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The three-dimensional generalized Lotka–Volterra systems

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Abstract. This work is devoted to the study of periodic solutions of generalized Lotka–Volterra systems in three dimensions. The group A_3 of isometries of the regular tetrahedron acts on the twelve-dimensional space of such systems. We find four components (modulo the action of A_3) of the center variety, i.e. the semi-algebraic subset consisting of systems with a family of periodic solutions (which degenerate at the center). We focus attention on the component which consists of systems X_0 with an invariant plane L_0 transversal to the coordinate axes and with a (partial) first integral on L_0. Under the assumption of normal hyperbolicity of X_0 along L_0 we begin study of limit cycles appearing for a perturbed system $X_0 + \varepsilon X_1$ (in the Lotka–Volterra class). The linearization of the problem leads to the problem of zeroes of certain Poincaré–Melnikov integrals of a new type. We determine the asymptotic behavior of these integrals near two critical values of the partial first integral of the restricted system $X_0|_{L_0}$. We also study the situation near the intersection of different components of the center variety.

1. *Introduction*

The aim of this paper is to answer a certain question of S. van Gils. When the second author visited the University of Enschede in 1993, van Gils presented him the following system

$$\dot{x} = x(-ay + az - u), \quad \dot{y} = y(-az + ax - u), \quad \dot{z} = z(-ax + ay - u), \tag{1.1}$$

where $u = x + y + z - 1$; he told the second author that he learned about it from E. Zeeman.

It is easy to see that the plane $L_0 = \{u = 0\}$ is invariant and that the system restricted to L_0 is Hamiltonian with $H = axy(1 - x - y)$ as the Hamilton function. Moreover, when $0 < |a| < 1$ the system is normally hyperbolic along the plane L_0 (in the sense of [**HPS**]). This means that when we perturb (1.1), then the invariant surface will still exist as a slight perturbation of L_0. However, the restricted system will not (in general) be integrable and limit cycles will appear.

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Van Gils' question was how many limit cycles one can obtain when one perturbs (1.1) in the class of generalized Lotka–Volterra systems. Van Gils also asked about developing a general theory to deal with this kind of problems (a generalization of the XVIth Hilbert problem).

The first steps in this theory were made by Leszczyński with the second author in [**LZ**] and by both the present authors in [**BZ1, BZ2**]. The following systems in $\mathbb{R}^2 \times (\mathbb{R}^\nu, 0)$ were investigated

$$\begin{aligned} \dot{x} &= H_y + z \cdot R + \varepsilon P, \\ \dot{y} &= -H_x + z \cdot S + \varepsilon Q, \\ \dot{z} &= Az + \varepsilon b. \end{aligned} \tag{1.2}$$

Here ε is a small parameter and $H(x, y)$, $P(x, y; z; \varepsilon)$, $Q(x, y; z; \varepsilon)$, $R(x, y; z)$, $S(x, y; z)$, $A(x, y; z)$, $b(x, y; z; \varepsilon)$ are analytic (scalar or vector or matrix) functions, polynomial in (x, y). The *normal variation matrix* $A_0 = A(x, y; 0)$ satisfies the normal hyperbolicity condition along the invariant plane $z = 0$ (for $\varepsilon = 0$), which means the hyperbolic splitting $A_0 = A^{\mathrm{s}} \oplus A^{\mathrm{u}}$ with contraction of A^{s} and expansion of A^{u} stronger than the action of the derivative of the Hamiltonian flow maps along $z = 0$.

For $\varepsilon \neq 0$ the invariant surface takes the form $z = \varepsilon g(x, y) + O(\varepsilon^2)$, where g satisfies the *normal variation equation*

$$\dot{g} = A_0 g + b, \tag{1.3}$$

and is (in general) a transcendental (vector) function.

The infinitesimal condition for limit cycles on the perturbed surface is given by the equation

$$I(h) = \int_{H=h} Q\,dx - P\,dy + \int_{H=h} g \cdot (S\,dx - R\,dy) = 0. \tag{1.4}$$

The first of the contour integrals in (1.4) is the usual Abelian integral (see [**Ar1**]), but the second contour integral cannot be (in general) reduced to an Abelian integral. We shall call the integral (1.4) the *first-order Poincaré–Melnikov integral*.

In [**LZ**] the case with quadratic Hamiltonian $H = x^2 + y^2$ was considered, while in [**BZ1**] we treated the case of elliptic Hamiltonian $H = y^2 - x^3 + 3x$ and in [**BZ2**] the Hamiltonian was general. Linear estimates in degrees of P, Q, R, S, b for the number of zeroes of $I(h)$ were obtained for a given matrix A_0; in [**LZ**] only zeroes in a bounded interval $(0, M)$ were counted and in [**BZ1**] the assumption $A_0 \equiv$ constant was imposed.

The present work is the next in our study of the multi-dimensional XVIth Hilbert problem and is concentrated on systems of the form

$$\begin{aligned} \dot{x} &= x(a + bx + cy + dz), \\ \dot{y} &= y(a' + b'x + c'y + d'z), \\ \dot{z} &= z(a'' + b''x + c''y + d''z). \end{aligned} \tag{1.5}$$

Following Arnold [**Ar1, Ar2**] we call the system (1.5) the *(three-dimensional) generalized Lotka–Volterra system*. By the Lotka–Volterra system people usually mean the system (1.5) with $b = c' = d'' = 0$ (see [**SW, CL**]). In the domain $x, y, z \geq 0$ it describes evolution of a logistic system of competing species, intensively studied by Lotka [**Lo**]

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and Volterra [**Vo**]. In fact, in [**Vo**] system (1.5) (and even more general systems) were considered. Kolmogorov [**Ko**] also investigated two-dimensional predator–prey systems; so sometimes Lotka–Volterra systems are called Kolmogorov systems. We refer the reader to the articles of Cairó [**Ca**] and of Hofbauer and Sigmund [**HSi, Si**] for other applications of generalized Lotka–Volterra systems.

The system (1.1) is a particular case of (1.5). In fact, we started our investigations from (1.1), but we discovered that the space of integrals (1.4), arising from perturbations of (1.1), is one-dimensional, generated by an Abelian integral, and a generic perturbation does not generate limit cycles

Therefore the right approach is to consider the whole twelve-dimensional space **LV** of generalized Lotka–Volterra systems and apply to it tools and methods developed for two-dimensional polynomial vector fields.

For example, in the twelve-dimensional space of quadratic planar vector fields one considers the subspace of systems with a center—it is a real semi-algebraic variety, called the *center variety* (see [**Zo2**]), and consists of four components (where the lower index denotes the codimension):

- Q_3^{H} of Hamiltonian systems;
- Q_3^{R} of time-reversible systems;
- Q_3^{LV} of generalized Lotka–Volterra systems;
- Q_4 of systems with a rational first integral of the form $f^3 g^{-2}$.

Next, one perturbs a system from the center variety and looks for arising limit cycles. The linearized condition leads to the problem of zeroes of the first-order Poincaré–Melnikov integrals, which take the form of Abelian integrals

$$I(h) = \int_{H=h} M(Q\,dx - P\,dy) = \int_{H=h} \frac{Q\,dx - P\,dy}{f_1 \cdots f_r}, \tag{1.6}$$

where $H = f_1^{a_1} \cdots f_r^{a_r}$ is the first integral (of Darboux type) and $M = H/f_1 \cdots f_r$ is the integrating multiplier. Sometimes the first-order Poincaré–Melnikov integrals vanish identically, e.g. when the initial system lies in the intersection of two components of the center variety. It was proved in [**Zo2**] that the integral (1.6) can have at most two zeroes for quadratic perturbations of the system from Q_3^{LV} (as expected by the codimension). Iliev [**Il**] calculated the number of zeroes of the third-order Poincaré–Melnikov integral arising when we start from a system in $Q_3^{\mathrm{LV}} \cap Q_3^{\mathrm{R}} \cap Q_3^{\mathrm{H}}$. The Hamiltonian case was solved by Gavrilov [**Ga**].

It seems that the present work is the first where an analogous theory is developed for the three-dimensional generalized Lotka–Volterra systems. There recently appeared some works concerning invariant algebraic surfaces and first integrals for the Lotka–Volterra systems (see [**CL**] and [**SW**], for example) and for the generalized Lotka–Volterra systems (see [**Ca**], where 366 cases of Darboux first integrals were found). Unfortunately, the authors of [**CL**] and [**Ca**] use rather standard methods (whose origins go back to Darboux) and give little help in our problem. For example, Cairó did not use the group A_3 of order 24 of projective orbital symmetries which permute the invariant planes $x = 0$, $y = 0$, $z = 0$ and the plane at infinity in $\mathbb{RP}^3 = \{(x_1 : x_2 : x_3 : x_4)\}$; in [**Ca**] only the permutations of the variables x, y, z are considered.

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We present (in §3 below) the 12 components of the center variety in the space **LV** of Lotka–Volterra systems (1.5). We agree to say that a system X_0 from **LV** has a *center* if it has a family of periodic solutions degenerating to a single point p_*, the center. Moreover, we consider only the case when the center p_* does not lie on any coordinate plane; the latter cases are reduced to the (well-investigated) two-dimensional generalized Lotka–Volterra systems. The group A_3 acts on the 12 components of the center variety with four orbits. Therefore, we have four components of the center variety in $\mathcal{LV} = \mathbf{LV}/A_3$. The components in **LV** are denoted by $\mathbf{LV}_i^{\text{plane}}$, $\mathbf{LV}_i^{\text{Darb}}$, $\mathbf{LV}^{\text{Hopf}}$ and the components in $\mathcal{LV}$ are denoted by $\mathcal{LV}_0^{\text{plane}}$, $\mathcal{LV}_1^{\text{plane}}$, $\mathcal{LV}^{\text{Darb}}$, $\mathcal{LV}^{\text{Hopf}}$.

One component which we denote by $\mathbf{LV}_0^{\text{plane}}$ consists of systems $X_0 \in \mathbf{LV}$ having an invariant plane $L_0 = \{kx + ly + mz + n = 0\}$ where $klmn \neq 0$, i.e. L_0 is transversal to the coordinate axes and avoids the origin. Then the restricted system $X_0|_{L_0}$ has a first integral of the form $H = |x|^\alpha |y|^\beta |z|^\gamma|_{L_0}$. When studying perturbations $X_\varepsilon = X_0 + \varepsilon X_1$ of $X_0 \in \mathbf{LV}_0^{\text{plane}}$ we shall usually (but not always) assume normal hyperbolicity of X_0 along L_0, which leads to further restrictions on X_0.

There are $\binom{4}{2} = 6$ components $\mathbf{LV}_1^{\text{plane}}, \dots, \mathbf{LV}_6^{\text{plane}}$ of the center variety in **LV**, which define one component $\mathcal{LV}_1^{\text{plane}}$ in $\mathcal{LV} = \mathbf{LV}/A_3$. One representative component, e.g. $\mathbf{LV}_1^{\text{plane}}$, consists of systems X_0 with an invariant plane $z = z_0$ and a partial first integral $G = |x|^\alpha |y|^\beta |kx + ly + m|^\gamma$ for $X_0|_{z=z_0}$.

The third case of a center consists of $\binom{4}{3} = 4$ components $\mathbf{LV}_i^{\text{Darb}}$, $i = 1, \dots, 4$ in **LV**, giving one component $\mathcal{LV}^{\text{Darb}}$ in $\mathcal{LV}$. One representative component, e.g. $\mathbf{LV}_1^{\text{Darb}}$, consists of systems X_0 with a global first integral of Darboux type $F = |x|^\alpha |y|^\beta |z|^\gamma |kx+ly+mz|^\delta$, $\delta = -\alpha - \beta - \gamma$. The integral F depends essentially on two variables x/z and y/z. So it has a line l_F of critical points. The singular point p_* of X_0 lies on the line l_F and has a local invariant center manifold $W^{\text{c}}(p_*)$. Thus $F|_{W^{\text{c}}(p_*)}$ is a first integral for $X_0|_{W^{\text{c}}(p_*)}$ and p_* is the center.

The last case of a center gives an A_3-invariant component $\mathbf{LV}^{\text{Hopf}}$ which consists of X_0 with global Darboux integral $E = |x|^\alpha |y|^\beta |z|^\gamma$. Any such X_0 has a line ℓ of non-isolated singular points. One obtains a family $Y_\rho = X_0|_{E=\rho}$ of two-dimensional systems on the surfaces $E = \rho$ with singular points $p_\rho = \ell \cap \{E = \rho\}$. It turns out that the singular point p_ρ changes stability for exactly one value ρ_* of the parameter. By the Hopf theorem there is a one-parameter family of periodic solutions to X_0 near the point $p_* = p_{\rho_*}$ (the center).

The components $\mathcal{LV}_0^{\text{plane}}$ and $\mathcal{LV}^{\text{Darb}}$ (each in codimension three in $\mathcal{LV}$) intersect in $\mathcal{LV}$ along a subvariety $\mathcal{LV}_0^{\text{plane}} \cap \mathcal{LV}^{\text{Darb}}$ of codimension two in $\mathcal{LV}_0^{\text{plane}}$ and in $\mathcal{LV}^{\text{Darb}}$ (non-transversally). The van Gils system (1.1) belongs to this intersection, it has the global first integral $F = xyz/(x + y + z)^3$. This explains why the space of first order Poincaré–Melnikov integrals associated with its perturbations is one-dimensional. Moreover, an additional analysis of the integrals (1.4) shows that the generator of this space is an Abelian integral of the same form as some integrals arising in cubic Lotka–Volterra systems considered in bifurcation theory in [**Zo1**] (see §4.6 below). The results of [**Zo1**] imply that this Abelian integral does not have zeroes away from the critical values of the partial first integral. Therefore (1.1) was not the best choice to start a new theory.

The components $\mathbf{LV}_0^{\text{plane}}$ and $\mathbf{LV}^{\text{Hopf}}$ intersect (also non-transversally) along the subvariety $\mathbf{LV}_0^{\text{plane}} \cap \mathbf{LV}^{\text{Hopf}}$ of codimension one in $\mathbf{LV}_0^{\text{plane}}$. Any X_0 from this intersection

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has two global first integrals $E = |x|^{\alpha}|y|^{\beta}|z|^{\gamma}$ and $D = |x|^{\alpha'}|y|^{\beta'}|z|^{\gamma'}|kx+ly+mz+n|^{\delta'}$. This means that X_0 has a two-parameter family of periodic solutions, which fill an open domain in the phase space.

The same occurs, i.e. the existence of two functionally independent first integrals, for the intersections $\mathcal{LV}^{\mathrm{Hopf}} \cap \mathcal{LV}_1^{\mathrm{plane}}$ and $\mathcal{LV}^{\mathrm{Hopf}} \cap \mathcal{LV}^{\mathrm{Darb}}$.

The components $\mathcal{LV}^{\mathrm{Darb}}$ and $\mathcal{LV}_1^{\mathrm{plane}}$ intersect along irreducible sub-variety $\mathcal{LV}^{\mathrm{Darb}} \cap \mathcal{LV}_1^{\mathrm{plane}} \subset \mathcal{LV}$. In **LV** we have a set of 12 components on which the group A_3 acts transitively.

Other intersections between the components of the center variety are empty.

The geometry of the center variety in **LV** looks very pretty. Also it is not difficult to write down the formulas for the first-order Poincaré–Melnikov integrals associated with perturbations of systems with a center. However, the analysis of these integrals has turned out very difficult.

