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# STOCHASTIC ORDERINGS BETWEEN $p$-SPACINGS OF GENERALIZED ORDER STATISTICS FROM TWO SAMPLES

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The concept of generalized order statistics was introduced as a unified approach to a variety of models of ordered random variables. The purpose of this article is to investigate conditions on the distributions and the parameters on which the generalized order statistics are based to establish the likelihood ratio ordering of general $p$-spacings and the hazard rate and the dispersive orderings of (normalizing) simple spacings from two samples. We thus strengthen and complement some results in Franco, Ruiz, and Ruiz [7] and Belzunce, Mercader, and Ruiz [5]. This article is a continuation of Hu and Zhuang [10].

## 1. INTRODUCTION

Generalized order statistics (GOSs) have been of interest during the last few years because they are more flexible in statistical modeling and inference (see Cramer and Kamps [6] and references therein). The concept of GOSs was introduced by Kamps [11,12] as a unified approach to a variety of models of ordered random variables (r.v.s), which was defined through uniform GOSs.

DEFINITION 1.1: (see Kamps [11]): *Let* $n \in \mathbb{N}$, $k \geq 1$, $m_1, \ldots, m_{n-1} \in \Re$, *and* $M_r = \sum_{j=r}^{n-1} m_j$, $1 \leq r \leq n-1$, *be parameters such that* $\gamma_{r,n} = k + n - r + M_r \geq 1$ *for all* $r = 1, \ldots, n-1$, *and let* $\tilde{m} = (m_1, \ldots, m_{n-1})$ *if* $n \geq 2$ ($\tilde{m}$ *arbitrary if* $n = 1$). *If the r.v.'s* $U(r, n, \tilde{m}, k)$, $r = 1, \ldots, n$, *possess a joint density of the form*

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$$f_{U(1,n,\tilde{m},k),\ldots,U(n,n,\tilde{m},k)}(u_1,\ldots,u_n) = k\left(\prod_{j=1}^{n-1}\gamma_{j,n}\right)\left(\prod_{i=1}^{n-1}(1-u_i)^{m_i}\right)(1-u_n)^{k-1}$$

*on the cone* $0 \le u_1 \le u_2 \le \cdots \le u_n < 1$ *of* $\Re^n$*, then they are called uniform GOSs. Now let F be an arbitrary distribution function. The r.v.'s,*

$$X(r,n,\tilde{m},k) = F^{-1}(U(r,n,\tilde{m},k)), \qquad r = 1,\ldots,n,$$

*are called the GOSs based on F, where* $F^{-1}$ *is the inverse of F defined by* $F^{-1}(u) = \sup\{x : F(x) \le u\}$ *for* $u \in [0,1]$*. In the particular case* $m_1 = \cdots = m_{n-1} = m$*, the above r.v.'s are denoted by* $U(r,n,m,k)$ *and* $X(r,n,m,k)$*,* $r = 1,\ldots,n$*, respectively.*

Throughout this article, we consider the special case of GOSs: $m_1 = \cdots = m_{n-1} = m$, in which the marginal distribution and density functions of the $r$th GOS have closed forms, and $\gamma_{r,n} = k + (n-r)(m+1)$ for each $r$.

Ordinary order statistics of a random sample from a distribution $F$ are a particular case of GOSs when $k = 1$ and $m_r = 0$ for all $r = 1,\ldots,n-1$. When $k = 1$ and $m_r = -1$ for all $r = 1,\ldots,n-1$, then we get the first $n$ record values from a sequence of independent and identically distributed (i.i.d.) r.v.'s with distribution $F$. Choosing the parameters appropriately, several other models of ordered r.v.'s are seen to be particular cases (see Kamps [12], Balakrishnan, Cramer, and Kamps [2], and Belzunce, Mercader, and Ruiz [5]).

Let $X$ and $Y$ be two nonnegative r.v.'s with associated distribution functions $F$ and $G$, and let $X(r,n,m,k)$ and $Y(r,n,m,k)$, $r = 1,\ldots,n$, be GOSs based on $F$ and $G$, respectively. For a given positive integer $p$, $p \le n$, let

$$V_{r,n}^{(p)} = X(r+p-1,n,m,k) - X(r-1,n,m,k), \qquad r = 1,\ldots,n-p+1, \quad \textbf{(1)}$$

$$\widetilde{V}_{r,n} = \gamma_{r,n}[X(r,n,m,k) - X(r-1,n,m,k)], \qquad r = 1,\ldots,n, \quad \textbf{(2)}$$

denote respectively the $p$-spacings and the normalizing (simple) spacings of the GOSs $\{X(i,n,m,k),\ i = 1,\ldots,n\}$. Here, $X(0,n,m,k) \equiv 0$. For $p = 1$, 1-spacings are simple spacings in the literature. Similarly, define

$$W_{r,n}^{(p)} = Y(r+p-1,n,m,k) - Y(r-1,n,m,k), \qquad r = 1,\ldots,n-p+1, \quad \textbf{(3)}$$

$$\widetilde{W}_{r,n} = \gamma_{r,n}[Y(r,n,m,k) - Y(r-1,n,m,k)], \qquad r = 1,\ldots,n. \quad \textbf{(4)}$$

Stochastic comparisons of ordinary order statistics, record values, and their spacings have been studied extensively by many authors during the last 10 years. Because of the similarity of some structural properties among ordinary order statistics, record values, and GOSs, it is natural and interesting to study stochastic properties of GOSs and their spacings. Some recent works on this subject are by Franco, Ruiz, and Ruiz [7], Khaledi [15], Hu and Zhuang [8–10], Khaledi and Kochar [17], and Belzunce et al. [5]. For the comparison of spacings of GOSs from

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two samples, Belzunce et al. [5] in their Theorem 4.4 proved that if $X \leq_{\mathrm{hr}} Y$ and $m \geq -1$ and if either $X$ or $Y$ is DFR (decreasing failure rate), then

$$V_{r,n}^{(p)} \leq_{\mathrm{st}} W_{s,l}^{(p)} \quad \text{whenever } p \geq 1, s \geq r \text{ and } s - r \geq l - n, \tag{5}$$

where $\leq_{\mathrm{hr}}$ and $\leq_{\mathrm{st}}$ denote the hazard rate and the usual stochastic orders, respectively (the formal definitions of the stochastic orders and of aging notion mentioned in this section can be found in Section 2). Franco et al. [7] proved that if $X \leq_{\mathrm{lr}} Y$ and $m \geq 0$ and if either $X$ or $Y$ is DFR, then

$$V_{r,n}^{(1)} \leq_{\mathrm{hr}} [\leq_{\mathrm{disp}}] W_{s,l}^{(1)} \quad \text{whenever } s \geq r \text{ and } s - r \geq l - n \tag{6}$$

and

$$\widetilde{V}_{r,n} \leq_{\mathrm{hr}} [\leq_{\mathrm{disp}}] \widetilde{W}_{s,l} \quad \text{whenever } s \geq r \text{ and } s - r \geq l - n. \tag{7}$$

