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NORMAL SPLIT DIVISORS IN RATIONAL HOMOGENEOUS SPACES

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Abstract. We show that a divisor in a rational homogenous variety with split normal sequence is the preimage of a hyperplane section in either the projective space or a quadric.

§1. Introduction

Let X be a rational homogeneous space, and let $i : M \to X$ be a submanifold of dimension at least two.[1] We say that the M is normal split in X if the normal sequence

$$0 \to T_M \xrightarrow{\mathrm{d}i} T_X \otimes \mathcal{O}_M \to N_{M/X} \to 0$$

admits a splitting morphism $s : T_X \otimes \mathcal{O}_M \to T_M$ such that $s \circ \mathrm{d}i = \mathrm{id}_{T_M}$. It is easy to see that T_M is then globally generated and therefore M is itself a homogeneous space. In our paper [5] we used the contact structure of $\mathbb{P}(T_X)$ to show that M is actually *rational homogeneous*, that is, M has no abelian factor. If one specifies the structure of X, much more can be shown:

- If $X \simeq \mathbb{P}^n$, then M is a linear subspace [16].
- If $X \simeq Q^n \subset \mathbb{P}^{n+1}$ is a quadric, then M is a linear space or a linear section of X [8].
- If X is an irreducible Hermitian symmetric space, then so is M [4].

In view of these surprisingly short lists, it is tempting to believe that normal split submanifolds (and their embeddings) can be classified up to projective equivalence. In this article, we build on an ingenious argument of Beauville and Mérindol [1] to obtain a vast generalization of the divisorial case in van de Ven's and Jahnke's theorem.

THEOREM 1.1. *Let X be a rational homogeneous space, and let $M \subset X$ be a smooth prime divisor that is normal split. Then, there exists a locally trivial fibration $\varphi : X \to U$ such that $U \simeq \mathbb{P}^n$ or $U \simeq Q^n$ and M is the pullback of a smooth divisor $N \subset U$ that is normal split.*

Moreover, if $n \geq 3$, then N is a hyperplane section and if $n = 2$, then N is either a hyperplane section or $Y = \mathbb{P}^1 \times \mathbb{P}^1$ and there is a non-negative integer a such that N has bidegree $(1, a)$ or $(a, 1)$.

The most difficult case in the proof of this statement is when the divisor M is ample. In this case, we have $X = U$ and our goal is to show that X is the projective space or a smooth quadric. The key observation is that the divisor $M \subset X$ satisfies the technical conditions in the paper [1], and therefore, it is the fixed locus of an involution σ on X. In particular, we have a two-to-one cover $X \to X/\sigma$. If the Picard number of X is 1, we

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[1] See Remark 4.1 for a discussion of the case of curves.

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conclude using a result of Hwang and Mok [7]. If the Picard number is at least 2, we study the Mori contractions on X to obtain a contradiction (unless $\dim X = 2$ and X is a quadric).

For the general situation, note that a prime divisor M in a rational homogeneous space X always defines a basepoint-free linear system and therefore induces a fibration

$$\varphi := \varphi_{|M|} : X \to U$$

onto a variety of dimension $n := \kappa(X, M)$. Let $N \subset U$ be the smooth ample divisor such that $M = \varphi^{-1}(N)$. We show that $N \subset U$ normal split and conclude with a case distinction based on our analysis of the ample case.

§2. Notation and basic facts

We work over the complex numbers, for general definitions, we refer to [6]. Varieties will always be supposed to be irreducible and reduced. We use the terminology of [3], [9] for birational geometry and notions from the minimal model program. We follow [12] for algebraic notions of positivity.

If X is a projective variety, then $\mathrm{Aut}^\circ(X)$ denotes the connected component of $\mathrm{Aut}(X)$ containing the identity.

A fibration is a surjective projective morphism $f : X \to Y$ with connected fibers between normal varieties such that $\dim X > \dim Y$. The fibration is locally trivial with fiber F if for every $y \in Y$, there exist an analytic neighborhood $U \subset Y$ such that $f^{-1}(U)$ is biholomorphic to $U \times F$.

For a submanifold $i\colon M \hookrightarrow X$ of a complex manifold X, we will simply denote the tangent map by

$$\mathrm{d}i : T_M \to T_X \otimes \mathcal{O}_M.$$

We recall the three basic examples of normal split submanifolds.

Example 2.1. Let $\tau : X \to Y$ be a degree two branched covering between complex manifolds. Then, the ramification divisor $R \subset X$ is normal split. In fact, the tangent map $\mathrm{d}\tau : T_X \to \tau^* T_Y$ has rank $\dim X - 1$ along R and its image is $\tau^* T_B$, where $B \subset Y$ is the branch divisor. Since $\tau|_R$ is an isomorphism onto B, the map

$$T_X \otimes \mathcal{O}_R \to \tau^* T_B \simeq T_R$$

defines a splitting.

Example 2.2. Let $\tau : X \to Y$ be a fibration between complex manifolds, and let $\sigma : Y \to X$ be a section. Then, $M := \sigma(Y) \subset X$ is normal split. Indeed, since $\tau \circ \sigma = \mathrm{id}_Y$, it is clear that the restriction of the tangent map T_φ to M defines a splitting.

