Grassmannia 9

The on-shell BCFW recursion formulas (7.39) and (7.61) have taught us that tree superamplitudes and loop-integrands of *planar* $\mathcal{N}=4$ SYM can be written as

$$\mathcal{A}_n^{L\text{-loop}} = \mathcal{A}_{n,\text{MHV}}^{\text{tree}} \times Y_n^{L\text{-loop}}\,, \tag{9.1}$$

where $Y_n^{L\text{-loop}}$ is a dual conformal invariant. In the N^KMHV sector, Y_n^{tree} is a sum of K products of 5-brackets (or R-invariants in Section 4.5), as found in Sections 5.3 and 5.4. At loop-level, $Y_n^{L\text{-loop}}$ is a linear combination of dual conformal invariant integrands.

The MHV tree amplitude prefactor in (9.1) serves an important purpose: since it is dual conformal *covariant* with homogeneous dual conformal inversion-weight of the external particles – as in (5.39) – it generates the necessary dual conformal "anomaly" that modifies the dual conformal generators in such a way that they become part of the level 1 generators of a Yangian symmetry. Hence, as described in Section 4.5, the MHV factor in (9.1) is essential for Yangian symmetry.

There are several interesting points to consider:

1. The color-ordered planar $\mathcal{N}=4$ SYM superamplitudes have ***cyclic symmetry*** in the labels of the external states. However, in the dual conformal BCFW representations (7.39) and (7.61), the cyclic symmetry is completely obscured. This is not surprising, because these recursion formulas are based on shifts of two adjacent external lines: making two lines special breaks the cyclic symmetry. In the pursuit of happiness and manifest symmetries, we may ask if there is a formalism for the planar $\mathcal{N}=4$ SYM superamplitudes in which both the (dual) conformal symmetry *and* the cyclic symmetry are manifest? This suggestion will guide us in Section 9.1.
2. In Section 8.3, we gave examples of how individual Leading Singularity diagrams can be understood as BCFW terms of tree amplitudes. Since each BCFW term is Yangian invariant, as indicated in the recursion formula (7.39), this implies that the Leading Singularities are also Yangian invariant. Thus, understanding the ***most general Yangian invariants*** is a step towards gaining control of the planar superamplitudes in $\mathcal{N}=4$ SYM at *any* loop order. Of course, one still needs to understand how to put the Yangian invariants together to obtain a given superamplitude; read on.
3. Super-BCFW recursion for $\mathcal{N}=4$ SYM amplitudes can be based on any choice of two shifted external momenta. Different choices can give very different representations of the same amplitude, in particular with distinct spurious poles. For the amplitude to be *local*, i.e. free of spurious poles, the residues of the spurious poles must cancel in the sum of BCFW diagrams. Thus, the ***equivalence of two different BCFW representations*** is intimately related to ***locality***. Each BCFW diagram is Yangian invariant, so by

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understanding how to enforce locality in a Yangian invariant way, it turns out that the equivalence between the different BCFW representations can be trivialized.

Interestingly, the above three points can be addressed jointly. The strategy is to find a way to generate the most general Yangian invariant rational function from a formula in which cyclic symmetry is manifest. That's our job now, so let's get to work.

9.1 Yangian invariance and cyclic symmetry

The level 0 Yangian generators are the superconformal generators studied in Section 5.1. In Section 5.2, we introduced the supertwistors $\mathcal{W}_i^{\mathsf{A}} = ([i|^a,\ |\tilde{\mu}_i\rangle^{\dot{a}},\ \eta_{iA})$ in order to linearize the action of the superconformal generators. The Grassmann components of the supertwistor are simply the on-shell superspace coordinates η_{iA}, and $|\tilde{\mu}_i\rangle^{\dot{a}}$ is the Fourier conjugate coordinate of $|i\rangle^{\dot{a}}$. A function f of on-shell momentum space spinor-helicity variables is Fourier transformed to the (super)twistor space as

$$\int \left[\prod_{i=1}^{n} d^2|i\rangle\right] f\big([i|, |i\rangle, \eta_i\big)\, e^{i\sum_{j=1}^{n}\langle j\tilde{\mu}_j\rangle} \equiv \tilde{f}(\mathcal{W}_i^{\mathsf{A}})\,. \tag{9.2}$$

In supertwistor space, the level-0 superconformal generators are

$$G^{\mathsf{A}}{}_{\mathsf{B}} = \sum_{i=1}^{n} G_i^{\mathsf{A}}{}_{\mathsf{B}} = \sum_{i=1}^{n} \mathcal{W}_i^{\mathsf{A}} \frac{\partial}{\partial \mathcal{W}_i^{\mathsf{B}}}\,, \tag{9.3}$$

and the level-1 generators can be written in bi-local form as (Section 5.3)

$$\sum_{i<j}^{n} (-1)^{|\mathsf{C}|}\big[G_i^{\mathsf{A}}{}_{\mathsf{C}}\, G_j^{\mathsf{C}}{}_{\mathsf{B}} - (i \leftrightarrow j)\big]\,. \tag{9.4}$$

Our aim is a cyclic invariant formula that generates Yangian invariant rational functions; these are the building blocks for N^KMHV superamplitudes of planar $\mathcal{N}=4$ SYM. Let us try to motivate the construction, step by step. To start with, note that the level 0 generators (9.3) act on the supertwistor variables as $SL(2,2|4)$ linear transformations. Any delta function $\delta^{4|4}$ whose argument is a linear combination of the supertwistors is invariant under the linear $SL(2,2|4)$ transformation: for example

$$\delta^{4|4}\Big(\sum_{i=1}^{n} C_i\, \mathcal{W}_i^{\mathsf{A}}\Big) \equiv \delta^2\Big(\sum_{i=1}^{n} C_i [i|^a\Big)\, \delta^2\Big(\sum_{i=1}^{n} |i\rangle^{\dot{a}} C_i\Big)\, \delta^{(4)}\Big(\sum_{i=1}^{n} C_i\, \eta_{iA}\Big)\,, \tag{9.5}$$

with some arbitrary auxiliary coefficients $C_i \in \mathbb{C}$. For a level 0 generator $G^{\mathsf{A}}{}_{\mathsf{B}}$ with $\mathsf{A} \neq \mathsf{B}$, this is because it transforms the argument of one of the delta functions to that of another delta function; schematically

$$x\frac{\partial}{\partial y}\delta(x)\delta(y) = x\,\delta(x)\,\delta'(y) = 0\,. \tag{9.6}$$

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For "diagonal" generators, such as $G^1{}_1$, invariance follows only after using the tracelessness condition, i.e. writing the generators in the form analogous to (5.19).

For an $\mathrm{N}^K\mathrm{MHV}$ superamplitude, we need Yangian invariants that are Grassmann polynomials of degree $4(K+2)$; so it is natural to take

$$k \equiv K + 2 \tag{9.7}$$

products of (9.5). Note that k counts the number of negative helicity gluons in the pure gluon amplitude. To avoid having k identical delta functions, we introduce k sets of the auxiliary variables $C_{\mathsf{a}i}$ labeled by a new index $\mathsf{a} = 1, 2, \ldots, k$. So now we have a Grassmann degree $4(K+2)$ object

$$\prod_{\mathsf{a}=1}^{k} \delta^{4|4}\Big(\sum_{i=1}^{n} C_{\mathsf{a}i}\,\mathcal{W}_i^{\mathsf{A}}\Big) \tag{9.8}$$

that is $SL(2,2|4)$ invariant. The parameters $C_{\mathsf{a}i}$ are sometimes called ***link variables*** [23].

The $n \times k$ parameters $C_{\mathsf{a}i}$ are arbitrary, so to remove the dependence on them let us integrate (9.8) over all $C_{\mathsf{a}i}$. This has the further benefit of making the integrated result cyclically invariant: a permutation of the $\mathcal{W}_i$s is compensated by a permutation of the $C_{\mathsf{a}i}$s. This is a change of integration variable with unit Jacobian. However, it is not clear what measure we should use when integrating over the $C_{\mathsf{a}i}$s. So let us allow for a general cyclically invariant function $f(C)$ and write our candidate "generating function" as

$$\int d^{k\times n}C\; f(C) \prod_{\mathsf{a}=1}^{k} \delta^{4|4}\Big(\sum_{i=1}^{n} C_{\mathsf{a}i}\,\mathcal{W}_i^{\mathsf{A}}\Big)\,. \tag{9.9}$$

The integral of the $k \times n$ complex parameters is intended to be carried out as a contour integral. The choice of contour is a very important and physically relevant aspect that will be discussed in Section 9.3.