The purpose of this article is to investigate conditions on the distributions and the parameters on which the GOSs are based to establish relationship (5) in the likelihood ratio order and to obtain (6) and (7) for $m \geq -1$. We thus extend some recent results on ordinary order statistics and record values. This article is a continuation of Hu and Zhuang [10] which is devoted to stochastic properties of $p$-spacings of GOSs from one sample. In Section 2, we recall the definitions of some stochastic orders and of some aging notions and we give some useful lemmas, which will be used in Section 3, where the main results are given.

Throughout, the terms "increasing" and "decreasing" mean "nondecreasing" and "nonincreasing," respectively. All ratios are well defined, and $a/0$ is understood to be $\infty$ whenever $a > 0$. All integrals and expectations are implicitly assumed to exist whenever they are written.

## 2. PRELIMINARIES

### 2.1. Stochastic Orders and Aging Notions

Some stochastic orders and aging notions that will be used in this article are recalled in the following two definitions.

DEFINITION 2.1: *Let $X$ and $Y$ be two r.v.'s with respective survival functions $\bar{F}$ and $\bar{G}$. We say that $X$ is smaller than $Y$*

- *in the usual stochastic order, denoted by $X \leq_{\mathrm{st}} Y$, if $\bar{F}(t) \leq \bar{G}(t)$ for all $t$*
- *in the hazard rate order, denoted by $X \leq_{\mathrm{hr}} Y$, if $\bar{G}(t)/\bar{F}(t)$ is increasing in $t$*
- *in the likelihood ratio order, denoted by $X \leq_{\mathrm{lr}} Y$, if $X$ and $Y$ have respective density functions (or mass functions) $f$ and $g$ and if $g(t)/f(t)$ is increasing in $t$*
- *in the dispersive order, denoted by $X \leq_{\mathrm{disp}} Y$, if $F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$ whenever $0 \leq \alpha < \beta \leq 1$.*

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The relationships among these orders are shown in the following (see Shaked and Shanthikumar [22] and Müller and Stoyan [20]):

$$X \leq_{\mathrm{lr}} Y \Rightarrow X \leq_{\mathrm{hr}} Y \Rightarrow X \leq_{\mathrm{st}} Y.$$

DEFINITION 2.2: *Let $X$ be a nonnegative r.v. with distribution function $F$. $X$ or $F$ is said to be*

- *ILR (increasing likelihood ratio) [DLR (decreasing likelihood ratio)] if its density function $f(x)$ exists and is logconcave [logconvex] in $x \in \Re_+$*
- *IFR (increasing failure rate) [DFR (decreasing failure rate)] if $\bar{F}(x)$ is logconcave [logconvex] in $x \in \Re_+$.*

*Remark 2.1*: Bagai and Kochar [1] proved that if $X \leq_{\mathrm{hr}} Y$ and either $X$ or $Y$ is DFR, then $X \leq_{\mathrm{disp}} Y$.

*Remark 2.2*: If $f$ is logconcave, then $F$ and $\bar{F}$ are also logconcave (see Barlow and Proschan [3, p. 77]). If $f$ is logconvex, then $\bar{F}$ is also logconvex while $F$ is logconcave (see Sengupta and Nanda [21]). Therefore, ILR $\Rightarrow$ IFR and DLR $\Rightarrow$ DFR.

## 2.2. Some Useful Lemmas

The following lemmas are useful in deriving the main results of this article.

LEMMA 2.1 (Misra and van der Meulen [19]): *Let $\Theta$ be a subset of the real line $\Re$ and let $X$ be a nonnegative r.v. having a distribution function belonging to the family $\mathcal{P} = \{H(\cdot|\theta), \theta \in \Theta\}$, which satisfies*

$$H(\cdot|\theta_1) \leq_{\mathrm{st}} [\geq_{\mathrm{st}}]\, H(\cdot|\theta_2) \quad \text{whenever } \theta_1, \theta_2 \in \Theta \text{ and } \theta_1 < \theta_2.$$

*Let $\Psi(x,\theta)$ be a real-valued function defined on $\Re \times \Theta$. Then*

*(i) $\mathbb{E}_\theta[\Psi(X,\theta)]$ is increasing in $\theta$ if $\Psi(x,\theta)$ is increasing in $\theta$ and increasing [decreasing] in $x$;*

*(ii) $\mathbb{E}_\theta[\Psi(X,\theta)]$ is decreasing in $\theta$ if $\Psi(x,\theta)$ is decreasing in $\theta$ and decreasing [increasing] in $x$.*

To state the next lemmas, we define the function $\delta_m : [0,1) \to \Re$, $m \in \Re$, by

$$\delta_m(x) = \begin{cases} \dfrac{1}{m+1}[1-(1-x)^{m+1}], & m \neq -1 \\ -\log(1-x), & m = -1. \end{cases} \tag{8}$$

LEMMA 2.2: *Let $X$ be a nonnegative r.v. with distribution function $F(\cdot)$ and hazard rate function $\lambda(\cdot)$. If $X$ is DLR and $m \geq 0$, or if $X$ is DLR, $\lambda(x)$ is logconvex in $x \in \Re_+$ and $m \in [-1,0)$, then*

$$\psi_{x_1,x_2}(u) \equiv \frac{\delta_m(F(x_2+u)) - \delta_m(F(u))}{\delta_m(F(x_1+u)) - \delta_m(F(u))}$$

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*is increasing in* $u \in \Re_+$ *for fixed* $x_2 > x_1 \geq 0$.

