_{k\to\infty}\Delta(\pi^k(4))$ is finite. Let L_1 be the sum

$$\begin{aligned}
L_1 &= \exp(\gamma(3))\cdot\exp(-\gamma(4))\cdot\lim_{k\to\infty}\Delta(\Pi^k(4)) \\
&= \exp(\gamma(3))\cdot\exp(-\gamma(4))\cdot\sum_{k=1}^{\infty}\exp(\gamma(u_1u_2\cdots u_k)) \\
&= \sum_{k=1}^{\infty}\exp(\gamma(3))\cdot\exp(-\gamma(4))\cdot\exp(\gamma(u_1))\cdot\exp(\gamma(u_2u_3\cdots u_k)); \quad u_1 = 4 \\
&= \sum_{k=1}^{\infty}\exp(\gamma(u_1u_2\cdots u_k)); \quad u_1 = 3.
\end{aligned}$$

Remark 8.1. Observe that $u = u_1u_2\cdots u_n\cdots$ with $u_1 = 3$ instead of $u_1 = 4$ is the itinerary of the positive orbit of $\beta_2(\alpha)$ with respect to R. In the following we will keep $u_1 = 3$.

Because of Proposition 7.3 we may denote by L_2 the sum

$$L_2 = \lim_{k\to\infty}\Delta(\Pi^k(1)) = \sum_{k=1}^{\infty}\exp(\Gamma(v_1v_2\cdots v_k)),$$

where $v = v_1v_2\cdots v_n\cdots$ is the itinerary of the positive orbit of $0 = \beta_0(\xi)$ with respect to $F = R^{-1}$.

We shall now define an atomic measure ν on $[0, 1)$. Let us take

$$\begin{aligned}
L &= \delta_1L_1 + \delta_2L_2 + \delta_1 + \delta_2, \quad \delta_1, \delta_2 > 0 \\
z &= 0 = \beta_0(\alpha) \\
t &= \beta_2(\alpha).
\end{aligned}$$

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So we define ν on the positive orbit of t by

$$\nu(\{R^n t\}) = \begin{cases} \dfrac{\delta_1}{L}, & n = 0 \\ \dfrac{\delta_1}{L} \cdot \exp(\gamma(u_1 u_2 \cdots u_n)), & n \geq 1, \end{cases}$$

and on the negative orbit of z (the positive orbit relative to $F = R^{-1}$) by

$$\nu(\{F^n z\}) = \begin{cases} \dfrac{\delta_2}{L}, & n = 0 \\ \dfrac{\delta_2}{L} \cdot \exp(\gamma(v_1 v_2 \cdots v_n)), & n \geq 1. \end{cases}$$

Since $z \in \mathcal{O}(\beta_1(\alpha))$ and R has no periodic points, we see that

$$\{R^n t \mid n \geq 0\} \cap \{F^n z \mid n \geq 0\} = \emptyset$$

and so ν is well defined. It is clear that ν is a probability measure. We know from Keane [**Kea**] that an IET is minimal if and only if it is positively minimal (i.e. positive orbits are dense). Since T is minimal, it follows that ν is positive on open sets.

We now define a map $\phi : [0, 1) \to [0, 1)$ by

$$\phi(x) = \nu([0, x)); \quad \phi(0) = \nu(\{0\}).$$

Since ν is positive an open sets, ϕ is strictly increasing and so it has countable many discontinuities. It is also clear that ϕ is right semi-continuous. So the set

$$G = [0, 1) \setminus \phi([0, x))$$

is the union of countable many intervals of the type $[a, b)$, which we will call *gaps*.

Define the map $h : [0, 1) \to [0, 1)$ by

$$\begin{aligned} h &= \phi^{-1} \quad \text{on } [0, 1) \setminus G; \\ h(x) &= \phi^{-1}(b) \quad a \leq x < b, \text{ for each gap } [a, b) \subset G. \end{aligned}$$

Since ϕ^{-1} is also right semi-continuous and increasing, it follows that h is continuous, non-decreasing and onto. Since the gaps come from the atomic measure ν at the orbits of z and t, let us denote these gaps by

$$\begin{aligned} G_n &= h^{-1}(R^{n-1} t), \quad n \geq 1, \\ G_{-n} &= h^{-1}(R^{-n} z), \quad n \geq 0. \end{aligned}$$

If we denote by $|I|$ the length of an interval I, then

$$|G_n| = \begin{cases} \delta_1/L \cdot \exp(\gamma(u_1 u_2 \cdots u_{n-1})), & n \geq 2 \\ \delta_1/L, & n = 1 \\ \delta_2/L, & n = 0 \\ \delta_2/L \cdot \exp(\Gamma(v_1 v_2 \cdots v_{|n|})), & n \leq -1. \end{cases}$$

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LEMMA 8.2. *With the above notation, we have:*
(a) $G_{n+1} = h^{-1}Rh(G_n)$ *for every* $n \neq 0$;
(b) $\sum_{n=-\infty}^{\infty} |G_n| = 1$.

