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BIVARIATE SHOCK MODELS DRIVEN BY THE GEOMETRIC COUNTING PROCESS

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Abstract

We consider shock models governed by the bivariate geometric counting process. By assuming the competing risks framework, failures are due to one of two mutually exclusive causes (shocks). We obtain and study some relevant functions, such as failure densities, survival functions, probability of the cause of failure, and moments of the failure time conditioned on a specific cause. Such functions are specified by assuming that systems or living organisms fail at the first instant in which a random threshold is reached by the sum of received shocks. Under this failure scheme, various cases arising for suitable choices of the random threshold are provided too.

Keywords: Shock model; geometric counting process; competing risks; failure distribution; random lifetime; hazard rate

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Secondary 62N05; 60G09

1. Introduction

Shock models are adequate to describe the failure mechanism of systems, living organisms, or general items that are subject to stress or shocks during their working period. In this context, a device is initially active and it receives random shocks according to a counting process. In addition, the number of shocks that cause the failure of the device is itself random. Many contributions regarding shock models have been developed in different areas of the literature. We refer the reader to [7, 22, 24, 32]. Moreover, aiming to deal with more general settings, multivariate shock models have been studied in various fields as well (see [5, 6, 21, 29, 33, 34, 37]).

Often shock models follow the Poisson process as the underlying counting process, since it represents a good compromise between a realistic representation of the phenomena and mathematical tractability of the model (see, for instance, [12, 19, 23]). However, in many practical situations, specific random occurrences of events in time cannot be adequately described by the Poisson process, since the assumption of independence between its increments is not feasible. Therefore, alternative counting processes to the Poisson one have been considered in view of managing shock models (see [9, 10, 20, 25]).

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The geometric counting process deals with random occurrences between events that are not independent, and it represents another suitable alternative to the Poisson process (see [8, 17]).

In this article, we study bivariate shock models in which shocks are governed by two homogeneous geometric counting processes, with possibly different intensities. Specifically, we consider the bivariate geometric counting process, which is a two-dimensional mixed Poisson process with mixing distribution given by the maximum of two exponential distributions. We face the problem when the intensities of baseline Poisson processes are shared, leading to the dependence between the marginal geometric counting processes.

The competing risks framework is also considered, by taking into account that failures are due to a single cause (risk). This model is of considerable interest in various fields (see [4, 11] as classical references). In reliability theory and survival analysis, Balakrishnan and Castilla [3] developed robust inference for one-shot device test data with Weibull lifetimes under the competing risks scenario. Di Crescenzo and Longobardi [13] proposed a version of the well-known 'new better than used' and 'new worse than used' ageing properties for random lifetimes within a specific cause of failure. Moreover, Di Crescenzo and Meoli [15] studied relations with copula and time-transformed exponential models. Competing risks are a common issue also in medical research. For instance, in cancer studies, both treatment-related death and disease recurrence are important outcomes. A classic example comes from studies on tumor-causing agents. Altshuler [2] proposed a method using Poisson processes to model these competing events, estimating biological activity over time. Kim [27] introduced statistical methods to analyze competing risks, by providing comparisons with their respective counterparts in standard survival analysis.

Since competing risks are noteworthy for the description of the failure of a complex system subject to different types of risk, we refer to observable pairs (T, δ), where T is a nonnegative random variable, namely, *random lifetime*, that describes the time of failure of an item, while δ describes the (exclusive) cause or type of failure. Di Crescenzo and Longobardi [14] considered a formulation of the competing risks model within the framework of stochastic shock models governed by a bivariate Poisson process. Recently, Di Crescenzo and Meoli [16] and Soni *et al.* [36] investigated different generalized versions of bivariate Poisson processes.