In particular, if $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $M \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, a)|$ is a smooth curve, then $M \subset X$ is normal split.

Example 2.3. Let $f : X \to Y$ be a fibration between rational homogeneous manifolds, and let F be a fiber of f. Then, F is normal split. Indeed, since X is homogeneous, f is smooth, and its fibers are all biholomorphic. By the Fischer–Grauert theorem, f is locally trivial and thus F is normal split. In particular, F is rational homogeneous, for example, by [5].

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§3. Normal split divisors

The goal of this section is to show the following key step toward Theorem 1.1

THEOREM 3.1. *Let X be a rational homogeneous space of dimension $n \geq 3$, and let $M \subset X$ be a smooth ample divisor that is normal split. Then, either $X \simeq \mathbb{P}^n$ or $X \simeq Q^n$. In both cases, M is a hyperplane section.*

3.1. Involutions and the case of Picard rank 1

We reformulate results in [1] in the following proposition.

PROPOSITION 3.2. *Let X be a rational homogeneous space of dimension $n \geq 3$ and let $M \subset X$ be a smooth ample divisor that is normal split. Assume that $(X, \mathcal{O}_X(M))$ is not $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Then, there is an involution of X with fixed locus M.*

Proof. By [14, Theorem 3.11] the ring of global sections $\bigoplus_{k\in\mathbb{N}} H^0(X, \mathcal{O}_X(kM))$ is generated in degree one, so M is very ample and defines an embedding

$$i : X \hookrightarrow \mathbb{P}^N$$

such that $M = X \cap H$ with $H \subset \mathbb{P}^N$ a hyperplane. By [14, Theorem 3.11(i), Remark 3.12] the variety $X \subset \mathbb{P}^N$ is an intersection of quadrics, so we can apply [1, corollaire 2, remarque (3)] to get the statement. For the convenience of the reader, let us sketch their remarkable argument: note first that the splitting map $s_M : T_X \otimes \mathcal{O}_M \to T_M$ induces a map $N_{M/X} \to T_X \otimes \mathcal{O}_M$. Since $M \in |\mathcal{O}_X(1)|$, the splitting of the normal sequence is thus given by a non-zero morphism $j : \mathcal{O}_M(1) \to T_X \otimes \mathcal{O}_M$. Consider the exact sequence

$$0 \to T_X(-1) \otimes \mathcal{O}_M \to T_{\mathbb{P}^N}(-1) \otimes \mathcal{O}_M \xrightarrow{u} N_{X/\mathbb{P}^N}(-1) \otimes \mathcal{O}_M \to 0.$$

The section j induces a section $s \in H^0(M, T_{\mathbb{P}^N}(-1) \otimes \mathcal{O}_M)$ such that $u(s) = 0$. Now consider the twisted Euler sequence restricted to M:

$$0 \to \mathcal{O}_M(-1) \to \mathcal{O}_M^{\oplus N+1} \to T_{\mathbb{P}^N}(-1) \otimes \mathcal{O}_M \to 0.$$

Since $\dim M \geq 2$, we have by Kodaira's theorem $H^0(M, \mathcal{O}_M(-1)) = H^1(M, \mathcal{O}_M(-1)) = 0$. Thus, the restriction map

$$H^0(M, \mathcal{O}_M^{\oplus N+1}) \to H^0(M, T_{\mathbb{P}^N}(-1) \otimes \mathcal{O}_M)$$

is an isomorphism. Repeating the argument for the twisted Euler sequence, we obtain an isomorphism

$$H^0(\mathbb{P}^N, T_{\mathbb{P}^N}(-1)) \cong H^0(M, T_{\mathbb{P}^N}(-1) \otimes \mathcal{O}_M).$$

Thus, up to linear coordinate change, the section s is the restriction of an Euler vector field $l(x)\frac{\partial}{\partial X_0}$ with l a linear form.

Step 1. We claim that if $x \in M$, then the projective tangent space $\mathbb{P}(T_{X,x}) \subset \mathbb{P}^N$ passes through the point $p := (1 : 0 : \cdots : 0)$.

Let $V \subset \mathbb{P}^N$ be the vanishing set of the Euler vector field, that is, the union of the hyperplane $l = 0$ and a point. Note that $M \not\subset V$ since otherwise $s = 0$. In order to show the claim, we can assume without loss of generality that $x \in (M \setminus V)$, the general case will just follow by passing to the limit.

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We can choose homogeneous polynomials $F_1, \dots, F_k$ vanishing on X such that $\mathbb{P}(T_{X,x}) = \cap_{j=1}^{k} \mathbb{P}(T_{V(F_j),x})$, where $\mathbb{P}(T_{V(F_j),x}) \subset \mathbb{P}^N$ is the hyperplane defined by the equation $\sum_{i=0}^{N} \frac{\partial F_j}{\partial X_i}(x) X_i = 0$. It is clearly sufficient to show that

$$p \in \mathbb{P}(T_{V(F_j),x}).$$

Yet this is equivalent to showing that $\frac{\partial F_j}{\partial X_0}(x) = 0$. Since $x \notin V$, the vector field $l(x)\frac{\partial}{\partial X_0}$ does not vanish in a neighborhood of x. Yet the condition $u(s) = 0$ is equivalent $s \in H^0(M, T_X(-1) \otimes \mathcal{O}_M)$, so the derivation $\frac{\partial F_j}{\partial X_0}(x)$ vanishes.