PROOF: Denote by $l_m(x)$ the derivative of $\delta_m(F(x))$. Then

$$l_m(x) = [\bar{F}(x)]^m f(x) = [\bar{F}(x)]^{1+m} \lambda(x).$$

Since the logconvexity of $f$ implies the logconvexity of $\bar{F}$, it follows that $l_m(x)$ is logconvex in $x \in \Re_+$ under the assumption of the lemma. Observe that

$$\delta_m(F(x+u)) - \delta_m(F(x)) = \int_0^\infty l_m(u+v) 1_{\{0 \leq v \leq x\}} \, dv, \tag{9}$$

where $1_A$ is the indicator function of set $A$, and that the logconvexity of $l_m(x)$ in $x \in \Re_+$ is equivalent to $l_m(u+v)$ is $\mathrm{TP}_2$ in $(u,v) \in \Re_+^2$. For the definition of $\mathrm{TP}_2$, one can refer to Karlin [13]. Here it should be mentioned that the logconvexity of a function $h$ does not in general imply that $h(x+y)$ is $\mathrm{TP}_2$ in $(x,y) \in \mathcal{X} \times \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are some subsets of $\Re$, if $h$ has a special support of the form $[a,\infty)$ with $a \neq 0$. Also, $1_{\{0 \leq v \leq x\}}$ is $\mathrm{TP}_2$ in $(v,x) \in \Re_+^2$. By applying the basic composition formula to (9) (see Karlin [13, p. 17]) or by Lemma 2 in Karlin and Proschan [14], we conclude that $\delta_m(F(x+u)) - \delta_m(F(x))$ is $\mathrm{TP}_2$ in $(x,u) \in \Re_+^2$. This means that $\psi_{x_1,x_2}(u)$ is increasing in $u \in \Re_+$ for $0 \leq x_1 < x_2$. ■

LEMMA 2.3: *Let* $X$ *and* $Y$ *be two nonnegative r.v.'s with respective distribution functions* $F(\cdot)$ *and* $G(\cdot)$ *and hazard rate functions* $\lambda(\cdot)$ *and* $\kappa(\cdot)$. *If* $X \leq_{\mathrm{lr}} Y$ *and* $m \geq 0$, *or if* $X \leq_{\mathrm{hr}} Y$, $\kappa(x)/\lambda(x)$ *is increasing in* $x \in \Re_+$, *and* $m \in [-1,0)$, *then*

*(i)* $\delta_m(G(x))/\delta_m(F(x))$ *is increasing in* $x \in \Re_+$

*(ii)*

$$\phi(x,u) \equiv \frac{\delta_m(G(x+u)) - \delta_m(G(u))}{\delta_m(F(x+u)) - \delta_m(F(u))}$$

*is increasing in* $(x,u) \in \Re_+^2$.

PROOF: We only give the proof of part (ii) since part (i) follows from part (ii) directly by noting that $\delta_m(G(x))/\delta_m(F(x)) = \phi(x,0)$.

Let $h_1(x)$ and $h_2(x)$ denote the respective derivatives of $\delta_m(F(x))$ and $\delta_m(G(x))$, and define

$$\Psi(i,x,u) = \int_0^\infty h_i(w) 1_{\{u \leq w \leq x+u\}} \, dw, \qquad (i,x,u) \in \{1,2\} \times \Re_+^2. \tag{10}$$

An argument similar to that in the proof of Lemma 2.2 yields that

$$\phi(x,u) = \frac{\Psi(2,x,u)}{\Psi(1,x,u)}, \qquad (x,u) \in \Re_+^2.$$

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It is well known that if $X \leq_{\mathrm{hr}} Y$ and $\kappa(x)/\lambda(x)$ is increasing, then $X \leq_{\mathrm{lr}} Y$ (see Belzunce, Lillo, Ruiz, and Shaked [4, Lemma 3.5]). Since the likelihood ratio order implies the hazard rate order, it follows that, under the assumptions of the lemma,

$$\frac{h_2(x)}{h_1(x)} = \left[\frac{\bar{G}(x)}{\bar{F}(x)}\right]^m \frac{g(x)}{f(x)} = \left[\frac{\bar{G}(x)}{\bar{F}(x)}\right]^{1+m} \frac{\kappa(x)}{\lambda(x)}$$

is increasing in $x \in \Re_+$; that is, $h_i(w)$ is $\mathrm{TP}_2$ in $(i,w) \in \{1,2\} \times \Re_+$. It can be checked that $1_{\{u \leq w \leq x+u\}}$ is both $\mathrm{TP}_2$ in $(w,x) \in \Re_+^2$ for each $u \in \Re_+$, and $\mathrm{TP}_2$ in $(w,u) \in \Re_+^2$ for each $x \in \Re_+$. Again, by applying the basic composition formula to (10) or by Lemma 2 in Karlin and Proschan [14], we get that $\Psi(i,x,u)$ is $\mathrm{TP}_2$ in $(i,x) \in \{1,2\} \times \Re_+$ for each $u \in \Re_+$, and $\mathrm{TP}_2$ in $(i,u) \in \{1,2\} \times \Re_+$ for each $x \in \Re_+$. Therefore, $\phi(x,u)$ is increasing in $(x,u) \in \Re_+^2$. ■

At the end of this section, we give two examples of r.v.'s satisfying the assumptions in Lemmas 2.2 and 2.3.

*Example 2.1*: For $\alpha \in (0,1]$, let $X_\alpha$ be a r.v. with the following Weibull distribution function:

$$F_\alpha(x) = 1 - e^{-x^\alpha}, \qquad x \in \Re_+.$$

The corresponding hazard rate and density functions of $X_\alpha$ are respectively given by

$$\lambda_\alpha(x) = \alpha x^{\alpha-1}, \qquad x \in \Re_+,$$

$$f_\alpha(x) = \alpha x^{\alpha-1} e^{-x^\alpha}, \qquad x \in \Re_+.$$

It can be checked that $\lambda_\alpha(x)$ and $f_\alpha(x)$ are both logconvex in $x \in \Re_+$ for $\alpha \in (0,1]$.

*Example 2.2*: Consider a r.v. $X$ with the density function $f(x)$ given by

$$f(x) = \theta\alpha e^{-\alpha x} + (1-\theta)\beta e^{-\beta x}, \qquad x \geq 0,$$

where $\theta \in (0,1)$ and $\beta > \alpha > 0$. Since the logconvexity is closed under mixtures (see Barlow and Proschan [3, p. 102]), it follows that $X$ is DLR, *and hence X is* DFR. Now let $Y$ be an exponential r.v. with parameter $\alpha$. Then it is easy to see that $X \leq_{\mathrm{lr}} Y$, $\kappa(t)/\lambda(t)$ is increasing in $t \in \Re_+$, and both $g$ and $\kappa$ are logconvex.