When the level 1 generators are considered, it turns out that there is a unique choice of $f(C)$ such that (9.9) is Yangian invariant. We will not repeat the argument here, but refer you to [122, 123]. The unique function that gives (9.9) full Yangian symmetry is

$$f(C) = \frac{1}{M_1 M_2 \cdots M_n}\,, \tag{9.10}$$

where M_i is the ith ordered minor of the $k \times n$ matrix $C_{\mathsf{a}i}$: this is the determinant of the $k \times k$ submatrix whose first column is the ith column of $C_{\mathsf{a}i}$, specifically

$$M_i \equiv \epsilon^{\mathsf{a}_1\,\mathsf{a}_2\,\cdots\,\mathsf{a}_k} C_{\mathsf{a}_1 i} C_{\mathsf{a}_2, i+1} \cdots C_{\mathsf{a}_k, i+k-1}\,, \tag{9.11}$$

with $i = 1, 2, \ldots, n$. One goes around cyclically when reaching the end of the C-matrix. For example, the $n = 5$ and $k = 2$ matrix

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} \end{pmatrix} \tag{9.12}$$

gives $M_1 = C_{11}C_{22} - C_{12}C_{21}$ and $M_5 = C_{15}C_{21} - C_{11}C_{25}$.

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At this stage, we have learned that the integral

$$\int \frac{d^{k\times n}C}{M_1 M_2 \cdots M_n} \prod_{\mathsf{a}=1}^{k} \delta^{4|4}\Big(\sum_{l=1}^{n} C_{\mathsf{a}l}\mathcal{W}_l^{\mathsf{A}}\Big) \tag{9.13}$$

is Yangian invariant and has cyclic symmetry.

Before declaring victory, there are some loose ends to tie. *First*, a minor issue (yes, that's a pun) is that if M_i contains columns that are not strictly increasing due to cyclicity (for example $M_{n-1} = \cdots C_{\mathsf{a}n}C_{\mathsf{a}1}\cdots$) then the proof of Yangian invariance goes through only on the support of the bosonic delta functions in (9.13). *Second*, a major issue is that the integral we so proudly wrote down in (9.13) is not at all well-defined – it is divergent. To see this, note that the product of delta functions is invariant under a $GL(k)$ rotation of the k a-indices. The minors only respect $SL(k)$ transformations: $GL(1)$ takes $C_{\mathsf{a}i} \to tC_{\mathsf{a}i}$, hence $M_i \to t^k M_i$, but this excess weight is canceled by the Jacobian of $d^{k\times n}C$. Thus the integral has $GL(k)$ symmetry. To define a proper integral we need to "gauge fix" the $GL(k)$ redundancy. We indicate the need to gauge fix $GL(k)$ by writing

$$\tilde{\mathcal{L}}_{n,k}(\mathcal{W}_i) = \int \frac{d^{n\times k}C_{\mathsf{a}i}}{GL(k)\prod_{j=1}^{n} M_j} \prod_{\mathsf{a}=1}^{k} \delta^{4|4}\Big(\sum_{l=1}^{n} C_{\mathsf{a}l}\mathcal{W}_l^{\mathsf{A}}\Big). \tag{9.14}$$

It turns out [122, 123] that for given n and k, $\mathcal{L}_{n,k}$ *is the unique cyclically invariant integral-expression that generates all Yangian invariants!* We are going to give examples in the following sections. The formula (9.14) was first introduced by Arkani-Hamed, Cachazo, Cheung, and Kaplan [23], who at the time conjectured that it produces all Leading Singularities of planar $\mathcal{N} = 4$ SYM. A similar integral formula was presented by Mason and Skinner [60] based on momentum supertwistors $\mathcal{Z}$, as opposed to the "regular" supertwistors $\mathcal{W}$, thus interchanging the role of the ordinary superconformal and dual superconformal symmetries. The twistor and momentum twistor versions of the Grassmannian integral are directly related (Arkani-Hamed *et al*. [60]). For a streamlined proof and more recent review of the Grassmannian formulation, see Elvang *et al*. [60].

We used supertwistors $\mathcal{W}_i^{\mathsf{A}} = ([i|^a,\ |\tilde{\mu}_i\rangle^{\dot{a}}|\,\eta_{iA})$ to emphasize superconformal and Yangian symmetry in the construction above. However, since we are more familiar with scattering amplitudes in momentum space $([i|^a,\ |i\rangle^{\dot{a}}|\,\eta_{iA})$, we are going to inverse-Fourier transform all $|\tilde{\mu}_i\rangle$ in (9.14) back to $|i\rangle$. This is conveniently done in a gauge-fixing of the $GL(k)$ symmetry where the first $k\times k$ block of $C_{\mathsf{a}i}$ is the unit matrix. For example for $n = 7$ and $k = 3$, we have

$$C = \begin{pmatrix} 1 & 0 & 0 & c_{14} & c_{15} & c_{16} & c_{17} \\ 0 & 1 & 0 & c_{24} & c_{25} & c_{26} & c_{27} \\ 0 & 0 & 1 & c_{34} & c_{35} & c_{36} & c_{37} \end{pmatrix}. \tag{9.15}$$

In this gauge, (9.14) becomes

$$\int \frac{d^{(n-k)\times k}c}{\prod_{j=1}^{n} M_j} \prod_{\mathsf{a}=1}^{k} \delta^2\Big([\mathsf{a}| + \sum_{l=k+1}^{n} c_{\mathsf{a}l}[l|\Big)\,\delta^2\Big(|\tilde{\mu}_{\mathsf{a}}\rangle + \sum_{l=k+1}^{n} |\tilde{\mu}_l\rangle\, c_{\mathsf{a}l}\Big)\,\delta^{(4)}\Big(\eta_{\mathsf{a}} + \sum_{l=k+1}^{n} c_{\mathsf{a}l}\,\eta_l\Big). \tag{9.16}$$

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Performing the inverse-Fourier transform $\int d^2|\tilde{\mu}_j\rangle\, e^{-i\langle j\,\tilde{\mu}_j\rangle}$ for each $j = 1, \ldots, n$ gives

$$\mathcal{L}_{n,k}\big([i|, |i\rangle, \eta_i\big) = \int \frac{d^{(n-k)\times k}c}{\prod_{j=1}^{n} M_j} \left[\prod_{\mathsf{a}=1}^{k} \delta^2\Big([\mathsf{a}| + \sum_{l=k+1}^{n} c_{\mathsf{a}l}[l|\Big)\, \delta^{(4)}\Big(\eta_{\mathsf{a}} + \sum_{l=k+1}^{n} c_{\mathsf{a}l}\,\eta_l\Big)\right] \times \left[\prod_{i=k+1}^{n} \delta^2\Big(|i\rangle - \sum_{\mathsf{a}=1}^{k} |\mathsf{a}\rangle c_{\mathsf{a}i}\Big)\right]. \tag{9.17}$$

▶ **Exercise 9.1**

Fill out the details of the inverse Fourier transformation to derive (9.17).

The representation (9.17) is central in the next section where we study the geometric interpretation of $\mathcal{L}_{n,k}$. In Section 9.3, we show that familiar amplitude expressions can be derived from $\mathcal{L}_{n,k}$.