Step 2. We have $p \notin M$. We argue by contradiction. By Step 1, we have $p \in \mathbb{P}(T_{X,x})$ for every $x \in M$. Since $M = X \cap H$ is smooth, we have $\mathbb{P}(T_{M,x}) = \mathbb{P}(T_{X,x}) \cap H$ for every $x \in M$. Thus, we have $p \in \mathbb{P}(T_{M,x})$ for every $x \in M$. Take now $H_1, \dots, H_{\dim M - 1}$ general hyperplane sections passing through p. Then, $C := M \cap H_1 \cap \cdots H_{\dim M - 1}$ is a smooth curve such that $p \in \mathbb{P}(T_{C,x})$ for every $x \in C$. By [6, Chapter IV, Proposition 3.9] the curve C is a line and therefore $(X, \mathcal{O}_X(M)) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, a contradiction to our assumption.

Step 3. Conclusion. Let $\sigma : \mathbb{P}^N \to \mathbb{P}^N$ be the involution fixing pointwise the point p and the hyperplane H. Let $Q \subset \mathbb{P}^N$ be a quadric that contains X, then $\sigma(Q) = Q$ by [1, Corollary 1]. Since X is an intersection of quadrics, we obtain that $\sigma(X) = X$ and therefore the sought involution is $\sigma_X := \sigma|_X$. We are left to show that the fixed locus of σ_X is M, that is, we have to show that $p \notin X$. Yet again by [1, Corollary 1] a quadric Q with $X \subset Q$ and $p \in Q$ is singular in p. Thus, X would be an intersection of quadric cones with vertex the point p, so itself a cone. Yet X is smooth, so we get again $(X, \mathcal{O}_X(M)) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, the final contradiction. □

It is well-known that if $B \subset \mathbb{P}^N$ is a smooth hypersurface, the variety B is rational homogeneous if and only if $\deg B \leq 2$. We now combine this basic fact with a difficult result of Hwang and Mok.

COROLLARY 3.3. *Let X be a rational homogeneous space of dimension n and Picard number 1. Let $\tau\colon X \to Y$ be a degree two finite morphism ramified along a smooth divisor H of X. Then, X is a smooth quadric.*

Proof. By [7, Main Theorem] the manifold Y is isomorphic to $\mathbb{P}^n$. The ramification divisor is a rational homogeneous space because it is normal split in X. Thus, the branch locus $B \simeq R \subseteq \mathbb{P}^n$ is homogeneous and therefore of degree at most two. Moreover, B is divisible by 2 in $\mathrm{Pic}(\mathbb{P}^n)$, so B is a smooth quadric in $\mathbb{P}^n$. The covering τ is thus induced by the data $\mathcal{O}_{\mathbb{P}^n}(H)^{\otimes 2} \simeq \mathcal{O}_{\mathbb{P}^n}(B)$, where H is the hyperplane class. The ramification formula $K_X = \tau^*(-(n+1)H + H)$ implies that $-K_X$ is divisible by n. By the Kobayashi–Ochiai criterion, this shows that X is a quadric. □

COROLLARY 3.4. *Let X be a rational homogeneous space of dimension $n \geq 3$ of Picard number 1, and let $M \subset X$ be a smooth ample divisor that is normal split. Then, either $X \simeq \mathbb{P}^n$ or $X \simeq Q^n$ and M is a hyperplane section.*

Proof. If $X \simeq \mathbb{P}^n$, this is van den Ven's theorem, so assume $X \not\simeq \mathbb{P}^n$. By Proposition 3.2, there is an involution $\sigma\colon X \to X$ whose fixed locus is M. By Corollary 3.3, the variety X is a smooth quadric and M is a hyperplane section. □

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3.2. Case of higher Picard rank

The goal of this section is to show that in the setup of Theorem 3.1, the Picard number of X is 1. If $\rho(X) > 1$, Proposition 3.2 still provides an involution $\sigma : X \to X$, but the theorem of Hwang and Mok [7] does not apply. Thus, we study the Mori fibrations on X and their interaction with the involution σ to obtain a contradiction. Note that Lau [11] generalized [7] to rational homogeneous spaces $X = G/P$ with G a *simple* Lie group, thereby providing an alternative approach to this setting.