## 3. MAIN RESULTS

Throughout this section, let $X$ and $Y$ be two nonnegative r.v.'s with associated distribution functions $F$ and $G$, densities $f$ and $g$, and hazard rate functions $\lambda$ and $\kappa$, respectively. Let $X(r,n,m,k)$ and $Y(r,n,m,k)$, $r = 1,\ldots,n$, be GOSs based on $F$ and $G$, respectively.

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From Lemma 3.3 of Kamps [12], it follows that for each $r = 1, \ldots, n$, the marginal density function of the $r$th GOS $X(r, n, m, k)$ based on $F$ is given by

$$f_{X(r,n,m,k)}(x) = \frac{c_{r-1,n}}{(r-1)!}[\bar{F}(x)]^{\gamma_{r,n}-1}[\delta_m(F(x))]^{r-1}f(x), \tag{11}$$

where $c_{r-1,n} = \prod_{j=1}^{r}\gamma_{j,n}$, $\gamma_{n,n} = k$, $M_n = 0$, and $\delta_m$ is defined by (8). Let $f_{r,n}^{(p)}(x)$, $F_{r,n}^{(p)}(x)$, and $\bar{F}_{r,n}^{(p)}(x)$ denote the respective density, distribution, and survival functions of $V_{r,n}^{(p)}$, $r = 1, \ldots, n - p + 1$. Clearly, $f_{1,n}^{(p)}(x) = f_{X(p,n,m,k)}(x)$ given by Eq. (11). From Lemma 3.5 of Kamps [12], it follows that

$$f_{r,n}^{(p)}(x) = \frac{c_{r+p-2,n}}{(r-2)!(p-1)!}\int_0^\infty [\bar{F}(u)]^m f(u)[\delta_m(F(u))]^{r-2}[\bar{F}(x+u)]^{\gamma_{r+p-1,n}-1} \times [\delta_m(F(x+u)) - \delta_m(F(u))]^{p-1}f(x+u)\,du \tag{12}$$

and, hence,

$$\bar{F}_{r,n}^{(p)}(x) = \frac{c_{r+p-2,n}}{(r-2)!(p-1)!}\int_0^\infty [\bar{F}(u)]^m f(u)[\delta_m(F(u))]^{r-2}[\bar{F}(x+u)]^{\gamma_{r+p-1,n}} \times \left\{\int_0^1 [\delta_m(1-\bar{F}(x+u)z) - \delta_m(F(u))]^{p-1}z^{\gamma_{r+p-1,n}-1}\,dz\right\}du \tag{13}$$

for $r = 2, \ldots, n - p + 1$ and $x \geq 0$.

Similarly, let $g_{r,n}^{(p)}(x)$, $G_{r,n}^{(p)}(x)$, and $\bar{G}_{r,n}^{(p)}(x)$ denote the respective density, distribution, and survival functions of $W_{r,n}^{(p)}$, $r = 1, \ldots, n - p + 1$.

### 3.1. Likelihood Ratio Ordering

THEOREM 3.1: *Let $X$ and $Y$ be two nonnegative r.v.'s with corresponding distribution functions $F$ and $G$ and hazard rate functions $\lambda$ and $\kappa$, and let $X(r, n, m, k)$ and $Y(r, n, m, k)$, $r = 1, \ldots, n$, be GOSs based on $F$ and $G$, respectively. Then*

$$V_{r,n}^{(p)} \leq_{\mathrm{lr}} W_{r,n}^{(p)} \quad \textit{for } r = 1, \ldots, n - p + 1, \tag{14}$$

*if either of the following assumptions [(A1) or (A2)] holds:*

- *(A1) $X \leq_{\mathrm{lr}} Y$, either $X$ or $Y$ is DLR, and $m \geq 0$*
- *(A2) $X \leq_{\mathrm{hr}} Y$, $\kappa(x)/\lambda(x)$ is increasing, either $X$ is DLR and $\lambda$ is logconvex or $Y$ is DLR and $\kappa$ is logconvex, and $m \in [-1, 0)$.*

PROOF: To prove (14), it suffices to prove that for fixed $r = 1, \ldots, n - p + 1$,

$$\Delta_1(\theta) = \frac{g_{r,n}^{(p)}(\theta)}{f_{r,n}^{(p)}(\theta)}$$

is increasing in $\theta \in \Re_+$.

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For $r = 1$, it follows from (11) that

$$\Delta_1(\theta) = \left[\frac{\bar{G}(\theta)}{\bar{F}(\theta)}\right]^{\gamma_{1,n}-1} \frac{g(\theta)}{f(\theta)}$$

is increasing in $\theta$ since $\gamma_{1,n} \geq 1$, Assumption (A2) implies $X \leq_{\mathrm{lr}} Y$, and $X \leq_{\mathrm{lr}} Y$ implies $X \leq_{\mathrm{hr}} Y$.

For $r = 2, \ldots, n-p+1$, it follows from (12) that

$$\Delta_1(\theta) = \mathbb{E}_\theta[\Psi_1(\theta, U_1)], \qquad \theta \in \Re_+,$$

where

$$\Psi_1(\theta,u) = \left[\frac{\bar{G}(u)}{\bar{F}(u)}\right]^m \frac{g(u)}{f(u)} \left[\frac{\delta_m(G(u))}{\delta_m(F(u))}\right]^{r-2} \left[\frac{\bar{G}(\theta+u)}{\bar{F}(\theta+u)}\right]^{\gamma_{r+p-1,n}-1} \times \left[\frac{\delta_m(G(\theta+u)) - \delta_m(G(u))}{\delta_m(F(\theta+u)) - \delta_m(F(u))}\right]^{p-1} \frac{g(\theta+u)}{f(\theta+u)}$$

and the nonnegative r.v. $U_1$ has a distribution function belonging to the family $\mathcal{P}_1 = \{H_1(\cdot|\theta), \theta \in \Re_+\}$ with corresponding densities given by

$$h_1(u|\theta) = d_1(\theta)[\bar{F}(u)]^m f(u) \delta_m^{r-2}(F(u))[\delta_m(F(\theta+u)) - \delta_m(F(u))]^{p-1} \times [\bar{F}(\theta+u)]^{\gamma_{p+r-1,n}-1} f(\theta+u), \qquad u \in \Re_+;$$

here $d_1(\theta)$ is the normalizing constant. Now consider three cases.