Proof. (a) Suppose $n > 0$. Since $h(G_{-n}) = R^{-n}z$ we have that $Rh(G_{-n}) = R^{-n+1}z$, so

$$h^{-1}Rh(G_{-n}) = h^{-1}(R^{-n+1}z) = G_{-n+1}.$$

On the other hand, since $h(G_n) = R^{n-1}t$ we have that $Rh(G_n) = R^n t$ and so

$$h^{-1}Rh(G_n) = h^{-1}(R^n t) = G_{n+1}.$$

(b)

$$\begin{aligned}\sum_{n=-\infty}^{\infty} |G_n| &= \frac{1}{L}\left[\delta_2 \sum_{n=1}^{\infty} \exp\Gamma(v_1 v_2 \cdots v_n) + \delta_2 + \delta_1 + \delta_4 \sum_{n=1}^{\infty} \exp\gamma(u_1 u_2 \cdots u_n)\right] \\ &= \frac{1}{L}[\delta_2 L_2 + \delta_2 + \delta_1 + \delta_1 L_1) = 1.\end{aligned}$$

□

Define the map $T : [0, 1) \to [0, 1)$ in the following way:

$$\begin{aligned}T(G_n) &= G_{n+1} \quad \text{linearly, for each gap } G_n; \\ T(x) &= h^{-1}Rh(x) \quad \text{if } x \in [0, 1) \setminus G.\end{aligned}$$

It is clear that T is one-to-one and it is onto by Lemma 8.2.

PROPOSITION 8.3. *There exists intervals* X_i, $i = 1, 2, 3, 4, 5$, *such that:*
(a) X_i *is half open,* $\bigcup_{i=1}^{5} X_i = [0, 1)$ *and* $X_i \cap X_j = \emptyset$ *for* $i \neq j$;
(b) T' *is constant on each* X_i.

Proof. (a) Let us take $\overline{\alpha} = (\overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3, \overline{\alpha}_4, \overline{\alpha}_5)$ defined by

$$\begin{aligned}\overline{\alpha}_1 &= \phi(0) = \frac{\delta_1}{L} \\ \overline{\alpha}_2 &= \phi(\beta_1(\alpha)) - \phi(0) \\ \overline{\alpha}_3 &= \phi(\beta_2(\alpha)^-) - \phi(\beta_1(\alpha)), \quad \text{where } \phi(\beta_2(\alpha)^-) = \lim_{x \to \beta_2(\alpha)^-} \phi(x) \\ \overline{\alpha}_4 &= \phi(\beta_3(\alpha)) - \phi(\beta_2(\alpha)^-) \\ \overline{\alpha}_5 &= 1 - \phi(\beta_3(\alpha)).\end{aligned}$$

Now take $X_i = [\beta_{i-1}(\overline{\alpha}),\ \beta_i(\overline{\alpha}))$, $i = 1, 2, 3, 4, 5$.

(b) Since $X_1 = G_0$ and T is linear on G_0, then

$$T' = \frac{|T(G_0)|}{|G_0|} = \frac{|G_1|}{|G_0|} = \frac{\delta_1}{\delta_2} \quad \text{on } X_1.$$

Now take a gap $G_n \subset X_i$, $i \neq 1$; then $G_n \neq G_0$. If $n > 0$ we have $h(G_n) = R^{n-1}t$ and $u_n = i - 1$, so

$$T' = \frac{|T(G_n)|}{|G_n|} = \frac{|G_{n+1}|}{|G_n|} = \frac{\exp(\gamma(u_1 u_2 \cdots u_n))}{\exp(\gamma(u_1 u_2 \cdots u_{n-1}))} = \exp(\gamma(i-1)) \quad \text{on } G_n.$$

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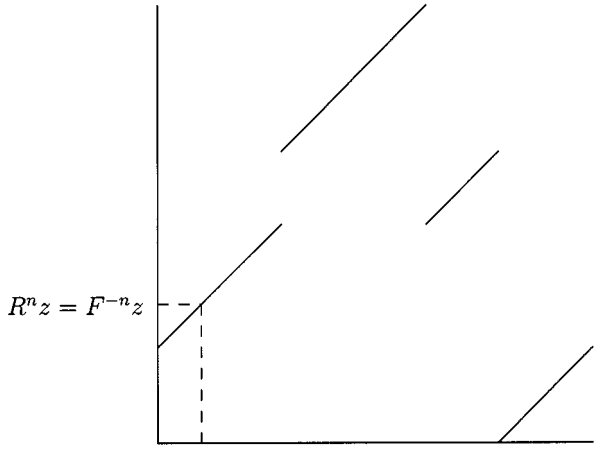


FIGURE 1.

