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Three-manifolds as branched covers

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Abstract. This note describes a new, and possibly more informative, proof of the fact that any closed orientable 3-manifold is a branched cover over the 3-sphere.

It is a long established theorem of Alexander(1) that, if M is a closed orientable 3-manifold, M is a branched cover over S^3 with branching index at most two. This means that there is a function $f\colon M \to S^3$ and a link L in S^3 such that

$$f\colon M - f^{-1}L \to S^3 - L$$

is a covering, and, when restricted to a neighbourhood of any component of $f^{-1}L$, f is either an embedding or a standard branched double covering map of $D^2 \times S^1$ by $D^2 \times S^1$. Here is an elementary proof of this fact, which yields a little more information, and which takes for its starting point the fact that M can be obtained from S^3 by surgery.

A preparatory lemma will assert the existence of a certain branched cover of $D^2 \times S^1$ by $(S^1 \times I) \times S^1$. There is a classical method of obtaining branched covers of planar regions by 'cutting and cross-joining'. In particular, an $S^1 \times I$ is obtained by cutting two copies of D^2 along the two copies of a single interior arc and cross-joining. If $D^2 \times S^1$ is obtained from $D^2 \times I$ by identifying $(x, 0)$ with $(-x, 1)$ and the above operation is performed on each $D^2 \times t$ where $t \in I$, then the resultant space is $(S^1 \times I) \times S^1$. In effect this $(S^1 \times I) \times S^1$ has been obtained by taking two copies of $D^2 \times S^1$ and cutting and cross-joining along a Möbius strip. This is the branched covering now to be constructed in some detail; its interest lies in the behaviour of the covering map on the two boundary components of $(S^1 \times I) \times S^1$.

Lemma. *Let S^1 be the boundary of the unit disc D^2 in the complex plane C. There exists a branched double covering map*

$$p\colon (S^1 \times I) \times S^1 \to D^2 \times S^1$$

such that

(i) $p((e^{i\theta}, 1), e^{i\phi}) = (e^{i\theta}, e^{i\phi})$,

(ii) $p((e^{i\theta}, 0), e^{i\phi}) = (e^{i(\phi-\theta)}, e^{i\phi})$,

(iii) *the branching locus is* $\{(\pm \tfrac{4}{5}e^{i\theta}, e^{i2\theta});\ 0 \leqslant \theta \leqslant \pi\}$, *a simple closed curve in* $D^2 \times S^1$.

Proof. Let $f\colon C - \{0\} \to C$ be defined by $f(z) = 2(z + z^{-1})$. This function is a branched double cover branched over the two points 4 and -4, and

$$f(re^{i\theta}) = 2(r + r^{-1})\cos\theta + i2(r - r^{-1})\sin\theta.$$

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Let A be the annulus $\{z; \frac{1}{2} \leqslant |z| \leqslant 2\}$ and let $E = fA$. Then E is the elliptically shaped region defined by

$$E = \{5r\cos\theta + \mathrm{i}3r\sin\theta;\ -\pi \leqslant \theta \leqslant \pi,\ 0 \leqslant r \leqslant 1\},$$

and $f\colon A \to E$ is still a branched double cover. In terms of this parametrization of E, a homeomorphism $\beta\colon E \to D^2$ is defined by

$$\beta(5r\cos\theta + i3r\sin\theta) = r\,e^{i\theta}.$$

Another homeomorphism $\alpha\colon S^1 \times I \to A$ is given by the formula

$$\alpha(e^{i\theta}, s) = 2^{-1}(3s+1)\,e^{i\theta}.$$

Now consider the composition of the following functions,

$$(S^1 \times I) \times I \xrightarrow{\hat{\alpha}} A \times I \xrightarrow{f\times 1} E \times I \xrightarrow{\hat{\beta}} D^2 \times I,$$

where $\hat{\alpha}$ and $\hat{\beta}$ are homeomorphisms defined by

$$\hat{\alpha}(x,t) = (e^{-it\pi}\alpha x, t)$$

$$\hat{\beta}(y,t) = (e^{it\pi}\beta y, t).$$

The fact that $f(-z) = -f(z)$ implies that $\hat{\beta}(f\times 1)\,\hat{\alpha}$ is compatible with identifying $(x, 0)$ with $(x, 1)$ in $(S^1 \times I) \times I$ and $(z, 0)$ with $(z, 1)$ in $D^2 \times I$. By means of these identifications $\hat{\beta}(f\times 1)\,\hat{\alpha}$ gives a function

$$p\colon (S^1 \times I) \times S^1 \to D^2 \times S^1,$$

where the second and third S^1's are parametrized by t, $0 \leqslant t \leqslant 1$, with 0 and 1 identified. It can easily be verified from the above formulae that

$$p(e^{i\theta}, 1, t) = (e^{i\theta}, t),$$

$$p(e^{i\theta}, 0, t) = (e^{2\pi i t - i\theta}, t).$$

Identification of ϕ with $2\pi t$ yields properties (i) and (ii). Note that $f\times 1$ is a branched double cover, branched over the two line intervals $\pm 4 \times I$. Thus p is also a branched double cover, branched over the image (under the identification) of $\hat{\beta}(\pm 4 \times I)$ in $D^2 \times S^1$, and

$$\hat{\beta}(\pm 4 \times I) = \{(\pm\tfrac{4}{5}\,e^{it\pi}, t);\ 0 \leqslant t \leqslant 1\}.$$