*Case 1*: Suppose that $X \leq_{\mathrm{lr}} Y$, $X$ is DLR, and $m \geq 0$. It is seen that

- $\Psi_1(u,\theta)$ is increasing in $(u,\theta) \in \Re_+^2$ by using Lemma 2.3 and the fact that $X \leq_{\mathrm{lr}} Y$ implies $X \leq_{\mathrm{hr}} Y$
- $H_1(\cdot|\theta_1) \leq_{\mathrm{lr}} H_1(\cdot|\theta_2)$ for $\theta_2 \geq \theta_1 \geq 0$ by using Lemma 2.2 and the fact that the logconvexity of $f$ implies the logconvexity of $\bar{F}$ (see the proof of Theorem 4.1 in Hu and Zhuang [10]).

Then, by Lemma 2.1(i), we conclude that $\Delta_1(\theta)$ is increasing in $\theta \in \Re_+$.

*Case 2*: Suppose that $X \leq_{\mathrm{lr}} Y$, $Y$ is DLR, and $m \geq 0$. An argument similar to the preceding paragraphs can establish that $\Delta_1^{-1}(\theta) = f_{r,n}^{(p)}(\theta)/g_{r,n}^{(p)}(\theta)$ is decreasing in $\theta \in \Re_+$ by Lemma 2.1(ii).

*Case 3*: Suppose assumption (A2) holds. Then rewrite $\Psi_1(u,\theta)$ and $h_1(u|\theta)$ as follows:

$$\Psi_1(\theta,u) = \left[\frac{\bar{G}(u)}{\bar{F}(u)}\right]^{1+m} \frac{\kappa(u)}{\lambda(u)} \left[\frac{\delta_m(G(u))}{\delta_m(F(u))}\right]^{r-2} \left[\frac{\bar{G}(\theta+u)}{\bar{F}(\theta+u)}\right]^{\gamma_{r+p-1,n}-1} \times \left[\frac{\delta_m(G(\theta+u)) - \delta_m(G(u))}{\delta_m(F(\theta+u)) - \delta_m(F(u))}\right]^{p-1} \frac{g(\theta+u)}{f(\theta+u)},$$

$$h_1(u|\theta) = d_1(\theta)[\bar{F}(u)]^{1+m} \lambda(u) \delta_m^{r-2}(F(u))[\delta_m(F(\theta+u)) - \delta_m(F(u))]^{p-1} \times [\bar{F}(\theta+u)]^{\gamma_{p+r-1,n}-1} f(\theta+u), \qquad u \in \Re_+.$$

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The rest of the proof is similar to those of Cases 1 and 2 and hence omitted. This completes the proof of the theorem. ■

*Remark 3.1*: Belzunce et al. [5] in their Theorem 4.5 established that under the same assumptions as Theorem 3.1,

$$(V_{1,n}^{(1)}, V_{2,n}^{(1)}, \ldots, V_{n,n}^{(1)}) \leq_{\mathrm{lr}} (W_{1,n}^{(1)}, W_{2,n}^{(1)}, \ldots, W_{n,n}^{(1)}), \tag{15}$$

where $\leq_{\mathrm{lr}}$ denotes the multivariate likelihood ratio order (see Shaked and Shanthikumar [22, Sect. 4.E]). Since the multivariate likelihood ratio order is closed under marginalization (see [22, Thm. 4.E.3(b)]), it follows from (15) that

$$V_{r,n}^{(1)} \leq_{\mathrm{lr}} W_{r,n}^{(1)} \quad \text{for } r = 1, \ldots, n. \tag{16}$$

Franco et al. [7] proved (16) under the assumption (A1). Our Theorem 3.1 generalizes (16) from simple spacings to general $p$-spacings.

*Remark 3.2*: Theorem 4.1 in Hu and Zhuang [10] states that if $X$ is DLR and $m \geq 0$, then

$$V_{r,n}^{(p)} \leq_{\mathrm{lr}} V_{r+1,n}^{(p)} \quad \text{for } r = 1, \ldots, n-p, \tag{17}$$

$$V_{r,n+1}^{(p)} \leq_{\mathrm{lr}} V_{r,n}^{(p)} \quad \text{for } r = 1, \ldots, n-p+1, \tag{18}$$

$$V_{r,n}^{(p)} \leq_{\mathrm{lr}} V_{r+1,n+1}^{(p)} \quad \text{for } r = 1, \ldots, n-p+1. \tag{19}$$

However, the proofs of Theorem 4.1 in Hu and Zhuang [10] are also valid for the case when $X$ is DLR, its hazard rate function $\lambda(x)$ is also logconvex in $x \in \Re_+$, and $m \in [-1,0)$ by using Lemma 2.2. Therefore, from (17)–(19) we have the following corollary.

COROLLARY 3.1: *Let $X$ be a nonnegative r.v. with distribution function $F$ and hazard rate function $\lambda$, and let $X(r,n,m,k)$, $r = 1, \ldots, n$, be GOSs based on $F$. If $X$ is DLR and $m \geq 0$, or if $X$ is DLR, $\lambda(x)$ is logconvex in $x \in \Re_+$, and $m \in [-1,0)$, then*

$$V_{r,n}^{(p)} \leq_{\mathrm{lr}} V_{s,l}^{(p)} \quad \textit{whenever } p \geq 1,\ s \geq r, \textit{ and } s - r \geq l - n.$$

Combining Corollary 3.1 with Theorem 3.1, we have the following.

COROLLARY 3.2: *Under the same assumptions as those of Theorem 3.1,*

$$V_{r,n}^{(p)} \leq_{\mathrm{lr}} W_{s,l}^{(p)} \quad \textit{whenever } p \geq 1, s \geq r, \textit{ and } s - r \geq l - n.$$

By choosing $m = -1$ and $m = 0$ in Corollaries 3.1 and 3.2, we get the following interesting results concerning the likelihood ratio orderings of record values and ordinary order statistics.

COROLLARY 3.3: *Let $X_{L(1)}$ and $X_{L(2)}, \ldots$ be record values based on a sequence of i.i.d. nonnegative r.v.'s with distribution function $F$. If $X$ is DLR and its hazard rate $\lambda(t)$ is logconvex, then*

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$$X_{L(p+r)} - X_{L(r)} \leq_{\mathrm{lr}} X_{L(p+s)} - X_{L(s)} \quad \textit{whenever } p \geq 1, 0 \leq r \leq s,$$

*where* $X_{L(0)} = 0$.

