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On a system of hyperstable frequency locking persistence under white noise

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Abstract. This paper studies a special system of N-coupled oscillators under white noise. It is shown that we can get a positive recurrent diffusion process by winding the solution of this system on a cylinder along a special direction. Thus defining the rotation number as the limit of the frequency of the oscillator, we can prove that the rotation number exists and the frequency locking is successful no matter how large the white noise is.

1. *Introduction*

The dynamical behaviour of various systems of coupled oscillators has caused much interest in recent years. Such systems are used in describing the behaviour of real physical phenomena such as the motion of Josephson junction arrays [**2**] and the modelling of oscillating chemical reactions [**3**]. They are also used in neural networks for pattern recognition to simulate our brains' binding ability which connects separate information such as images, voices, colours, etc., [**6, 9**].

The systems of coupled oscillators with the property of frequency locking have uses in the segmentation of connected images, i.e. if each segment is represented by locally coupled oscillators and different segments are decoupled, then all oscillators within one segment go to a common limit frequency by the frequency-locking property. The segmentation can be done by choosing proper initial frequencies such that different segments have different limit frequencies [**5**]. The following special system was considered in [**4**]:

$$\begin{cases} \dot{\phi}_1 = -\sin\phi_1 + K(\phi_2 - \phi_1) + I \\ \dot{\phi}_2 = -\sin\phi_2 + K(\phi_1 + \phi_3 - 2\phi_2) \\ \qquad \cdots \\ \dot{\phi}_j = -\sin\phi_j + K(\phi_{j-1} + \phi_{j+1} - 2\phi_j) \\ \qquad \cdots \\ \dot{\phi}_N = -\sin\phi_N + K(\phi_{N-1} - \phi_N) \end{cases} \tag{1.1}$$

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where ϕ_j $(1 \leq j \leq N)$ is the phase angle of the jth oscillator, $I \geq 0$ and $K \geq 1$. This is a system of N oscillators with nearest-neighbour coupling on a line. It was shown in [**4**] that for any solution $\Phi(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_N(t))^T$ of (1.1) the limits $\lim_{t\to\infty} \phi_j(t)/t (j = 1, 2, \ldots, N)$ are convergent, identical and independent of the initial values of $\phi_j(0)$ $(j = 1, 2, \ldots, N)$. So in the definite case all the oscillators tend to oscillate with the same frequency after a sufficiently long period of time, that is to say, the frequency locking is successful. The rotation number of this system was defined as $\lim_{t\to\infty} \phi_1(t)/2\pi t$.

Dynamical systems of coupled oscillators are simulated by computers in practical use. If small stochastic perturbations cause the frequency-locking property of definite systems to disappear, the segmentation may fail because there are always some errors on computers. So it is necessary to consider the stochastic perturbed dynamical systems of coupled oscillators. Some simulations made on computers show that some systems which have the frequency-locking property in definite cases keep this property under small stochastic perturbations.

In this paper, a white noise is introduced to system (1.1). To be precise, we consider a stochastic system of coupled oscillators as follows:

$$\begin{cases} d\phi_1(t) = (-\sin\phi_1 + I_1 + K(\phi_2 - \phi_1))\,dt + D\,dW_1(t) \\ d\phi_2(t) = (-\sin\phi_2 + I_2 + K(\phi_1 + \phi_3 - 2\phi_2))\,dt + D\,dW_2(t) \\ \qquad \cdots \\ d\phi_j(t) = (-\sin\phi_j + I_j + K(\phi_{j-1} + \phi_{j+1} - 2\phi_j))\,dt + D\,dW_j(t) \\ \qquad \cdots \\ d\phi_N(t) = (-\sin\phi_N + I_N + K(\phi_{N-1} - \phi_N))\,dt + D\,dW_N(t), \end{cases} \tag{1.2}$$

where $\phi_j (1 \leq j \leq N)$ is the phase angle of the jth oscillator under white noise, $I_j \in \mathbf{R}$ $(j = 1, 2, \ldots, N)$, K and D are two positive constants and $\mathbf{W}(t) = (W_1(t), W_2(t), \ldots, W_N(t))^T$ is a Brownian motion on $\mathbf{R}^N$. If $D = 0$ and $I_2 = I_3 = \cdots = I_N = 0$ then system (1.2) is nothing but system (1.1).

