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A note on the real zeros of the basic confluent hypergeometric function

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1. *Introduction and notation.* Some years back, Slater (4) discussed the approximations, based on the expansion in series, for the cases ${}_1F_1(a;\ b;\ x) = 0$, when either of b and x or a and x are fixed. These approximations were based essentially on the well-known Newton's method of approximation and were helpful in the numerical evaluation of the small real zeros of the confluent hypergeometric function ${}_1F_1(a; b; x)$. In this note, we deal with the corresponding problem for the basic confluent hypergeometric function ${}_1\Phi_1(a;\ b;\ x)$.

Following the usual notation, we define the basic analogue of the confluent hypergeometric function ${}_1F_1(a;\ b;\ x)$ by means of the series

$$ {}_1\Phi_1(a; b; x) = \sum_{n=0}^{\infty} \frac{(a)_n x^n q^{\frac{1}{2}n(n+1)}}{(q)_n (b)_n}, \tag{1·1} $$

where $(a)_r = (q^a)_r = (1-q^a)(1-q^{a+1})\ldots(1-q^{a+r-1})$, $(a)_0 = 1, |q| < 1$.

The q-difference operator Δ is defined by

$$ \Delta f(x) \equiv \Delta_x f(x) = \frac{f(qx) - f(x)}{x(q-1)}, \quad (0 < q < 1). \tag{1·2} $$

A repeated application of the Δ-operator n times is denoted by $\Delta_x^{(n)}$ or simply by $\Delta^{(n)}$.

For a function of two variables we shall denote the basic partial derivatives by the symbols ϑ_x and ϑ_y respectively.

In what follows we take a, b, x and q to be real, so that $0 < q < 1$.

2. *The zeros in* 'a'. We express ${}_1\Phi_1(a+\epsilon;\ b;\ x)$ as a series in ϵ in order to find the values of 'a' for which ${}_1\Phi_1(a;\ b;\ x) = 0$, when b and x are fixed.

The q-form of Taylor's theorem (2) is given by

$$ \phi(x) = \phi(a) + (x-a)\,\Delta\phi(a) + \ldots + (x-a)(x-aq)\ldots(x-aq^{n-1})\,\frac{\Delta^{(n)}\phi(a)}{(q)_n} + \ldots, \tag{2·1} $$

and consequently the basic analogue of Newton's approximation formula takes the form

$$ \epsilon = (x-a) = -\phi(a) \Big/ \left[\sum_{n=0}^{\infty} \frac{a^n}{1-q^{n+1}} \Delta^{(n+1)}\phi(a) \right]. \tag{2·2} $$

Writing (2·1) with $\phi(x) = {}_1\Phi_1(a;\ b;\ x)$, we obtain

$$ {}_1\Phi_1(a+\epsilon;\ b;\ x) = {}_1\Phi_1(a;\ b;\ x) + \epsilon \sum_{r=0}^{\infty} \frac{a^r}{(1-q^{r+1})} \Delta^{(r+1)} \left\{ \sum_{n=0}^{\infty} \frac{(a)_n x^n q^{\frac{1}{2}n(n+1)}}{(q)_n (b)_n} \right\} + O(\epsilon^2). \tag{2·3} $$

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We know that (Hahn (1), equation (2·14))

$$\Delta_x^{(r)}(1-\kappa x)_s = (-)^r\kappa^r q^{\frac{1}{2}r(r-1)}[s]\,[s-1]\ldots[s-r+1]\,(1-q^r\kappa x)_{s-r},$$

where

$$[s] = \frac{(1-q^s)}{1-q}.$$

So that, neglecting ϵ^2 and higher powers of ϵ, (2·3) reduces to

$$\epsilon = \frac{{}_1\Phi_1(a+\epsilon;\, b;\, x) - {}_1\Phi_1(a;\, b;\, x)}{A(a)},$$

where

$$A(a) = \sum_{r=0}^{\infty} \frac{a^r}{(1-q^{r+1})} \sum_{n=0}^{\infty} \frac{x^n q^{\frac{1}{2}n(n+1)}}{(q)_n\,(b)_n}\,\Delta^{(r+1)}\{(a)_n\}$$

$$= \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{r+1}a^r q^{\frac{1}{2}r(r+1)+\frac{1}{2}n(n+1)}x^n[n]\,[n-1]\ldots[n-r]}{(1-q^{r+1})\,(q)_n(b)_n}\,(aq^{r+1})_{n-r-1}.$$

Thus, given a first approximation a_0 to a zero in 'a' of ${}_1\Phi_1(a;\, b;\, x)$, we have by the relation (2·2)

$$a_1 = a_0 - \frac{{}_1\Phi_1(a_0;\, b;\, x)}{A(a_0)},$$

as our second approximation and, in general, the repetition of the above argument gives

$$a_n = a_{n-1} - \frac{{}_1\Phi_1(a_{n-1};\, b;\, x)}{A(a_{n-1})}. \tag{2·4}$$

3. *The zeros in 'b'.* Next, to find for what values of b, ${}_1\Phi_1(a;\, b;\, x) = 0$, when a and x are fixed, let us expand ${}_1\Phi_1(a;\, b+\eta;\, x)$ in powers of η. Then, as in section 2

$${}_1\Phi_1(a;\, b+\eta;\, x) = {}_1\Phi_1(a;\, b;\, x) + \eta B(b) + O(\eta^2), \tag{3·1}$$

where

$$B(b) = \sum_{n=0}^{\infty} \frac{b^n}{1-q^{n+1}} \sum_{r=0}^{\infty} \frac{(a)_r x^r q^{\frac{1}{2}r(r+1)}}{(q)_r}\,\Delta^{(n+1)}\left\{\frac{1}{(b)_r}\right\}.$$

Since

$$\frac{1}{(1+x)_\alpha} = \sum_{j=0}^{\infty} \frac{(\alpha)_j\,(-x)^j}{(q)_j}, \quad (|x| < 1),$$

and

$$\Delta^{(r)}x^i = [i]\,[i-1]\ldots[i-r+1]\,x^{i-r}, \quad (r = 1, 2, \ldots, i),$$

it follows that

$$B(b) = \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{n=0}^{(j-1)} \frac{(a)_r x^r q^{\frac{1}{2}r(r+1)}\,(q^r)_j\,[j]\,[j-1]\ldots[j-n]}{(q)_r\,(q)_j(1-q^{n+1})}\,b^{j-1}, \quad (|b| < 1).$$

Thus the required cyclic process is given by

$$b_n = b_{n-1} - \frac{{}_1\Phi_1(a;\, b_{n-1};\, x)}{B(b_{n-1})}. \tag{3·2}$$

4. Slater (4) also gave an extension of analogous expansions for the Airy integrals given by Miller (3) by developing a series for a zero in terms of its first approximation. The problem of obtaining the q-analogue of such expansions does not appear to be possible.

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In this connexion, the following formulae may, however, be noted

$$\Delta_x\left\{\frac{\phi(x)}{\psi(x)}\right\} = \frac{\psi(qx)\,\Delta\phi(x) - \phi(qx)\,\Delta\psi(x)}{\psi(x)\,\psi(qx)}, \tag{4·1}$$

$$\Delta_t\{f(x,y)\} = \vartheta_x\{f(x,qy)\}\,\Delta_t x + \vartheta_y\{f(x,y)\}\,\Delta_t y, \tag{4·2}$$

which are fairly easy to verify and are useful.

(4·2) gives the q-analogue of the total differential coefficient of $f(x,y)$ with respect to t.

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