It is worthwhile pointing out that if the hazard rate function $\lambda$ of $F$ is logconcave, then

$$X_{L(p+r)} - X_{L(r)} \geq_{\mathrm{lr}} X_{L(p+s)} - X_{L(s)} \quad \text{whenever } p \geq 1, 0 \leq r \leq s$$

(see Corollary 4.2 in Hu and Zhuang [10]).

COROLLARY 3.4: *Let* $\{X_{L(n)}, n \geq 1\}$ *and* $\{Y_{L(n)}, n \geq 1\}$ *be record values based on two sequences of i.i.d. nonnegative r.v.'s with respective distribution functions F and G. If* $X \leq_{\mathrm{hr}} Y$ *and* $\kappa(x)/\lambda(x)$ *is increasing in x and if either X is DLR and* $\lambda(\cdot)$ *is logconvex or Y is DLR and* $\kappa(\cdot)$ *is logconvex, then*

$$X_{L(p+r)} - X_{L(r)} \leq_{\mathrm{lr}} Y_{L(p+s)} - Y_{L(s)} \quad \textit{whenever } p \geq 1,\ 0 \leq r \leq s,$$

*where* $X_{L(0)} = Y_{L(0)} = 0$.

Note that the epoch times of a nonhomogeneous Poisson process with intensity function $\lambda(t)$ are the record values of a sequence of i.i.d. nonnegative r.v.'s with the hazard rate being $\lambda(t)$, where $\int_t^\infty \lambda(u)\, du = \infty$ for all $t \in \Re_+$. Observing this fact, the special case of Corollary 3.4 with $p = 1$ is Corollary 4.7 in Belzunce et al. [4].

COROLLARY 3.5: *Let* $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ *and* $Y_{1:n} \leq Y_{2:n} \leq \cdots \leq Y_{n:n}$ *denote the corresponding ordinary order statistics of two random samples of size n from distribution functions F and G, respectively. Define*

$$V_{r,n}^{(p)} = X_{r+p-1:n} - X_{r-1:n}, \qquad r = 1, \ldots, n-p+1,$$

*and*

$$W_{r,n}^{(p)} = Y_{r+p-1:n} - Y_{r-1:n}, \qquad r = 1, \ldots, n-p+1.$$

*If* $X \leq_{\mathrm{lr}} Y$ *and if either X or Y is DLR, then*

$$V_{r,n}^{(p)} \leq_{\mathrm{lr}} W_{s,l}^{(p)} \quad \textit{whenever } p \geq 1, s \geq r, \textit{ and } s - r \geq l - n, \tag{20}$$

*where* $X_{0:n} = Y_{0,n} \equiv 0$.

Kochar [18] obtained Corollary 3.5 for the special case $p = 1$. In fact, he established the multivariate likelihood ratio ordering (15) between the random vectors of simple spacings of ordinary order statistics from two samples under the assumption that $X \leq_{\mathrm{lr}} Y$ and either $X$ or $Y$ is DLR. Xu and Li [23] proved (20) under the assumption that $X \leq_{\mathrm{lr}} Y$ and $X$ is DLR.

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### 3.2. Hazard Rate Ordering

In Theorem 3.1, if, instead, either $X$ or $Y$ is assumed to be DFR, then the result for simple spacings can be weakened from the likelihood ratio order to the hazard rate order (see Theorem 3.2). It is still an open problem whether the hazard rate orderings between the general $p$-spacings $V_{r,n}^{(p)}$ and $W_{r,n}^{(p)}$ hold under the same assumptions as those of Theorem 3.2.

THEOREM 3.2: *Let $X$ and $Y$ be two nonnegative r.v.'s with corresponding distribution functions $F$ and $G$ and hazard rate functions $\lambda$ and $\kappa$, and let $X(r,n,m,k)$ and $Y(r,n,m,k)$, $r=1,\ldots,n$, be GOSs based on $F$ and $G$, respectively. Then*

$$V_{r,n}^{(1)} \leq_{\mathrm{hr}} W_{r,n}^{(1)} \quad \text{for } r=1,\ldots,n \tag{21}$$

*if either one of the following assumptions [(A3) and (A4)] holds:*

*(A3) $X \leq_{\mathrm{lr}} Y$, either $X$ or $Y$ is DFR, and $m \geq 0$.*

*(A4) $X \leq_{\mathrm{hr}} Y$, $\kappa(x)/\lambda(x)$ is increasing, either $X$ or $Y$ is DFR and $m \in [-1,0)$.*

PROOF: We give the proof of the case that $m > -1$; the proof of the case $m = -1$ follows from the closure property of the hazard rate order under weak convergence. For $r=1$, the desired result (21) follows from Theorem 3.3 in Hu and Zhuang [9] or Theorem 3.3 in Franco et al. [7].

Now consider the case $r=2,\ldots,n$, and let $\lambda_{r,n}^{(1)}(\cdot)$ and $\kappa_{r,n}^{(1)}(\cdot)$ denote the hazard rate functions of $V_{r,n}^{(1)}$ and $W_{r,n}^{(1)}$, respectively. From (12) and (13), we get that

$$\lambda_{r,n}^{(1)}(x) = \gamma_{r,n}\mathbb{E}[\lambda(x+U_2)] \tag{22}$$

and

$$\kappa_{r,n}^{(1)}(x) = \gamma_{r,n}\mathbb{E}[\kappa(x+U_3)], \tag{23}$$

where the nonnegative r.v.'s $U_1$ and $U_3$ have distribution functions belonging to the families $\mathcal{P}_2 = \{H_2(\cdot|x), x \in \Re_+\}$ and $\mathcal{P}_3 = \{H_3(\cdot|x), x \in \Re_+\}$, respectively, with corresponding densities given by

$$h_2(u|x) = d_2(x)\bar{F}^m(u)f(u)[\delta_m(F(u))]^{r-2}[\bar{F}(x+u)]^{\gamma_{r,n}}, \qquad u \in \Re_+,$$
$$h_3(u|x) = d_3(x)\bar{G}^m(u)g(u)[\delta_m(G(u))]^{r-2}[\bar{G}(x+u)]^{\gamma_{r,n}}, \qquad u \in \Re_+;$$

here $d_2(x)$ and $d_3(x)$ are the normalizing constants. It can be checked that under assumption (A3) or (A4),

$$\begin{aligned}\frac{h_3(u|x)}{h_2(u|x)} &= \frac{d_3(x)}{d_2(x)}\left(\frac{\bar{G}(u)}{\bar{F}(u)}\right)^m \frac{g(u)}{f(u)}\left(\frac{\delta_m(G(u))}{\delta_m(F(u))}\right)^{r-2}\left(\frac{\bar{G}(x+u)}{\bar{F}(x+u)}\right)^{\gamma_{r,n}} \\ &= \frac{d_3(x)}{d_2(x)}\left(\frac{\bar{G}(u)}{\bar{F}(u)}\right)^{m+1} \frac{\kappa(u)}{\lambda(u)}\left(\frac{\delta_m(G(u))}{\delta_m(F(u))}\right)^{r-2}\left(\frac{\bar{G}(x+u)}{\bar{F}(x+u)}\right)^{\gamma_{r,n}}\end{aligned}$$