PROPOSITION 3.5. *Let X be a rational homogeneous space of Picard number at least two, and let $M \subset X$ be a smooth ample divisor that is normal split.*

- *If $\dim X = 2$ (and hence $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$), then M is a section of one of the rulings.*
- *If $\dim X \geq 3$, there exists an elementary Mori contraction $X \to Y$ induced by an extremal ray Γ such that $(K_X + M) \cdot \Gamma < 0$. Moreover, for each contraction of this type, we have $\dim X \geq 2 \dim Y$.*

Proof. Without any assumption on the dimension, observe first that $K_M = (K_X + M)|_M$ is not nef since the normal split divisor $M \subset X$ is rational homogeneous [5]. Applying the cone theorem to the Fano manifold X, we obtain that there exists an extremal ray Γ such that $(K_X + M) \cdot \Gamma < 0$, denote the corresponding contraction by $f : X \to Y$.

1st case. Assume that $\dim X - \dim Y = 1$. Then, the general fiber F is $\mathbb{P}^1$, so $K_X \cdot F = -2$ and $M \cdot F > 0$ implies that $M \cdot F = 1$. Thus, M is a rational section of f. Since M is homogeneous, the birational morphism $M \to Y$ is an isomorphism, hence, M is even a section. By [10, Appendix, Lemma (1)] this implies that $\dim X \leq 2$. Thus, we have $\dim X = 2$ and M is a section of the ruling f.

2nd case. Assume that $\dim X - \dim Y > 1$. Note that this implies $\dim X > 2$ since otherwise we would have $\rho(X) = 1$. Let F be a general fiber of F. Since $(F \cap M) \subset F$ is an ample divisor of dimension at least one, it is connected. Thus, f induces a fibration $f|_M : M \to Z$. Since M is rational homogeneous, the fibration $f|_M$ has maximal rank. Thus, a theorem of Sommese [15, Proposition V] implies that $\dim X \geq 2 \dim Y$. □

LEMMA 3.6. *Let X be a rational homogeneous space and let $f : X \to Z$ be a fibration. Then, for a fiber F, one has $\rho(F) = \rho(X) - \rho(Z)$.*

Proof. Since X is rational homogeneous, the fibration f is smooth. Thus, by [2, Theorem 2.2] the local system $R^1 f_* \mathbb{G}_m \otimes \mathbb{Q}$ is a local system on Z with finite monodromy. Since Z is simply connected, it is a trivial local system. The statement follows. □

Let us recall that if X is $\mathbb{Q}$-factorial and $X \to Y$ is a finite surjective morphism between normal varieties, then Y is $\mathbb{Q}$-factorial [9, Lemma 5.16].

LEMMA 3.7. *Let X be a $\mathbb{Q}$-factorial normal projective variety. Let $\sigma : X \to X$ be a finite order automorphism and let $\tau : X \to Y := X/\langle\sigma\rangle$ be the induced Galois cover.*

1. *If $\sigma_* : N_1(X)_{\mathbb{R}} \to N_1(X)_{\mathbb{R}}$ leaves each extremal ray of $\overline{NE}(X)$ invariant, then σ_* is the identity.*
2. *If $\sigma_* : N_1(X)_{\mathbb{R}} \to N_1(X)_{\mathbb{R}}$ is the identity map, then $\tau_* : N_1(X)_{\mathbb{R}} \to N_1(Y)_{\mathbb{R}}$ induces an isomorphism $\overline{NE}(X) \to \overline{NE}(Y)$.*

Proof. Fix a $k \in \mathbb{N}$ such that $\sigma^k = \mathrm{id}$.

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We start with (1). Let R be an extremal ray in $\overline{NE}(X)$, then by assumption $\sigma_*(R) = R$. Choose $0 \neq v \in R$, then v is an eigenvector with eigenvalue $\lambda \in \mathbb{R}$. Since $\sigma^k = id$, we have $\sigma_* v = \pm v$. Yet $\sigma_* R \subseteq \overline{NE}(X)$, so we have $\sigma_* v = v$. The extremal rays of $\overline{NE}(X)$ span $N_1(X)_{\mathbb{R}}$, therefore, σ_* is the identity.

As for (2), we first prove that $\tau^*\colon N^1(Y)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$ is an isomorphism. The map τ^* is always injective. To show that it is surjective, let $[D] \in N^1(X)_{\mathbb{R}}$. Since σ_* and therefore σ^* is the identity, we have

$$[kD] = \left(\sum_0^{k-1} (\sigma^i)^* [D] \right) = \tau^* \tau_* [D],$$

proving that $D = \tau^* \tau_* [1/kD]$.

Moreover, τ^* induces an isomorphism between the nef cones of X and Y. Passing to the dual cones, we get the claim. □

REMARK 3.8. Let $\tau : X \to Y$ be a double cover between projective manifolds of dimension $n \geq 4$. Then, τ induces an isomorphism from the ramification divisor $R \subset X$ onto the branch locus $B \subset Y$. Assume that R (or equivalently B) is an ample divisor. Since $n > 3$, we know by the Lefschetz hyperplane theorem that

$$\rho(X) = \rho(R) = \rho(B) = \rho(Y).$$

We will frequently use the following technical statement.