Recall that in the planar quadratic case the center variety was found by Dulac and Kapteyn by calculating only the focus quantities, which is equivalent to the calculation of the first several terms in the Taylor expansion of the higher-order Melnikov functions. We tried to make the calculations in the Lotka–Volterra case analogous, but the task turned out to be highly difficult. The formulas (obtained by a computer) became too complicated to make further conclusions. In particular, we do not have a rigorous proof that we have found all of the cases of the center.

We calculated only the first two terms in expansions of the linear Melnikov integrals near the center and near the boundary of the triangle for perturbations of systems from $\mathbf{LV}_0^{\mathrm{plane}}$. One reason for the difficulties is the fact that the center variety is very large; after dilations of the coordinates and time, the component $\mathbf{LV}_0^{\mathrm{plane}}$ is five-dimensional (while the components $Q_i^{\#}$ are essentially two-dimensional or less).

Moreover, there are no methods to treat Melnikov type integrals, which are not Abelian integrals along cycles in algebraic curves. The long ago stated problem of zeroes of the integral $\int (Q\,dx - P\,dy)/xy(1-x-y)$ along $x^{\alpha}y^{\beta}(1-x-y)^{\gamma} = h$, formulated by Arnold (see [**Ar2**]), remains unsolved.

Nevertheless the problems stated here seem to be very interesting. We hope that they will be investigated further.

Let us discuss some works of other authors devoted to relation between the integrability and the dynamics of three-dimensional Lotka–Volterra systems.

Most of them are devoted to the so-called competitive case, when $a, a', a'' > 0$ and $b, b', \ldots, d'' < 0$ in (1.5). Hirsch [**Hi**] proved the existence of a so-called carrying simplex Σ (in the positive orthant), which is attracting for the whole orthant. Zeeman and Zeeman [**ZZ1, ZZ2**] investigated the influence of the geometry of Σ onto the dynamics, e.g. it turns out that there is a global attracting point when Σ is convex.

Mierczyński [**Mi**] proved that if there exists a first integral, then the ω-limit set of any non-periodic trajectory is an equilibrium point. In the works of Duarte *et al.* [**DFO**] and Bustamante and Hojman [**BH**] a kind of Lagrangian formalism is developed (but for some extended system). In [**DFO**] the Lyapunov function $V = x - x_* \log x + y - y_* \log y + z - z_* \log z$ for dissipative systems was introduced and in [**BH**] situations with invariant volume $x^{\alpha}y^{\beta}z^{\gamma}\,dx\,dy\,dz$ were studied.

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Examples of limit cycles in the three-dimensional case were found by Zeeman [**Ze**] (one cycle), by Hofbauer and So [**HSo**] (two cycles) and by Xiao and Li [**XL**] (two cycles). These cycles are obtained in the same way as in §5; one cycle appears in the Hopf's bifurcation and the existence of another cycle follows from the study of the return map near the boundary of the carrying simplex. In [**DFO**] a concrete four-dimensional completely integrable example is studied in detail.

Very intriguing are the works of the physicists Arneodo *et al.* [**ACT**] and Schnabl *et al.* [**SSFS**] with numerical calculations suggesting the existence of a chaotic attractor in some three-dimensional Lotka–Volterra systems. Unfortunately these papers are far from being rigorous. We briefly discuss this topic at the end of this paper.

The paper is organized as follows. In §2 we recall known results about the two-dimensional generalized Lotka–Volterra systems. In §3 we give a precise description of the center variety in the space of three-dimensional Lotka–Volterra systems. In §4 we define the first-order Poincaré–Melnikov integrals associated with perturbations X_ε of systems X_0 from the center variety. We prove that generic perturbations of systems from $\mathcal{LV}_0^{\text{plane}} \cap \mathcal{LV}^{\text{Darb}}$ do not create limit cycles (Theorem 4.2). Section 5 is devoted to the study of the behavior of perturbations X_ε of systems from $\mathcal{LV}_0^{\text{plane}}$ near the triangle contour in the invariant surface L_ε and to calculations of focus quantities near the focus p_* in the regime of very strong normal hyperbolicity. In §6 we discuss the possibility of the existence of other components of the center variety.

2. *The planar generalized Lotka–Volterra systems*

A general planar generalized Lotka–Volterra system has four singular points: the origin, two points on the coordinate axes and a point $q_* = (x_*, y_*)$ outside the axes. For non-generalized Lotka–Volterra systems the two points on the axes lie at infinity.

We restrict our attention to the case with four singular points. Thus, after rescaling the coordinates, we assume that the singular points on the axes are $(1, 0)$ and $(0, 1)$. Therefore, we are dealing with vector fields $Y(x, y)$ of the form

$$\dot{x} = x[Ay - B(1 - x - y)], \quad \dot{y} = y[C(1 - x - y) + Dx], \tag{2.1}$$

or of the form

$$\dot{x} = x[\widetilde{A}y + \widetilde{B}(1 - x)], \quad \dot{y} = y[\widetilde{C}(1 - y) + \widetilde{D}x]. \tag{2.2}$$

The following result was proved by Bautin [**Ba**].

THEOREM 2.1.

(a) *The following conditions are equivalent:*
  (i) *the line $x + y = 1$ is invariant;*
  (ii) $\operatorname{div} Y(q_*) = 0$*;*
  (iii) *there is a first integral of the form*

$$H = |x|^\alpha |y|^\beta |1 - x - y|^\gamma; \tag{2.3}$$

  (iv) $A + D = 0$ *in (2.1);*
  (v) $\widetilde{A} + \widetilde{B} + \widetilde{C} + \widetilde{D} = 0$ *in (2.2).*

(b) *The vector field Y does not have limit cycles.*

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Proof. The systems with the first integral (2.3) take the form

$$\dot{x} = x[\gamma y - \beta(1 - x - y)], \quad \dot{y} = y[\alpha(1 - x - y) - \gamma x]. \tag{2.4}$$

We see then that (iii), (iv) and (v) are equivalent. Moreover, $A = \gamma$, $B = \beta$ and $C = \alpha$ in (2.1).

Assume that the singular point q_* lies in the first quadrant, the other cases are analogous. Multiplying the field (2.1) by the Dulac function $\Phi = x^{C/A-1}y^{B/A-1}$ and taking the divergence, we find that for $x, y > 0$

$$\operatorname{div}(\Phi Y) = \frac{B}{A}(A + D)x^{C/A}y^{B/A-1}.$$

From this the other statements of Theorem 2.1 follow. In particular, if $A + D \neq 0$, then the divergence of ΦY does not change sign in the first quadrant and the system (2.1) cannot have limit cycles. □

Remark 2.2. The equation $A + D = 0$ (i.e. $\operatorname{tr} dY(q_*) = 0$) and the inequality $\det dY(q_*) > 0$ define the *center variety* in the space of two-dimensional generalized Lotka–Volterra systems. The fields Y_0 from the center variety have the form (2.4) and have center at the critical point

$$q_* = (x_*, y_*) = \left(\frac{\alpha}{\alpha + \beta + \gamma}, \frac{\beta}{\alpha + \beta + \gamma}\right).$$

We have

$$\det dY_0(q_*) = \alpha\beta\gamma(\alpha + \beta + \gamma)^{-3}. \tag{2.5}$$

Therefore, if the center lies in the first quadrant, then all of the numbers $\alpha, \beta, \gamma, \alpha + \beta + \gamma$ have the same sign and the triangle $\{x > 0, y > 0, x + y < 1\}$ is filled with periodic solutions of Y_0; we call it the *center* locus of q_*. Otherwise, the center locus is unbounded.

COROLLARY 2.3. *Consider perturbations of the integrable system Y_0 of the form (2.4) with a center. Then the space of first-order Poincaré–Melnikov integrals associated with them is one-dimensional and is generated by the Abelian integral*

$$I_0 = \int_{H=h} |x|^\alpha |y|^\beta |1 - x - y|^{\gamma-1}\, dx = \pm h \int \frac{dx}{|1 - x - y|}. \tag{2.6}$$

Proof. Assume $x_*, y_* > 0$. We write $A = \gamma$, $B = \beta$, $C = \alpha$ and $D = -\gamma + \varepsilon$ in (2.1) for a perturbation Y_ε of Y_0; here α, β, γ may be slightly different to those for Y_0. The increment of the first integral function (2.3) along an integral curve of Y_ε after period equals

$$\Delta H = \int \dot{H}\, dt = \varepsilon \int \frac{\partial H}{\partial y} xy\, dt \approx \varepsilon \int_{H=h} x^\alpha y^\beta (1 - x - y)^{\gamma-1}\, dx. \qquad \square$$

The above results about the quadratic planar Lotka–Volterra systems will often be used in our study of the three-dimensional case. However, we will also need something more.

We shall also use some results about certain generalized Lotka–Volterra systems of third degree. They are of the form

$$\begin{aligned} \dot{x} &= x[\gamma y - \beta(1 - x - y) + \varepsilon P(x, y)], \\ \dot{y} &= y[\alpha(1 - x - y) - \gamma x + \varepsilon Q(x, y)], \end{aligned} \tag{2.7}$$

where ε is a small parameter and P, Q are quadratic polynomials.

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(The systems (2.7) appear in bifurcation theory. They are amplitude systems for deformations of vector fields in $(\mathbb{R}^4, 0)$ with singular point at 0 whose eigenvalues are purely imaginary $\pm i\omega_1, \pm i\omega_2$ and without resonances. See [**Zo1**] for example.)

The group (isomorphic to $\mathbb{R}^6$) of changes of the form $(x, y, dt) \to (x(1+\varepsilon(ax+by)), y(1+\varepsilon(cx+dy)), dt(1+\varepsilon(ex+fy)))$ acts on (2.7) modulo $O(\varepsilon^2)$. Using this action one can show (see [**Zo1**]) that there are essentially two perturbations and the space of first-order Poincaré–Melnikov integrals is two-dimensional. We have the formula

$$\Delta H = \varepsilon_1 I_1(h) + \varepsilon_2 I_2(h) + O(\varepsilon^2) \tag{2.8}$$

for the increment of H along a trajectory of (2.7) after period. Here I_1 and I_2 are some special Abelian integrals (e.g. I_1 can be chosen as I_0 in (2.6)) and $\varepsilon_1, \varepsilon_2$ are small parameters, of order $\sim\varepsilon$. In [**Zo1**] the following result was proved (see [**KG**] and [**ZK**] for other proofs).

THEOREM 2.4. *For any $(\varepsilon_1, \varepsilon_2) \neq 0$ the function $h \to \varepsilon_1 I_1(h) + \varepsilon_2 I_2(h)$ has at most one zero (counting with multiplicities). In other words, for generic perturbations the generalized cubic Lotka–Volterra system (2.7) can have at most one limit cycle.*

We do not provide the proof of this theorem. We shall use it in the following situation. Suppose that we know (by some other arguments) that there is a separatrix connection between the two saddles near $(1, 0)$ and $(0, 1)$ for the system (2.7); such a connection together with the suitable intervals in the axes forms a limit contour.

COROLLARY 2.5. *If the perturbation $\varepsilon(xP\partial_x + yQ\partial_y)$ in (2.7) is generic and such that there exists the separatrix connection between saddles on the axes, then the system (2.7) does not have limit cycles (only the limit contour).*

3. *The space of three-dimensional Lotka–Volterra systems: its symmetries and the center variety*

3.1. *Symmetries.* Any generalized Lotka–Volterra system (1.5) defines a field of directions (a foliation) of the projective space $\mathbb{RP}^3$ (with the homogeneous coordinates $(x_1 : x_2 : x_3 : x_4)$, thus $x = x_1/x_4$, $y = x_2/x_4$, $z = x_3/x_4$). It easy to see that in the chart $s = x_4/x_3 = 1/z$, $v = x_1/x_3 = x/z$, $w = x_2/x_3 = y/z$ the new system divided by s has the same form (1.5). The change

$$(x, y, z) \to (s, v, w) \tag{3.1}$$

defines a projective automorphism of $\mathbb{RP}^3$ which preserves the planes $x_1 = 0$, $x_2 = 0$ and exchanges the planes $x_3 = 0$ and $x_4 = 0$. Therefore, it defines an automorphism of the twelve-dimensional space **LV** of systems (1.5). The change (3.1) together with permutations of the affine coordinates x, y, z generate a group of symmetries of the space **LV**. This group is isomorphic to the symmetric group $S(4)$, but we prefer to treat it as A_3, i.e. the group of isometries of a regular tetrahedron (the Coxeter group generated by three reflections).

We denote

$$\mathcal{LV} = \mathbf{LV}/A_3$$

with the quotient map $\mathbf{LV} \xrightarrow{\pi} \mathcal{LV}$.

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We also use the action of the real torus $(\mathbb{R}^*)^3 \simeq (\mathbb{R}^*)^4/\mathbb{R}^*$ of changes $(x_1 : x_2 : x_3 : x_4) \to (\lambda_1 x_1 : \lambda_2 x_2 : \lambda_3 x_3 : \lambda_4 x_4)$, $\lambda_i \neq 0$. Thus we have an action of the semi-direct product

$$A_3 \leftthreetimes (\mathbb{R}^*)^3. \tag{3.2}$$

There is also the action of $\mathbb{R}^*$ by changes of time, but we use it sporadically.

We do not consider all of the systems from the space **LV**. We delete from **LV** certain algebraic subsets. Namely, we define the space $\mathbf{LV}^*$ as consisting of $X \in \mathbf{LV}$ which satisfy the following conditions.

1. The foliation $\mathcal{F}$ of $\mathbb{RP}^2$ defined by any X is a foliation of degree 2, i.e. in any affine chart the corresponding vector field cannot be divided by a linear (or quadratic) function.
2. Each coordinate line $x_i = x_j = 0$ contains at least three geometrically distinct singular points of $\mathcal{F}$.
3. There is a singular point p_* outside the coordinate planes (this point may be non-isolated).

The space $\mathbf{LV}^*$ is invariant with respect to the action of the group (3.2). Denote $\mathcal{LV}^* = \mathbf{LV}^*/A_3$ and $\pi^* : \mathbf{LV}^* \to \mathcal{LV}^*$ the quotient map.

Modulo the action of $(\mathbb{R}^*)^3$ any element of $\mathcal{LV}^*$ has a representative $X \in \mathbf{LV}^*$ of the form

$$\dot{x} = x(ay + bz + \lambda u), \quad \dot{y} = y(cz + dx + \mu u), \quad \dot{z} = z(ex + fy + \nu u), \tag{3.3}$$

where

$$u = x + y + z - 1$$

and $a, b, c, d, e, f, \lambda, \mu, \nu$ are real parameters.

We denote by LV the space of systems (3.3).

3.2. *The center variety.* In the following four theorems we define four components LV_0^{plane}, LV_1^{plane}, LV^{Darb} and LV^{Hopf} of the center variety in LV and describe their main properties. These components define corresponding components in $\mathcal{LV}^*$ (and also in $\mathcal{LV}$ and in **LV**).