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is increasing in $u \in \Re_+$ for each fixed $x$ by Lemma 2.3; that is, $U_2 \leq_{\mathrm{lr}} U_3$ for each $x \in \Re_+$. Note that $X \leq_{\mathrm{hr}} Y$ implies $\kappa(t) \leq \lambda(t)$ for all $t \in \Re_+$. Therefore, if $X$ is DFR, then

$$\lambda^{(1)}_{r,n}(x) \geq \gamma_{r,n}\mathbb{E}[\lambda(x+U_3)] \geq \gamma_{r,n}\mathbb{E}[\kappa(x+U_3)] = \kappa^{(1)}_{r,n}(x),$$

and if $Y$ is DFR, then

$$\lambda^{(1)}_{r,n}(x) \geq \gamma_{r,n}\mathbb{E}[\kappa(x+U_2)] \geq \gamma_{r,n}\mathbb{E}[\kappa(x+U_3)] = \kappa^{(1)}_{r,n}(x).$$

This completes the proof of the theorem. ■

Corollary 4.3 in Hu and Zhuang [10] states that if $X$ is DFR and $m \geq -1$, then

$$V^{(p)}_{r,n} \leq_{\mathrm{lr}} V^{(p)}_{s,l} \quad \text{whenever } p \geq 1,\ s \geq r, \text{ and } s-r \geq l-n.$$

Combining this with Theorem 3.2, we have the following.

COROLLARY 3.6: *Under the same assumptions as those in Theorem 3.2,*

$$V^{(1)}_{r,n} \leq_{\mathrm{hr}} W^{(1)}_{s,l} \quad \textit{whenever } s \geq r \textit{ and } s-r \geq l-n.$$

It is known that simple spacings of GOSs from a DFR distribution are also DFR; see Kamps [11, Thm. 1.15] or Hu and Zhuang [10, Thm. 3.1(1)]. Combining this result and Remark 2.1 with Corollary 3.6, we get the next corollary.

COROLLARY 3.7: *Under the same assumptions as those in Theorem 3.2,*

$$V^{(1)}_{r,n} \leq_{\mathrm{disp}} W^{(1)}_{s,l} \quad \textit{whenever } s \geq r \textit{ and } s-r \geq l-n.$$

*Remark 3.3*: Franco et al. [7] proved Theorem 3.2 and Corollaries 3.6 and 3.7 for GOSs under assumption (A3) by using a different method.

Theorem 2.1 of Kochar [18] on ordinary order statistics is the special case of Theorem 3.2 by choosing $k = 1$ and $m = 0$. Choosing $k = 1$ and $m = -1$ in Corollaries 3.6 and 3.7, we get the next result concerning stochastic comparisons of simple spacings of record values in the hazard rate and the dispersive orders, which was proved by Belzunce et al. [4, Thm. 4.4] under the additional assumption that both $X$ and $Y$ are DFR.

COROLLARY 3.8: *Let $\{X_{L(n)}, n \geq 1\}$ and $\{Y_{L(n)}, n \geq 1\}$ be as defined in Corollary 3.4. If $X \leq_{\mathrm{hr}} Y$ and $\kappa(x)/\lambda(x)$ is increasing in $x$ and if either $X$ or $Y$ is DFR, then*

$$X_{L(r+1)} - X_{L(r)} \leq_{\mathrm{hr}} Y_{L(s+1)} - Y_{L(s)} \quad \textit{whenever } 0 \leq r \leq s$$

*and*

$$X_{L(r+1)} - X_{L(r)} \leq_{\mathrm{disp}} Y_{L(s+1)} - Y_{L(s)} \quad \textit{whenever } 0 \leq r \leq s,$$

*where $X_{L(0)} = Y_{L(0)} = 0$.*

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Franco et al. [7] also proved that

$$\widetilde{V}_{r,n} \leq_{\mathrm{hr}} [\leq_{\mathrm{disp}}] \widetilde{W}_{s,l} \quad \text{whenever } s \geq r \text{ and } s - r \geq l - n, \tag{24}$$

under assumption (A3), which contains Theorem 2.2 and Corollary 2.2 in Khaledi and Kochar [16] on ordinary order statistics as its special case. We close this section with a comparison result of normalizing spacings of GOSs in the hazard rate and dispersive orders under assumption (A4).

THEOREM 3.3: *Let $X(r,n,m,k)$, $Y(r,n,m,k)$, and assumptions (A3) and (A4) be as defined in Theorem 3.2. If either assumption (A3) or (A4) is satisfied, then (24) holds.*

PROOF: We only give the proof of the hazard rate order in (24) under assumption (A4). Assume that $s \geq r$ and $s - r \geq l - n$. Then $\gamma_{r,n} \geq \gamma_{s,l}$. Let $\tilde{\lambda}_{r,n}(\cdot)$ and $\tilde{\kappa}_{r,n}(\cdot)$ denote the hazard rate functions of $\widetilde{V}_{r,n}$ and $\widetilde{W}_{r,n}$, respectively.