LEMMA 3.9. *Let X be a rational homogeneous space, and let*

$$f_i : X \to Z_i$$

be two distinct contractions of extremal rays on X, for $i = 1, 2$. Then, the map

$$f := f_1 \times f_2 : X \to Z_1 \times Z_2$$

is $Aut^\circ(X)$-equivariant and an embedding, that is, an isomorphism onto its image.

Proof. The fibrations f_i are $\mathrm{Aut}^\circ(X)$-equivariant by Blanchard's lemma, so it is clear that f is also equivariant. For every point $(z_1, z_2) \in f(X)$, we have $f^{-1}((z_1, z_2)) = f_1^{-1}(z_1) \cap f_2^{-1}(z_2)$. Since the extremal rays are distinct, their fibers intersect in at most finitely many points. This implies that $f\colon X \to f(X)$ is finite. Since $\mathrm{Aut}^\circ(X)$ acts transitively on X, the equivariance of f implies that $f(X)$ is smooth and $f : X \to f(X)$ is étale. Since $f(X)$ is rationally connected and smooth, it is simply connected. Thus, the étale map $f : X \to f(X)$ is an isomorphism. □

LEMMA 3.10. *Let X be a rational homogeneous space, and let $M \subset X$ be a smooth ample divisor. Assume that there is a finite morphism of degree two $\tau\colon X \to Y$ ramified along M such that σ_* is not the identity on $NE(X)$, where $\sigma \in Aut(X)$ is the automorphism of the cover τ. Then, $X \cong \mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. The Mori cone of a Fano manifold is generated by its extremal rays and σ_* acts by permutation on them. If the permutation is trivial, then σ_* is the identity by Lemma 3.7(1).

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Thus, we can assume that there exists an extremal ray Γ_1 such that $\Gamma_2 := \sigma_*\Gamma_1$ is distinct from Γ_1. Let $f_i\colon X \to Z_i$ for $i = 1,2$ be the two $Aut^\circ(X)$-equivariant fibrations induced by the two extremal rays of $NE(X)$. By Lemma 3.9, the map

$$f := f_1 \times f_2 : X \to Z_1 \times Z_2$$

is an isomorphism onto its image.

By the rigidity lemma, there is an isomorphism σ_Z making the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ Z_1 & \xrightarrow[\sigma_Z]{} & Z_2 \end{array}$$

Consider now the involution

$$\begin{array}{rcl} i\colon Z_1 \times Z_2 & \to & Z_1 \times Z_2 \\ (z_1, z_2) & \mapsto & (\sigma_Z^{-1} z_2, \sigma_Z z_1). \end{array}$$

Let f be the embedding of Lemma 3.9. The image $f(M) \simeq M \subset Z_1 \times Z_2$ is contained in the fixed locus of i.

Moreover, i is conjugated by the isomorphism $(id, \sigma_Z^{-1})\colon Z_1 \times Z_2 \to Z_1 \times Z_1$ to the involution

$$\begin{array}{rcl} \iota : Z_1 \times Z_1 & \to & Z_1 \times Z_1 \\ (z_1, z_2) & \mapsto & (z_2, z_1). \end{array}$$

The fixed locus of ι is the diagonal $\Delta \simeq Z_1$, so we obtain that $\dim M \leq \dim Z_1 \leq \dim X - 1$. Since M is a divisor in X, we have equality and $M \simeq Z_1$. In particular, we have $\rho(M) = \rho(Z_1) = \rho(X) - 1$.

By the Lefschetz hyperplane theorem, the restriction map $N^1(X) \to N^1(M)$ is injective if $\dim X \geq 3$. Thus, we obtain $\dim X \leq 2$ and therefore $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ (note that the case of a projective space is excluded by the condition that σ_* is not the identity). □

LEMMA 3.11. *Let X be a rational homogeneous space of dimension $n \geq 4$, and let $M \subset X$ be a smooth ample divisor. Assume that there is a finite morphism of degree two $\tau\colon X \to Y$ ramified along M such that σ_* is the identity on $NE(X)$, where $\sigma \in Aut(X)$ is the automorphism of the cover τ.*

Then, $-K_X - M$ is ample.

Proof. Since $n \geq 4$, we know by Remark 3.8 that $\rho(X) = \rho(Y)$.

Since M is rational homogeneous, it is clear that $-K_M = (-K_X - M)|_M$ is ample, but it is not so clear for $-K_X - M$ itself. Arguing by contradiction, we set $0 < \lambda \leq 1$ such that $-K_X - \lambda M$ is a nef, but non-ample divisor class. Note that in general $\lambda \in \mathbb{R}$, but since the Mori cone of the Fano manifold X is rational polyhedral, we have $\lambda \in \mathbb{Q}$. Moreover, by the cone theorem, there exists an extremal ray Γ such that $(-K_X - \lambda M) \cdot \Gamma = 0$, and we denote by

$$f : X \to T$$

its contraction. Since X is rational homogeneous, the fibration is of fiber type and a locally trivial fibration with general fiber F with $\rho(F) = 1$ (by Lemma 3.6).