Any homeomorphism of $S^1 \times S^1$ to itself is determined up to isotopy by the automorphism it induces on $H_1(S^1 \times S^1) \cong (Z \oplus Z)$. The map p of the lemma is the identity on one boundary component of $S^1 \times I \times S^1$ and is described by the matrix $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ on the other; this follows at once from (i) and (ii).

In (2), (3), and (4) the following result is proved. Let M be a closed orientable 3-manifold, then there exists a set of disjoint solid tori (copies of $D^2 \times S^1$) $T_1, T_2, \ldots, T_n$ contained in M, and a set of disjoint solid tori $U_1, U_2, \ldots, U_n$ contained in S^3, such that there is a homeomorphism $h\colon M - \bigcup_1^n \mathring{T}_i \to S^3 - \bigcup_1^n \mathring{U}_i$. The U_i can be taken to be regular neighbourhoods of unknotted, but probably linking, simple closed curves in S^3, and $h(\partial T_i) = \partial U_i$. Further, generators of the first homology groups can be chosen so that, in $H_1(\partial T_i)$ the generator $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ corresponds to a curve which bounds a disc in T_i, in

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$H_1(\partial U_i)$ the generators $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ correspond to curves which bound discs in U_i and $S^3 - \mathring{U}_1$ respectively, and, for each i, $h_*: H_1(\partial T_i) \to H_1(\partial U_i)$ is given by the matrix $\begin{pmatrix}1 & -1\\1 & 0\end{pmatrix}$.

THEOREM (*Alexander*). *Any closed orientable* 3*-manifold is a branched cover over the* 3*-sphere with branching index at most two.*

Proof. The proof uses the above notation. It is required to construct a branched covering map $f: M \to S^3$. Let $f\left|\left(M - \bigcup_1^n \mathring{T}_i\right)\right.$ be h. $f\,|\,T_i$ is defined in the following way. Let $\partial T_i \times I$ be a collar of ∂T_i in T_i with $\partial T_i \times 1$ identified with ∂T_i; the generators chosen for $H_1(\partial T_i \times 1)$ transfer in an obvious way to generators of $H_1(\partial T_i \times O)$. Similarly let $\partial U_i \times I$ be a collar of ∂U_i in U_i. Now, identifying $\partial U_i \times I$ with $(S^1 \times I) \times S^1$ and U_i with $D^2 \times S^1$ in a generator preserving way, let $f\,|\,\partial T_i \times I$ be

$$p((f\,|\,\partial T_i) \times 1)\colon \partial T_i \times I \to U_i,$$

where p is the branched double cover of the lemma. Then $f\,|\,\partial T_i \times O$ is a homeomorphism of $\partial T_i \times O$ on to ∂U_i inducing a homology isomorphism given by

$$\begin{pmatrix}-1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}.$$

Thus f maps a curve $\left(\text{corresponding to } \begin{pmatrix}1\\0\end{pmatrix}\right)$ in $\partial T_i \times O$ which bounds a disc in

$$T_i - (\partial T_i \times (0, 1])$$

to a curve $\left(\text{corresponding to } \begin{pmatrix}0\\1\end{pmatrix}\right)$ in ∂U_i which bounds a disc in $S^3 - \mathring{U}_i$. Hence f can be extended over the rest of T_i by a homeomorphism of $T_i - (\partial T_i \times I)$ on to $S^3 - U_i$.

Note. The branched cover $f: M \to S^3$ has the following properties.

(i) There is a link L in S^3, the branching locus, with n components $L_1, \ldots, L_n$ such that $f: M - f^{-1}L \to S^3 - L$ is an $(n+1)$-fold covering.

(ii) If $N(L_i)$ is a small regular neighbourhood of L_i then, for each i, $f^{-1}N(L_i)$ consists of n disjoint solid tori; $n-1$ of these are mapped homeomorphically by f on to $N(L_i)$, on the other f is equivalent to the standard branched double cover g: $D^2 \times S^1 \to D^2 \times S^1$ defined by $g(z, t) = (z^2, t)$.

(iii) The L_i are unknotted in S^3. Further, L can be obtained by taking a link L' of the type described in (4) and, for each component L_i' of L', by letting L_i be the boundary of an isotopically standard Möbius strip within a small regular neighbourhood of L_i' with L_i' as its centre curve.

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