For $r = 2, \ldots, n$, it follows from (22) and (23) that

$$\tilde{\lambda}_{r,n}(x) = \mathbb{E}\left[\lambda\left(\frac{x}{\gamma_{r,n}} + U_2\right)\right], \qquad \tilde{\kappa}_{s,l}(x) = \mathbb{E}\left[\kappa\left(\frac{x}{\gamma_{s,l}} + U_4\right)\right],$$

where $U_2$ is defined in the proof of Theorem 3.2 and $U_4$ has a distribution function belonging to the families $\mathcal{P}_4 = \{H_4(\cdot|x), x \in \Re_+\}$ with corresponding densities given by

$$h_4(u|x) = d_4(x)\bar{G}^m(u)g(u)[\delta_m(G(u))]^{s-2}[\bar{G}(x+u)]^{\gamma_{s,l}}, \qquad u \in \Re_+;$$

here $d_4(x)$ is the normalizing constant. Then

$$\frac{h_4(u|x)}{h_2(u|x)} = \frac{d_4(x)}{d_2(x)}\left(\frac{\bar{G}(u)}{\bar{F}(u)}\right)^{m+1}\frac{\kappa(u)}{\lambda(u)}\left(\frac{\delta_m(G(u))}{\delta_m(F(u))}\right)^{r-2} \times \left(\frac{\bar{G}(x+u)}{\bar{F}(x+u)}\right)^{\gamma_{s,l}}\frac{[\delta_m(G(u))]^{s-r}}{[\bar{F}(x+u)]^{\gamma_{r,n}-\gamma_{s,l}}}$$

is increasing in $u$ for each fixed $x \in \Re_+$ by Lemma 2.3 and the fact that $\delta_m(\cdot)$ is increasing; that is, $U_2 \leq_{\mathrm{lr}} U_4$ for each $x$. A similar argument to that in the proof of Theorem 3.2 yields that $\tilde{\lambda}_{r,n}(x) \geq \tilde{\kappa}_{s,l}(x)$ for each $x \in \Re_+$ and $r \geq 2$.

For $r = 1$, it follows from (11) that

$$\tilde{\lambda}_{1,n}(x) = \lambda\left(\frac{x}{\gamma_{1,n}}\right), \qquad x \in \Re_+.$$

If $X$ is DFR, then

$$\tilde{\lambda}_{1,n}(x) \geq \lambda\left(\frac{x}{\gamma_{s,l}}\right) \geq \mathbb{E}\left[\lambda\left(\frac{x}{\gamma_{s,l}} + U_4\right)\right] \geq \mathbb{E}\left[\kappa\left(\frac{x}{\gamma_{s,l}} + U_4\right)\right] = \tilde{\kappa}_{s,l}(x),$$

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and if $Y$ is DFR, then

$$\tilde{\lambda}_{1,n}(x) \geq \kappa\left(\frac{x}{\gamma_{1,n}}\right) \geq \mathbb{E}\left[\kappa\left(\frac{x}{\gamma_{s,l}} + U_4\right)\right] = \tilde{\kappa}_{s,l}(x).$$

This means that (24) holds for $r = 1$. Therefore, we complete the proof. ■

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*References*

1. Bagai, I. & Kochar, S.C. (1986). On tail ordering and comparison of failure rates. *Communications in Statistics: Theory and Methods* 15: 1377–1388.
2. Balakrishnan, N., Cramer, E., & Kamps, U. (2001). Bounds for means and variances of progressive type II censored order statistics. *Statistics & Probability Letters* 54: 301–315.
3. Barlow, R.E. & Proschan, F. (1981). *Statistical theory of reliability and life testing*. Silver Spring, MD: To begin with.
4. Belzunce, F., Lillo, R.E., Ruiz, J.M., & Shaked, M. (2001). Stochastic comparisons of nonhomogeneous processes. *Probability in the Engineering and Informational Sciences* 15: 199–224.
5. Belzunce, F., Mercader, J.A., & Ruiz, J.M. (2005). Stochastic comparisons of generalized order statistics. *Probability in the Engineering and Informational Sciences* 19: 99–120.
6. Cramer, E. & Kamps, U. (2001). Sequential $k$-out-of-$n$ systems. In N. Balakrishnan, & C.R. Rao (eds.), *Handbook of statistics: Advances in reliability*, Vol. 20, Amsterdam: Elsevier, pp. 301–372.
7. Franco, M., Ruiz, J.M., & Ruiz, M.C. (2002). Stochastic orderings between spacings of generalized order statistics. *Probability in the Engineering and Informational Sciences* 16: 471–484.
8. Hu, T. & Zhuang, W. (2006). Stochastic comparisons of $m$-spacings. *Journal of Statistical Planning and Inference* 136: 33–42.
9. Hu, T. & Zhuang, W. (2005). A note on stochastic comparisons of generalized order statistics. *Statistics & Probability Letters* 72: 163–170.
10. Hu, T. & Zhuang, W. (2005). Stochastic properties of $p$-spacings of generalized order statistics. *Probability in the Engineering and Informational Sciences* 19: 257–276.
11. Kamps, U. (1995). *A concept of generalized order statistics*. Stuttgard: Teubner.
12. Kamps, U. (1995). A concept of generalized order statistics. *Journal of Statistical Planning and Inference* 48: 1–23.
13. Karlin, S. (1968). *Total positivity*. Stanford, CA: Stanford University Press.
14. Karlin, S. & Proschan, F. (1960). Pólya type distributions of convolutions. *Annals of Mathematical Statistics* 31: 721–736.
15. Khaledi, B.-E. (2004). Some new results on stochastic orderings between generalized order statistics. *Journal of Iranian Statistical Society* (to appear).
16. Khaledi, B.-E. & Kochar, S. (1999). Stochastic orderings between distributions and their sample spacings—II. *Statistics & Probability Letters* 44: 161–166.
17. Khaledi, B.-E. & Kochar, S. (2005). Dependence orderings for generalized order statistics. *Statistics & Probability Letters* 73: 357–367.
18. Kochar, S.C. (1999). On stochastic orderings between distributions and their sample spacings. *Statistics & Probability Letters* 42: 345–352.
19. Misra, N. & van der Meulen, E.C. (2003). On stochastic properties of $m$-spacings. *Journal of Statistical Planning and Inference* 115: 683–697.

https://doi.org/10.1017/S0269964806060281 Published online by Cambridge University Press

20. Müller, A. & Stoyan, D. (2002). *Comparison methods for stochastic models and risks*. West Sussex, UK: Wiley.
21. Sengupta, D. & Nanda, A.K. (1999). Log-concave and concave distributions in reliability. *Naval Research Logistics* 46: 419–433.
22. Shaked, M. & Shanthikumar, J.G. (1994). *Stochastic orders and their applications*. New York: Academic Press.
23. Xu, M. & Li, X. (2004). Likelihood ratio order and DFRA aging property of sample's $m$-spacings. Technical report, School of Mathematics and Statistics, Lanzhou University, Lanzhou, China.

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