We will prove that the frequency-locking property of system (1.1) is invariant under white noise, i.e. for any solution $\Phi(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_N(t))^T$ of (1.2), $(\phi_j(t)/t)$ $(j = 1, 2, \ldots, N)$ are convergent and the limits are identical and independent of the initial values of $\phi_j(0)$ $(j = 1, 2, \ldots, N)$ no matter how large D is. So the frequency locking is still successful in this case and we can define the rotation number of the system as $V = \lim_{t\to\infty} \phi_1(t)/2\pi t$.

Obviously, the solution $\{\Phi(t)\}_{t\geq 0}$ of (1.2) is a diffusion process on $\mathbf{R}^N$. In §2, we prove that a positive recurrent diffusion process can be obtained by winding $\{\Phi(t)\}_{t\geq 0}$ on a cylinder along a special direction. In §3, we define the rotation number and prove its existence by ergodicity.

2. *Positive recurrence*

Rewrite the system in matrix form

$$d\Phi = (\mathbf{F}(\Phi(t)) - K\mathbf{M}\Phi(t))\,dt + D\,d\mathbf{W}(t), \tag{2.1}$$

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in which

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix}, \quad \mathbf{F}(\Phi) = \begin{pmatrix} -\sin\phi_1 + I_1 \\ -\sin\phi_2 + I_2 \\ \vdots \\ -\sin\phi_N + I_N \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 1 \end{pmatrix}.$$

Here $\Phi \in \mathbf{R}^N$, $\mathbf{F}$ is a non-linear mapping from $\mathbf{R}^N$ to $\mathbf{R}^N$ and $\mathbf{M}$ is a $N \times N$ tridiagonal matrix.

Obviously, $\mathbf{M}$ can be diagonalized by an orthonormal matrix $\mathbf{P} = (p_{ij})_{N\times N}$.

$$\mathbf{P}^T\mathbf{M}\mathbf{P} = \mathbf{D} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_N \end{pmatrix} \tag{2.2}$$

with $p_{j1} = 1/\sqrt{N}$, $\lambda_j = 4\sin^2[(j-1)\pi/2N]$, $j = 1, 2, \dots, N$. Let

$$\Phi = \mathbf{P}\Psi, \tag{2.3}$$

where $\Psi = (\psi_1, \psi_2, \dots, \psi_N)^T$. (2.1) becomes

$$d\Psi(t) = (\mathbf{G}(\Psi(t)) - K\mathbf{D}\Psi(t))\,dt + D\,d\mathbf{B}(t), \tag{2.4}$$

in which

$$\mathbf{G}(\Psi) = \begin{pmatrix} G_1(\Psi) \\ G_2(\Psi) \\ \vdots \\ G_N(\Psi) \end{pmatrix} = \mathbf{P}^T\mathbf{F}(\Phi) = \begin{pmatrix} \sum_{i=1}^N p_{i1}(I_i - \sin\phi_i) \\ \sum_{i=1}^N p_{i2}(I_i - \sin\phi_i) \\ \vdots \\ \sum_{i=1}^N p_{iN}(I_i - \sin\phi_i) \end{pmatrix},$$

$$\phi_i = \sum_{j=1}^N p_{ij}\psi_j, \quad i = 1, 2, \dots, N,$$

and $\mathbf{B}(t) = (B_1(t), B_2(t), \dots, B_N(t))^T$ is still a Brownian motion on $\mathbf{R}^N$.