The article is organized as follows. Section 2 is devoted to the description of the two-dimensional shock model within the competing risks framework, by our taking into account that failures are due to a single cause. The study is carried out in a general configuration, i.e. underlying counting processes and the failure pattern are not fixed. We also provide sufficient conditions for the independence between the random failure time T and the cause of failure δ. In Section 3 we assume that the shocks are driven by two homogeneous geometric counting processes $N_1(t)$ and $N_2(t)$ by providing the explicit expression of some relevant functions. In Section 4 failures are assumed to occur at the first instant in which $N_1(t) + N_2(t) = M$, where M is a random counting number, independent of $(N_1(t), N_2(t))$. Special choices of the random threshold are considered too. Some final remarks are then given in Section 5.

Throughout this article the terms *increasing* and *decreasing* are used as meaning 'non-decreasing' and 'nonincreasing', respectively. Moreover, $\mathbb{R}^+$ denotes the set of positive real numbers, $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$, $\mathbb{N}$ denotes the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2. Bivariate Shock Models within Competing Risks

In this section we describe and study the bivariate shock model driven by (generic) counting processes within the competing risks framework. In particular, we provide sufficient conditions for which the failure time and the cause of failure are independent.

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Let T be a random lifetime with an absolutely continuous distribution that describes the failure time of an item (system or a living organism) having survival function (SF) defined as $\overline{F}_T(t) = \mathbb{P}(T > t)$ for $t \geq 0$. We assume that the item is initially active, i.e. $\overline{F}_T(0) = 1$. Moreover, it fails in a certain instant, so that $\lim_{t\to+\infty} \overline{F}_T(t) = 0$. Let $\delta = i$ if the failure occurs due to a shock of type i for $i = 1, 2$. We denote by $N_i(t)$, for $i = 1, 2$, the counting process representing the number of shocks of type i occurring in $[0, t]$.

We can partition the state space $\mathbb{N}_0^2$ of $(N_1(t), N_2(t))$ into nonempty subsets

$$S_k := \{(x_1, x_2) \in \mathbb{N}_0^2 : \Psi(x_1, x_2) = k\}, \qquad k \in \mathbb{N}_0, \tag{1}$$

called *failure sets*, where $\Psi : \mathbb{N}_0^2 \longrightarrow \mathbb{N}$ is a given suitable function that creates a partition of the state space. We assume that S_0 includes the numbers of nonfatal shocks; namely, if $(N_1(t), N_2(t)) \in S_0$ then the shocks that arrived up to time t do not cause a failure.

We denote by M the integer-valued random variable that represents the index of which failure set S_k, $k \in \mathbb{N}$, contains the numbers of fatal shocks, independent of $(N_1(t), N_2(t))$. In other words, $M = k$ if and only if the system fails at the first instant $t > 0$ in which $(N_1(t), N_2(t)) \in S_k$. For $k \in \mathbb{N}$ the *risky sets* are defined as

$$\begin{aligned} R_k^{(1)} &:= \{(x_1, x_2) \in \mathbb{N}_0^2 \setminus S_k : (x_1 + 1, x_2) \in S_k\}, \\ R_k^{(2)} &:= \{(x_1, x_2) \in \mathbb{N}_0^2 \setminus S_k : (x_1, x_2 + 1) \in S_k\}, \end{aligned} \tag{2}$$

where $R_k^{(i)}$, $i = 1, 2$, contains all states of $\mathbb{N}_0^2$ leading to a failure at time t at the next occurrence of a shock of type i, given that $M = k$. Note that, from (2), the failures are due to a single cause. In addition, we remark that in general $R_k^{(i)}$ is different from S_{k-1}. Under these assumptions, the failure time can be denoted as

$$T = \inf\{t \geq 0 : \Psi\,(N_1(t), N_2(t)) = M\}.$$

The probability density function (PDF) of T is expressed as

$$f_T(t) = f_1(t) + f_2(t), \qquad t \geq 0,$$

where f_i, for $i = 1, 2$, is the *failure density* (FD), also named *subdensity function*, defined by

$$f_i(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}\{T \leq t,\ \delta = i\}, \qquad t \geq 0. \tag{3}$$

