An appraisal of Lalanne's nomogram

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A number of methods are available for the solution of cubic equations. Amongst these, and possibly less well known than those based on algebraic or trigonometrical procedures, is an elegant graphical solution devised by Lalanne [1]. His approach and the resulting nomogram are appraised in this paper.

The general cubic equation in the variable X

$$X^3 + a_1X^2 + a_2X + a_3 = 0 \tag{1}$$

is considered. In developing a solution, this form is invariably simplified by substituting

$$X = x - \frac{a_1}{3} \tag{2}$$

to give

$$x^3 + px + q = 0 \tag{3}$$

where

$$p = a_2 - \frac{a_1^2}{3} \tag{4}$$

and

$$q = \frac{2a_1^3}{27} - \frac{a_1a_2}{3} + a_3. \tag{5}$$

Lalanne's tactic was to see (3) as a linear relationship between the variables p and q in which, for a particular value of x, x^3 was 'the constant term' and x itself was a simple multiplier on p. With x specified, the relationship plots, on p, q axes, as a straight line. The family of such lines shown in Figure 1 for a range of x values, comprises the nomogram. By locating the intersection of the p, q coordinates for a particular cubic equation, the corresponding x value(s) (i.e. the root(s) of the equation) can readily be read off.

A prominent feature of the nomogram is that over a considerable area (in fact for all positive p values and for much of the negative p space) there is only one set of x lines and therefore only one root of the equation is obtained. However, within a chevron-shaped region straddling the negative p-axis, three x lines pass through each point, giving three roots for the cubic equation. It is also clear that, in cases where there are three real roots, one will be of opposite sign to q and two will be of the same sign. A single real root will always be of opposite sign to q.

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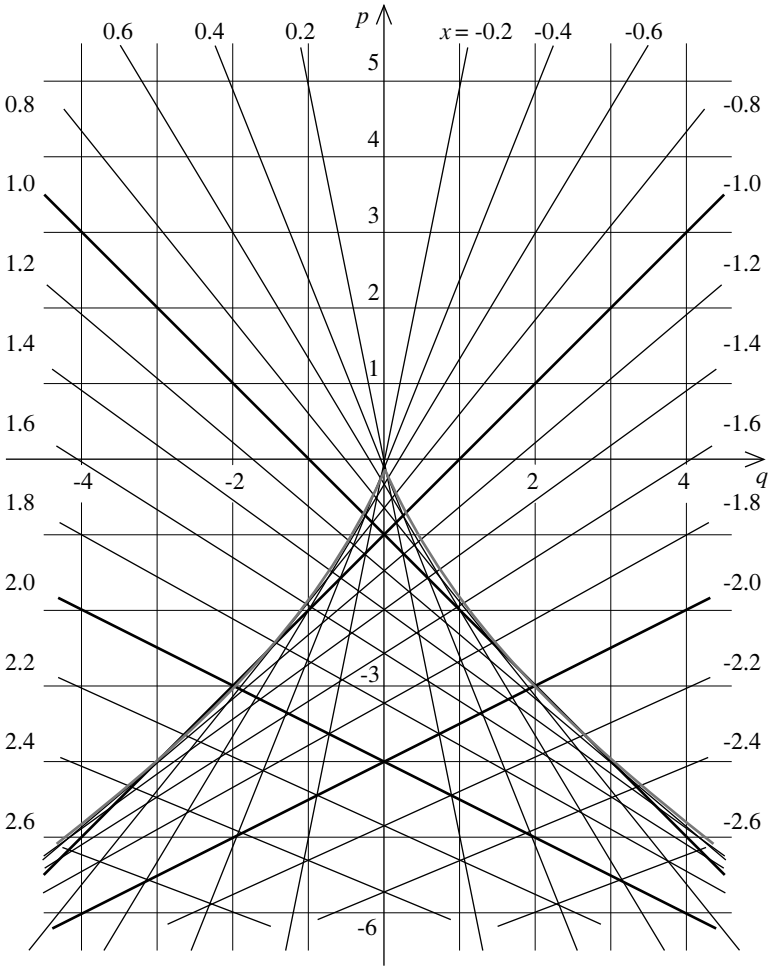


FIGURE 1: Lalanne's nomogram

The form of the curve separating the three-root zone from the one-root zone is of interest. In graphical terms, if the function $y = x^3 + px^2 + q$ is plotted as a graph on the cartesian coordinates x, y, then the real roots of (3) are the x coordinates of the points of intersection of the curve with the x-axis. It is clear that as x approaches infinity so does y, and that as x approaches minus infinity then again so does y. Therefore the curve must intersect the x-axis at least once and the cubic equation (3) must have at least one real root. Also, there may or may not be two turning points on the curve. If there are not two such points, then it will have only one intersection with the x-axis and (3) will have only one real root. But if the curve does have two turning points and they are on opposite sides of the x-axis, then there will be two additional intersections, making three real roots

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in all. This condition is conveniently expressed as the requirement that the product of the y values corresponding to the two turning points is negative. It follows that for three real roots

$$q^2 + \frac{4p^3}{27} \leqslant 0. \tag{6}$$

(The two curves representing this relationship are shown in Figure 1.) Otherwise there is only one real root.