We want to prove the existence of a rotation number (a precise definition will be given in §3) by the ergodic theory of Markov processes. This requires proving the positive recurrency of the Markov processes. Since $\{\Psi(t)\}_{t\geq 0}$ is an orthonormal transformation of $\{\Phi(t)\}_{t\geq 0}$, we can consider $\{\Psi(t)\}_{t\geq 0}$ instead of $\{\Phi(t)\}_{t\geq 0}$. The drift coefficient of each ψ_j $(j = 1, 2, \dots, N)$ is composed of a non-linear bounded term and a linear term $-K\lambda_j\psi_j$. ψ_j $(j = 2, \dots, N)$ would be positive recurrent as $\lambda_j > 0$, but ψ_1 would not be positive recurrent as $\lambda_1 = 0$. So $\{\Psi(t)\}_{t\geq 0}$ would not be positive recurrent. To avoid this problem, we adopt a technique as follows.

Notice that $\mathbf{G}(\Psi) - K\mathbf{D}\Psi$ is $2\sqrt{N}\pi$ periodic in ψ_1 because of $\lambda_1 = 0$, so we can wind $\{\Psi(t)\}_{t\geq 0}$ on a cylinder. To do this, let

$$\widetilde{\psi}_1(t) = \psi_1(t) \bmod 2\sqrt{N}\pi, \quad \widetilde{\psi}_j(t) = \psi_j(t), \quad j = 2, \dots, N,$$
$$\widetilde{\Psi}(t) = (\widetilde{\psi}_1(t), \widetilde{\psi}_2(t), \dots, \widetilde{\psi}_N(t))^T. \tag{2.5}$$

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Then $\{\widetilde{\Psi}(t)\}_{t\geq 0}$ is a diffusion process on the cylinder $S^1 \times \mathbf{R}^{N-1}$ ($S^1 = [0, 2\sqrt{N}\pi]$ with 0 and $2\sqrt{N}\pi$ identical) [**1**] and

$$d\widetilde{\Psi}(t) = (\mathbf{G}(\widetilde{\Psi}(t)) - K\mathbf{D}\widetilde{\Psi}(t))\,dt + D\,d\widetilde{\mathbf{B}}(t), \tag{2.6}$$

where $\{\widetilde{\mathbf{B}}(t)\}_{t\geq 0}$ is a BM on $S^1 \times \mathbf{R}^{N-1}$.

In this section, we will prove the following theorem.

THEOREM 2.1. *$\{\widetilde{\Psi}(t)\}_{t\geq 0}$ is a positive recurrent diffusion process on $S^1 \times \mathbf{R}^{N-1}$.*

To prove Theorem 2.1, we only need to prove the existence of an invariant distribution of $\{\widetilde{\Psi}(t)\}_{t\geq 0}$. To do this, we need the following two lemmas.

LEMMA 2.1. *For $\mathbf{x} = (x_1, x_2, \ldots, x_N)^T \in S^1 \times \mathbf{R}^{N-1}$,*

$$\sup_{t\geq 0} E\left(\sum_{i=2}^{N} (\widetilde{\psi}_i^{\mathbf{x}}(t))^2\right) < +\infty, \tag{2.7}$$

where $\widetilde{\Psi}^{\mathbf{x}}(t) = (\widetilde{\psi}_1^{\mathbf{x}}(t), \widetilde{\psi}_2^{\mathbf{x}}(t), \ldots, \widetilde{\psi}_N^{\mathbf{x}}(t))^T$ is the solution to (2.6) with $\widetilde{\Psi}(0) = \mathbf{x}$.

Proof. For $i = 2, \ldots, N$, since

$$d\widetilde{\psi}_i(t) = (G_i(\widetilde{\Psi}(t)) - K\lambda_i\widetilde{\psi}_i(t))\,dt + D\,d\widetilde{B}_i(t), \tag{2.8}$$

we have

$$\begin{aligned} &d(e^{\lambda_i K t}\widetilde{\psi}_i(t)) \\ &\quad = (\lambda_i K e^{\lambda_i K t}\widetilde{\psi}_i(t) + e^{\lambda_i K t}G_i(\widetilde{\Psi}(t)) - \lambda_i K e^{\lambda_i K t}\widetilde{\psi}_i(t))\,dt + De^{\lambda_i K t}\,d\widetilde{B}_i(t) \\ &\quad = e^{\lambda_i K t}G_i(\widetilde{\Psi}(t))\,dt + De^{\lambda_i K t}\,d\widetilde{B}_i(t). \end{aligned} \tag{2.9}$$