THEOREM 3.1. *The systems (3.3) with the plane*

$$L_0 = \{u = 0\}$$

invariant form a linear subspace given by the equations

$$a + d = b + e = c + f = 0. \tag{3.4}$$

Any such X_0 restricted to the plane L_0 has the partial first integral

$$H = |x|^\alpha |y|^\beta |z|^\gamma|_{L_0} = |x|^\alpha |y|^\beta |1 - x - y|^\gamma, \tag{3.5}$$

(where $\alpha = -d$, $\beta = b$, $\gamma = a$) and has center at the critical point

$$p_* = \left(\frac{\alpha}{\alpha+\beta+\gamma}, \frac{\beta}{\alpha+\beta+\gamma}, \frac{\gamma}{\alpha+\beta+\gamma}\right)$$

provided that $\det d(X_0|_{L_0}) = \alpha\beta\gamma(\alpha+\beta+\gamma)^{-3} > 0$.

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Therefore, the latter inequality and (3.4) define the subvariety $LV_0^{\text{plane}} \subset LV$*, a component of the center variety of codimension three.*

Proof. We have

$$\dot{u} = (\lambda x + \mu y + \nu z)u + (a+d)xy + (b+e)xz + (c+f)yz, \tag{3.6}$$

which gives (3.4). The other statements follow from Theorem 2.1(a) and the observation that $X_0|_{L_0}$ has the line $L_0 \cap \{z=0\} = \{x+y-1=z=0\}$ invariant. □

Remark 3.2. Each of the equations (3.4) separately means the necessary condition for the center of a suitable two-dimensional system. For example, the condition $a+d=0$ means the center condition on the plane $z=0$. Recall, however, that we have agreed not to include in the center variety the cases of a center in the coordinate planes.

THEOREM 3.3. *The space* LV_1^{plane} *of systems* $X_0 \in LV$ *with the plane* $z=1$ *invariant and center* $p_* \in L_1$ *of* $X_0|_{z=1}$ *forms the semi-algebraic subset defined by the equations*

$$e+\nu = f+\nu = b\mu(\lambda-\mu-d) + c\lambda(\mu-\lambda-a) = 0 \tag{3.7}$$

and the inequality $\det d(X_0|_{z=1})(p_*) > 0$.

Therefore, LV_1^{plane} *defines the component* $\mathcal{LV}_1^{\text{plane}}$ *of the center variety of codimension three in* $\mathcal{LV}$*. This component corresponds to six components* $\mathbf{LV}_i^{\text{plane}}$*,* $i=1,\dots,6$*, of the center variety in* LV*, consisting of vector fields with an invariant plane of the form* $x_i = \text{constant} \cdot x_j$ *in* $\mathbb{RP}^3$.

Proof. The first two conditions in (3.7) mean that the plane $z=1$ is invariant, thus $\dot{z} = \nu z(z-1)$. Putting $z=1$ in the system (3.3) we get

$$\dot{x} = x[b+\lambda x + (a+\lambda)y] = x[b(1-x_1) + (a+\lambda)(-c/\mu)y_1]$$
$$\dot{y} = y[c+(d+\mu)x + \mu y] = y[c(1-y_1) + (d+\mu)(-b/\lambda)x_1]$$

for $x_1 = -\lambda x/b$ and $y_1 = -\mu y/c$. The third condition in (3.7) is the condition from the point (a).(v) of Theorem 2.1. □

THEOREM 3.4. *The space* LV_1^{Darb} *of systems* $X_0 \in LV$ *with the plane* $x+y=1$ *invariant is defined by the equations*

$$a+\lambda = c+\mu = a+d = 0. \tag{3.8}$$

Each such system has the global first integral

$$F = |x|^\alpha |y|^\beta |1-x-y|^\delta, \tag{3.9}$$

where $\alpha=\mu$, $\beta=\lambda$, $\delta=a$*, and if, additionally,* $\alpha\beta\delta(\alpha+\beta+\delta) > 0$*, then the system has center* p_* *at the line*

$$\left\{ x = \frac{\alpha}{\alpha+\beta+\delta}, y = \frac{\beta}{\alpha+\beta+\delta} \right\}$$

of critical points of F.

Therefore, $LV_1^{\text{Darb}} \subset LV$ *defines the component* $\mathcal{LV}^{\text{Darb}}$ *of codimension* 3 *of the center variety in* $\mathcal{LV}$*. This corresponds to four components* $\mathbf{LV}_i^{\text{Darb}}$*,* $i=1,2,3,4$*, of the center variety in* $\mathbf{LV}$*, consisting of* X_0 *with an invariant plane of the form* $\{Ax_i + Bx_j + Cx_k = 0\}$ *and with the global first integral* $|x_1|^\alpha |x_2|^\beta |x_3|^\gamma |Ax_i + Bx_j + Cx_k|^{-\alpha-\beta-\gamma}$.

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Proof. We have

$$\frac{d}{dt}(x+y-1) = (\lambda x+\mu y)(x+y-1) + (a+\lambda)xz + (c+\mu)yz + (a+d)xy,$$

what gives the first statement of the theorem.

The condition $\alpha\beta\delta(\alpha+\beta+\delta) > 0$ is the condition that F has local extremum along the line $\{x = x_*, y = y_*\}$. This line forms an invariant manifold (stable or unstable) for the singular point p_*, provided that $dX_0(p_*)$ has non-zero eigenvalues. The point p_* has a center manifold $W^c = W^c(p_*)$ and the restriction $X_0|_{W^c}$ has the first integral $F|_{W^c}$ and hence the center at p_*.

In the case when one eigenvalue of $dX_0(p_*)$ is zero the whole line $\{x = x_*, y = y_*\}$ is singular for X_0. Here we refer the reader to the proof of Theorem 3.7 below. □

Remark 3.5. The systems $X_0 \in LV$ with an invariant plane of the form $kx+ly+mz = 0$ (i.e. through the origin) also represent elements of the component $\mathcal{LV}^{\text{Darb}}$; we denote by LV_2^{Darb} the corresponding component in LV. They have the global first integral

$$F = |x|^\alpha |y|^\beta |z|^\gamma |kx+ly+mz|^{-\alpha-\beta-\gamma}.$$

The latter function also depends essentially on only two variables x/z and y/x and has a whole line of critical points. The conditions for this case are $\lambda = \mu = \nu$ (i.e. the linear part is homogeneous) and one condition for the center at infinity (i.e. for an additional invariant line in the plane $x_4 = 0$).

Remark 3.6. One can write down a differential equation for the center manifold $W^c(p_*) = \{z = \theta(x, y)\}$, the center locus of p_*:

$$\dot\theta = [(e+\nu)x + (f+\nu)y - 1]\theta + \nu\theta^2. \tag{3.10}$$

When restricted to the phase curves $F(x, y) = \sigma$ this is a Riccati equation with periodic coefficients, i.e. $\dot\theta = A(t)\theta + \nu\theta^2$, $A(t+T) \equiv A(t)$, where $T = T(\sigma)$ is a period. We look for periodic solutions, $\theta(t+T) \equiv \theta(t)$.

For this one considers the monodromy map $M : \{(\theta, t) : t = t_0\} \to \{(\theta, t) : t = t_0+T\}$ after one period. This map is fractional–linear, $M(\theta) = (p\theta+q)/(r\theta+s)$, where p, q, r, s depend on σ and t_0. The map $M = M_{t_0,\sigma}$ has two fixed points $\theta = 0$ and $\theta = \vartheta(t_0, \sigma)$ for generic (t_0, σ); this holds at least for σ close to the critical value $\sigma_* = F(p_*)$. For some (isolated) values of (t_0, σ) the fixed point $\vartheta(t_0, \sigma)$ escapes to infinity, or becomes equal to zero; the latter may occur only at a singular point of the field X_0.

The surface

$$W^c_{\text{glob}}(p_*) = \{x = x(t, \sigma), y = y(t, \sigma), z = \vartheta(t, \sigma)\}$$

where $x(t, \sigma)$, $y(t, \sigma)$ are solutions of the planar system and $\vartheta(t, \sigma)$ is the (non-zero) periodic solution to (3.10) is the extension of the local center manifold $W^c_{\text{loc}}(p_*)$.

Note that after the change $\eta = 1/\theta$ the equation (3.10) is the same as the normal variation equation (1.3). The normal hyperbolicity condition corresponds to the condition $\int_0^T A(t)\,dt \neq 0$ (for any σ); in such a case $\vartheta(t, \sigma)$ is always non-zero.

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THEOREM 3.7. *The systems $X_0 \in LV$ with non-isolated critical points form a subspace given by the equations*

$$\begin{vmatrix} 0 & a & b \\ d & 0 & c \\ e & f & 0 \end{vmatrix} = \alpha\lambda + \beta\mu + \gamma\nu = 0, \tag{3.11}$$

where $\alpha = -df$, $\beta = -ae$, $\gamma = ad$. They have the global first integral

$$E = |x|^{\alpha}|y|^{\beta}|z|^{\gamma} \tag{3.12}$$

and a line

$$\ell = \{by + cz + \lambda u = cz + dx + \mu u = ex + fy + \nu u = 0\} \tag{3.13}$$

of non-isolated critical points. The point $p_* = \ell \cap \{\lambda x + \mu y + \nu z = 0\}$ is the center for X_0 provided that $\det(dX_0|_{E=E(p_*)}) > 0$.

Therefore, the latter inequality and the equations (3.11) define the component LV^{Hopf} of the center variety of codimension two in LV and A_3-invariant component $\mathbf{LV}^{\mathrm{Hopf}}$ in $\mathbf{LV}$.

Proof. If a system X_0 has non-isolated critical points, then these singular points form a straight line. (They cannot form a surface, because X_0 is not divisible by a linear function.) However, this means that the functions $f_1 = ay + bz + \lambda u$, $f_2 = cz + dx + \mu u$, $f_3 = ex + fy + \nu u$ are dependent. Since they constitute the cofactors of the (Darboux) polynomials x, y, z (e.g. $\dot{x} = xf_1$), then there exists a first integral of the form (3.12). The equations for the exponents are

$$(\alpha, \beta, \gamma)\begin{pmatrix} 0 & a & b \\ d & 0 & c \\ e & f & 0 \end{pmatrix} = (0, 0, 0)$$

(for the coefficients before x, y, z) and

$$\Lambda \stackrel{\mathrm{df}}{=} \alpha\lambda + \beta\mu + \gamma\nu = 0 \tag{3.14}$$

(for the coefficient before u).

To prove the existence of a center we consider the one-parameter family $Y_\rho = X_0|_{\{E=\rho\}}$ of two-dimensional vector fields. Each system Y_ρ has singular point $p_\rho = \ell \cap \{E = \rho\}$. Since one eigenvalue of $dX_0(p_\rho)$ is zero, we have $\operatorname{div} Y_\rho(p_\rho) = \operatorname{div} X_0(p_\rho)$. The function $(\operatorname{div} X_0)|_\ell$ is linear and equals

$$(\operatorname{div} X_0)|_\ell = (\lambda x + \mu y + \nu z)|_\ell. \tag{3.15}$$

There are two possibilities:

(i) $(\operatorname{div} X_0)|_\ell \not\equiv 0$; and

(ii) $(\operatorname{div} X_0)|_\ell \equiv 0$.

In the case (i) we denote by p_* the intersection point of the line ℓ with the plane $\lambda x + \mu y + \nu z = 0$ and denote $\rho_* = E(p_*)$. We claim that if $\det dY_{\rho_*}(p_*) > 0$, *then p_* is a center.*

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However, this follows from the classical theorem of Hopf about bifurcations of periodic solutions.

Assume that we have a family Y_ρ, $\rho \in (\mathbb{R}, 0)$ of planar vector fields of the form

$$\dot{\zeta} = (a(\rho) + ib(\rho))\zeta + O(|\zeta|^2),$$

where $\zeta = \zeta_1 + i\zeta_2 \in (\mathbb{C}, 0) \simeq (\mathbb{R}^2, 0)$, $b(0) > 0$ and $a(0) = 0$, $a'(0) \neq 0$. Then there exists a curve $C : (\mathbb{R}_+, 0) \to (\mathbb{R} \times \mathbb{R}, 0) \subset \mathbb{C} \times \mathbb{R}$, $\tau \to (\zeta_1(\tau), \rho(\tau))$ such that for any τ the trajectory γ_τ of $Y_{\rho(\tau)}$ through $\zeta_1(\tau)$ is periodic.

This is the original Hopf formulation. We refer the reader to [**MM**] for the proof. (Note that in bifurcation theory people usually assume a non-degeneration of third-order terms in (3.15) and call the corresponding deformation the Andronov–Hopf bifurcation, see [**Ar1**].)

Consider the case (ii). Here the Hopf theorem does not work. Assume that the critical line ℓ intersects the coordinate planes $x = 0$, $y = 0$ and $z = 0$ at the points q_1, q_2 and q_3, respectively. If $(\operatorname{div} X_0)|_\ell \equiv 0$, then also $\operatorname{div}(X_0|_{x=0})(q_1) = 0$, $\operatorname{div}(X_0|_{y=0})(q_2) = 0$ and $\operatorname{div}(X_0|_{z=0})(q_3) = 0$. However, this means that the corresponding planar systems are integrable (by Theorem 2.1) and have additional invariant lines. This in order implies that the plane $x + y + z = 1$ is invariant for X_0. Therefore, we have the case LV_0^{plane} of center. We can assume the system

$$\dot{x} = x(\gamma y - \beta z + \lambda u), \quad \dot{y} = y(\alpha z - \gamma x + \mu u), \quad \dot{z} = z(\beta x - \alpha y + \nu u); \tag{3.16}$$

with $\dot{u} = (\lambda x + \mu y + \nu z)u$ and $\Lambda = \alpha\lambda + \beta\mu + \gamma\nu = 0$.

The condition for a first integral of the form $x^{\alpha'} y^{\beta'} z^{\gamma'} u^{-\delta'}$ leads to the equation

$$(\alpha', \beta', \gamma', \delta') \begin{pmatrix} 0 & \gamma & -\beta & -\lambda \\ -\gamma & 0 & \alpha & -\mu \\ \beta & -\alpha & 0 & -\nu \\ \lambda & \mu & \nu & 0 \end{pmatrix} = (0, 0, 0, 0). \tag{3.17}$$

(This equation can be treated as dual to the equation for equilibrium points of (3.16).) The antisymmetric matrix in (3.17) has rank 2 and the space of solutions to (3.17) is two-dimensional with the basis: $(\alpha, \beta, \gamma, 0)$, $(0, -\nu, \mu, \alpha)$. This means that the system (3.16) has two independent first integrals and the center is formed by a two-parameter family of periodic solutions.

In order to finish the proof of this theorem we should consider also the cases when the critical line takes some non-generic position, e.g. is parallel to a coordinate plane. These cases also lead to an additional first integral and are treated in §4.8 below. □

Remark 3.8. The first condition in (3.10), i.e.

$$ace + bdf = 0, \tag{3.18}$$

has the following dynamic interpretation.

The critical point $(1, 0, 0)$ has the eigenvalues: λ (with the eigenvector along the x-axis), d (i.e. $\dot{y} = y(d + \cdots)$) and e (i.e. $\dot{z} = z(e + \cdots)$). Similarly the point $(0, 1, 0)$ has the eigenvalues: μ (along the y-axis), f (in a direction transversal to $z = 0$) and a (in a direction transversal to $x = 0$). The corresponding eigenvalues for the point $(0, 0, 1)$ are ν, b, c.

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Suppose now that there exists some invariant surface $\widetilde{L}$ through these three points and that the first octant cuts a (nonlinear) triangle in $\widetilde{L}$ with vertices being saddle points for the restricted vector field $X|_{\widetilde{L}}$. The boundary of this triangle is a polycycle Γ composed of three saddle connections: Γ_1 between $(0, 0, 1)$ and $(0, 1, 0)$; Γ_2 between $(0, 1, 0)$ and $(0, 0, 1)$; and Γ_3 between $(0, 0, 1)$ and $(1, 0, 0)$ (see Figure 1).