Note that f_i, for $i = 1, 2$, is not a proper PDF. Indeed, since $\delta = i$, when the failure occurs due to a shock of type i, we have

$$\mathbb{P}(\delta = i) = \int_0^{+\infty} f_i(t)\,\mathrm{d}t, \qquad i = 1, 2. \tag{4}$$

From (3) and (4), we can obtain the moments of the failure time conditioned on the cause of failure for $s \geq 1$ as

$$\mathbb{E}(T^s \mid \delta = i) = \frac{1}{\mathbb{P}(\delta = i)} \int_0^{+\infty} t^s f_i(t)\,\mathrm{d}t, \qquad i = 1, 2. \tag{5}$$

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To express the FDs f_i for $i=1,2$ in terms of the joint probability distribution of $(N_1(t), N_2(t))$, we introduce the following hazard rates:

$$\begin{aligned} r_1(x_1,x_2;t) &= \lim_{\tau\to 0^+} \frac{1}{\tau}\mathbb{P}[N_1(t+\tau)=x_1+1, N_2(t+\tau)=x_2 | N_1(t)=x_1, N_2(t)=x_2], \\ r_2(x_1,x_2;t) &= \lim_{\tau\to 0^+} \frac{1}{\tau}\mathbb{P}[N_1(t+\tau)=x_1, N_2(t+\tau)=x_2+1 | N_1(t)=x_1, N_2(t)=x_2], \end{aligned} \tag{6}$$

with $(x_1,x_2)\in\mathbb{N}_0^2$ and $t\geq 0$. Hence, given that x_1 shocks of type 1 and x_2 shocks of type 2 occurred in $[0,t]$, $r_i(x_1,x_2;t)$ gives the intensity of the occurrence of a shock of type i immediately after t, with $i=1,2$. Note that $r_1(x_1,x_2;t)+r_2(x_1,x_2;t)$ is the hazard rate of a shock of any type.

From the assumptions stated above, the item fails at time t due to a shock of type i if the two-dimensional counting process $(N_1(t), N_2(t))$ takes values in $R_k^{(i)}$ at time t and a shock of type i occurs immediately after. Hence, conditioning on M and recalling (2) and (6), for $i=1,2$, we have

$$f_i(t) = \sum_{k=1}^{+\infty} p_M(k) \sum_{(x_1,x_2)\in R_k^{(i)}} \mathbb{P}[N_1(t)=x_1, N_2(t)=x_2]\, r_i(x_1,x_2;t), \qquad t\geq 0, \tag{7}$$

where $p_M(k)=\mathbb{P}(M=k)$, for $k\in\mathbb{N}$, denotes the probability distribution of M. Therefore, by using (7), we can express (4) and (5) in terms of the joint probability distribution of $(N_1(t), N_2(t))$.

We recall that a multivariate counting process $\{\mathbf{N}(t), t\in\mathbb{R}_0^+\}$ has the multinomial property if the identity

$$\begin{aligned} &\mathbb{P}\left[\bigcap_{j=1}^{m}\{\mathbf{N}(t_j)-\mathbf{N}(t_{j-1})=\mathbf{n}_j\}\right] \\ &= \left(\prod_{i=1}^{\ell}\frac{\left(\sum_{j=1}^m n_j^{(i)}\right)!}{\prod_{j=1}^m n_j^{(i)}!}\prod_{j=1}^{m}\left(\frac{t_j-t_{j-1}}{t_m}\right)^{n_j^{(i)}}\right)\mathbb{P}\left[\left\{\mathbf{N}(t_m)=\sum_{j=1}^{m}\mathbf{n}_j\right\}\right] \end{aligned} \tag{8}$$

holds for all $m\in\mathbb{N}$ and $t_0,t_1,\ldots,t_m\in\mathbb{R}_0^+$ with $0=t_0<t_1<\cdots<t_m$, and for all $\mathbf{n}_j\in\mathbb{N}_0^\ell$, $j\in\{1,\ldots,m\}$; see Section 2.2 in [38]. Hereafter, we characterize the expression for the FDs given in (7) by assuming that the corresponding bivariate counting process has the multinomial property.