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Since σ_* acts as the identity on $N_1(X)$, the image of the extremal ray $\tau_*\Gamma$ is an extremal ray in $NE(Y)$ by Lemma 3.7(2), and we denote its contraction by

$$g : Y \to U.$$

By the rigidity lemma, we have an induced morphism $\bar{\tau} : T \to U$ such that

$$\begin{array}{ccc} X & \xrightarrow{\tau} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ T & \xrightarrow[\bar{\tau}]{} & U \end{array}$$

commutes. Since $\rho(X) = \rho(Y)$, we have

$$\rho(T) = \rho(X) - 1 = \rho(Y) - 1 = \rho(U).$$

Thus, the morphism $\bar{\tau}$ is finite, and we claim that it is actually an isomorphism. We argue by contradiction. Since τ is two-to-one, it is clear that $\bar{\tau}$ is then also two-to-one. In what follows, we denote by $G = g^{-1}(u)$ a general g-fiber.

1st case. Assume that the fibers of f and g have dimension one. In this case $G \simeq \mathbb{P}^1$. Since $\bar{\tau}^{-1}(u)$ consists of two points, the preimage $\tau^{-1}(G)$ has two connected components $F_1 \cup F_2$ mapping isomorphically onto G. Yet $-K_X \cdot G = 2 = -K_Y \cdot F_i$ since they are general fibers of the fibrations. Moreover, by the ramification formula, $-K_X = \tau^*(-K_Y) - M$. Since $\tau|_{F_i}$ is an isomorphism, we obtain $M \cdot F_i = 0$, a contradiction to the ampleness of M.

2nd case. Assume that the fibers of f and g have dimension at least two. In this case, the restriction of g to $M \subset Y$ has connected fibers, since the intersection $M \cap G \subset G$ is an ample divisor. Since M is rational homogeneous (something that we do not know about Y), we obtain that $g|_M : M \to U$ is smooth and U is smooth and rational homogeneous by Blanchard's lemma.

The morphism $\bar{\tau}\colon T \to U$ is a finite morphism of degree two between rational homogeneous, and in particular Fano, varieties. Therefore, it cannot be étale.

Since $\bar{\tau}$ is of degree two and not étale, there exists by purity of branch, a prime divisor $B \subset U$ such that $\bar{\tau}^*B$ is not reduced. Thus, $f^*\bar{\tau}^*B$ has a non-reduced irreducible component. Since g is an elementary Mori contraction, the preimage $g^{-1}B$ is irreducible. As a consequence of the Graber–Harris–Starr theorem, the irreducible divisor g^*B is reduced. Yet $\tau^*g^*B = f^*\bar{\tau}^*B$ is not reduced, so g^*B is an irreducible component of the branch locus of τ. Yet τ ramifies exactly along the ample divisor M, a contradiction.

This finishes the proof of the claim that $\bar{\tau}$ is an isomorphism.

Thus, for any point $u \in U$, we have an induced degree two cover

$$\tau_F : F := f^{-1}(\bar{\tau}^{-1}(u)) \to G := g^{-1}(u)$$

that ramifies along $F \cap M$. Choose $u \in U$ general so that G is smooth. The fiber F is rational homogeneous by Example 2.3. Since $\rho(F) = 1$, we can apply Corollary 3.3 to obtain that F is a quadric Q^d and $G \simeq \mathbb{P}^d$. Moreover, the divisor $(F \cap M) \subset F$ is a hyperplane section, that is, an element of $|\mathcal{O}_Q(1)|$.

If $\dim F \geq 2$, this gives an immediate contradiction: by assumption, $K_F + \lambda M_F$ is trivial. Yet for a quadric $-K_Q \simeq \mathcal{O}_Q(\dim Q)$, so we obtain $\lambda = \dim Q > 1$, a contradiction.

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Thus, we have $\dim F = 1$, in this case, the ramification formula yields

$$2 = -K_X \cdot F = \tau^*(-K_Y) \cdot F - M \cdot F = -K_Y \cdot 2G - M \cdot F = 4 - M \cdot F,$$

and therefore $M \cdot F = 2$. Since $X \to T$ is a $\mathbb{P}^1$-bundle and and $M \to T$ is finite (since M is rational homogeneous), a result of Kollár [10, Appendix, Lemma (2)] implies $\dim T \leq 2$, the final contradiction. □

PROPOSITION 3.12. *Let X be a rational homogeneous space of dimension $n \geq 4$, and let $M \subset X$ be a smooth ample divisor. Assume that there is a finite morphism of degree two $\tau\colon X \to Y$ ramified along M such that σ_* is the identity on $NE(X)$, where $\sigma \in Aut(X)$ is the automorphism of the cover τ. Then, $\rho(X) = 1$.*

Proof. Assume by contradiction that $\rho(X) \geq 2$. Let Γ be any extremal ray in $\overline{\mathrm{NE}}(X)$, and let $f : X \to T$ be its contraction.

1st case. Assume first that $\dim X - \dim T = 1$. By Lemma 3.11, we have $(-K_X - M) \cdot F > 0$, which implies that

$$2 = -K_X \cdot F > M \cdot F \geq 1.$$

Thus, $M \cdot F = 1$, and M being rational homogeneous by example 2.1 and by [5], it is a section of f. Yet M is ample, so Kollár's result [10, Appendix, Lemma (1)] gives $\dim T \leq 1$, a contradiction.