Then the return map near this polycycle takes the form

$$\xi \to A\xi^{\kappa} + \cdots, \tag{3.19}$$

where ξ is a local parameter along a local section transversal to Γ and

$$\kappa = \left(-\frac{e}{d}\right)\left(-\frac{a}{f}\right)\left(-\frac{c}{b}\right). \tag{3.20}$$

Therefore, the condition (3.18) means that

$$\kappa = 1.$$

It is the first necessary condition for the center in $\widetilde{L}$. The next condition is $A = 1$.

Remark 3.9. The invariant manifold with a family of periodic solutions, i.e. the center locus, from the thesis of Theorem 3.7 is only well defined locally. It would be interesting to see how large it is. It can probably be prolonged up to the coordinate planes, i.e. to the four planes in the projective space.

4. *Poincaré–Melnikov integrals*

The *Poincaré–Melnikov integrals* arise when one perturbs a system for which some solutions are known. In one such situation one has an autonomous vector field with a homoclinic loop; after periodic in time perturbation, the separatrices are split and a chaotic regime is created. In another situation, considered in this paper, a family of periodic solutions is given; after perturbation there remain only finitely many limit cycles.

For the perturbed system it is natural to study the equations in variations with respect to parameters. Integration of the latter equations leads to some contour integrals, which we call the Poincaré–Melnikov integrals (in the literature they are also are met under other names, like Pontryagin or Arnold).

When one considers variations linear in parameters, then one deals with the *first-order Poincaré–Melnikov integrals*. However, there are known situations, when the first-order Poincaré–Melnikov integrals are not sufficient to determine the dynamics; due to some geometrical reasons they can vanish (cf. Theorem 4.2 below). In that case one should consider higher-order equations in variations and higher-order Poincaré–Melnikov integrals. In fact, in this paper we do not study the higher-order Poincaré–Melnikov integrals, but we want to underline their importance in bifurcation theory.

In this section we define the first-order Poincaré–Melnikov integrals associated with perturbations of systems X_0 with center of type $\mathcal{LV}_0^{\text{plane}}$, $\mathcal{LV}_1^{\text{plane}}$ and $\mathcal{LV}^{\text{Darb}}$. However, we make additional restrictions onto the unperturbed systems. In order that the Poincaré–Melnikov integrals really control arising limit cycles we assume the normal hyperbolicity condition.

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The component $\mathcal{LV}^{\text{Hopf}}$ is different to other components of the center variety. Here the periodic solutions are not given in an explicit form and we cannot impose any normal hyperbolicity condition. It turns out that the problem of limit cycles here is related to the problem of singular perturbations (e.g. relaxational oscillations, delay of the loss of stability, etc.).

4.1. *The center with invariant plane transversal to axes.* Here $X_0 \in LV_0^{\text{plane}}$ and the perturbed vector field $X_\varepsilon = X_0 + \varepsilon_1 X_1 + \varepsilon_2 X_2 + \varepsilon_3 X_3$ is taken in the form

$$\begin{aligned} \dot{x} &= x[\gamma(1+\varepsilon_3)y - \beta z + \lambda u], \\ \dot{y} &= y[\alpha(1+\varepsilon_1)z - \gamma x + \mu u], \\ \dot{z} &= z[\beta(1+\varepsilon_2)x - \alpha y + \nu u], \end{aligned} \tag{4.1}$$

$u = x + y + z - 1$. We have

$$\dot{u} = (\lambda x + \mu y + \nu z)u + \alpha\varepsilon_1 yz + \beta\varepsilon_2 xz + \gamma\varepsilon_3 xy. \tag{4.2}$$

We can also assume that $\alpha, \beta, \gamma > 0$ and that the center of the unperturbed system lies in the triangle $\Delta \subset L_0 = \{u = 0\}$ bounded by the planes $x = 0$, $y = 0$, $z = 0$. Indeed, in the homogeneous coordinates of $\mathbb{RP}^3$ the first integral takes the form

$$(|x_1|^\alpha |x_2|^\beta |x_3|^\gamma |x_4|^{-\alpha-\beta-\gamma})|_{x_1+x_2+x_3=x_4}$$

and the center p_* lies in one of the four triangles in the projective plane $\mathbb{RP}^2 = \{x_1 + x_2 + x_3 = x_4\}$ cut by the planes $x_i = 0$. After applying a projective transformation one obtains the desired situation.

After projection of the plane L_0 to the (x, y)-plane we can easily find the eigenvalues of the singular points of X_0 at the vertices of the triangle. They are presented in Figure 1 (where λ, μ, ν are the eigenvalues in the direction transversal to L_0).

The *normal hyperbolicity condition* means the following:

(i) The 'normal variation matrix' $A(x, y) = \lambda x + \mu y + \nu(1 - x - y)$ has constant sign in the triangle $\Delta = \{x \geq 0, y \geq 0, x + y \leq 1\}$; and

(ii) $|\lambda| > \max(\beta, \gamma)$, $|\mu| > \max(\alpha, \gamma)$, $|\nu| > \max(\alpha, \beta)$, (i.e. the normal hyperbolicity at the vertices of Δ).

Like in [**BZ1**] one shows that the perturbed system X_ε has an invariant surface L_ε of class C^1 which takes the form

$$u = \varepsilon_1 g_1(x, y) + \varepsilon_2 g_2(x, y) + \varepsilon_3 g_3(x, y) + o(|\varepsilon|),$$

where the functions g_j satisfy the equations

$$\mathcal{D} g_j = b_j, \tag{4.3}$$

where $b_1 = \alpha y(1 - x - y)$, $b_2 = \beta x(1 - x - y)$, $b_3 = \gamma xy$ and

$$\mathcal{D} = \frac{d}{dt} - \hat{A}(t)$$

is the *connection* and $\hat{A}(t) = A|_{\{H=h\}}$. $\mathcal{D}$ is treated as a connection in the trivial one-dimensional bundles over the curves $\{H = h\} = \{x^\alpha y^\beta (1 - x - y)^\gamma = h\}$, or as a

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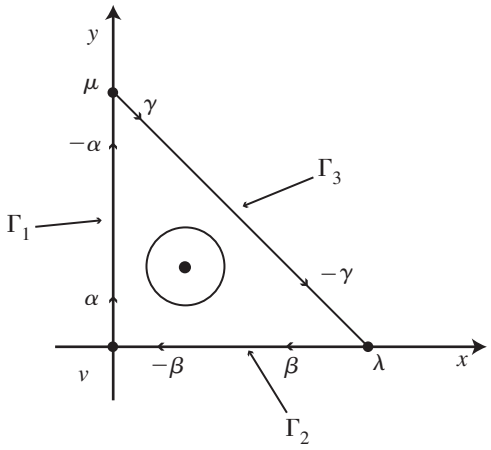


FIGURE 1. Eigenvalues of the singular points of X_0 at the vertices of the triangle.

connection in the module $\mathbb{R}[x, y]$ of polynomials; d/dt is the derivative along the vector field X_0.

In [**BZ1**] there are integral formulas for solutions to (3.3). When the normal hyperbolicity condition holds these solutions are unique.

The first-order Poincaré–Melnikov integral takes the form

$$I(h) = \oint_{H=h} My[\alpha\varepsilon_1(1-x-y)\,dx - \gamma\varepsilon_2 x\,dy] + \oint (\varepsilon_1 g_1 + \varepsilon_2 g_2 + \varepsilon_3 g_3) M[(\mu+c)y\,dx - (\lambda+b)x\,dy], \tag{4.4}$$

where $H = x^\alpha y^\beta (1-x-y)^\gamma$ is the (partial) first integral and

$$M = x^{\alpha-1} y^{\beta-1} (1-x-y)^{\gamma-1}$$

is the integrating factor. Above the first contour integral is the known Abelian integral, equal to $h \cdot \oint \omega/x(1-x-y)$, $\omega = \alpha\varepsilon_1(1-x-y)\,dx - \gamma\varepsilon_2 x\,dy$, but along a non-algebraic curve (in general). The second contour integral is the linear combination of integrals of the form

$$\Psi(\hat{b}_i, \hat{c}) = \oint_{H=h} \hat{g}_i(t)\hat{c}(t)\,dt, \tag{4.5}$$

where $\hat{g}_i(t) = g_i|_{H=h}$ are solutions to (4.3). Also $\hat{b}_i(t) = b_i|_{H=h}$ and $\hat{c}(t)\,dt = M[(\mu+c)y\,dx - (\lambda+b)x\,dy]|_{H=h}$.

We have the following property (see [**BZ2**]).

If $\hat{g}_i \in \mathbb{C}[x, y]|_{H=h}$, i.e. $\hat{b}_i \in \operatorname{Im}\mathcal{D}(\mathbb{C}[x, y]|_{H=h})$, or $\hat{c} \in \operatorname{Im}\mathcal{D}^*(\mathbb{C}[x, y]|_{H=h})$, where $\mathcal{D}^* = -d/dt - \hat{A}$ is the conjugate connection (i.e. $\int \mathcal{D}\hat{d} \cdot \hat{c}\,dt = \int \hat{d} \cdot \mathcal{D}^*\hat{c}\,dt$), then the integral (4.5) is the usual Abelian integral.

Note also that, by Corollary 2.3, the Abelian integral in (4.4) can be reduced to constant $\cdot\, h \oint dx/(1-x-y)$.

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4.2. *The center with invariant plane $z = 1$.* Here $X_0 \in LV_1^{\text{plane}}$ and $X_\varepsilon = X_0 + \varepsilon_1 X_1 + \varepsilon_2 X_2 + \varepsilon_3 X_3$. After renaming the parameters of X_0 we can assume the system

$$\begin{aligned} \dot{x} &= x[\gamma(1+\varepsilon_1)y - \beta(1-x-y) + Az_1], \\ \dot{y} &= y[\alpha(1-x-y) - \gamma x + Bz_1], \\ \dot{z}_1 &= z_1(\nu + \nu z_1) + \varepsilon_2 x(1+z_1) + \varepsilon_2 y(1+z_1), \end{aligned} \tag{4.6}$$

where $z_1 = z - 1$. We can also assume that $\alpha, \beta, \gamma > 0$. Then the normal variation equations for the perturbed invariant surface $L_\varepsilon = \{z_1 = \varepsilon_1 g_1 + \varepsilon_2 g_2 + o(|\varepsilon|)\}$ take the form

$$\dot{g}_i = \nu g_i + b_i, \quad b_1 = x, \quad b_2 = y. \tag{4.7}$$

Although the equations (4.7) definitely look simpler than (4.3), their solutions are also transcendental functions.

The first-order Poincaré–Melnikov integral associated with (4.6) is given by the formula

$$I(h) = -\gamma\varepsilon_1 \oint_{H=h} Mxy\,dy + \oint_{H=h} (\varepsilon_1 g_1 + \varepsilon_2 g_2) M[By\,dx - Ax\,dy]. \tag{4.8}$$

4.3. *The center with global Darboux integral.* Here $X_0 \in LV_1^{\text{Darb}}$ and X_ε is of the form

$$\begin{aligned} \dot{x} &= x[\gamma(1+\varepsilon_1)y - \beta(1-x-y) + \varepsilon_2 z], \\ \dot{y} &= y[\alpha(1-x-y) - \gamma x + \varepsilon_3 z], \\ \dot{z} &= z(p + qx + ry) + \nu z^2. \end{aligned} \tag{4.9}$$

The first-order Poincaré–Melnikov integral equals

$$I(h) = \gamma\varepsilon_1 \oint_{H=h} Mxy\,dy + \oint_{H=h} gM[\varepsilon_3 y\,dx - \varepsilon_2 x\,dy]. \tag{4.10}$$

The function g, i.e. $\hat{g}(t) = g|_{H=h}$, satisfies the equation

$$d\hat{g}/dt = A(t)\hat{g} + \nu\hat{g}^2, \quad A(t) = (p + qx + ry)|_{H=h}.$$

The solution is given by

$$\hat{g}(t) = \frac{1 - \exp(-\int_0^T A(s)\,ds)}{\nu \int_{t-T}^t \exp(\int_t^s A(\tau)\,d\tau)\,ds}. \tag{4.11}$$

This formula follows from the formula $(1 - \mathbf{M}_t)^{-1} \int_{t-T}^t \mathbf{F}(t,s)\hat{b}(s)\,ds$ for the unique solution to the linear equation $\dot{w} = -A(t)w + \hat{b}(t)$ with the 'fundamental matrix' $\mathbf{F}$ and the 'monodromy matrix' $\mathbf{M}_t = \mathbf{F}(t+T, t)$, where T is the period.

We see that (4.10) and (4.11) constitute a new form of the Poincaré–Melnikov integral (probably not studied before).

The natural analogue of the normal hyperbolicity condition in the context of (4.9) states that the monodromy 'matrix' $\mathbf{M}_t = \mathbf{M}_0 = \int_0^T A(s)\,ds \neq 1$ for any value h of the first integral $H = x^\alpha y^\beta (1-x-y)^\gamma$. This guarantees that $\hat{g}(t)$ would not be equal to zero. However, it does not guarantee that $g(x, y)$ would be bounded. In general, solutions to the Riccati equations often escape to infinity (with first-order poles).

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When the escaping to infinity occurs, the approximation of the return map by means of the first-order Poincaré–Melnikov integral ceases to be correct. However, at the same moment the corresponding trajectory of the unperturbed field X_0 ceases to be periodic; it lies in the boundary of the center locus.

Here we stop the discussion of this case.

4.4. *The Hopf's center.* Since the unperturbed field X_0 has non-isolated singularities, the problem of limit cycles after perturbation $X_\varepsilon = X_0 + \varepsilon_1 X_1 + \varepsilon_2 X_2$ is very delicate. The right theory to deal with this kind of problems is the theory of relaxational oscillations. For a reference we cite the book [**AAIS**].

Indeed, we introduce the local coordinates $(x, y, \rho) \in (\mathbb{R}^3, (x_*, y_*, \rho_*))$ where $\rho = x^{\alpha/\gamma} y^{\beta/\gamma} z$ (for the global first integral $x^\alpha y^\beta z^\gamma$). Then we get the system

$$\dot{x} = f_1(x, y, \rho; \varepsilon), \quad \dot{y} = f_2(x, y, \rho; \varepsilon), \quad \dot{\rho} = \varepsilon g(x, y, \rho; \varepsilon), \tag{4.12}$$

where $\varepsilon = |\varepsilon|$. We see that the variables (x, y) are the 'fast' variables and ρ is the 'slow' variable. Such systems are typical in the theory of relaxational oscillations.

We know also that for $\varepsilon = 0$ the fast two-dimensional subsystems $Y_\rho : \dot{x} = f_1(x, y, \rho; 0), \dot{y} = f_2(x, y, \rho; 0)$ have singular point p_ρ with non-real eigenvalues (focus or center). Therefore, we can apply the theory of normal forms (a variant of the Poincaré–Dulac normal form, see [**Ar1**]) to the system (4.12). The result should be the following system

$$\dot{r} = Ar\rho_1 + Br^3 + \cdots, \quad \dot{\rho}_1 = \varepsilon(C + D\rho_1 + Er^2 + \cdots), \tag{4.13}$$

where r is the amplitude of oscillations of the fast new variables, $\rho_1 = \rho - \rho_* + \cdots$, $A = A(\varepsilon)$, $B = B(\varepsilon)$, $C = C(\varepsilon)$, $D = D(\varepsilon)$, $E = E(\varepsilon)$ are constants and '$\cdots$' denotes terms of higher order in r and ρ_1.