9.2 The Grassmannian

It is very convenient to view the $n \times k$ matrices $C_{\mathsf{a}l}$ in (9.14) as k n-component vectors that define a k-plane in $\mathbb{C}^n$. The space of all k-planes in an n-dimensional space is called the ***Grassmannian*** $\mathrm{Gr}(k, n)$. The formula (9.14) for $\tilde{\mathcal{L}}_{n,k}$ is therefore naturally viewed as a cyclic invariant integral over all k-planes in the Grassmannian. Since any non-degenerate linear transformation of the k n-vectors gives the same plane, there is a natural $GL(k)$ invariance. It is precisely the same $GL(k)$ redundancy we encountered previously in the discussion of the integral (9.14): the Grassmannian integral (9.14) is well-defined only when "gauge fixing" the $GL(k)$ redundancy. Because of the $GL(k)$ redundancy, the dimensions of the Grassmannian $\mathrm{Gr}(k, n)$ are $k \times n - k^2 = k(n-k)$.

With this geometric picture in mind, let us now examine the bosonic delta functions in the gauge-fixed expression (9.17) for $\mathcal{L}_{n,k}$. They enforce the constraints

$$\sum_{i=1}^{n} C_{\mathsf{a}i}\,[i|^a = 0\,, \qquad \sum_{i=1}^{n} \tilde{C}_{\mathsf{a}'i}\,\langle i| = 0\,, \tag{9.18}$$

where C and $\tilde{C}$ are $k \times n$ and $(n-k) \times n$ matrices respectively, so $\mathsf{a}' = k+1, \ldots, n$. They are explicitly given as

$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 & c_{1,k+1} & \cdots & c_{1n} \\ 0 & 1 & \cdots & 0 & c_{2,k+1} & \cdots & c_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & c_{kk+1} & \cdots & c_{kn} \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} -c_{1,k+1} & \cdots & -c_{k,k+1} & 1 & 0 & \cdots & 0 \\ -c_{1,k+2} & \cdots & -c_{k,k+2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -c_{1n} & \cdots & -c_{kn} & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{9.19}$$

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An important feature is that

$$C\,\tilde{C}^T = \sum_{i=1}^{n} C_{\mathsf{a}i}\,\tilde{C}_{\mathsf{a}'i} = 0\,. \tag{9.20}$$

▶ **Exercise 9.2**

Construct $\tilde{C}$ associated with (9.15) and check that (9.20) holds.

We can view $\tilde{C}$ as $(n-k)$ n-vectors spanning an $(n-k)$-plane in n dimensions. The condition (9.20) states that the $(n-k)$-plane defined by $\tilde{C}$ is the orthogonal complement of the k-plane defined by C.

In this notation, we can reinstate the $GL(k)$ redundancy and write our momentum space Grassmannian integral (9.17) as

$$\mathcal{L}_{n,k} = \int \frac{d^{n\times k}C}{GL(k)\prod_{j=1}^{n} M_j} \left[\prod_{\mathsf{a}=1}^{k} \delta^2\Big(\textstyle\sum_i C_{\mathsf{a}i}\,[i|\Big)\delta^{(4)}\Big(\sum_i C_{\mathsf{a}i}\,\eta_{iA}\Big)\right] \times \left[\prod_{\mathsf{a}'=k+1}^{n} \delta^2\Big(\textstyle\sum_i \tilde{C}_{\mathsf{a}'i}\,\langle i|\Big)\right], \tag{9.21}$$

with the understanding that $\tilde{C}$ is defined as the complement to C in the sense of (9.20).

Now a geometric picture is emerging of the meaning of the constraints (9.18). The collection of the n $|i\rangle$s defines a 2-plane in an n-dimensional space,

$$\begin{pmatrix} |1\rangle^{\dot{1}} & |2\rangle^{\dot{1}} & \cdots & |n\rangle^{\dot{1}} \\ |1\rangle^{\dot{2}} & |2\rangle^{\dot{2}} & \cdots & |n\rangle^{\dot{2}} \end{pmatrix}. \tag{9.22}$$

Similarly the $[i|$s define a 2-plane in an n-dimensional space.

The constraints (9.18) say that the 2-plane spanned by the $[i|$s is orthogonal to the k-plane C and that the 2-plane defined by the $|i\rangle$s is orthogonal to the $(n-k)$-plane $\tilde{C}$. This is illustrated in Figure 9.1. Since $\tilde{C}$ and C are orthogonal complements, we immediately conclude that $\tilde{C}$ must contain the 2-plane $[i|$ while C must contain $|i\rangle$. This in turn tells us that the 2-plane of $[i|$ must be orthogonal to the 2-plane of $|i\rangle$, i.e.

$$\sum_{i=1}^{n} |i\rangle[i| = 0\,. \tag{9.23}$$

This is just the statement that the external momenta satisfy momentum conservation. Thus, seemingly out of nowhere, the cyclic- and Yangian-invariant generating function $\mathcal{L}_{n,k}$ "knows" about momentum conservation!

As we have just shown, the bosonic delta functions in (9.21) give non-vanishing results only on the support of momentum conservation, $\delta^4(P)$. With this in mind, let us count the number of "free" integration variables in (9.21), i.e. the number of $c_{\mathsf{a}i}$s *not* localized by the bosonic delta functions. After gauge fixing $GL(k)$, we have a total of $k\times(n-k)$

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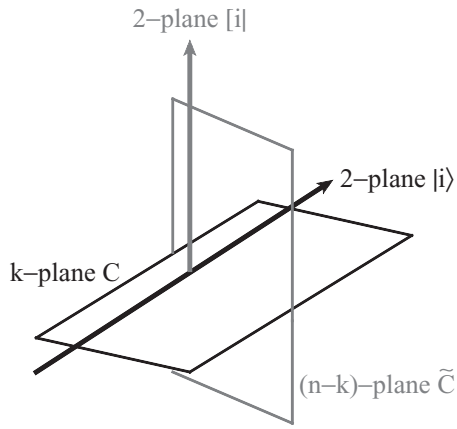


Fig. 9.1 The geometry of planes in the Grassmannian. The k-plane C and $(n-k)$-plane $\tilde{C}$ are orthogonal complements, so the constraint $\sum_i C_{\mathsf{a}i}[i| = 0$ in (9.18) means that the 2-plane spanned by the n $[i|$s is orthogonal to C and hence must be contained in $\tilde{C}$. Similarly, C contains the 2-plane spanned by the $|i\rangle$s. It follows from the geometry that the 2-planes spanned by $[i|$ and $|i\rangle$, respectively, are orthogonal, but that is exactly the statement of momentum conservation $\sum_{i=1}^{n} |i\rangle[i| = 0$.

$c_{\mathsf{a}i}$-variables. There are $[2k + 2(n-k)] = 2n$ bosonic delta functions, but this includes momentum conservation $\delta^4(P)$, so four of the delta functions do not localize any $c_{\mathsf{a}i}$-variables. Therefore, a total of

$$\#(\text{integration variables}) = [k \times (n-k) - (2n-4)] = (k-2)(n-k-2) \qquad (9.24)$$

$c_{\mathsf{a}i}$-variables are left to be integrated.

In the MHV sector, $k = 2$ so we learn from (9.24) that the integral (9.21) is fully localized by the bosonic delta functions. For generic k and n, $\mathcal{L}_{n,k}$ is a multi-dimensional integral that localizes on the poles in the minors; we work out an explicit example in the next section. For $k = 0$, the integral vanishes since it is proportional to $\delta^2(|i\rangle)$ which does not have support for generic momenta. This is simply the statement that the "all-plus" gluon amplitude vanishes in $\mathcal{N} = 4$ SYM. For $k = 1$, the last delta function in (9.21) forces all of the $|i\rangle$s to be proportional to each other, but this lacks support for generic momenta, with the exception of special kinematics for $n = 3$. Not surprisingly, this says that the all-plus-and-one-minus gluon amplitudes vanish in $\mathcal{N} = 4$ SYM for $n > 3$.

The counting of integration variables in (9.24) is invariant under $k \to (n-k-2)$. This corresponds to a flip of what we identify as positive and negative helicity, i.e. which states are associated with highest/lowest Grassmann weight. Indeed, for $k = n-2$, the (super)amplitude is anti-MHV, so it makes sense that the Grassmannian integral is localized completely by the bosonic delta functions, just as it is for the MHV sector $k = 2$.