Proposition 1. *If* $\mathbf{N}(t)=(N_1(t),N_2(t))$ *has the multinomial property, then*

$$f_i(t) = \sum_{k=1}^{+\infty} p_M(k) \sum_{(x_1,x_2)\in R_k^{(i)}} (x_i+1)t^{x_1+x_2} \lim_{\tau\to 0^+} \frac{\mathbb{P}[N_i(t+\tau)=x_i+1, N_j(t+\tau)=x_j]}{(t+\tau)^{x_1+x_2+1}} \tag{9}$$

for $i=1,2$, $j=3-i$, *and* $t\geq 0$.

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Proof. We prove the result for $i = 1$, since in a similar way it follows for $i = 2$. From (6), we have

$$
\begin{aligned}
&r_1(x_1, x_2; t) \\
&= \lim_{\tau \to 0^+} \frac{1}{\tau} \frac{\mathbb{P}\left[N_1(t+\tau) = x_1 + 1, N_2(t+\tau) = x_2, N_1(t) = x_1, N_2(t) = x_2\right]}{\mathbb{P}\left[(N_1(t), N_2(t)) = (x_1, x_2)\right]} \\
&= \lim_{\tau \to 0^+} \frac{1}{\tau} \frac{\mathbb{P}\left[\{(N_1(t), N_2(t)) = (x_1, x_2)\} \cap \{(N_1(t+\tau) - N_1(t), N_2(t+\tau) - N_2(t)) = (1, 0)\}\right]}{\mathbb{P}\left[(N_1(t), N_2(t)) = (x_1, x_2)\right]} \\
&= (x_1 + 1) t^{x_1 + x_2} \lim_{\tau \to 0^+} \frac{\mathbb{P}\left[(N_1(t+\tau), N_2(t+\tau)) = (x_1 + 1, x_2)\right]}{(t+\tau)^{x_1 + x_2 + 1}\, \mathbb{P}\left[(N_1(t), N_2(t)) = (x_1, x_2)\right]},
\end{aligned}
$$

having used in the last equality the multinomial property given in (8) for $\ell = m = 2$. Hence, the result follows by applying the latter expression in (7). □

Since $\{T > t\} = \{\Psi\,(N_1(t), N_2(t)) < M\}$ for all $t \geq 0$, by conditioning on $(N_1(t), N_2(t))$, from (1) we obtain

$$
\overline{F}_T(t) = \sum_{k=0}^{+\infty} \overline{F}_M(k) \sum_{(x_1, x_2) \in S_k} \mathbb{P}[N_1(t) = x_1, N_2(t) = x_2], \qquad t \geq 0, \tag{10}
$$

where $\overline{F}_M(k)$, for $k \in \mathbb{N}_0$, is the SF of M such that $\overline{F}_M(0) = 1$.

Remark 1. Let us assume that two random thresholds M_1 and M_2 are ordered in the usual stochastic sense; that is, $\overline{F}_{M_1}(k) \leq \overline{F}_{M_2}(k)$ for all $k \in \mathbb{N}_0$ (denoted as $M_1 \leq_{\mathrm{st}} M_2$). Therefore, from (10), if the corresponding random failure times T_1 and T_2 are conditioned on the same pair of counting processes, according to the same failure scheme, then $T_1 \leq_{\mathrm{st}} T_2$. For more details about stochastic orders, we refer the reader to [35]. Moreover, see [31] for stochastic comparisons between random lifetimes in the context of shock models driven by univariate counting processes.

Hereafter we provide sufficient conditions for which the random failure time T and the cause of failure δ are independent. We refer to [26] for a characterization of the independence between time and cause of failure for a general item.