We know only that $A(0) \neq 0$ (e.g. when $X_0 \in LV^{\text{Hopf}} \setminus LV_0^{\text{plane}}$). The other constants $B(0), C(0), D(0), E(0)$ should be calculated. We have not performed these calculations, leaving this task to further investigations. We conjecture only that these constants are non-zero and hence there is at most one limit cycle near p_*.

Since our above argumentation was rather general, we did not specify the perturbation X_ε of X_0.

Remark 4.1. The system (4.13) with $\dot{\rho}_1 = \varepsilon \equiv$ constant > 0 and $A(0) > 0$, $B(0) < 0$ describes the phenomenon of delay of stability of the equilibrium point, described in [**AAIS**]. When the initial divergence $\lambda = A\,(\rho_1|_{t=0}) < 0$ is fixed, i.e. independent of ε, and $(r|_{t=0}) > 0$ is fixed, then $r(t)$ quickly approaches the equilibrium $r = 0$ and remains there until the divergence $A\rho_1(t)$ takes the value $-\lambda > 0$. At that moment $r(t)$ jumps to the amplitude of the fast system.

Note, however, that the case of 'flowing parameter' $\rho_1 = \rho_1(0) + \varepsilon t$ is degenerate for the three-dimensional system (4.13) and does not produce limit cycles.

Another question concerning perturbations of systems from LV^{Hopf} is the problem of periodic solutions of not small amplitude. This is related to the problem of global center locus stated in Remark 3.9 above. Unfortunately, at the moment we do not know how

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to attack this question rigorously. This problem is probably related with the problem of existence of chaos (see also the discussion in the last section).

4.5. *The intersection* $\mathcal{LV}_0^{\text{plane}} \cap \mathcal{LV}^{\text{Darb}}$. Here we prove the following result.

THEOREM 4.2. *Let* $X_0 \in \mathcal{LV}_0^{\text{plane}} \cap \mathcal{LV}^{\text{Darb}}$. *Then the space of first-order Poincaré–Melnikov integrals (4.4) is one-dimensional. Its generator* $J(h)$ *has the property that it vanishes only at the endpoints of the interval* $[0, h_*]$, $h_* = H(p_*)$ *of its definition.*

This means that a generic one-parameter perturbation $X_{\varepsilon(\delta)}$, $\delta \in (\mathbb{R}, 0)$ *does not have limit cycles.*

Proof. We assume that X_0 has the global Darboux first integral $G = x^\alpha y^\beta z^\gamma (x + y + z)^{-\alpha-\beta-\gamma}$; we denote the corresponding component LV_2^{Darb} in LV. (The van Gils system (1.1) is such.) Thus, $H = G|_{L_0} = x^\alpha y^\beta (1 - x - y)^\gamma$. We assume also that $\alpha > 0, \beta > 0$ and $\gamma = 1$.

Since the plane $x + y + z = 0$ is invariant, the linear part of X_0 at the origin must be homogeneous, i.e. $\lambda = \mu = \nu$. This and the condition $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$ in (4.1) define the intersection of the corresponding components LV_2^{Darb} and LV_0^{plane} in LV. From this it follows also that the tangent spaces to LV_0^{plane} and LV_2^{Darb} at X_0 span a hyperplane in $T_{X_0} LV$, what gives the first statement of Theorem 4.2.

Since $\lambda = \mu = \nu$, we get $\hat{A}(t) = \lambda \equiv$ constant in (4.3) and hence $\mathcal{D} = d/dt - \lambda$ and $\mathcal{D}^* = -d/dt - \lambda$. We use the formalism described in §4.1. The generalized Abelian integral in (4.4) takes the form

$$\oint_{H=h} (\varepsilon_1 g_1 + \varepsilon_2 g_2 + \varepsilon_3 g_3)\omega, \tag{4.14}$$

where

$$\begin{aligned}\omega &= x^{\alpha-1} y^{\beta-1}[(\lambda+\alpha) y\, dx - (\lambda - \beta) x\, dy] \\ &= x^\alpha y^\beta [-\lambda(\alpha+\beta+1)(1-x-y) - (\beta x - \alpha y - \lambda)]\, dt \\ &= \hat{c}(t)\, dt.\end{aligned}$$

Now direct calculations show that

$$\mathcal{D}^* H = -\lambda x^\alpha y^\beta (1 - x - y), \quad \mathcal{D}^* x^\alpha y^\beta = x^\alpha y^\beta (\beta x - \alpha y - \lambda).$$

Therefore,

$$\hat{c}(t) = \mathcal{D}^*[(\alpha + \beta + 1)H - x^\alpha y^\beta] = \mathcal{D}^* \hat{d}(t)$$

and

$$\Psi(\hat{b}_i, \hat{c}) = \int \hat{g}_i \hat{c}\, dt = \int \hat{g}_i \cdot \mathcal{D}^* \hat{d}\, dt = \int \mathcal{D}\hat{g}_i \cdot \hat{d}\, dt = \int \hat{b}_i \hat{d}\, dt.$$

Next we have

$$\hat{d}(t)\, dt = x^\alpha y^\beta \{[\alpha(1 - x - y) - x]\, dt + [\beta(1 - x - y) - y]\, dt\} = x^\alpha y^\beta (x\, dy - y\, dx)$$

and the integral (4.14) takes the form

$$\int x^{\alpha-1} y^{\beta-1} (\varepsilon_1 b_1 + \varepsilon_2 b_2 + \varepsilon_3 b_3)(x\, dy - y\, dx),$$

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with $b_1 = \alpha y(1-x-y)$, $b_2 = \beta x(1-x-y)$, $b_3 = xy$. This is an Abelian integral corresponding to the perturbation

$$M^{-1}X_{\mathrm{H}} + (\varepsilon_1 b_1 + \varepsilon_2 b_2 + \varepsilon_3 b_3)(x\partial_x + y\partial_y),$$

where $M = x^{\alpha-1}y^{\beta-1}$ and $X_{\mathrm{H}} = H_y\partial_x - H_x\partial_y$ is the Hamiltonian vector field.

Now we are in a situation of a cubic Lotka–Volterra perturbation of an integrable planar Lotka–Volterra vector field. We know also that the saddle connection cannot be destroyed (since the plane $z = 0$ is invariant for the spacial system).

Therefore, the remaining statements of Theorem 4.2 follow from Theorem 2.4 and Corollary 2.5 in §2. □

Of course, Theorem 4.2 cannot give an exact estimate for the number of limit cycles in an arbitrary perturbation X_ε. Such an estimate requires second-order analysis of the return map (with respect to $|\varepsilon|$). It is not difficult to write down the corresponding formulas for the invariant surface L_ε (under the normal hyperbolicity condition) and the second-order Poincaré–Melnikov integrals. However, we do not present them here.

4.6. *The intersection* $\mathcal{LV}_1^{\mathrm{plane}} \cap \mathcal{LV}^{\mathrm{Darb}}$. Here the perturbed system takes the form

$$\begin{aligned}\dot{x} &= x[\gamma(1+\varepsilon_1)y - \beta(1-x-y) + \varepsilon_2 z]\\ \dot{y} &= y[\alpha(1-x-y) - \gamma x + \varepsilon_3 z]\\ \dot{z} &= z[\nu(z-1) + \varepsilon_4 x + \varepsilon_5 y],\end{aligned}$$

i.e. $X_0 \in LV_1^{\mathrm{plane}} \cap LV_1^{\mathrm{Darb}}$. We see that the invariant surface takes the form

$$z = g(x) + O(|\varepsilon|) = 1 + O(|\varepsilon|)$$

and the first-order Poincaré–Melnikov integral (4.10) equals

$$\int x^{\alpha-1}y^{\beta-1}(1-x-y)^{\gamma-1}[\varepsilon_1 xy\,dx - \varepsilon_2 x\,dy + \varepsilon_3 y\,dx].$$

By Corollary 2.3 it is proportional to the integral $I_0 = \int x^{\alpha-1}y^{\beta-1}(1-x-y)^{\gamma-1}\,dx \neq 0$. Therefore we have the following result.

THEOREM 4.3. *The space of first-order Poincaré–Melnikov integrals associated with perturbations of a field from* $\mathcal{LV}_1^{\mathrm{plane}} \cap LV^{\mathrm{Darb}}$ *is one-dimensional and is generated by the integral* $I_0(h)$*, which does not vanish for* $h \in [0, h_*)$*. This means that a generic one-parameter perturbation* $\widetilde{X}_\delta$*,* $\delta \in (\mathbb{R}, 0)$ *does not create limit cycles.*

4.7. *The intersection* $\mathcal{LV}_0^{\mathrm{plane}} \cap \mathcal{LV}^{\mathrm{Hopf}}$. We know from the proof of Theorem 3.7 that this is the case with two functionally independent first integrals

$$E = x^\alpha y^\beta z^\gamma, \quad F = x^{\alpha'}y^{\beta'}z^{\gamma'}u. \tag{4.15}$$

(Here we assume that the center lies in the first octant.) An open domain in the phase space is filled with periodic solutions

$$\gamma_{\rho,\sigma} = \{E = \rho, F = \sigma\}$$

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to the unperturbed system X_0. Note also that, contrary to the case of generic element from $\mathcal{LV}^{\mathrm{Hopf}}$, the center locus is defined globally.

The perturbed system takes the form

$$\begin{aligned} \dot{x} &= x[\gamma(1+\varepsilon_3)y - \beta z + (\lambda+\varepsilon_4)u], \\ \dot{y} &= y[\alpha(1+\varepsilon_1)z - \gamma x + \mu u], \\ \dot{z} &= z[\beta(1+\varepsilon_2)x - \alpha y + \nu u]. \end{aligned} \tag{4.16}$$

The linearization of the problem of limit cycles for (4.16) leads to consideration of the following vector-valued first-order Poincaré–Melnikov integral

$$\begin{aligned} I(\rho,\sigma) = \alpha\varepsilon_1 \begin{pmatrix} \int E_y yz\,dt \\ \int F_y yz\,dt \end{pmatrix} + \beta\varepsilon_2 \begin{pmatrix} \int E_z zx\,dt \\ \int F_z xz\,dt \end{pmatrix} \\ + \gamma\varepsilon_3 \begin{pmatrix} \int E_x xy\,dt \\ \int F_x xy\,dt \end{pmatrix} + \varepsilon_4 \begin{pmatrix} \int E_x xu\,dt \\ \int F_x xu\,dt \end{pmatrix}. \end{aligned} \tag{4.17}$$

Here the integrations run along the curves $\gamma_{\rho,\sigma}$ parametrized by the time t of the unperturbed system. The condition for existence of a family $\gamma_{\rho,\sigma}(\varepsilon)$ of limit cycles of X_ε which tend to $\gamma_{\rho,\sigma}$ as $\varepsilon \to 0$ is the vanishing of I at (ρ,σ).

We have not investigated the integral (4.17) in detail. However, we have some observations, which show how this case is distinct from the two-dimensional case.

Example 4.4. Let the $X_0 \in LV_0^{\mathrm{plane}} \cap LV^{\mathrm{Hopf}}$ and its one-parameter perturbation be the following

$$\begin{aligned} \dot{x} &= x[(1+\delta)(\gamma y - \beta z) + \lambda u], \\ \dot{y} &= y[\alpha z - \gamma x + \mu u], \\ \dot{z} &= z[\beta x - \alpha y + \nu u], \end{aligned} \tag{4.18}$$

$\delta \in (\mathbb{R}, 0)$. Assume also that $\alpha+\beta+\gamma = 1$ and recall that $\Lambda = \alpha\lambda + \beta\mu + \gamma\nu = 0$.

The perturbation (4.18) lies in the subspace $T_{X_0}LV_0^{\mathrm{plane}} + T_{X_0}LV^{\mathrm{Hopf}}$ of the tangent space $T_{X_0}LV$. This is because the vector field $X_0 + \delta x(\gamma y - \beta z + \lambda u)\partial_x$ belongs to LV^{Hopf} and the vector field $X_0 - \delta\lambda xu\partial_x$ belongs to LV_0^{plane}.

The system (4.18) has the singular point $p_* = (\alpha, \beta, \gamma)$ and

$$A \stackrel{\mathrm{df}}{=} dX(p_*) = \begin{pmatrix} \alpha\lambda & \alpha\lambda + \alpha\gamma_1 & \alpha\lambda - \alpha\beta_1 \\ \beta\mu - \beta\gamma & \beta\mu & \beta\mu + \beta\alpha \\ \gamma\nu + \gamma\beta & \gamma\nu - \gamma\alpha & \gamma\nu \end{pmatrix},$$

where $\beta_1 = (1+\delta)\beta$ and $\gamma_1 = (1+\delta)\gamma$. The necessary condition for center is that the characteristic polynomial $P(x) = -x^3 + a_1x^2 + A_2x + a_3$ of A takes the form $P(x) = -(x^2+\omega^2)(x-\eta)$, i.e.

$$L_1 \stackrel{\mathrm{df}}{=} a_3 - a_1a_2 = 0. \tag{4.19}$$

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The quantity L_1 is the three-dimensional analogue of the first Poincaré–Lyapunov focus quantity.

In our case we have $a_1 = \operatorname{tr} A = \alpha\lambda + \beta\mu + \gamma\nu = 0$ and (after some calculations) we get

$$a_3 = \det A = -(\alpha^2\beta\gamma\lambda)\cdot\delta + O(\delta^2).$$

We see that the center condition (4.19) is linear with respect to the perturbation. However, when one begins calculations of the Poincaré–Melnikov integrals, namely the first term in their Taylor expansion at $(\rho_*, \sigma_*) = (E(p_*), F(p_*))$, then one obtains zero; we have done such calculations. (In the two-dimensional case perturbations from a subspace spanned by tangent spaces of two components of the center variety always give first-order Poincaré–Melnikov integrals identically equal to zero.)

The explanation of this phenomenon relies on the fact that the quantity L_1 measures the increment (after the period) of another first integral, Φ, which has different nature than E and F. Namely, by definition, Φ is the amplitude of oscillations, it measures the distance to the line ℓ of critical points. On the other hand, the surfaces $E = \rho_*$ and $F = \sigma_*$ (through p_*) are transversal to ℓ and are tangent one to the other at p_*. The system (E, F) of first integrals degenerates at p_*.

There is one more question concerning the unperturbed field $X_0 \in LV_0^{\text{plane}} \cap LV^{\text{Hopf}}$. This X_0 can be approximated by fields from LV_0^{plane} and from LV^{Hopf} as well: $X_0 = \lim_{\theta\to 0} \widetilde{X}_\theta = \lim_{\vartheta\to 0} \widetilde{\widetilde{X}}_\vartheta$ where $\widetilde{X}_\theta \in LV_0^{\text{plane}} \setminus LV^{\text{Hopf}}$ and $\widetilde{\widetilde{X}}_\vartheta \in LV^{\text{Hopf}} \setminus LV_0^{\text{plane}}$. The fields $\widetilde{X}_\theta$ have the center manifold $W^{\text{c}}_\theta(p_*) = \{u = 0\}$ with the limit equal $\{u = 0\}$. However, we do not know what is the limit

$$\lim_{\vartheta\to 0} \widetilde{W}^{\text{c}}_\vartheta(p_*)$$

of the local center loci $\widetilde{W}^{\text{c}}_\vartheta(p_*)$ of the fields $\widetilde{\widetilde{X}}_\vartheta$. (Note that the center manifolds $W^{\text{c}}_\vartheta(p_*)$ for $\widetilde{\widetilde{X}}_\vartheta$ constitute whole neighborhoods of p_*.)