Perhaps you have noticed that the dimension $(k-2)(n-k-2)$ of the $\mathcal{L}_{n,k}$-integral is also the dimension of the Grassmannian $\mathrm{Gr}(k-2, n-4)$. This is not a coincidence. The bosonic delta functions enforce that $C_{\mathsf{a}i}$ contains the 2-plane $|i\rangle$, so using $GL(k)$ redundancy we can choose the first two rows of $C_{\mathsf{a}i}$ to be $|i\rangle$. The $2k-4$ delta functions (where the -4 comes from momentum conservation) impose that the remaining $(k-2)$-plane in

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n-dimensions is orthogonal to $|j]$. This places $2(k-2)$ constraints on $C_{\mathsf{a}i}$. Taking the $GL(k-2)$ redundancy into account, the number of free variables in $C_{\mathsf{a}i}$ is therefore $n(k-2)-(k-2)^2-2(k-2)=(k-2)(n-k-4)$. This is the dimension of the Grassmannian $\mathrm{Gr}(k-2,n-4)$.

So far we have constructed a cyclic and Yangian invariant integral $\mathcal{L}_{n,k}$ in the Grassmannian, but we have not really done anything with it. In fact, other than showing that it captures momentum conservation, we have given you little reason to believe that there is any connection to scattering amplitudes in planar $\mathcal{N}=4$ SYM. So now we had better show how it works.

9.3 Yangian invariants as residues in the Grassmannian

We carry out the Grassmannian integral (9.21) in the simplest cases to illustrate how the familiar MHV and NMHV superamplitudes appear.

9.3.1 MHV amplitudes

By the counting (9.24), the bosonic delta functions completely localize the integral (9.21) for the MHV sector ($k=2$). Furthermore, the geometric description in the previous section tells us that for MHV, the bosonic delta functions exactly encode conservation of 4-momentum on the n external states: $\delta^4(P)$. What about the Grassmann delta function in (9.21)? Well, when $k=2$, C defines a 2-plane, and since the 2-plane $|i\rangle$ must be contained in C, we can simply identify the two 2-planes; up to a $GL(2)$ transformation we therefore have

$$\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \end{pmatrix} = \begin{pmatrix} |1\rangle^{\dot{1}} & |2\rangle^{\dot{1}} & \cdots & |n\rangle^{\dot{1}} \\ |1\rangle^{\dot{2}} & |2\rangle^{\dot{2}} & \cdots & |n\rangle^{\dot{2}} \end{pmatrix}. \tag{9.25}$$

Using this explicit representation of the $C_{\mathsf{a}i}$s, the Grassmann delta function becomes the familiar statement of supermomentum conservation,

$$\prod_{\mathsf{a}=1}^{2} \delta^{(4)}\Big(\textstyle\sum_i C_{\mathsf{a}i}\,\eta_i\Big) = \prod_{\dot{a}=1}^{2} \delta^{(4)}\Big(\textstyle\sum_i |i\rangle^{\dot{a}}\,\eta_i\Big) = \delta^{(8)}\big(\tilde{Q}\big)\,. \tag{9.26}$$

Finally, consider the minors in (9.21):

$$M_i = \epsilon^{\mathsf{ab}}\, C_{\mathsf{a}i}\, C_{\mathsf{b},i+1} = -\epsilon_{\dot{a}\dot{b}}\, |i\rangle^{\dot{a}}\, |i+1\rangle^{\dot{b}} = -\langle i, i+1\rangle\,. \tag{9.27}$$

Putting everything together, we find

$$\mathcal{L}_{n,2} = (-1)^n \frac{\delta^{(8)}(\tilde{Q})\,\delta^4(P)}{\prod_{i=1}^{n}\langle i, i+1\rangle} = (-1)^n \mathcal{A}^{\mathrm{MHV}}_{n,\mathrm{tree}}\,. \tag{9.28}$$

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So for $k = 2$, the cyclic invariant integral $\mathcal{L}_{n,2}$ nicely produces the MHV tree amplitude (up to an overall convention-dependent sign).

▷ *Example.* Did the argument above go a little fast? Fair enough, let us evaluate $\mathcal{L}_{n,2}$ in full detail, starting with the gauged-fixed expression (9.17), which for $k = 2$ gives

$$\mathcal{L}_{n,2} = \int \frac{d^{(n-2)\times 2}c}{M_1 \cdots M_n} \left[\prod_{\mathsf{a}=1}^{2} \delta^2\Big([\mathsf{a}| + \sum_{l=3}^{n} c_{\mathsf{a}l}[l|\Big)\, \delta^{(4)}\Big(\eta_{\mathsf{a}} + \sum_{l=3}^{n} c_{\mathsf{a}l}\,\eta_l\Big)\right] \times \left[\prod_{i=3}^{n} \delta^2\Big(|i\rangle - |1\rangle c_{1i} - |2\rangle c_{2i}\Big)\right]. \tag{9.29}$$

The last set of delta functions localize the $2(n-2)$ components c_{1i} and c_{2i}:

$$\delta^2\Big(|i\rangle - |1\rangle c_{1i} - |2\rangle c_{2i}\Big) = \frac{1}{\langle 12\rangle}\,\delta\left(c_{1i} - \frac{\langle i2\rangle}{\langle 12\rangle}\right)\,\delta\left(c_{2i} - \frac{\langle i1\rangle}{\langle 21\rangle}\right). \tag{9.30}$$

Thus, on the support of these delta functions, the first four bosonic delta functions in (9.29) give

$$\delta^2\Big([1| + \sum_{l=3}^{n} c_{1l}[l|\Big)\,\delta^2\Big([2| + \sum_{l=3}^{n} c_{2l}[l|\Big) = \langle 12\rangle^2\,\delta^4(P)\,. \tag{9.31}$$

This is how the momentum conservation delta function appears.

Likewise for the Grassmann delta function: on the support of (9.30) it gives

$$\prod_{\mathsf{a}=1}^{2} \delta^{(4)}\Big(\eta_{\mathsf{a}} + \sum_{l=3}^{n} c_{\mathsf{a}l}\,\eta_l\Big) = \frac{1}{\langle 12\rangle^4}\,\delta^{(8)}(\tilde{Q})\,. \tag{9.32}$$

Finally, we evaluate the minors M_i. With the help of the Schouten identity we find

$$M_1 = 1\,,\quad M_2 = \frac{\langle 23\rangle}{\langle 12\rangle}\,,\quad M_3 = -\frac{\langle 34\rangle}{\langle 12\rangle}\,,\quad M_4 = -\frac{\langle 45\rangle}{\langle 12\rangle}\,,\quad \ldots\,,\, M_n = -\frac{\langle n1\rangle}{\langle 12\rangle}\,, \tag{9.33}$$

and hence

$$\prod_{i=1}^{n} M_i = \frac{(-1)^n}{\langle 12\rangle^n}\left(\prod_{i=1}^{n}\langle i, i+1\rangle\right). \tag{9.34}$$

Inserting everything into (9.29) we indeed obtain $(-1)^n$ times the MHV tree superamplitude, just as in (9.28). ◁

▶ Exercise 9.3

Derive (9.30)–(9.34). Then plug the results into (9.29) to verify that all powers of $\langle 12\rangle$ cancel.

The Grassmannian integral $\mathcal{L}_{n,k}$ has given a unique result, $\mathcal{A}^{\text{MHV}}_{n,\text{tree}}$, for $k = 2$. Given that the Grassmannian integral produces all possible Yangian invariants [122, 123], it must be

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that all MHV superamplitudes have the same Leading Singularities in planar $\mathcal{N}=4$ SYM, up to a sign, *to all loop-orders*. We have already seen a non-trivial manifestation of this in Section 8.2 (and again in Section 8.3), where the Leading Singularities of the 1-loop and 2-loop 4-point superamplitudes were found to be the MHV tree superamplitude. We now know that even if we were to take the 234-loop MHV superamplitude and solve the on-shell constraints that localize the $234 \times 4 = 936$ loop-momenta, the result of the Leading Singularities will again be MHV tree superamplitudes! No other Yangian invariants are available at MHV order.