Theorem 1. *In* (1), *let us assume* $\Psi(x_1, x_2) = \Psi(x_2, x_1)$ *for all* $(x_1, x_2) \in \mathbb{N}_0^2$. *If*

$$
(N_1(t), N_2(t)) =_{\mathrm{st}} (N_2(t), N_1(t)) \quad \text{for all } t \geq 0, \tag{11}
$$

then T *and* δ *are independent, where* $=_{\mathrm{st}}$ *means equality in law.*

Proof. Under the stated hypothesis on Ψ, from (2) it is easy to see that $(x_1, x_2) \in R_k^{(1)}$ if and only if $(x_2, x_1) \in R_k^{(2)}$, for any $(x_1, x_2) \in \mathbb{N}_0^2$. Therefore, from (7), by using (11), it follows that $f_1(t) = f_2(t)$ for all $t \geq 0$. Hence, from (4) we have $\mathbb{P}(\delta = 1) = \mathbb{P}(\delta = 2) = 1/2$, which allows us to obtain for $i = 1, 2$

$$
\mathbb{P}(\delta = i) f_T(t) = f_i(t), \qquad t \geq 0,
$$

and the proof is completed. □

From this point on we assume that shocks are driven by homogeneous geometric counting processes.

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3. Results under the Geometric Counting Process

In this section we assume that the two kinds of shock affect the system according to homogeneous geometric counting processes $N_1(t)$ and $N_2(t)$, respectively. The definition of the geometric counting process can be stated as follows.

Definition 1. Let $\{N^{(\alpha)}(t), t \in \mathbb{R}_0^+\}$ be a Poisson process with intensity α. Let U be an exponential distribution with mean $\lambda \in \mathbb{R}^+$. The process $\{N(t), t \in \mathbb{R}_0^+\}$ having marginal distributions for $x \in \mathbb{N}_0$ given by

$$\mathbb{P}[N(t) = x] = \int_0^{+\infty} \mathbb{P}\big[N^{(\alpha)}(t) = x\big] \mathrm{d}U(\alpha), \qquad t \in \mathbb{R}_0^+,$$

is named geometric counting process with intensity λ.

We now extend the geometric counting process to the bivariate case.

Definition 2. Let $\{N_1^{(\alpha)}(t), t \in \mathbb{R}_0^+\}$ and $\{N_2^{(\alpha)}(t), t \in \mathbb{R}_0^+\}$ be independent Poisson processes, both having intensity $\alpha \in \mathbb{R}^+$. Let U be the distribution of $\max\{T_1, T_2\}$, where T_i has an exponential distribution with mean $\frac{1}{\lambda_i} \in \mathbb{R}^+$ for $i = 1, 2$. The bivariate process $\boldsymbol{N}(t) = (N_1(t), N_2(t))$, for $t \in \mathbb{R}_0^+$, having joint distribution for $x_1, x_2 \in \mathbb{N}_0$ given by

$$\mathbb{P}[N_1(t) = x_1, N_2(t) = x_2] = \int_{\mathbb{R}^+} \mathbb{P}\big[N_1^{(\alpha)}(t) = x_1, N_2^{(\alpha)}(t) = x_2\big] \mathrm{d}U(\alpha), \qquad t \in \mathbb{R}_0^+, \tag{12}$$

is named bivariate geometric counting process, where the intensity of N_i is $1/\lambda_i$, for $i = 1, 2$.

Remark 2. By taking into account Lemma 3.1.1 in [38], the bivariate geometric counting process in (12) has the multinomial property given in (8). Then it satisfies the assumptions of Proposition 1.

For the next result we recall that the gamma function is defined as

$$\Gamma(z) := \int_0^{+\infty} t^{z-1} \mathrm{e}^{-t} \mathrm{d}t, \tag{13}$$

with $Re(z) > 0$ (see, for instance, the equation 6.1.1 in [1]).

Proposition 2. *Under the assumptions of Definition 2, if T_1 and T_2 are independent, then*

$$\begin{aligned} \mathbb{P}[N_1(t) = x_1, N_2(t) = x_2] &= \binom{x_1+x_2}{x_1} t^{x_1+x_2} \\ &\times \left[\frac{\lambda_1}{(2t+\lambda_1)^{x_1+x_2+1}} + \frac{\lambda_2}{(2t+\lambda_2)^{x_1+x_2+1}} - \frac{\lambda_1+\lambda_2}{(2t+\lambda_1+\lambda_2)^{x_1+x_2+1}} \right] \end{aligned} \tag{14}$$

for $x_1, x_2 \in \mathbb{N}_0$.