4.8. *Other intersections.* As should be expected, the intersection $\mathcal{LV}^{\text{Hopf}} \cap \mathcal{LV}^{\text{Darb}}$ consists of systems with two independent global first integrals. These systems can be taken in the form

$$\dot{x} = x[\gamma y - \beta(1 - x - y)], \quad \dot{y} = y[\alpha(1 - x - y) - \gamma x], \quad \dot{z} = z[ex + fy + g] \tag{4.20}$$

with the integrals $G = x^\alpha y^\beta (1 - x - y)^\gamma$ and $E = x^{\alpha'} y^{\beta'} z$ (i.e. from $LV^{\text{Hopf}} \cap LV_1^{\text{Darb}}$). Here $ex_* + fy_* + g = 0$ for

$$(x_*, y_*) = \left(\frac{\alpha}{\alpha + \beta + \gamma}, \frac{\beta}{\alpha + \beta + \gamma}\right),$$

the critical point of G. The first-order Poincaré–Melnikov integrals are vector-valued, as in (4.17). However, we do not encounter phenomena as in Example 4.4 above.

The systems from $\mathcal{LV}^{\text{Hopf}} \cap \mathcal{LV}_1^{\text{plane}}$ are of the form (4.20) with $e = f = g = 0$.

Note that the systems (4.20) have the property that the z-line contains only one singular point, the origin (e.g. for $g \neq 0$). Therefore, the latter intersections lie outside the space $\mathcal{LV}^*$ defined in §3.1.

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It is clear that the intersections $\mathbf{LV}_i^{\text{plane}} \cap \mathbf{LV}_j^{\text{plane}}$, $i, j = 0, \dots 6$ (in $\mathbf{LV}$) are empty; because the center cannot lie on the intersection of two invariant planes.

Also the intersections $\mathbf{LV}_i^{\text{Darb}} \cap \mathbf{LV}_j^{\text{Darb}}$ are empty. Indeed, let one of the Darboux integrals be $F = x^\alpha y^\beta (1 - x - y)^\gamma$ with $\alpha, \beta, \gamma > 0$. There are two possibilities for the additional invariant plane (modulo the A_3-action): (i) $y + z = 1$; or (ii) $kx + ly + mz = 0$. Standard calculations show that $\dot{y} = \mu y(y - 1)$ and $\alpha = -\mu$, $\gamma = \mu$ in the case (i). In the case (ii) the origin has equal eigenvalues and $\beta = -\alpha$ (by restriction of F to $z = 0$).

5. *Asymptotic of the Poincaré–Melnikov integrals*

The whole section is devoted to the study of the asymptotic behavior of the function $I(h)$ defined in (4.4) and associated with perturbation (4.1) of a system from LV_0^{plane}. The argument h varies in the interval $[0, h_*]$ of values of the partial first integral $H = x^\alpha y^\beta z^\gamma|_{u=0}$. We study the behavior of $I(h)$ near the endpoints $h = 0$ and $h = h_*$. We assume $\alpha, \beta, \gamma > 0$ and the normal hyperbolicity condition in the form

$$\lambda > \beta, \gamma; \quad \mu > \alpha, \gamma; \quad \nu > \alpha, \beta. \tag{5.1}$$

5.1. *The limit triangle.* The level curve $H(x, y) = 0$ defines the boundary of the triangle $\Delta = \{x, y, z \geq 0\} \cap L_0 = \{x, y, 1 - x - y \geq 0\}$ (when we identify the invariant plane L_0 with the (x, y)-plane). After perturbation there also exists a (nonlinear) invariant triangle $\Delta_\varepsilon = \{x, y, \phi_\varepsilon(x, y) \geq 0\}$ with boundary $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ consisting of saddle connections. Therefore, the return map takes the form

$$\xi \to C\xi^\kappa(1 + o(1)) \quad \text{as } \xi \to 0. \tag{5.2}$$

Here ξ is some parametrization of a section S to the saddle connection Γ_3 (see Figure 1); we assume $\xi = H^{1/\gamma}|_S$. The exponent $\kappa = 1 + O(\varepsilon)$ and the constant $C = 1 + O(\varepsilon)$. Compare also the formula (3.19) above; we refer the reader also to [**Zo3**] where limit cycles near the triangle Γ were investigated for general quadratic (planar) perturbation.

THEOREM 5.1. *Assume the conditions (5.1). Then in (5.2) we have:*

(a) $\kappa = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + O(|\varepsilon|^2)$;

(b) $C|_{\kappa=0} = 1 - \varepsilon_1(\Lambda/\alpha)A(\mu/\alpha, \nu/\alpha) - \varepsilon_2(\Lambda/\beta)A(\nu/\beta, \lambda/\beta) - \varepsilon_3(\Lambda/\gamma)A(\lambda/\gamma, \mu/\gamma) + O(|\varepsilon|^2)$, *where* $\Lambda = \alpha\lambda + \beta\mu + \gamma\nu$ *is the eigenvalue of* $dX_0(p_*)$ *in the unstable direction and*

$$A(a, b) = \int_0^1 \int_0^1 \frac{u^a v^{b-1}}{1 - uv}\, du\, dv. \tag{5.3}$$

In particular, the Poincaré–Melnikov integral $I(h)$ *has the following asymptotic as* $h \to 0$

$$I(h) = -(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)h \log h - (C - 1)h + o(h). \tag{5.4}$$

Proof. The point (a) is just the infinitesimal version of the formula (3.19) from §3.

To calculate the coefficient C we use the fact that, when $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$, then the function

$$H_\varepsilon = x^\alpha y^{\beta(1-\varepsilon_1)} z^{\gamma(1+\varepsilon_3)}$$

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satisfies

$$\dot{H}_\varepsilon = \Lambda H_\varepsilon u + O(|\varepsilon|^2)$$

(derivative along the perturbed field (4.1)). Therefore, we can choose the parameter ξ (in the section S to the connection Γ_3) as $\xi = (H_\varepsilon^{1/\gamma(1+\varepsilon_3)})|_S$. Hence, the increment of ξ along an orbit equals

$$\Delta\xi = \Lambda\xi \int_{H_\varepsilon=0} u\,dt + O(|\varepsilon|^2) = \Lambda\xi \int_{H_\varepsilon=0} (\varepsilon_1 g_1 + \varepsilon_2 g_2 + \varepsilon_3 g_3)\,dt + O(|\varepsilon|^2). \quad (5.5)$$

The integral $\int \left(\sum \varepsilon_i g_i\right) dt$ is calculated separately on each segment $\Gamma_1 \subset \{x = 0\}$, $\Gamma_2 \subset \{y = 0\}$ and $\Gamma_3 \subset \{z = 0\}$. In each segment we solve the normal variation equation (4.3), i.e. $\dot{g}_i = (\lambda x + \mu y + \nu(1-x-y))g_i + b_i$, $b_1 = \alpha y(1-x-y)$, $b_2 = \beta x(1-x-y)$, $b_3 = \gamma xy$.

Take the segment $\Gamma_1 = \{(x, y) : x = 0, y \in [0, 1]\}$. Since $b_{2,3}|_{\Gamma_1} \equiv 0$, we have $g_{2,3}|_{\Gamma_1} \equiv 0$. The function $g_1|_{\Gamma_1} = g_{10}(y)$ should vanish at the endpoints $y = 0$ and $y = 1$. As

$$\left.\frac{d}{dt}\right|_{\Gamma_1} = \alpha y(1-y)\frac{d}{dy},$$

we get the equation

$$\alpha y(1-y)g'_{10} = (\mu y + \nu(1-y))g_{10} + \alpha y(1-y)$$

with the general solution

$$g_{10}(y) = y^{\nu/\alpha}(1-y)^{-\mu/\alpha} \int_{y_0}^{y} \tau^{-\nu/\alpha}(1-\tau)^{\mu/\alpha}\,d\tau.$$

With the assumption $\mu > \alpha$, $\nu > \alpha$ the requirement $g_{10}(0) = g_{10}(1) = 0$ forces the choice $y_0 = 1$.

Therefore,

$$g_{10}(y) = -y^{\nu/\alpha}(1-y)^{-\mu/\alpha} \int_{y}^{1} \tau^{-\nu/\alpha}(1-\tau)^{\mu/\alpha}\,d\tau$$

and in (5.5) we get

$$\int_{\Gamma_1} \left(\sum \varepsilon_i g_i\right) dt = -\frac{\varepsilon_1}{\alpha} \iint_{0\le y\le\tau\le 1} y^{\nu/\alpha-1}(1-y)^{-\mu/\alpha-1}\tau^{-\nu/\alpha}(1-\tau)^{\mu/\alpha}\,d\tau\,dy.$$

Let us transform the above double integral

$$A(a, b) = \iint_{0\le t\le\tau\le 1} \left(\frac{1-\tau}{1-t}\right)^a \left(\frac{t}{\tau}\right)^b \frac{d\tau\,dt}{t(1-t)},$$

$a = \mu/\alpha, b = \nu/\alpha$, using the substitution $u = (1-\tau)/(1-t), v = t/\tau$ (with $\tau = (1-u)/(1-uv), t = v(1-u)/(1-uv)$ and $|\partial(u, v)/\partial(\tau, t)| = (1-uv)/\tau(1-t)$). We find the formula (5.3).

The contributions to $\Delta\xi$ in (5.5) arising from other segments of Γ are given by cyclic permutations of the coordinates and the parameters.

(In the case $\lambda, \mu, \nu < 0$ we get the formula $C - 1 = \varepsilon_1(\Lambda/\alpha)A(-\mu/\alpha, -\nu/\alpha) + \varepsilon_2(\Lambda/\beta)A(-\nu/\beta, -\lambda/\beta) + \varepsilon_3(\Lambda/\gamma)A(-\lambda/\gamma, -\mu/\gamma) + \cdots$). □

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The integral (5.3) is similar to some integrals giving values of the Riemann's zeta function at integer points. However, the function $A(a,b)$ is transcendental and we did not succeed in its application in the problem of zeroes of the integral $I(h)$. However, there exist surprisingly many cases (i.e. lines in the (a,b)-plane) when the integral $A(a,b)$ can be computed explicitly.

PROPOSITION 5.2. *There are the following particular formulas for* $A(a,b)$.

(a) $A(a,a+1)=\sum_{n=1}^{\infty}(1/(a+n)^2)=\zeta(2,a+1)$, *where* $\zeta(s,v)=\sum_{n\geq 0}(v+n)^{-s}$ *is the generalized zeta-function (see* [**BE**]*).*

(b) $A(a,b)=\sum_{n\geq 0}(1/(a+n+1)(b+n))=(1/(a-b+1))\sum_{n\geq 0}(1/(b+n)-1/(a+n+1))$ *and, in particular,*

$$A(a,a)=\frac{1}{a},\quad A(a,a-1)=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{a+1}\right),\dots$$

$$A(a,a-k)=\frac{1}{k}\left(\frac{1}{a}+\frac{1}{a-1}+\cdots+\frac{1}{a-k}\right),$$

$$A(a,a+2)=\frac{1}{a+1},\quad A(a,a+3)=\frac{1}{2}\left(\frac{1}{a+1}+\frac{1}{a+2}\right),\dots$$

(c) $A(a,-a)=(-\pi/2(a+1))\cot(\pi a)$, $-1<a<0$.

Proof. (c) We apply the change

$$u=\xi\frac{1+\eta}{1+\xi},\quad v=\eta\frac{1+\xi}{1+\eta}\quad\left(\text{with }\left|\frac{\partial(u,v)}{\partial(\xi,\eta)}\right|=\frac{1-uv}{(1+\xi)(1+\eta)}\right)$$

and we obtain the integral

$$\iint_{\xi,\eta\geq 0,\xi\eta\leq 1}\frac{\xi^a}{(1+\xi)^{2a-2}}\frac{(1+\eta)^{2a}}{\eta^{a+1}}\,d\xi\,d\eta.$$

The next change $\xi=1/\xi'$, $\eta=1/\eta'$ gives an integral of the same function, but over the domain $\{\xi,\eta\geq 0,\xi\eta>1\}$. Therefore,

$$\begin{aligned}A(a,-a)&=\frac{1}{2}\int_0^{\infty}\frac{\xi^a}{(1+\xi)^{2a-2}}\int_0^{\infty}\frac{(1+\eta)^{2a}}{\eta^a}\\&=\frac{1}{2}B(1+a,1+a)B(-a,-a)\\&=\frac{1}{2}\frac{(\Gamma(1+a)\Gamma(-a))^2}{(2a+2)\Gamma(1+2a)\Gamma(-2a)}\\&=\frac{1}{4(1+a)}\left(\frac{\pi}{\sin(-\pi a)}\right)^2\frac{\sin(-2\pi a)}{\pi}.\end{aligned}$$

□

Remark 5.3. From Proposition 5.2(b) it follows that for $\lambda=\mu=\nu$ (i.e. the intersection $LV_0^{\text{plane}}\cap LV_2^{\text{Darb}}$) we get $C-1=\Lambda\lambda(\varepsilon_1+\varepsilon_2+\varepsilon_3+O(|\varepsilon|^2))=O(|\varepsilon|^2)$ when $\kappa=0$. It is what is expected.

However, when we intersect LV_0^{plane} with LV_1^{Darb}, i.e. with the invariant plane $x+y=1$, then the conditions $\lambda-\beta=\alpha+\mu=0$ (see (3.8)) mean losing the normal hyperbolicity property. Moreover, in this case some of the quantities $A(a,b)$ become infinite.

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5.2. *The Poincaré–Lyapunov quantities.* Recall that for a planar vector field $\dot\zeta = a_1\zeta + \cdots, \zeta \in (\mathbb{C}, 0)$ the focus quantities are defined as

$$L_i = 2\mathrm{Re}\, a_i, \quad i = 1, 2, \ldots$$

in the Poincaré–Dulac normal form $\dot\zeta_1 = \zeta_1(a_1 + a_2|\zeta_1|^2 + a_3|\zeta_1|^4 + \cdots)$. Thus L_1 is the trace of the linear part (divergence) at $\zeta = 0$.

In the case of the perturbation (4.1) of the system $X_0 \in LV_0^{\mathrm{plane}}$ we consider the planar system $Y_\varepsilon = X_\varepsilon|_{L_\varepsilon}$. Therefore, the first task is to expand the equation $u = \sum \varepsilon_i g_i + \cdots$ for L_ε into the Taylor series at the singular point (x_*, y_*).

The following observation highly simplifies the calculations.

LEMMA 5.4. *If $\varepsilon_1+\varepsilon_2+\varepsilon_3 = 0$, i.e. $\kappa = O(|\varepsilon|^2)$ in Theorem 5.1, then $\sum \varepsilon_i b_i(x_*, y_*) = 0$ and $\sum \varepsilon_i g_i(x_*, y_*) = 0$ in the normal variation equation. This means that the surface L_ε passes through p_* (in linear approximation).*

We take hence a two-parameter perturbation of the form

$$\begin{aligned} \dot x &= x[(1+\delta_1)(\gamma y - \beta z) + \lambda u], \\ \dot y &= y[\alpha z - \gamma x + \mu u], \\ \dot z &= z[(1+\delta_0)(\beta x - \alpha y) + \nu u], \end{aligned} \tag{5.6}$$

$\delta_{0,1} \in (\mathbb{R}, 0)$.