9.3.2 6-point NMHV amplitudes

Let us now move on to a slightly more complicated – hence more exciting – example, the 6-point NMHV amplitude.[1] With $k=3$ and $n=6$, the counting formula (9.24) reveals that the Grassmannian integral $\mathcal{L}_{6,3}$ involves just one non-trivial integration. To evaluate it, we choose the gauge

$$\begin{pmatrix} c_{21} & 1 & c_{23} & 0 & c_{25} & 0 \\ c_{41} & 0 & c_{43} & 1 & c_{45} & 0 \\ c_{61} & 0 & c_{63} & 0 & c_{65} & 1 \end{pmatrix} . \tag{9.35}$$

The c-variables are labeled such that the bosonic delta functions in (9.21) can be written as:

$$\delta^2\left([\,\bar{i}\,| + \sum_j c_{\bar{i}j}\,[j|\right) , \quad \delta^2\left(|j\rangle - \sum_{\bar{i}} |\,\bar{i}\,\rangle\, c_{\bar{i}j}\right) , \tag{9.36}$$

where $\bar{i}=2,4,6$ and $j=1,3,5$. Since the integral is 1-dimensional, there must be a 1-parameter family of solutions that solve the delta function constraints (9.36). Indeed, if $c^*_{\bar{i}j}$ is a solution, then

$$\hat{c}_{\bar{i}j}(\tau) = c^*_{\bar{i}j} + \tfrac{1}{4}\,\tau\;\epsilon_{\bar{i}\bar{j}\bar{k}}\,\langle\bar{j}\bar{k}\rangle\;\epsilon_{jkl}[kl]\,, \tag{9.37}$$

is also a solution for any τ. Here $\epsilon_{\bar{i}\bar{j}\bar{k}}$ is a Levi-Civita symbol for the indices $\bar{i}=2,4,6$, and similarly for ϵ_{jkl}. There are implicit sums over repeated labels in (9.37). That $\hat{c}_{\bar{i}j}(\tau)$ is a solution can be seen from the result that the τ dependence drops out from the constraints in (9.36) due to the Schouten identity:

$$\begin{aligned}[\,\bar{i}\,|^a + \sum_j \hat{c}_{\bar{i}j}(\tau)[j|^a &= \tfrac{1}{4}\,\tau\;\epsilon_{\bar{i}\bar{j}\bar{k}}\langle\bar{j}\bar{k}\rangle \sum_j \epsilon_{jkl}[j|^a[kl] \\ &= \tfrac{1}{2}\,\tau\;\epsilon_{\bar{i}\bar{j}\bar{k}}\langle\bar{j}\bar{k}\rangle\big([1|^a[35] + [3|^a[51] + [5|^a[13]\big) = 0\,. \end{aligned} \tag{9.38}$$

We can now remove the bosonic delta functions by localizing the integral on the solution to the constraints (9.36) such that the remaining integral is over the 1-dimensional parameter

[1] For a much simpler evaluation of the NMHV residues in momentum twistor space, see Elvang *et al*. [60].

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τ. That gives

$$\begin{aligned}\mathcal{L}_{6,3} &= \int \frac{d^9 c_{\bar{i}j}}{M_1 \cdots M_n}\Big[\prod_j \delta^2\Big(|j\rangle - \sum_{\bar{i}} |\,\bar{i}\,\rangle\, c_{\bar{i}j}\Big)\Big] \\ &\quad \times \Big[\prod_{\bar{i}} \delta^2\Big([\,\bar{i}\,| + \sum_j c_{\bar{i}j}\,[j|\Big)\delta^{(4)}\Big(\eta_{\bar{i}} + \sum_j c_{\bar{i}j}\eta_j\Big)\Big] \\ &= \delta^4(P)\int \frac{d^9 c_{\bar{i}j}\, d\tau}{M_1 \cdots M_n}\,\delta^9\Big(c_{\bar{i}j} - \hat{c}_{\bar{i}j}(\tau)\Big)\prod_{\bar{i}}\delta^{(4)}\Big(\eta_{\bar{i}} + \sum_j c_{\bar{i}j}\eta_j\Big) \\ &= \delta^4(P)\int \frac{d\tau}{\hat{M}_1 \cdots \hat{M}_n}\prod_{\bar{i}}\delta^{(4)}\Big(\eta_{\bar{i}} + \sum_j \hat{c}_{\bar{i}j}\eta_j\Big)\,. \end{aligned} \tag{9.39}$$

The "hat" indicates dependence on τ via (9.37). In the gauge (9.35), the minors are

$$\begin{aligned} &\hat{M}_1 = \hat{c}_{43}\hat{c}_{61} - \hat{c}_{41}\hat{c}_{63}\,, \quad \hat{M}_3 = \hat{c}_{23}\hat{c}_{65} - \hat{c}_{25}\hat{c}_{63}\,, \quad \hat{M}_5 = \hat{c}_{21}\hat{c}_{45} - \hat{c}_{25}\hat{c}_{41}\,, \\ &\hat{M}_2 = -\hat{c}_{63}\,, \qquad\qquad \hat{M}_4 = -\hat{c}_{25}\,, \qquad\qquad \hat{M}_6 = -\hat{c}_{41}\,. \end{aligned} \tag{9.40}$$

At this stage, there appears to be no a priori prescription of which contour to pick in the τ-plane. Each minor $\hat{M}_i$ has a simple pole in τ, so there are six different residues that we denote $\{M_i\}$. Let us focus on the pole in $\hat{M}_4$. This means that τ is evaluated at τ_* such that $\hat{c}_{25}(\tau_*) = 0$. We can make the calculation simpler by choosing the origin for τ such that $\hat{M}_4 = 0$ for $\tau = 0$; in other words, we choose $\hat{c}^*_{25} = 0$. Let us use the constraints (9.36) to solve for the eight other $\hat{c}^*_{\bar{i}j}$s. From

$$|5\rangle - |4\rangle c^*_{45} - |6\rangle c^*_{65} = 0, \qquad [2| + c^*_{21}[1| + c^*_{23}[3| = 0\,, \tag{9.41}$$

we deduce

$$c^*_{45} = \frac{\langle 56\rangle}{\langle 46\rangle}, \quad c^*_{65} = \frac{\langle 45\rangle}{\langle 46\rangle}, \quad c^*_{21} = -\frac{[23]}{[13]}, \quad c^*_{23} = -\frac{[12]}{[13]}\,. \tag{9.42}$$

And this in turn allows us to solve

$$[4| + c^*_{41}[1| + c^*_{43}[3| + c^*_{45}[5| = 0, \qquad [6| + c^*_{61}[1| + c^*_{63}[3| + c^*_{65}[5| = 0\,, \tag{9.43}$$

to find

$$c^*_{41} = -\frac{\langle 6|4{+}5|3]}{\langle 46\rangle[13]}, \quad c^*_{43} = \frac{\langle 6|4{+}5|1]}{\langle 46\rangle[13]}, \quad c^*_{61} = \frac{\langle 4|5{+}6|3]}{\langle 46\rangle[13]}, \quad c^*_{63} = -\frac{\langle 4|5{+}6|1]}{\langle 46\rangle[13]}\,. \tag{9.44}$$

▶ **Exercise 9.4**

Use the above results for $c^*_{\bar{i}j}$ to show that the unused constraints in (9.36) give $\delta^4(P)$.

We can now substitute the solutions $c^*_{\bar{i}j}$ into the minors (9.40) and the Grassmann delta functions to obtain the residue of the integral (9.39) of the pole $1/M_4$, denoted by $\{M_4\}$. For simplicity, consider a particular component amplitude, namely the gluon amplitude with helicity assignments $(+,-,+,-,+,-)$. For this amplitude, the coefficient from the

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Grassmann delta functions is just 1. Taking into account the extra factor of $\langle 46\rangle[13]$ coming from $\hat{c}_{25} = -\tau\langle 46\rangle[13]$, we find

$$\{M_4\} = \frac{\langle 46\rangle^4[13]^4}{\langle 4|5{+}6|1]\langle 6|4{+}5|3][21][23]\langle 54\rangle\langle 56\rangle P_{456}^2}\,. \tag{9.45}$$

▶ **Exercise 9.5**

Show that $\hat{M}_1\big|_{\tau=0} = \frac{P_{456}^2}{\langle 46\rangle[13]}$. Evaluate the other minors at $\tau = 0$ and use them to derive the result (9.45) for the residue at $\tau = 0$.