Hence

$$\widetilde{\psi}_i^{\mathbf{x}}(t) = x_i e^{-\lambda_i K t} + e^{-\lambda_i K t}\int_0^t e^{\lambda_i K u}G_i(\widetilde{\Psi}^{\mathbf{x}}(u))\,du + D\widetilde{U}_i(t), \tag{2.10}$$

where $\widetilde{U}_i(t) = e^{-\lambda_i K t}\int_0^t e^{\lambda_i K u}\,d\widetilde{B}_i(u)$ is a Ornstein–Uhlenbeck process. As

$$G_i(\Psi) = \left|\sum_{j=1}^{N} p_{ji}\left(I_j - \sin\left(\sum_{l=1}^{N} p_{jl}\psi_l\right)\right)\right| \leq N\left(\sup_{1\leq j\leq N}|I_j| + 1\right) \triangleq C_0 \tag{2.11}$$

and

$$E(\widetilde{U}_i(t))^2 = e^{-2\lambda_i K t}\int_0^t e^{2\lambda_i K u}\,du = \frac{e^{-2\lambda_i K t}(e^{2\lambda_i K t} - 1)}{2\lambda_i K}, \tag{2.12}$$

we have

$$\begin{aligned} E(\widetilde{\psi}_i^{\mathbf{x}}(t))^2 &\leq 3\left(x_i^2 e^{-2\lambda_i K t} + C_0^2 e^{-2\lambda_i K t}\left(\int_0^t e^{\lambda_i K u}\,du\right)^2 + D^2 E(U_i(t))^2\right) \\ &= 3\left(x_i^2 e^{-2\lambda_i K t} + \left(\frac{C_0}{\lambda_i K}\right)^2 + \frac{D^2 e^{-2\lambda_i K t}(e^{2\lambda_i K t} - 1)}{2\lambda_i K}\right). \end{aligned} \tag{2.13}$$

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It follows that

$$\sup_{t\geq 0} E\left(\sum_{i=2}^{N}(\widetilde{\psi}_i^{\mathbf{x}}(t))^2\right) < \infty. \qquad \square$$

Now denoting the transition probability of $\{\widetilde{\Psi}(t)\}_{t\geq 0}$ by $\{p(u,\mathbf{x},d\mathbf{y}), t \geq 0\}$, we have the following.

LEMMA 2.2. *For $\mathbf{x} \in S^1 \times \mathbf{R}^{N-1}$, the family $\{t^{-1}\int_0^t p(u,\mathbf{x},d\mathbf{y})\,du,\ t > 0\}$ is tight.*

Proof. For arbitrary $C > 0$,

$$\begin{aligned}
&\frac{1}{t}\int_0^t P\left(u,\mathbf{x},\left\{\mathbf{y}:\left(\sum_{i=2}^{N} y_i^2\right)^{1/2} > C\right\}\right)du \\
&\quad= \frac{1}{t}\int_0^t P\left(\left\{\sum_{i=2}^{N}(\widetilde{\psi}_i^{\mathbf{x}}(u))^2 > C\right\}\right)du \\
&\quad\leq \frac{1}{t}\int_0^t \frac{E\left(\sum_{i=2}^{N}(\widetilde{\psi}_i^{\mathbf{x}}(u))^2\right)}{C^2}\,du \quad \text{(by Chebyshev's inequality)} \\
&\quad\leq \frac{1}{C^2}\sup_{t\geq 0} E\left(\sum_{i=2}^{N}(\widetilde{\psi}_i^{\mathbf{x}}(t))^2\right). \qquad (2.14)
\end{aligned}$$