The first focus quantity L_1 will be computed without any restrictions onto $\alpha, \beta, \ldots, \nu$. For other quantities we assume that the system $Y_0 = X_0|_{L_0}$ is Hamiltonian and we assume *strong normal hyperbolicity*, i.e. we put

$$\alpha = \beta = \gamma = 1, \quad \Lambda = \lambda + \mu + \nu \to \infty. \tag{5.7}$$

We introduce the variables (which are of order $O(1)$ as $\Lambda \to \infty$)

$$\xi = \xi_0 = \tfrac12(\lambda - \mu), \quad \xi_1 = \tfrac12(\mu - \nu), \quad \eta = \eta_0 = \tfrac12(\lambda + \mu) - \nu, \quad \eta_1 = \tfrac12(\mu + \nu) - \lambda. \tag{5.8}$$

We have

$$\begin{gathered} \lambda = \tfrac13\Lambda + \tfrac13\eta + \xi, \quad \mu = \tfrac13\Lambda + \tfrac13\eta - \xi, \quad \nu = \tfrac13\Lambda - \tfrac23\eta, \\ \xi_1 = \tfrac12(\eta - \xi), \quad \eta_1 = \tfrac{-1}{2}(\eta + 3\xi). \end{gathered} \tag{5.9}$$

THEOREM 5.5. *Assume the perturbation (5.6).*

(a) *We have*

$$\begin{aligned} L_1 = {} & \frac{\alpha\beta\gamma/(\alpha+\beta+\gamma)}{\Lambda^2 + \alpha\beta\gamma(\alpha+\beta+\gamma)} \\ & \times \{\delta_0[\nu\gamma(\alpha+\beta+\gamma) - \Lambda(\lambda - \mu + \gamma)] + \delta_1[\lambda\alpha(\alpha+\beta+\gamma) - \Lambda(\mu - \nu + \alpha)]\}. \end{aligned} \tag{5.10}$$

(b) *If, additionally, we assume (5.7), then in the notations (5.8) we have*

$$L_1 = C_1 \cdot \left[\delta_0\left(\frac{\xi_0}{\Lambda} + \frac{\eta_0}{\Lambda^2}\right) + \delta_1\left(\frac{\xi_1}{\Lambda} + \frac{\eta_1}{\Lambda^2}\right)\right], \tag{5.11}$$

$$L_2 = C_2 L_1 + \frac{C_3}{\Lambda^3}[\delta_0 P(\xi_0, \eta_0) + \delta_1 P(\xi_1, \eta_1)] + O\left(\frac{1}{\Lambda^4}\right), \tag{5.12}$$

$$P(\xi, \eta) = 5\xi - 3\xi^2 + \eta^2 + 3\xi^3 + \xi\eta^2.$$

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The determinant $|\partial(L_1, L_2)/\partial(\delta_0, \delta_1)|$ *equals*

$$\frac{C_4}{\Lambda^4}\eta_0\eta_1\eta_2 + O\left(\frac{1}{\Lambda^5}\right), \tag{5.13}$$

where $\eta_2 = \frac{1}{2}(\nu+\lambda)-\mu$ *(here* C_1, C_2, C_3, C_4 *are constants depending on* Λ *and non-zero for* $\Lambda = \infty$*).*

This means that for (ξ, η) *fixed and such that* $\eta_0\eta_1\eta_2 \neq 0$ *and for* $\Lambda \to \infty$ *the system (5.6) can have at most one small amplitude limit cycle. In particular, the component* LV_0^{plane} *is isolated in the center variety in* LV*.*

Proof. It is enough to determine dependence of L_j on δ_i separately. The calculations for δ_1 differ only by cyclic permutations of variables. So, assume that $\delta_0 = \delta$ and $\delta_1 = 0$ in (5.6).

(a) In the coordinates $x_1 = x - x_*$, $y_1 = y - y_*$ (with $(x_*, y_*) = 1/(\alpha+\beta+\gamma)(\alpha, \beta)$) the normal variation equation (for g_0 in $u = \delta_0 g_0(x, y) + o(\delta_0)$ for the invariant surface) takes the form

$$\frac{\partial g_0}{\partial x}\alpha[\beta x_1 + (\beta+\gamma)y_1] - \frac{\partial g_0}{\partial y}\beta[(\alpha+\gamma)x_1 + \alpha y_1] = \Lambda g_0 + \gamma(\beta x_1 - \gamma y_1)$$

+ nonlinear terms. This equation has the solution

$$g_0 = \frac{\gamma}{\Lambda^2 + \alpha\beta\gamma(\alpha+\beta+\gamma)}[-\beta(\Lambda + \alpha(\alpha+\beta+\gamma))x_1 + \alpha(\Lambda - \beta(\alpha+\beta+\gamma))y_1]$$

+ nonlinear terms.

Substituting δg_0 in the place of u (with $z = 1 - x - y + u$) in the planar system and taking the divergence gives

$$L_1 = \frac{\delta}{\alpha+\beta+\gamma}\left[\alpha(\lambda-\beta)\frac{\partial g_0}{\partial x}(p_*) + \beta(\mu+\alpha)\frac{\partial g_0}{\partial y}(p_*)\right],$$

what leads to (5.10).

(b) Assume $\alpha = \beta = \gamma = 1$. Then the formula (5.11) is just the rewritten formula (5.10).

Introduce the coordinates

$$u = x_1 + y_1, \quad v = x_1 - y_1;$$

(thus $x = \frac{1}{3} + \frac{1}{2}u + \frac{1}{2}v$, $y = \frac{1}{3} + \frac{1}{2}u - \frac{1}{2}v$). Then the (unperturbed) Hamiltonian system becomes

$$\dot{u} = -\tfrac{1}{3}v + uv, \quad \dot{v} = u + \tfrac{3}{2}u^2 - \tfrac{1}{2}v^2$$

with the Hamiltonian

$$\widetilde{H} = \tfrac{1}{2}u^2 + \tfrac{1}{6}v^2 + \tfrac{1}{2}u^3 - \tfrac{1}{2}uv^2. \tag{5.14}$$

The normal variation equation for $g(u, v) = g_0(x, y)$ takes the form $\dot{g} = (\frac{1}{3}\Lambda + \eta u + \xi v)g + (\frac{1}{3} - u)v$, or

$$\tfrac{1}{3}\Lambda g = (u - \tfrac{1}{3})v + [g'_u v(u - \tfrac{1}{3}) + g'_v(u + \tfrac{3}{2}u^2 - \tfrac{1}{2}v^2) - (\eta u + \xi v)g].$$

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Writing g in the series $g_1\Lambda^{-1} + g_2\Lambda^{-2} + \cdots$, we find successively the coefficients-polynomials g_j. We have

$$\begin{aligned}
g_1 &= -v + 3uv,\\
g_2 &= -3u + (\tfrac{9}{2}u^2 + 3\eta uv + \tfrac{3}{2}(2\xi-1)v^2) + (\tfrac{27}{2}u^3 - 9\eta u^2 v + \tfrac{9}{2}(1-2\xi)uv^2),\\
g_3 &= 3v + (18\eta u^2 + 27(\xi-1)uv - 3\eta v^2)\\
&\quad + (-27\eta u^3 - \tfrac{9}{2}(9\xi + 2\eta^2)u^2 v + 9\eta(3-2\xi)uv^2 + \tfrac{9}{2}(1-2\xi)v^3)\\
&\quad + (-81\eta u^4 + (162 - \tfrac{243}{2}\xi + 27\eta^2)u^3 v + 54\eta(\xi-1)u^2v^2 + (27\xi^2 - \tfrac{27}{2}\xi)uv^3),\\
g_4 &= (81\xi - \tfrac{135}{2})u^2 + (\tfrac{45}{2} - 36\xi)v^2 - (\tfrac{243}{2} + 81\eta^2)u^3\\
&\quad + (-\tfrac{81}{2} + 243\xi - 162\xi^2 + 27\eta^2)uv^2 + \cdots .
\end{aligned} \tag{5.15}$$

The '$\cdots$' above means other monomials in g_4, which are of no use for us.

The perturbed planar system becomes

$$\begin{aligned}
\dot u &= -\widetilde H'_v + \delta\left\{\left[\frac{1}{3}(\Lambda+\eta)\left(\frac{2}{3}+u\right) + (\xi-1)v\right]\left[\frac{g_1}{\Lambda} + \frac{g_2}{\Lambda^2} + \cdots\right]\right\},\\
\dot v &= \widetilde H'_v + \delta\left\{\left[\frac{1}{3}(\Lambda+\eta)v + (\xi-1)\left(\frac{2}{3}+u\right)\right]\left[\frac{g_1}{\Lambda} + \frac{g_2}{\Lambda^2} + \cdots\right]\right\}.
\end{aligned} \tag{5.16}$$

The focus quantities, when taken modulo $O(\delta^2)$, are defined as the Taylor expansion coefficients of the Abelian integral associated with the perturbation (5.16). This Abelian integral is the sum of powers of Λ times other Abelian integrals,

$$I(h) = J_0(h)\Lambda^0 + J_1(h)\Lambda^{-1} + \cdots \tag{5.17}$$

From (5.16) we get

$$J_0(h) = \int_{\widetilde H = h} \tfrac{1}{3}(\tfrac{2}{3}+u)(3u-1)v\,dv - \tfrac{1}{3}(3u-1)v^2\,du = \int f(u)v^2\,du.$$

It is zero, because by (5.14) v^2 is a single-valued function of u. Next, we have

$$\begin{aligned}
J_1 &= \int [\tfrac{1}{3}(\tfrac{2}{3}+u)g_2 + (\tfrac{1}{3}\eta(\tfrac{2}{3}+u) + (\xi-1)v)g_1]\,dv\\
&\quad - [\tfrac{1}{3}vg_2 + (\tfrac{1}{3}\eta v + (\xi-1)(\tfrac{2}{3}+u))g_1]\,du,\\
J_2 &= \int [\tfrac{1}{3}(\tfrac{2}{3}+u)g_3 + (\tfrac{1}{3}\eta(\tfrac{2}{3}+u) + (\xi-1)v)g_2]\,dv\\
&\quad - [\tfrac{1}{3}vg_3 + (\tfrac{1}{3}\eta v + (\xi-1)(\tfrac{2}{3}+u))g_2]du.
\end{aligned} \tag{5.18}$$

The quantity L_1 is calculated using the parts $\int_{\widetilde H=h} u\,dv = \iint_{\widetilde H\le h} du\,dv$ and $\int v du = -\iint du\,dv$. It is proportional to the area of the domain $\{\widetilde H \le h\}$, equal constant $\cdot\, h + O(h^2)$. The coefficient before h^2 leads to a contribution into L_2, but this contribution is proportional to L_1.

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The quantity L_2, the coefficient before h^2, arises from the integrals $\int u^3\,dv = 3\iint u^2\,du\,dv \approx 6\sqrt{3}\pi h^2$, $\int uv^2\,dv = \iint v^2\,du\,dv \approx 6\sqrt{3}\pi h^2$, $\int uv^2\,du = -\iint u^2\,du\,dv$ and $\int v^3\,du = -3\iint v^2\,du\,dv$, but also from the integrals $\int u^2\,dv$, $\int uv\,dv = -\frac{1}{2}\int u^2\,dv$, $\int v^2\,du$ and $\int uv\,dv$. However, like J_0, the latter two vanish identically. The integral $\int u^2 dv$ equals $\int (2h - \frac{1}{3}v^2 - u^3 + uv^2)\,dv$ (see (5.14)), and hence equals $\int uv^2\,dv - \int u^3\,dv = O(h^3)$ (by the above calculations).

Therefore, in computation of L_2 modulo L_1 it is enough to consider in the perturbation (5.16) only the following terms

$$\dot{u} = \cdots + \delta(Au^3 + Buv^2), \quad \dot{v} = \cdots + \delta(Cu^2v + Dv^3),$$

then

$$L_2 = \text{constant} \cdot \delta \cdot (3A + 3B + C + 9D).$$

The computation of A, B, C, D is straightforward (but tedious) and leads to the formula (5.12).

Further calculations give (5.13).

The remaining statements of Theorem 5.5 are clear. The isolation of $\mathbf{LV}_0^{\text{plane}}$ means that this semi-algebraic subvariety of **LV** cannot be embedded into a higher dimensional component of the center variety. □

The formula (5.13) in Theorem 5.5 suggests that the quantities L_1 and L_2 become dependent (asymptotically as $\Lambda \to \infty$) when (ξ, η) belongs to one of the three lines $\eta_0 = \eta = 0$, $\eta_1 = -(\eta + 3\xi)/2 = 0$ and $\eta_2 = (\eta - 3\xi)/2 = 0$. A suspicion arises that each equation would define an intersection of LV_0^{plane} with some component of the center variety of new type. (Recall that the intersection $LV_0^{\text{plane}} \cap LV_2^{\text{Darb}}$ is given by $\xi = \eta = 0$; due to this the focus numbers vanish for $\xi = \eta = 0$.) We claim that it is not the case.

In order to convince the reader about this statement, we shall show that along the line $\eta = 0$ some other quantities L_k and L_m are independent, i.e.

$$\left| \frac{\partial(L_k, L_m)}{\partial(\delta_0, \delta_1)} \right| \Bigg|_{\eta=0} \not\equiv 0.$$

More precisely, we prove the independence of the Λ^{-1}th part of L_k, proportional to $\delta_0\xi_0 + \delta_1\xi_1$, and of the Λ^{-2}th part of some L_m.

For this we use the integrals J_1 and J_2, the Λ^{-1}th and Λ^{-2}th parts of the Abelian integral. These integrals can be extended to complex values of h and define multi-valued functions (elliptic functions). We look for their behavior as $h \to \infty$.

LEMMA 5.6.

(a) *We have* $J_1(h) = \xi \cdot K_1(h)$, *where*

$$K_1(h) \sim \text{constant} \cdot h^{5/3}, \quad h \to \infty.$$

(b) *We have* $J_2(h) = \eta K_2(h) + \xi^2 K_3(h) + \eta^2 K_4(h)$, *where*

$$K_2(h) \sim \text{constant} \cdot h^2, \quad h \to \infty.$$

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Proof. The fact that $J_i(h)|_{\xi=\eta=0} \equiv 0$ follows from the intersection of LV_0^{plane} and LV_2^{Darb}.

Since J_1 is linear in (ξ, η) (see (5.18)), we have $J_1 = \xi K_1 + \eta \widetilde{K}_1$. However, $\widetilde{K}_1(h)$ vanishes identically for the same reasons J_0 vanishes (see above).

Similarly, $J_2 = \eta K_2 + \xi \widetilde{K}_2 + \xi^2 K_3 + \eta^2 K_4 + \xi\eta K_5$, where $\widetilde{K}_2 \equiv K_5 \equiv 0$.