We could calculate the residues associated with each of the other minors similarly. $\{M_6\}$ and $\{M_2\}$ are just cyclic permutations of $\{M_4\}$ by two sites, so we have

$$\begin{aligned}\{M_6\} &= \frac{\langle 62\rangle^4[35]^4}{\langle 6|1{+}2|3]\langle 2|6{+}1|5][43][45]\langle 16\rangle\langle 12\rangle P_{612}^2}\,,\\ \{M_2\} &= \frac{\langle 24\rangle^4[51]^4}{\langle 2|3{+}4|5]\langle 4|2{+}3|1][65][61]\langle 32\rangle\langle 34\rangle P_{234}^2}\,.\end{aligned} \tag{9.46}$$

For the residues $\{M_1\},\ \{M_3\},\ \{M_5\}$, it is convenient to choose the gauge

$$\begin{pmatrix} 1 & c_{12} & 0 & c_{14} & 0 & c_{16}\\ 0 & c_{32} & 1 & c_{34} & 0 & c_{36}\\ 0 & c_{52} & 0 & c_{54} & 1 & c_{56}\end{pmatrix}\,. \tag{9.47}$$

Then following the same steps as before we find that the $\{M_1\}$ residue for the $(+,-,+,-,+,-)$ amplitude is

$$\{M_1\} = \frac{-\langle 6|2{+}4|3]^4}{\langle 1|5{+}6|4]\langle 5|6{+}1|2][23][34]\langle 56\rangle\langle 61\rangle P_{561}^2}\,. \tag{9.48}$$

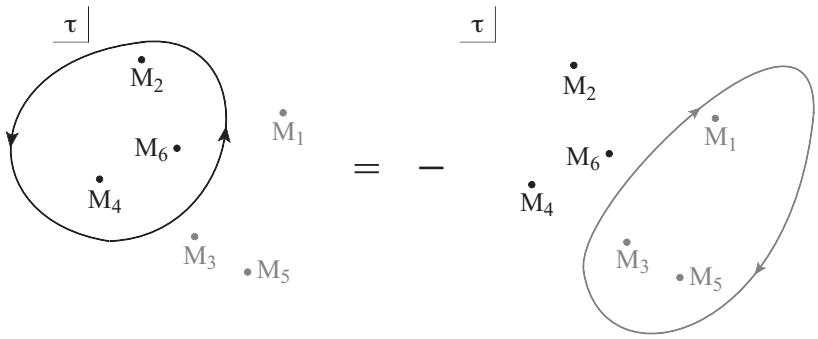
The other residues, $\{M_3\}$ and $\{M_5\}$, are obtained by relabeling the external states in (9.48).

We have now extracted six residues $\{M_i\}$ from (9.39) for a projection that corresponds to the helicity configuration $(+,-,+,-,+,-)$ of a gluon amplitude. But it is not yet clear what the residues have to do with the amplitude. Each of the $\{M_i\}$s contains spurious poles, such as $\langle 4|5{+}6|1]$ in $\{M_2\}$ and $\{M_4\}$. However, in the sum $\{M_2\} + \{M_4\}$, this spurious pole cancels. In fact, in the sum $\{M_2\} + \{M_4\} + \{M_6\}$ all three spurious poles – $\langle 4|5{+}6|1]$, $\langle 6|4{+}5|3]$, and $\langle 2|6{+}1|5]$ – cancel, so this is a local object. Your brain may even be tingling with the sensation that you have seen this combination before. Go back to look at Exercise 3.9: there we calculated the 6-point tree amplitude $A_6[1^+2^-3^+4^-5^+6^-]$ from a $[2,3\rangle$-BCFW shift and found that it was exactly

$$A_6[1^+2^-3^+4^-5^+6^-] = \{M_2\} + \{M_4\} + \{M_6\}\,. \tag{9.49}$$

In the BCFW construction, each of the three terms in (9.49) corresponds exactly to a BCFW diagram. Now we have also seen that each term can be understood as the residue of a pole

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The "tree contour" in the Grassmannian. It circles the residues of the poles $\{M_2\}$, $\{M_4\}$, $\{M_6\}$. Through contour deformation, the result is equivalent to minus the sum of $\{M_1\}$, $\{M_3\}$, $\{M_5\}$. Fig. 9.2

associated with the minor M_i in the cyclically invariant Grassmannian integral. So for $\mathcal{N} = 4$ SYM, individual BCFW terms are in one-to-one correspondence with the residues of the Grassmannian integral. Since the Grassmannian integral was constructed to produce Yangian invariants, we now understand that each super-BCFW term is a Yangian invariant.

Consider the contour that encircles the minors $\{M_2\}$, $\{M_4\}$, $\{M_6\}$. It is this contour that gives a Yangian invariant rational function that is local and free of spurious singularities. The statement of locality has become a choice of contour.

Through contour deformation, illustrated schematically in Figure 9.2, we have

$$\{M_2\} + \{M_4\} + \{M_6\} = -\{M_1\} - \{M_3\} - \{M_5\}\,. \tag{9.50}$$

This means that the tree amplitude $A_6[1^+2^-3^+4^-5^+6^-]$ can also be represented by (minus) the sum of $\{M_1\}$, $\{M_3\}$, and $\{M_5\}$. Indeed, this is the representation that one obtains from the BCFW shift $[3, 2\rangle$, the "parity conjugate" of the shift $[2, 3\rangle$ that produced the $\{M_2\}$, $\{M_4\}$, $\{M_6\}$ representation. Actually, this is a little too quick, because the shift $[3, 2\rangle$ would be an illegal $[+, -\rangle$-shift for the component-amplitude $A_6[1^+2^-3^+4^-5^+6^-]$. The correct statement is that the $\{M_1\}$, $\{M_3\}$, $\{M_5\}$ representation is the result of a $[3, 2\rangle$ BCFW *super*-shift recursion relation with a subsequent projection to the $(+, -, +, -, +, -)$ gluon helicity states.

The insight gained here is that the mysterious six-term identity (9.50) that arises from the equivalence of the two conjugate BCFW super-shifts $[2, 3\rangle$ and $[3, 2\rangle$ is simply a consequence of the residue theorem of the Grassmannian integral $\mathcal{L}_{n,k}$! Actually, the identity (9.50) is the 5-bracket six-term identity (5.63) projected to the $(+, -, +, -, +, -)$ gluon helicity states. In Chapter 10 we expose an underlying geometric interpretation of such identities.

The Grassmannian, the tree contour, and the twistor string

In Witten's *twistor string* [54], mentioned briefly in Section 5.2, the $\mathrm{N}^K\mathrm{MHV}$ superamplitudes in $\mathcal{N} = 4$ SYM are calculated as open string current algebra correlators integrated over the moduli space of degree $(K+1)$ curves in supertwistor space. It turns out that

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the so-called *RSV connected prescription* [124] for the twistor string has a direct relation to the BCFW recursion relations. Moreover, different BCFW representations are related via (higher-dimensional versions of) Cauchy's theorem [125]. Sounds similar to the properties of the Grassmannian integral, does it not? In fact, it can be shown that the tree contour in the Grassmannian precisely gives the RSV connected prescription for the twistor string.

Let us be more explicit about how this works. Recall the procedure for carrying out the Grassmann integral $\mathcal{L}_{n,k}$ in (9.21). Gauge fixing $GL(k)$ leaves $nk - k^2$ integration variables of which the bosonic delta functions fix $2k + 2(n-k) - 4 = 2n - 4$. (The -4 is momentum conservation.) So this leaves $nk - k^2 - (2n-4) = (k-2)(n-k-2)$ variables and the strategy is to fix those by localizing all of them on zeroes of the minors in the denominator of $\mathcal{L}_{n,k}$. Reversing the procedure, we ignore the delta functions to begin with and first localize on the minors: this leaves an integral with $nk - k^2 - (k-2)(n-k-2) = 2n-4$ variables. These will of course be fixed by the bosonic delta functions, but note that $2n-4$ is the dimension of the Grassmannian $\mathrm{Gr}(2,n)$. Thus, after localizing on the minors, the integral $\mathcal{L}_{n,k}$ naturally lives in $\mathrm{Gr}(2,n)$. Now the punchline is that precisely with the tree contour, reducing the $\mathrm{Gr}(k,n)$-integral to a $\mathrm{Gr}(2,n)$-integral results in the twistor string formula after a suitable Fourier transform [126]. Thus the underlying property that makes the tree contour special is that it localizes the $\mathrm{Gr}(k,n)$ Grassmannian integral to $\mathrm{Gr}(2,n)$ in a particular fashion that is intimately tied to locality and the twistor string. We will see this story repeat itself when we consider the Grassmannian formula for the 3-dimensional ABJM theory in Section 11.3. There are several papers in the literature on the relationship between amplitudes and the twistor string, see for example [54, 124–129].