If $n < 0$ we have $h(G_n) = R^n z = F^{-n} z$ and from Figure 1:
(1) if $R^n z = F^{-n} z \in I_1^\alpha$, then $F^{-n-1} z \in I_4^{\overline{\alpha}}$ and so $v_n = 4$;
(2) if $R^n z = F^{-n} z \in I_2^\alpha$, then $F^{-n-1} z \in I_1^{\overline{\alpha}}$ and so $v_n = 1$;
(3) if $R^n z = F^{-n} z \in I_3^\alpha$, then $F^{-n-1} z \in I_3^{\overline{\alpha}}$ and so $v_n = 3$;
(4) if $R^n z = F^{-n} z \in I_4^\alpha$, then $F^{-n-1} z \in I_2^{\overline{\alpha}}$ and so $v_n = 2$.
Thus we have $\exp \Gamma(v_n) = \exp(-\gamma(i-1))$, and so

$$\begin{aligned} T' &= \frac{|T(G_n)|}{|G_n|} = \frac{|G_{n+1}|}{|G_n|} = \frac{\exp(\Gamma(v_1 v_2 \cdots v_{|n|-1}))}{\exp(\Gamma(v_1 v_2 \cdots v_{|n|}))} \\ &= \exp(-\Gamma(v_n)) = \exp(\gamma(i-1)) \quad \text{on } G_n. \end{aligned}$$

Now, since the sum of the lengths of the gaps is one, $\sum_{n=-\infty}^{\infty} |G_n| = 1$, then

$$|X_i| = \sum_k |G_{n_k}|,$$

where the sum is over the k's for which $G_{n_k} \subset X_i$. Since $T' = \exp(\gamma(i-1))$ on each $G_{n_k} \subset X_i$ and T is continuous on X_i, then it follows that $T' = \exp(\gamma(i-1))$ on X_i, $i \neq 1$. □

Remark 8.4.
(1) The value of T' on each X_i is given by

$$\begin{aligned} \delta_1/\delta_2 &\text{ on } X_1 \\ \exp(a_0) &\text{ on } X_2 \\ \exp(b_0) &\text{ on } X_3 \\ \exp(-c_0) &\text{ on } X_4 \\ \exp(-d_0) &\text{ on } X_5. \end{aligned}$$

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(2) From Figure 1 we see also that T exchanges the intervals X_i according to the permutation $\overline{\pi} = (13)(25)$.

COROLLARY 8.5. *The map T above is a uniquely ergodic AIET such that X_1 is a wandering interval.*

Proof of Theorem 2.2. Let C be the connected component of $G^* \setminus G^s$ containing the eigenvector v which has as approximate value

$$(0.24624,\ 0.540134,\ -0.732246,\ -0.333821),$$

as seen in §6.

Given $\tilde{\gamma} \in C$, we can write $\tilde{\gamma} = w + cv$, where $w \in G^s$ and $c > 0$. Then, extending $\tilde{\gamma}$ in the usual way, we have

$$\begin{aligned} \gamma(\pi^n(i)) &= (w + cv)A^n r(i) \\ &= (w_n + c\lambda^n v)r(i) \\ &= c\lambda^n(w'_n + v)r(i), \end{aligned}$$

where $w_n \to 0$ as $n \to \infty$. So we can take n big enough so that $w'_{n+k} + v$ has components $(m_{n+k}(1),\ m_{n+k}(2),\ m_{n+k}(3),\ m_{n+k}(4))$ which satisfy

$$m_{n+k}(1) \le 0.25, \quad m_{n+k}(2) \le 0.55, \quad m_{n+k}(3) \ge -0.73, \quad m_{n+k}(4) \ge -0.33,$$

for all $k \ge 1$. If we take $C = C(n)$ such that

$$\Delta(\pi^n(i)) \le C, \quad i = 1,\ 2,\ 3,\ 4,$$

then Lemma 6.2 applies and, consequently, so does Proposition 6.3. The same can be said of Lemma 8.2 and Proposition 8.3. So the construction above goes exactly the same way, which gives us the theorem. □

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