As $h \to \infty$ and $|u|, |v|$ become large, the approximate equation for the elliptic curve $\{\widetilde{H} = h\} \subset \mathbb{C}^2$ becomes

$$u^3 - uv^2 \approx 2h,$$

i.e. $v \approx \sqrt{u^2 - 2h/u}$. Then the analytic continuation of the oval $\gamma_h = \{\widetilde{H} = h\} \cap \mathbb{R}^2$ becomes a 'large cycle'. With $u = (2h)^{1/3}U$, $v = (2h)^{1/3}V$ we get a cycle Γ in $\{U^3 - UV^2 = 1\} \subset \mathbb{C}^2$ and we find the formulas

$$K_1 \sim 11(2h)^{5/3} \int_\Gamma UV^3\, dU,$$

$$K_2 \sim 18(2h)^2 \int_\Gamma (9U^2 + 4V^2)U^2 V\, dU.$$

The latter contour integrals are non-zero. □

Let $\hat{J}_1 = \hat{J}_1(h; \xi, \eta)$ and $\hat{J}_2 = \hat{J}_2(h; \xi, \eta)$ be the Λ^{-1}th and Λ^{-2}th parts of the Abelian integral associated with the perturbation (5.6) for $\delta_0 = 0$, $\Delta\widetilde{H} = \delta_1(\hat{J}_1\Lambda^{-1} + \hat{J}_2\Lambda^{-2} + \cdots)$. Lemma 5.6 implies that for $(\xi, \eta) \neq (0, 0)$ and small the quantities $(\delta_0 J_1 + \delta_1 \hat{J}_1)\Lambda^{-1}$ and $(\delta_0 J_2 + \delta_1 \hat{J}_2)\Lambda^{-2}$ are independent as linear functions of δ_0, δ_1. This means that some coefficients of the Taylor expansions at $h = h_*$ of these integrals are independent. This is just the independence of the corresponding focus quantities.

6. *Other cases of center?*

This section is devoted to consider the question of eventuality of existence of other components of the center variety in **LV**. We present some arguments which suggest that there are no such components.

In the two-dimensional case there are two general mechanisms known for creating a center: *existence of a Darboux type first integral* and *reversibility*.

The first mechanism means that, if there exist enough invariant algebraic curves, then there exists a center. Generally, one needs $q = \frac{1}{2}n(n+1) + 1$ such curves, where n is the degree of the vector field (i.e. degree of the right-hand sides). (Here $\frac{1}{2}n(n+1)$ is the dimension of the space of polynomials of degree $\leq n - 1$, the cofactors.)

However, sometimes this number can be diminished. For example, for $n = 2$ we have $q = 4$, but when we know that there are three invariant algebraic curves (e.g. straight lines) avoiding one singular point 0, then their cofactors are linearly dependent and we have a first integral. If we know that the invariant curves are in a 'good' position (e.g. with transversal intersections between themselves and with the line at infinity), then a bound for the sum of their degrees is sufficient for the integrability.

The simplest case of reversibility is the so-called time-reversibility, which means invariance of the system with respect to a reflection in the phase space and reversion of direction of time. For example, the two-dimensional Lotka–Volterra Hamiltonian system $\dot{x} = x(x + 2y - 1)$, $\dot{y} = y(1 - 2x - y)$ admits three such symmetries.

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In examining the possibility of other cases of center we take one known component if the center variety, i.e. $\mathbf{LV}_i^{\text{plane}}$ or $\mathbf{LV}_i^{\text{Darb}}$ or $\mathbf{LV}^{\text{Hopf}}$, and we look for a possible intersection with the 'new' component.

Calculations of the focus quantities in the previous section indicate that the reversibility mechanism does not work here. The reason is that, although the initial system $X_0|_{L_0}$ can be time-reversible, the perturbed invariant surface L_ε cannot be chosen in a symmetric (or antisymmetric) way.

Therefore the next possibility is that some 'new' cases of center are associated with existence of some invariant algebraic surface S_0, which would support an integrable system. Such surface should have degree > 1, because the linear cases were considered above and in [**Ca**].

Suppose that a component consisting of fields X_0 with invariant surface S_0 intersects the component LV_1^{Darb} (with the global fist integral $x^\alpha y^\beta(1-x-y)^\gamma$). By the results of Section 4.4 the surface S_0 is given by $z = g(x, y)$, where g is the unique solution to the Riccati equation $\dot g = (p + qx + ry)g + \nu g^2$. Thus, S_0 is a rational surface of the form $z = P(x, y)/Q(x, y)$. The curve $\{z = P(x, y) = 0\}$ is an invariant curve for the planar Lotka–Volterra system at $z = 0$. Moreover, the degree of the function P/Q, which equals $\deg P - \deg Q$, should be < 1. After deformation X_ε of X_0 along the direction of the 'new' component there should exist an invariant surface S_ε, and $z = P_\varepsilon(x, y)/Q_\varepsilon(x, y)$ should also be rational. However, then we get the two-dimensional system

$$\dot x = x[Q_\varepsilon(a + bx + cy) + \varepsilon_1 P_\varepsilon],$$
$$\dot y = y[Q_\varepsilon(d + ex + fy) + \varepsilon_2 P_\varepsilon],$$

whose degree is too high to get some kind of Darboux integrability.

In all of the cases of intersections of two components of the center variety (including the supposed latter intersection) there was no ambiguity as to what kind of center is considered. We mean that it was the $1{:}-1$ resonance between a pair of non-real eigenvalues. However, the center conditions apply also to the $1{:}-1$ resonant saddles; in fact, we should take only the equation for center (not inequalities).

In dimension-three two such resonances can coexist. So we consider such a situation. Suppose that for X_0 there exists an invariant plane, assume that it is $z = 1$, with the partial first integral $F = |x|^\alpha|y|^\beta|1 - x - y|^\gamma$ on $z = 1$ and $\alpha, \gamma > 0, \beta < 0, \alpha + \beta + \gamma > 0$. This means that we have a saddle point $p_* = (x_*, y_*, 1)$ with $x_* > 0, y_* < 0$. Suppose now that X_0 exhibits another $1 : -1$ resonance between its eigenvalues, between one eigenvalue in the z-direction and one eigenvalue in the (x, y)-direction. One expects that the second resonance is associated with partial integrability of X_0 at an invariant surface S_0 transversal to $z = 1$. Then $S_0 \cap \{z = 1\}$ is an invariant algebraic curve through the saddle p_*, it is its separatrix.

It is not difficult to show that when the curve $F(x, y) = F(p_*)$ is reducible, with one algebraic component, then this component is a straight line: either $x + y = 0$ (then $\alpha + \beta = 0$) or $x = 1$ (then $\beta + \gamma = 0$). For this one should consider the curves $x^\alpha(1 - x - y)^\gamma = c(-y)^{-\beta}$ near the singular points $(0 : 0 : 1 : 1)$, $(0 : 1 : 1 : 1)$, $(0 : 1 : 0 : 0)$ and $(1 : 0 : 0 : 0)$ in $\mathbb{RP}^2 \subset \mathbb{RP}^3$.

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We see that the most probable seem to be the situation when the invariant surface S_0 is linear.

We finish this work by remarking that our rather thorough analysis did not reveal existence of any chaotic regime in 3D Lotka–Volterra systems. On the other hand, other specialists used to regard the existence of a chaotic attractor as established (see [**HSi, Si**]). There exist physical works [**ACT, SSFS**] with computer simulations revealing a chaotic regime.

The authors of the latter two papers refer to two mechanisms for the creation of chaos. One of them is a sequence of the period doubling bifurcations, like in the one-dimensional logistic family $x \to ax(1-x)$ (even figures in [**SSFS**] are surprisingly similar). The other is the so-called Shilnikov bifurcation, i.e. a singular point with a one-dimensional unstable separatrix W^{u}, with a two-dimensional stable manifold W^{s} corresponding to a pair of complex eigenvalues and the separatrix W^{u} ends-up in W^{s} (before bifurcation). Bustamante and Hojman [**BH**] noticed that there is a point in the space of parameters, from the family of systems used in [**ACT**], which corresponds to the Hopf's center $\mathbf{LV}^{\mathrm{Hopf}}$.

In an initial version of the paper we had formulated a conjecture that the three-dimensional generalized Lotka–Volterra systems are not chaotic and, moreover, that the number of their isolated periodic solutions is uniformly bounded. Due to remarks of Janusz Mierczyński and a referee (whom we are deeply grateful) we have changed our mind. As we have said before, the case $\mathbf{LV}^{\mathrm{Hopf}}$ (and its perturbations) is little understood by us. What we know is how to investigate limit cycles which appear near the Hopf's point p_* (see §4.4); the calculations are under way [**GSZ**]. Maybe away from the Hopf's point a scenario suggested by the physicists does indeed occur.

References

[ACT] A. Arneodo, P. Coullet and C. Tresser. Occurrence of strange attractors in three-dimensional Volterra equations. *Phys. Lett.* A **79**(4) (1980), 259–263.

[Ar1] V. I. Arnold. *Geometrical Methods in the Theory of Ordinary Differential Equations*. Springer, New York, 1993. (Translated from *Dopol'nitel'nye glavy teorii obyknovennykh differents'ialnykh uravnenij*. Nauka, Moskva, 1978.)

[Ar2] V. I. Arnold. Sur quelques problèmes de la théorie des systémes dynamiques. *Topol. Meth. Nonlin. Anal.* **4**(1) (1994), 209–225.

[AAIS] V. I. Arnold, V. S. Afraimovich, Yu. S. Il'yashenko and L. P. Shilnikov. *Bifurcation Theory and Catastrophe theory (Encyclopaedia of Math. Sci. Vol. 5, Dynamical Systems V)*. Springer, Berlin, 1994. (Translated from *Sovremennye Problemy Matematiki, Fundamental'nye Napravlenia*, t. 5, VINITI, Moskva, 1986.)

[BE] G. Bateman and A. Erdélyi. *Higher Transcendental Functions*. Vol. 1. McGraw-Hill, New York, 1953.

[Ba] N. N. Bautin. On periodic solutions of a system of differential equations. *Prikl. Mat. Mech.* **18** (1954), 128 (in Russian).

[BZ1] M. Bobieński and H. Żołądek. Limit cycles for multidimensional vector fields. The elliptic case. *J. Dynam. Control Syst.* **9**(2) (2003), 265–310.

[BZ2] M. Bobieński and H. Żołądek. Limit cycles for three dimensional polynomial vector fields. *Nonlinearity* **18**(1) (2005), 175–209.

https://doi.org/10.1017/S0143385704000902 Published online by Cambridge University Press

[BH] M. D. Bustamante and S. A. Hojman. Lagrangian structures, integrability and chaos for 3D dynamical equations. *J. Phys. A: Math. Gen.* **36**(1) (2003), 151–160.

[Ca] L. Cairó. Darboux first integrals and integrability of the 3D Lotka–Volterra systems. *J. Nonlin. Math. Phys.* **7**(4) (2000), 511–534.

[CL] L. Cairó and J. Llibre. Darboux integrability for 3D Lotka–Volterra systems. *J. Phys. A: Math. Gen.* **33**(12) (2000), 2395–2406.

[DFO] P. Duarte, R. L. Fernandes and W. M. Oliva. Dynamics of the attractor in the Lotka–Volterra equations. *J. Diff. Eq.* **149** (1998), 143–189.

[Ga] L. Gavrilov. The infinitesimal 16th Hilbert problem in the quadratic case. *Invent. Math.* **143**(3) (2001), 449–497.

[GSZ] Ł. J. Gołaszewski, P. Sławiński and H. Żołądek. Limit cycles in 3D Lotka–Volterra systems appearing after perturbation of Hopf center. *Preprint*, University of Warsaw, 2004.

[Hi] M. Hirsch. Systems of differential equations which are competitive or cooperative: III. Competing species. *Nonlinearity* **1** (1988), 51–71.

[HPS] M. Hirsch, C. Pugh and M. Shub. *Invariant Manifolds (Lecture Notes in Mathematics, 583)*. Springer, Berlin, 1977.

[HSi] J. Hofbauer and K. Sigmund. Evolutionary game dynamics. *Bull. Amer. Math. Soc.* **40**(4) (2003), 479–519.

[HSo] J. Hofbauer and J. W.-H. So. Multiple limit cycles for three dimensional Lotka–Volterra equations. *Appl. Math. Lett.* **7**(6) (1994), 65–70.

[Il] I. Iliev. The cyclicity of the period annulus of the quadratic Hamiltonian triangle. *J. Diff. Eq.* **128** (1996), 309–326.

[KG] E. Khorozov and S. van Gils. Uniqueness of limit cycles in planar vector fields which leave the axes invariant. In *Multiparameter bifurcation theory (Arcata 1985)*. *Contemp. Math.* **56** (1986), 117–129.

[Ko] A. N. Kolmogoroff. Sulla theorie di Volterra della Lotka per l'essistenza. *Giorn. Inst. Ital. Attuari* **7** (1936), 74–80.

[LZ] P. Leszczyński and H. Żołądek. Limit cycles appearing after perturbation of certain multidimensional vector fields. *J. Dynam. Diff. Eq.* **13**(4) (2001), 689–709.

[Lo] A. J. Lotka. *Elements of Mathematical Biology*. Dover, New York, 1958.

[MM] J. Marsden and M. McCracken. *The Hopf bifurcation and its applications*. Springer, New York, 1976.

[Mi] J. Mierczyński. Three-dimensional cooperative irreducible systems with a first integral. *Colloq. Math.* **76**(2) (1998), 181–190.

[SSFS] W. Schnabl, P. F. Stadler, C. Forst and P. Schuster. Full characterization of a strange attractor, Chaotic dynamics in low-dimensional replicator systems. *Physica* D **48** (1991), 65–90.

[Si] K. Sigmund, The population dynamics of conflict and cooperation. *Proceedings of the ICM*, Vol. 1. Universität Bielefeld, Bielefeld, 1998, pp. 487–506.

[SW] J. M. Strelcyn and S. Wojciechowski. A method of finding integrals for three-dimensional dynamical systems. *Phys. Lett.* A **133**(4-5) (1988), 207–212.

[Vo] V. Volterra. *Leçons sur la théorie mathématique de la lutte pour la vie*. Gauthier–Villars, Paris, 1931.

[XL] D. Xiao and W. Li. Limit cycles for the competitive three dimensional Lotka–Volterra system. *J. Diff. Eq.* **164** (2000), 1–15.

[ZK] A. Zegeling and R. Kooij. Uniqueness of limit cycle in polynomial systems with algebraic invariants. *Bull. Austral. Math. Soc.* **49** (1994), 7–20.

[Ze] M. L. Zeeman. Hopf bifurcations in competitive three-dimensional Lotka–Volterra systems. *Dynam. Stability Syst.* **8** (1993), 189–217.

[ZZ1] E. C. Zeeman and M. L. Zeeman. From local to global behavior in competitive Lotka–Volterra systems. *Trans. Amer. Math. Soc.* **355**(2) (2002), 713–734.

[ZZ2] E. C. Zeeman and M. L. Zeeman. An n-dimensional competitive Lotka–Volterra system is generically determined by the edges of its carrying simplex. *Nonlinearity* **15** (2002), 2019–2032.

[Zo1] H. Żołądek. Bifurcations of a certain family of planar vector fields tangent to axes. *J. Diff. Eq.* **67**(1) (1987), 1–55.

[Zo2] H. Żołądek. Quadratic systems with center and their perturbations. *J. Diff. Eq.* **109**(2) (1994), 223–273.

[Zo3] H. Żołądek. The cyclicity of triangles and segments inside quadratic systems. *J. Diff. Eq.* **112**(1) (1995), 137–159.

https://doi.org/10.1017/S0143385704000902 Published online by Cambridge University Press