The Grassmannian picture is interesting and has given us insight about locality, but there is perhaps a small stone in our shoe: we have not yet been able to see how the cancellation of spurious poles takes place within the tree superamplitude in a manifest fashion. There is a geometric story about how that happens and we draw it in Chapter 10.

9.4 From on-shell diagrams to the Grassmannian

The terms appearing in the BCFW expansion of the 6-point NMHV amplitude have now appeared in two distinct guises. We have seen in the previous section that they are given as residues of an integral over a Gr(3,6) Grassmannian manifold. In Section 8.3, they were the result of gluing on-shell cubic vertices together into on-shell diagrams. So can we make a connection between the Grassmannian and the on-shell diagrams? Yes, we can!

In Section 8.3, we did not explicitly compute any of the on-shell diagrams beyond the simplest box-diagram. The reason is simple: explicitly solving all momentum conservation constraints at each vertex is a complicated task because it is quadratic in spinor variables. In this section, we have seen that momentum conservation can be converted into a linear constraint with the aid of the Grassmannian. This means that if we convert all 3-point vertices in an on-shell diagram into Grassmannian integrals, then the momentum conservation

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constraints are just a set of linear equations. Let us see how this is done in practice. For the MHV 3-point amplitude, the Grassmannian integral is simply Gr(2,3):

$$\mathcal{A}_3^{\text{MHV}} = \int \frac{d^{2\times 3}C}{M_1 M_2 M_3}\, \delta^{2\times 2}\big(C_i[i|\big)\, \delta^{(4\times 2)}\big(C_i\eta_i\big)\, \delta^{2\times 1}\big(\tilde{C}_i\langle i|\big)\,, \tag{9.51}$$

where we have used the momentum space representation (9.21): C is a 2×3 matrix, and $\tilde{C}$ is its 1×3-dimensional orthogonal complement. For the anti-MHV 3-point amplitude, the analogue Grassmannian integral in Gr(1,3) is

$$\mathcal{A}_3^{\text{anti-MHV}} = \int \frac{d^{1\times 3}C}{M_1 M_2 M_3}\, \delta^{2\times 1}\big(C_i[i|\big)\, \delta^{(4\times 1)}\big(C_i\eta_i\big)\, \delta^{2\times 2}\big(\tilde{C}_i\langle i|\big)\,, \tag{9.52}$$

where now C is a 1×3-dimensional matrix and $\tilde{C}$ is its 2×3-dimensional orthogonal complement. To see that this indeed gives the correct 3-point amplitude, note that the first bosonic delta function requires that C is orthogonal to the 2-plane $[i|$, so this localizes the integral (up to an irrelevant overall rescaling) to

$$C = \Big(\, [23]\,,\ [31]\,,\ [12]\,\Big)\,. \tag{9.53}$$

Substituting this into (9.52), one indeed recovers the anti-MHV 3-point amplitude.

▶ Exercise 9.6

Given (9.53), determine a representation of $\tilde{C}_{\mathrm{a}i}$. Substitute the result into (9.52) to recover the 3-point amplitude. Note that all potential Jacobian factors can be fixed by dimension-counting and symmetry analysis.

In summary, the MHV and anti-MHV 3-point amplitudes can be viewed as providing 2×2 and 1×2 linear constraints respectively for the $[i|$s. For later convenience, we parameterize the Gr(2, 3) and Gr(1, 3) Grassmannians as follows:

$$\begin{matrix} b & c & a \\ 1 & 0 & \alpha_b \\ 0 & 1 & \alpha_c \end{matrix} \qquad \text{a} \leftarrow \bullet \begin{matrix} \nearrow \text{b} \\ \searrow \text{c} \end{matrix}\,, \tag{9.54}$$

$$\begin{matrix} a & b & c \\ 1 & \beta_b & \beta_c \end{matrix} \qquad \text{a} \rightarrow \circ \begin{matrix} \nearrow \text{b} \\ \searrow \text{c} \end{matrix}\,. \tag{9.55}$$

The arrows represent the particular gauge that we have chosen for each Grassmannian integral. Incoming lines on the 3-point vertex indicate that the corresponding columns in the Grassmannian are $GL(k)$-gauge-fixed to be the identity matrix. Outgoing lines correspond to unfixed columns. In this gauge, the $[i|$ parts of the bosonic delta functions

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are:

$$\text{MHV:}\quad \delta^2\big([b| + \alpha_b[a|\big)\,\delta^2\big([c| + \alpha_c[a|\big)\,, \qquad \text{anti-MHV:}\quad \delta^2\big([a| + \beta_b[b| + \beta_c[c|\big)\,. \tag{9.56}$$

For each vertex, the spinors of the incoming lines are expressed as a linear combination of those of the outgoing lines. One can perform a similar analysis for the fermionic delta functions and the bosonic delta functions of $\langle i|$. The analyses are exactly parallel, so we leave them implicit.

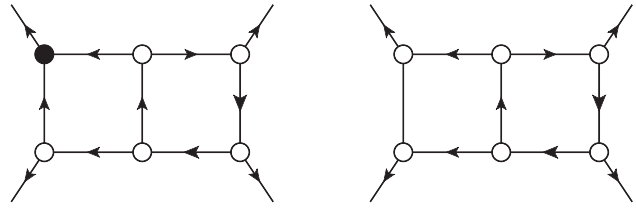
▶ **Exercise 9.7**

What does the bosonic delta function for $\langle i|$ look like? What constraints does it impose?

We are now ready to start gluing 3-point on-shell vertices together. Recall from (8.49) that each internal line in the on-shell diagram corresponds to an integral over the set of internal variables,

$$\int \frac{d^2|I\rangle\, d^2|I]\, d^4\eta_I}{U(1)}\,. \tag{9.57}$$

Since the spinors $\big(|I\rangle,\ |I]\big)$ also appear in the vertices on each end of the line, the bosonic delta functions of these vertices can be used to localize the integral (9.57). This can be made manifest in a graphical way. For each on-shell diagram, we decorate the lines with arrows following the rule that for each black vertex, there should be two incoming lines and one outgoing line, while for each white vertex there should be one incoming line and two outgoing lines, just as in (9.54), (9.55). One might wonder if it is always possible to find such decoration consistent throughout the on-shell diagram. For diagrams of *physical relevance* the answer is yes, since one can interpret the outgoing lines as $+$ helicity, incoming lines as $-$ helicity, and a consistent decoration is equivalent to consistent helicity assignments. For example, consider gluing six vertices together to form a double box diagram. We can have a consistent decoration if there is at least one different color vertex, but not if they are all the same:[2]



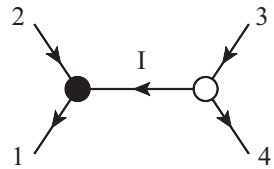
(9.58)

The lines of the second diagram cannot be consistently oriented. It follows from the previous discussion that the spinors of the internal line are completely determined by the outgoing

[2] You may find it a little peculiar that we show examples corresponding to on-shell diagrams for all-plus amplitudes; the point is just that they give simple illustrations of the consistency conditions.