By Lemma 2.1,

$$\sup_{t\geq 0} E\left(\sum_{i=2}^{N}(\widetilde{\psi}_i^{\mathbf{x}}(t))^2\right) < \infty,$$

therefore there exists C_ϵ such that

$$\frac{1}{t}\int_0^t P\left(u,\mathbf{x},\left\{\mathbf{y}:\left(\sum_{i=2}^{N} y_i^2\right)^{1/2} > C_\epsilon\right\}\right)du < \epsilon \qquad (2.15)$$

for given $\epsilon > 0$. It follows that

$$\left\{\frac{1}{t}\int_0^t p(u,\mathbf{x},d\mathbf{y})\,du : t > 0\right\}$$

is tight since

$$\left\{\mathbf{y}:\left(\sum_{i=2}^{N} y_i^2\right)^{1/2} \leq C_\epsilon\right\}$$

is a compact set in $S^1 \times \mathbf{R}^{N-1}$. $\square$

Proof of Theorem 2.1. It follows from the tightness of

$$\left\{\frac{1}{t}\int_0^t p(u,\mathbf{x},d\mathbf{y})\,du : t > 0\right\}$$

that there exists a sequence $t_n \to \infty$ and a probability measure π on $S^1 \times \mathbf{R}^{N-1}$ such that

$$\frac{1}{t_n}\int_0^{t_n} p(u,\mathbf{x},d\mathbf{y})\,du \xrightarrow{\text{weakly}} \pi(d\mathbf{y}) \quad \text{as } n \to \infty, \qquad (2.16)$$

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i.e. for every bounded continuous function $f : S^1 \times \mathbf{R}^{N-1} \to \mathbf{R}$,

$$\frac{1}{t_n}\int_0^{t_n} P_u f(\mathbf{x})\,du \longrightarrow \int_{S^1\times\mathbf{R}^{N-1}} f(\mathbf{y})\pi(d\mathbf{y}), \tag{2.17}$$

where $\{P_t : t \geq 0\}$ is the transition operator of $\{\widetilde{\Psi}(t)\}_{t\geq 0}$.

For $s > 0$, $P_s f$ is a bounded continuous function by the Feller property. Therefore, applying (2.17) to $P_s f$,

$$\begin{aligned}\frac{1}{t_n}\int_0^{t_n} P_{s+u} f(\mathbf{x})\,du &= \frac{1}{t_n}\int_0^{t_n} P_u(P_s f)(\mathbf{x})\,du \\ &\longrightarrow \int_{S^1\times\mathbf{R}^{N-1}} P_s f(\mathbf{y})\pi(d\mathbf{y}) \quad \text{as } n\to\infty.\end{aligned} \tag{2.18}$$

On the other hand,

$$\begin{aligned}\frac{1}{t_n}\int_0^{t_n} P_{s+u} f(\mathbf{x})\,du &= \frac{1}{t_n}\int_s^{t_n+s} P_u f(\mathbf{x})\,du \\ &= \frac{1}{t_n}\left(\int_0^{t_n} P_u f(\mathbf{x})\,du - \int_0^{s} P_u f(\mathbf{x})\,du + \int_{t_n}^{t_n+s} P_u f(\mathbf{x})\,du\right) \\ &\longrightarrow \int_{S^1\times\mathbf{R}^{N-1}} f(\mathbf{y})\pi(d\mathbf{y}) \quad \text{as } n\to\infty.\end{aligned} \tag{2.19}$$

It follows from (2.18) and (2.19) that

$$\int_{S^1\times\mathbf{R}^{N-1}} P_s f(\mathbf{y})\pi(d\mathbf{y}) = \int_{S^1\times\mathbf{R}^{N-1}} f(\mathbf{y})\pi(d\mathbf{y}), \tag{2.20}$$

i.e. if $\widetilde{\Psi}(0)$ has distribution π, then $Ef(\widetilde{\Psi}(t)) = Ef(\widetilde{\Psi}(0))$ for all $t \geq 0$. In other words, π is an invariant distribution. A diffusion process is either transient or recurrent and if there exists an invariant measure then it is recurrent. Now we have specified an invariant distribution of $\{\widetilde{\Psi}(t)\}_{t\geq 0}$, therefore $\{\widetilde{\Psi}(t)\}_{t\geq 0}$ is positive recurrent and π is the only invariant distribution [**7**]. □

It is shown in the proof of Theorem 2.1 that the weak limit of $t^{-1}\int_0^t p(u,\mathbf{x},d\mathbf{y})\,du$ is the only invariant distribution of $\{\widetilde{\Psi}(t)\}_{t\geq 0}$ for a fixed $\mathbf{x} \in S^1 \times \mathbf{R}^{N-1}$.