In using the nomogram for the solution of a specific equation, interpolation may be necessary and a particular result may be inexact. This is incidental. The value of the nomogram lies as much in the overall picture it affords as in its use for solving equations. It provides a 'landscape' view of the root characteristics for the full range of cubic equations, showing the extent of the two distinct solution regimes and the boundary between them. The effects of changes in any of the variables are clearly evident, notably the decreasing 'resolution' of the plot in the three-root region as p approaches zero. The information content is remarkable.

In bringing together strands of algebra and analytical geometry, the nomogram is elegant, concise and efficient, and it might be argued that it deserves to be more widely known. (It is not mentioned in the Penguin dictionary [2].) As an educational device, for example, it would seem to have very considerable attractions.

It is convenient to supplement the nomogram of Figure 1, for cases in which $(q^2 + 4p^3/27)$ is positive, by an additional graph from which the imaginary part of the two complex roots of (3) can be obtained.

The two complex roots of (3) (x_2 and x_3) can be derived as

$$\begin{aligned} x_2 &= -\frac{x_1}{2} + i\sqrt{-\frac{x_1^2}{4} - \frac{q}{x_1}} \\ x_3 &= -\frac{x_1}{2} - i\sqrt{-\frac{x_1^2}{4} - \frac{q}{x_1}} \end{aligned} \tag{7}$$

in which x_1 is the real root of (3) and i denotes $\sqrt{-1}$. (See Appendix.)

The imaginary part of the complex roots, I, is given by

$$I = \sqrt{-\frac{x_1^2}{4} - \frac{q}{x_1}}$$

from which

$$q = -x_1\left(\frac{x_1^2}{4} + I^2\right). \tag{8}$$

Several possibilities are available for the graphical development of (8), with or without (3). A graph consisting of x_1 contours on q, I^2 axes can be constructed from (8) alone and has the advantage that the x_1 contours are

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straight lines. This graph has been developed; the construction procedure was as follows. For a given x_1 value and two values of I^2, q is derived from (8). The x_1 contour is defined by the q, I^2 pairs. Then x_1 is changed and the process repeated. The resulting plot of $|x_1|$-contours on $|q|$, I^2-axes is shown in Figure 2. (Modular quantities are used for x_1 and q so that one graph covers both the q negative, x_1 positive and the q positive, x_1 negative combinations.)

The graph of I^2 versus I has been added adjacent to the I^2-axis in Figure 2 so that the graphical determination of I can be completed.

Having determined the real root, x_1, of (3) from Figure 1, and knowing q from (3), the derivation of I^2 and thence I from Figure 2 is straightforward. The two complex roots of (3), x_2 and x_3, follow from (7).

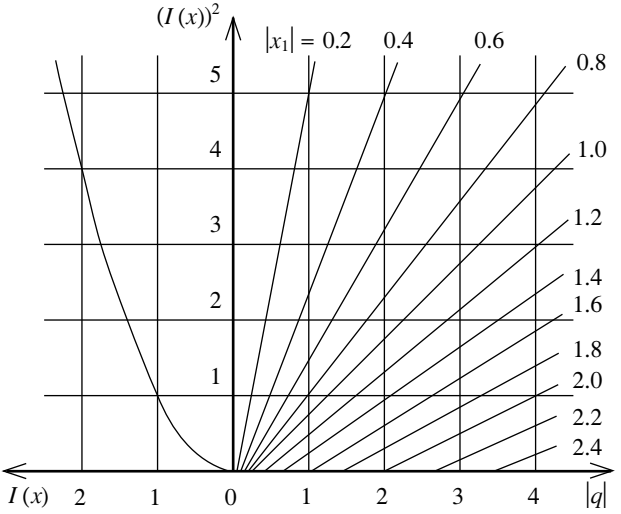


FIGURE 2

Appendix

Following what is essentially Cardano's treatment [2], (3) is transformed into a quadratic equation in a secondary variable, w^3, using the substitution

$$x = w - \frac{p}{3w}. \tag{A1}$$

The resulting quadratic is

$$\left(w^3\right)^2 + qw^3 - \frac{p^3}{27} = 0$$

which solves as

$$w^3 = -\frac{q}{2} \pm \frac{q}{2}\sqrt{1 + \frac{4p^3}{27q^2}}$$

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whence

$$w = \left[-\frac{q}{2} \pm \frac{q}{2}\sqrt{1 + \frac{4p^3}{27q^2}} \right]^{1/3} \tag{A2}$$

and the x values follow from (A1).

If $q^2 + 4p^3/27$ is positive (see Figure 1), then one of the roots of (3) (x_1 say) is real; the other two (x_2 and x_3) are complex.

In such a case, (3) may be expressed in the form

$$(x - x_1)(x^2 + gx + h) = 0 \tag{A3}$$

in which g and h are new coefficients.

Expressions for g and h in terms of x_1 and q are obtained by equating the coefficients of like powers of x in the expanded version of (A3) and (3). These are substituted for g and h in the quadratic part of (A3), which is then solved to give x_2 and x_3 in the form

$$\begin{aligned} x_2 &= -\frac{x_1}{2} + i\sqrt{-\frac{x_1^2}{4} - \frac{q}{x_1}} \\ x_3 &= -\frac{x_1}{2} - i\sqrt{-\frac{x_1^2}{4} - \frac{q}{x_1}} \end{aligned}. \tag{7}$$

References

1. I. Gratton-Guinness, *The Fontana history of mathematical sciences*, Fontana Press (1997).
2. D. Nelson (ed.), *The Penguin dictionary of mathematics*, (2nd edn.), Penguin Books (1998).

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