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lines of one of the vertices. For example, decorations of the diagram



(9.59)

dictate the bosonic delta functions for the square spinors to be

$$\delta^2\big([2|+\alpha_2[1|\big)\,,\quad \delta^2\big([I|+\alpha_I[1|\big)\,,\quad \delta^2\big([3|+\beta_4[4|+\beta_I[I|\big)\,. \tag{9.60}$$

The second bosonic delta function localizes the $\int d^2|I]$ integral, while the remaining two delta functions become

$$\delta^2\big([2|+\alpha_2[1|\big)\,,\quad \delta^2\big([3|+\beta_4[4|-\beta_I\alpha_I[1|\big)\,. \tag{9.61}$$

The delta functions in (9.61) can be combined to the form $\prod_{\mathsf{a}=1}^{2}\delta^2(C_{\mathsf{a}i}[i|)$ with the Gr(2,4) Grassmannian given as

$$C_{\mathsf{a}i}=\begin{array}{c}\begin{matrix}1 & 2 & 3 & 4\end{matrix}\\ \begin{pmatrix}\alpha_2 & 1 & 0 & 0\\ -\beta_I\alpha_I & 0 & 1 & \beta_4\end{pmatrix}\end{array}. \tag{9.62}$$

So gluing the two 3-point vertices together now gives a new Grassmannian integral

$$\int\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_I}{\alpha_I}\frac{d\beta_4}{\beta_4}\frac{d\beta_I}{\beta_I}\frac{1}{U(1)}\;\delta^{2\times 2}\big(C_i[i|\big)\,\delta^{(4\times 2)}\big(C_i\eta_i\big)\,\delta^{2\times 2}\big(\tilde{C}_i\langle i|\big)\,, \tag{9.63}$$

where the $C_{\mathsf{a}i}$ is identified in (9.62).

Note the leftover $1/U(1)$ in (9.63). We have been treating $(|I\rangle, |I])$ as independent variables, each being fixed by the bosonic delta functions. However, there remains a gauge-fixing functional that is present to remove the little-group redundancy. This functional is represented by this $1/U(1)$ factor. We do not need its explicit form, just remember that it can be used to fix an additional degree of freedom.

The above simple example generalizes to an arbitrary decorated on-shell diagram. A diagram with n_b black vertices and n_w white vertices contains $2\times(2n_b+n_w)$ constraints on the $|i]$s. If it has n_I internal lines, then the $2\times n_I$ integrations over the internal $|I]$s can be localized by these bosonic delta functions. Finally, there will be $2\times(2n_b+n_w-n_I)$ constraints left and they can be conveniently grouped into a degree $2\times k$ delta function $\prod_{\mathsf{a}=1}^{k}\delta^2(C_{\mathsf{a}i}[i|)$ where $k=(2n_b+n_w-n_I)$ and $n=3(n_w+n_b)-2n_I$. Note that by counting the Grassmann degrees of the black (2×4) and white (1×4) blobs minus the n_I internal Grassmann integrations $(n_I\times 4)$, this k is exactly the same k as in the N^{k+2}MHV classification. Thus each on-shell diagram corresponds to the following Grassmannian integral:

$$\int\left(\prod_{i=1}^{n_b}\frac{d\alpha_{i1}}{\alpha_{i1}}\frac{d\alpha_{i2}}{\alpha_{i2}}\right)\left(\prod_{i=1}^{n_w}\frac{d\beta_{i1}}{\beta_{i1}}\frac{d\beta_{i2}}{\beta_{i2}}\right)\left(\prod_{i=1}^{n_I}\frac{1}{U(1)_i}\right)\delta^{2\times k}\big(C_i[i|\big)\,\delta^{(4\times k)}\big(C_i\eta_i\big)\,\delta^{2\times(n-k)}\big(\tilde{C}_i\langle i|\big)\,. \tag{9.64}$$

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As we have seen, this Gr(k, n) Grassmannian integral is parameterized by an on-shell diagram decorated with arrows consistently throughout the diagram. The Grassmannian integral (9.64) is in a $GL(k)$-gauge-fixed form. Each vertex contains two degrees of freedom, but the n_I internal lines each leave a $1/U(1)$ gauge-fixing function, so the dimension of this Grassmannian integral is

$$\dim(C) = 2 \times n_v - n_I \,, \tag{9.65}$$

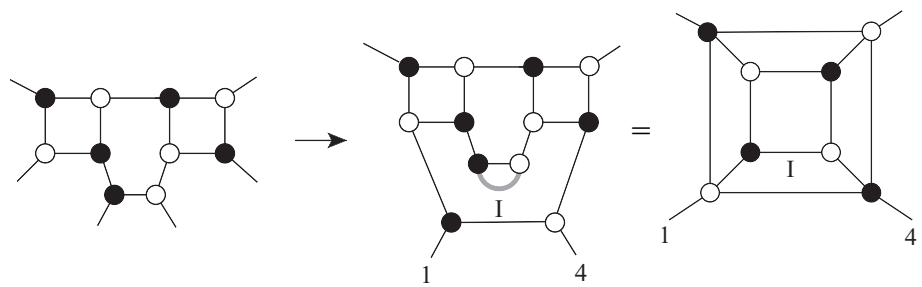
where the number of vertices is $n_v = n_b + n_w$. Using Euler's formula for a planar diagram $(n_f - n) - n_I + n_v = 1$, where n_f is the number of faces in a diagram, we find that the dimension of the Grassmannian integral corresponding to a particular on-shell diagram is

$$\dim(C) = n_f - 1 \,. \tag{9.66}$$

The total number of bosonic delta functions is $2 \times (k + n - k) - 4 = 2n - 4$. If $\dim(C) = 2n - 4$, then all the degrees of freedom in the integral are completely localized by the bosonic delta functions. This is the case for the on-shell diagrams that correspond to the BCFW terms in Section 8.3. We saw this in Section 9.3.2 for Gr(3,6): prior to solving the bosonic delta functions, each BCFW term is obtained by localizing on the zeroes of one of the minors. This is precisely the $9 - 1 = 8$-dimensional Grassmannian manifold indicated by the on-shell diagrams.

If $\dim(C) < 2n - 4$, then the bosonic delta functions over-constrain the external data and can only be satisfied in special kinematics. This is precisely the scenario for the example that led to (9.62), where $\dim(C) = 3 < 4$. From (9.62) one can readily read off what the special kinematics is: $[1| \sim [2|$.

What happens if $\dim(C) > 2n - 4$? It corresponds to a term in the BCFW representation of a loop amplitude, where the remaining integrations can be translated into the loop-momentum integration! For example, consider attaching a BCFW bridge on the forward limit of (8.55). In our calculation of the loop-recursion in Section 7.5, only the BCFW term that corresponds to the middle on-shell diagram of (8.55) has a non-vanishing forward limit contribution. If we attach a BCFW bridge to this middle diagram and perform a series of equivalence moves, we find:



(9.67)

The final diagram is precisely the 1-loop 4-point amplitude. One can count that with $n = 4$, $\dim(C) = 9 - 1 = 4 + (2n - 4)$, i.e. four integrals remain after solving all bosonic delta functions. These four extra integrals correspond to the integration over the four components of the loop-momentum. We can readily identify these extra components in the

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on-shell diagram: since the original tree-diagram contains no extra integration variable, the new degrees of freedom must arise from the procedure of taking the forward limit. This introduces a factor of $\int d^2|I\rangle\, d^2|I]\, d^4\eta_I / U(1)$ in (9.67); that is three integrations because the $U(1)$ is mod'ed out. The presence of the BCFW bridge introduces the fourth integration $\int \frac{dz}{z}$. The loop-momentum ℓ can then be written

$$\ell = |I\rangle[I| + z|1\rangle[4| \,. \tag{9.68}$$

It is quite remarkable that starting out with a fully on-shell construction of "on-shell diagrams" leads to a loop-integrand construction in which the loop-momentum is off-shell.

▶ **Exercise 9.8**

Use equivalence moves to prove the last two diagrams of (9.67) are equivalent to each other.

There is much more information in the connection between the Grassmannian and the on-shell diagrams, but we also have other fish to fry and birds to scare. If we have awoken your appetite for blob diagrams and Grassmannians and you are interested in learning more about their relations to permutations, stratifications, amalgamation, dimers, bipartite graphs, and quivers, you should take a look at reference [120] for more details.

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