From above, 0 is an eigenvalue of $\mathbf{M}$ and the dimension of its eigenspace is 1. The direction of the eigenvector of $\mathbf{M}$ corresponding to 0 is the only direction which would cause $\{\Psi(t)\}_{t\geq 0}$ to be not positive recurrent. So the diffusion process obtained by winding $\{\Psi(t)\}_{t\geq 0}$ on a cylinder along this direction is positive recurrent.

3. *Rotation numbers*

Definition 3.1. $\{\mathbf{X}(t)\}_{t\geq 0}$ is a diffusion process on $\mathbf{R}^N$. If $t^{-1}X_i(t) \xrightarrow{\text{a.e.}} x_i$ as $t \to \infty$, $i = 1, 2, \ldots, N$, then $V_i(\mathbf{X}) \stackrel{\triangle}{=} x_i/2\pi$ is called the rotation number of $\{\mathbf{X}(t)\}_{t\geq 0}$ along the ith direction. Furthermore, if $V_1(\mathbf{X}) = V_2(\mathbf{X}) = \cdots = V_N(\mathbf{X}) \stackrel{\triangle}{=} V(\mathbf{X})$, then $V(\mathbf{X})$ is called the rotation number of $\{\mathbf{X}(t)\}_{t\geq 0}$.

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The existence of a rotation number along the ith direction means that the ith oscillator has a limit frequency. The existence of a rotation number implies that all oscillators have a common limit frequency, i.e. the frequency locking is successful. If the rotation number of $\{\Psi(t)\}_{t\geq 0}$ exists, system (1.1) will keep the frequency-locking property under white noise and the segmentation can be done by computer.

Now we prove that the rotation number of $\{\Phi(t)\}_{t\geq 0}$ exists. To this end, we consider $\{\Psi(t)\}_{t\geq 0}$ at first.

LEMMA 3.1. $\lim_{t\to\infty} t^{-1}\psi_1(t)$ *exists and is independent of its initial value.*

Proof. It follows from (2.4) and (2.5) that

$$\begin{aligned}\psi_1(t) &= \psi_1(0) + \int_0^t G_1(\Psi(u))\,du + DB_1(t)\\ &= \psi_1(0) + \int_0^t G_1(\widetilde{\Psi}(u))\,du + DB_1(t).\end{aligned} \tag{3.1}$$

By the ergodicity of $\{\widetilde{\Psi}(t)\}_{t\geq 0}$, we have

$$\frac{1}{t}\int_0^t G_1(\widetilde{\Psi}(u))\,du \xrightarrow{\text{a.e.}} \int_{S^1\times\mathbf{R}^{N-1}} G_1(\mathbf{y})\pi(d\mathbf{y}), \tag{3.2}$$

where π is the invariant distribution of $\{\widetilde{\Psi}(t)\}_{t\geq 0}$. At the same time, we have

$$\frac{B_1(t)}{t} \xrightarrow{\text{a.e.}} 0 \quad \text{as } t\to\infty \tag{3.3}$$

by the iterated law of logarithm. Therefore

$$\begin{aligned}\frac{1}{t}\psi_1(t) &= \frac{1}{t}\psi_1(0) + \frac{1}{t}\int_0^t G_1(\widetilde{\Psi}(u))\,du + \frac{1}{t}DB_1(t)\\ &\xrightarrow{\text{a.e.}} \int_{S^1\times\mathbf{R}^{N-1}} G_1(\mathbf{y})\pi(d\mathbf{y}) \quad \text{as } t\to\infty.\end{aligned}$$

□

LEMMA 3.2. *The limit* $\lim_{t\to\infty}\psi_i(t)/t = 0$ *for* $i = 2,\dots,N$.