2nd case. Assume now that $\dim X - \dim T > 1$. Applying Proposition 3.5, we see that we even have $\dim X \geq 2 \dim T$. Since $\rho(X) \geq 2$, there are at least two elementary contractions f_1 and f_2, and their general fibers F_1 and F_2 intersect in at most finitely many points. By the first case and the estimate above implies that

$$\dim F_1 = \dim F_2 = \frac{1}{2} \dim X.$$

In particular, the morphism $(f_1, f_2)\colon X \to T_1 \times T_2$ is surjective and therefore an isomorphism by Lemma 3.9.

Since σ_* is the identity, for $i = 1, 2$, there are involutions σ_i for $i = 1, 2$ and a commutative diagram

$$\begin{array}{ccc} T_1 \times T_2 & \xrightarrow{\sigma} & T_1 \times T_2 \\ \downarrow{\scriptstyle p_i} & & \downarrow{\scriptstyle p_i} \\ T_i & \xrightarrow[\sigma_i]{} & T_i \end{array}$$

But this implies that the fixed locus of σ is contained in $p_1^{-1}Fix(\sigma_1) \cap p_2^{-1}Fix(\sigma_2)$, a contradiction because the fixed locus of σ is a non-empty ample divisor. □

We are left with the three-dimensional case.

PROPOSITION 3.13. *Let X be a rational homogeneous space of dimension three, and let $M \subset X$ be a smooth ample divisor that is normal split. Then, one has $\rho(X) = 1$.*

The following proof could be carried out without the classification of Mori and Mukai [13], but a degree computation in the spirit of [1, Proposition 3] gives a more streamlined argument.

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Proof. Assume by contradiction that $\rho(X) \geq 2$ and let $f_i \colon X \to T_i$ for $i = 1,2$ be two extremal contractions. Since the fibers of f_1 and f_2 intersect in finitely many points, there are, up to renumbering f_1 and f_2, two possibilities: either $\dim T_1 = 1$ and $\dim T_2 = 2$, or $\dim T_1 = \dim T_2 = 2$.

Since $\rho(X) \geq 2$, by Proposition 3.2, there is an involution σ whose fixed locus is M. We denote by Y the quotient of X by the action of σ, and by $\tau \colon X \to Y$ the degree two morphism ramified along M. Moreover, by [5] the divisor M is rational homogeneous.

1st case: Assume that $\dim T_1 = 1$ *and* $\dim T_2 = 2$. Since X has an elementary Mori contraction onto a curve, we have $\rho(X) = 2$. Thus, Lemma 3.9 implies that $X \simeq \mathbb{P}^1 \times \mathbb{P}^2$. Since the fibers of f_1 and f_2 have different dimensions, the involution σ_* acts as the identity and by Lemma 3.7, we have $\rho(Y) = \rho(X) = 2$.

The ample divisor M is an element of $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(a,b)|$ for some $a \geq 1, b \geq 1$. Since M is rational homogeneous, the induced fibration

$$f_1 : M \to \mathbb{P}^1$$

must be a $\mathbb{P}^1$-bundle structure. In particular, by the adjunction formula, we have $b \leq 2$. Since $K_M^2 = 8$ for every Hirzebruch surface, a short intersection computation allows to exclude the case $b = 2$. Thus, we have $b = 1$ and it is not difficult to see that a general element in $\mathcal{O}_X(a,1)$ with $a \in 2\mathbb{N}$ is indeed isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. We claim that such an $M \subset X$ is never normal split and argue by contradiction. Recall that there is a degree two cover

$$\tau : \mathbb{P}^1 \times \mathbb{P}^2 \to Y$$

that ramifies along M. By the ramification formula,

$$\tau^*(-K_Y) \simeq -K_X + M$$

is ample, so Y is Fano. Moreover, we compute

$$(-K_Y)^3 = \frac{1}{2}(-K_X + M)^3 = 8(2+a).$$

Since $(-K_Y)^3 \leq 64$ for any smooth Fano threefold [13, Corollary 11], we see that $a = 2$ or $a = 4$.

If $a = 2$, we have $(-K_Y)^3 = 32$, so by [13, Table 2] the threefold Y is a blow-up of $\mathbb{P}^3$. Yet we know that the Mori contractions of Y are fiber spaces, a contradiction.

If $a = 4$, we have $(-K_Y)^3 = 48$, so by [13, Table 2] the threefold Y is the flag manifold $\mathbb{P}(T_{\mathbb{P}^2})$. Yet for this threefold, both Mori contractions have one-dimensional fibers, a contradiction to $\dim T_1 = 1$.