Proof. For $i = 2,\dots,N$, since

$$d\psi_i(t) = (G_i(\Psi(t)) - K\lambda_i\psi_i(t))\,dt + D\,dB_i(t), \tag{3.4}$$

we have

$$d(e^{\lambda_i Kt}\psi_i(t)) = e^{\lambda_i Kt}G_i(\Psi(t))\,dt + De^{\lambda_i Kt}\,dB_i(t), \tag{3.5}$$

hence

$$\psi_i(t) = \psi_i(0)e^{-\lambda_i Kt} + e^{-\lambda_i Kt}\int_0^t e^{\lambda_i Ku}G_i(\Psi(u))\,du + DU_i(t), \tag{3.6}$$

where $U_i(t) = e^{-\lambda_i Kt}\int_0^t e^{\lambda_i Ku}\,dB_i(u)$ is a Ornstein–Uhlenbeck process. By (2.11)

$$\left|\frac{1}{t}e^{-\lambda_i Kt}\int_0^t e^{\lambda_i Ku}G_i(\Psi(u))\,du\right| \leq \frac{C_0}{\lambda_i Kt} \longrightarrow 0 \quad \text{as } t\to\infty. \tag{3.7}$$

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On the other hand,

$$\frac{1}{t}U_i(t) \xrightarrow{\text{a.e.}} 0 \quad \text{as } t \to \infty \tag{3.8}$$

by the iterated law of logarithm. It follows from (3.7) and (3.8) that

$$\begin{aligned}\frac{1}{t}\psi_i(t) &= \frac{1}{t}\psi_i(0)e^{-\lambda_i K t} + \frac{1}{t}e^{-\lambda_i K t}\int_0^t e^{\lambda_i K u}G_i(\Psi(u))\,du + \frac{D}{t}U_i(t)\\ &\xrightarrow{\text{a.e.}} 0 \quad \text{as } t \to \infty\end{aligned}$$

for $i = 2, \ldots, N$. □

Combining Lemmas 3.1 and 3.2, we have the following.

THEOREM 3.1. *The rotation number of* $\{\Psi(t)\}_{t\geq 0}$ *along the* i*th direction exists and is independent of its initial value for* $i = 1, 2, \ldots, N$. *Moreover,*

$$V_i(\Psi) = 0, \quad i = 2, \ldots, N. \tag{3.9}$$

Now the following theorem follows easily.

THEOREM 3.2. *The rotation number of* $\{\Phi(t)\}_{t\geq 0}$ *exists.*

Proof. Since $\Phi = \mathbf{P}\Psi$, by Theorem 3.1 and noticing that $p_{i1} = 1/\sqrt{N}$, $i = 1, 2, \ldots, N$, we have

$$\frac{\Phi(t)}{t} \xrightarrow{\text{a.e.}} \mathbf{P}\begin{pmatrix} V_1(\Psi) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{V_1(\Psi)}{\sqrt{N}} \\ \frac{V_1(\Psi)}{\sqrt{N}} \\ \vdots \\ \frac{V_1(\Psi)}{\sqrt{N}} \end{pmatrix}, \tag{3.10}$$

hence the rotation number of $\{\Phi(t)\}_{t\geq 0}$ exists by Definition 3.1. □

The above theorem asserts that the frequency locking persists under a white noise which could be very large.

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Appendix

There is no method with which to calculate the rotation number of system (1.2) for $N \geq 2$ at present. But the rotation number of the system in $N = 1$ case could help us to recognize some properties about it in the $N \geq 2$ case because of the frequency-locking property. The system in the $N = 1$ case is studied in detail in [**8**]. To be precise, system

$$d\phi(t) = (I - \sin\phi(t))\,dt + D\,dB(t) \tag{A.1}$$

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is considered, where $I \geq 0$ and $B(t)$ is a Brownian motion on $\mathbf{R}$. The rotation number of system (A.1) is continuous as $D \to 0$. Moreover, when the input frequency I is not greater than 1, the rotation number is 0 when $D = 0$ and is a positive number when $D > 0$ [**8**]. This is an interesting phenomenon.

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