2nd case: Assume that $\dim T_1 = \dim T_2 = 2$. The surfaces T_1 and T_2 are rational homogeneous of the same Picard rank. Since M is a rational homogeneous surface, we have $\rho(M) \leq 2$. We get by the Lefschetz theorem,

$$\rho(T_i) = \rho(X) - 1 \leq \rho(M) - 1 \leq 1.$$

Thus, $T_1 \cong T_2 \cong \mathbb{P}^2$. Now, the Mori–Mukai classification (or Lemma 3.9) implies that $X \simeq \mathbb{P}(T_{\mathbb{P}^2}) \subset \mathbb{P}^2 \times \mathbb{P}^2$.

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Since $M \subset X$ is ample, we have $M \in |\mathcal{O}_X(a,b)|$ with $a \geq 1, b \geq 1$. Again, recall that there is a degree two morphism $\tau\colon X \to Y$ ramified along M. The threefold Y is Fano and

$$(-K_Y)^3 = \frac{1}{2}(-K_X + M)^3 = \frac{3}{2}(2+a)(2+b)(4+a+b).$$

Since $a \geq 1, b \geq 1$, we get $(-K_Y)^3 \geq 81$, a contradiction. □

§4. Proof of Theorem 1.1

Proof. Our goal is to reduce Theorem 1.1 to Theorem 3.1 using a diagram chase in the spirit of [8, Section 1.4].

Since M is effective and X is homogeneous, the divisor M is base-point-free, and induces a morphism $\varphi\colon X \to U$. Moreover, there is a smooth ample divisor N on U such that $M = \varphi^{-1}N$. After taking the Stein factorization $\varphi = \nu \circ \psi$ and replacing φ with ψ and N with $\nu^{-1}N$, we may assume that φ is a fibration.

Step 1. We claim that N is normal split in U. Consider the diagram

$$\begin{array}{ccccc}
T_{M/N} & \xrightarrow{h} & T_{X/U} & & \\
\downarrow{\scriptstyle \alpha_M} & & \downarrow{\scriptstyle \alpha} & & \\
T_M & \underset{\tau_M}{\overset{s}{\rightleftarrows}} & T_X \otimes \mathcal{O}_M & \longrightarrow & N_{M/X} \\
\downarrow{\scriptstyle d\varphi_M} & & \downarrow{\scriptstyle d\varphi} & & \downarrow \\
\varphi^* T_N & \longrightarrow & \varphi^* T_U \otimes \mathcal{O}_M & \longrightarrow & \varphi^* N_{N/U}
\end{array}$$

where s gives the splitting of the normal sequence of M in X. We notice that h is an isomorphism. For a local section v of $\varphi^* T_U \otimes \mathcal{O}_M$, we want to define by diagram chase $\tilde{s}_U(v) = d\varphi(s(w))$, where w is a local section of T_X such that $d\varphi(w) = v$. To show that $\tilde{s}_U$ is well defined, we assume $w \in \ker(d\varphi)$. Let u be a local section of $T_{X/U}$ such that $\alpha(u) = w$ and u' such that $h(u') = u$. We want to prove that $d\varphi_M(s(w)) = 0$. We have

$$d\varphi_M(s(w)) = d\varphi_M(s(\alpha(h(u')))) = d\varphi_M(s(i(\alpha_M(u')))) = d\varphi_M(\alpha_M(u')) = 0.$$

We defined $\tilde{s}_U \in \mathrm{Hom}(\varphi^* T_N, \varphi^* T_U \otimes \mathcal{O}_M) \cong H^0(U, \varphi^*(T_U \otimes \mathcal{O}_N \otimes T_N^\vee))$ such that $\varphi^* \tau_N \circ \tilde{s}_U = id$. Since φ is a fibration and by the projection formula, this yields a section $s_U \in H^0(U, T_U \otimes \mathcal{O}_N \otimes T_N^\vee)$ such that $\tau_N \circ s_U = id$.

Step 2. Conclusion. We make a case distinction in terms of $n := \dim U$.

- $n = 1$; equivalently, $U \simeq \mathbb{P}^1$. Then, M is a fiber of the locally trivial fibration φ and obviously normal split.
- $n = 2$; equivalently, $U \simeq \mathbb{P}^2$ or $U \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Then, N is completely determined by Van de Ven's theorem and Proposition 3.5.
- Finally, if $n \geq 3$, we know by Theorem 3.1 that U is the projective space or a quadric, and N is a hyperplane section. □

Remark 4.1. In this article, we have focused on the case of divisors, that is, the smallest possible codimension. One might ask if the classification can also be carried out for normal split submanifolds $M \subset X$ of dimension one. In this case $M \simeq \mathbb{P}^1$, so the only question is to determine the embedding. Unfortunately, there are often many possible choices for the embedding: let $X = Q^n \subset \mathbb{P}^{n+1}$ be a smooth quadric and let $S \subset X$ be a linear section of

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codimension $n-2$, that is, $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is a smooth quadric surface in X. Then, S is normal split in X [8, Theorem 4.5]. Moreover, let $M \subset S$ be an irreducible element of $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,a)|$ for *any* $a \in \mathbb{N}$. Then, $M \subset S$ is normal split by Example 2.2, and therefore, $M \subset X$ is normal split by [8, Section 1.2].

In view of this example, we think that a more reasonable approach is to classify normal split submanifolds of minimal codimension.

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