Proof. Under the stated hypothesis, from (12) it holds that

$$\begin{aligned} &\mathbb{P}[N_1(t) = x_1, N_2(t) = x_2] \\ &= \int_{\mathbb{R}^+} \mathrm{e}^{-2\alpha t} \frac{(\alpha t)^{x_1+x_2}}{x_1! x_2!} \left[\lambda_1 \mathrm{e}^{-\lambda_1 \alpha} (1 - e^{-\lambda_2 \alpha}) + \lambda_2 \mathrm{e}^{-\lambda_2 \alpha} (1 - \mathrm{e}^{-\lambda_1 \alpha}) \right] \mathrm{d}\alpha \\ &= \frac{t^{x_1+x_2} \left[h(x_1+x_2; \lambda_1) + h(x_1+x_2; \lambda_2) - h(x_1+x_2; \lambda_1+\lambda_2) \right]}{x_1! x_2!}, \end{aligned}$$

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where

$$h(x;\lambda) := \int_0^{+\infty} \lambda \alpha^x \mathrm{e}^{-\alpha(2t+\lambda)} \mathrm{d}\alpha = \frac{\lambda \Gamma(x+1)}{(2t+\lambda)^{x+1}},$$

having set $u = \alpha(2t+\lambda)$ in the last equality and by recalling (13). Then, since $\Gamma(x+1) = x!$, for $x \in \mathbb{N}_0$, the thesis follows by straightforward calculations. □

Note that, under the assumptions of Proposition 2, (11) holds. Moreover, (14) can be rewritten as

$$\mathbb{P}\left[N_1(t) = x_1, N_2(t) = x_2\right] = \binom{x_1+x_2}{x_1} \frac{1}{2^{x_1+x_2}} \mathbb{P}\left[\tilde{N}^{(2\max\{T_1,T_2\})}(t) = x_1 + x_2\right],$$

where T_1 and T_2 are independent exponential random variables with means $1/\lambda_1$ and $1/\lambda_2$, respectively, and $\tilde{N}^{(\alpha)}(t)$ denotes a Poisson process with intensity $\alpha \in \mathbb{R}^+$ and that is independent of T_1 and T_2.

As a consequence of Remark 2 and Proposition 2, from (9) it follows that

$$\begin{aligned} f_i(t) = \sum_{k=1}^{+\infty} p_M(k) \sum_{(x_1,x_2)\in R_k^{(i)}} (x_1+x_2+1) \binom{x_1+x_2}{x_1} t^{x_1+x_2} \\ \times \left[\frac{\lambda_1}{(2t+\lambda_1)^{x_1+x_2+2}} + \frac{\lambda_2}{(2t+\lambda_2)^{x_1+x_2+2}} - \frac{\lambda_1+\lambda_2}{(2t+\lambda_1+\lambda_2)^{x_1+x_2+2}} \right] \end{aligned} \tag{15}$$

for $i = 1, 2$ and $t \geq 0$. Moreover, from (10) and (14), the SF of T becomes

$$\begin{aligned} \overline{F}_T(t) = \sum_{k=0}^{+\infty} \overline{F}_M(k) \sum_{(x_1,x_2)\in S_k} \binom{x_1+x_2}{x_1} t^{x_1+x_2} \\ \times \left[\frac{\lambda_1}{(2t+\lambda_1)^{x_1+x_2+1}} + \frac{\lambda_2}{(2t+\lambda_2)^{x_1+x_2+1}} - \frac{\lambda_1+\lambda_2}{(2t+\lambda_1+\lambda_2)^{x_1+x_2+1}} \right] \end{aligned} \tag{16}$$

for $t \geq 0$.

We now specify the expression for the probability of the cause of failure given in (4) under the hypothesis that shocks are governed by the bivariate geometric counting process having probability law given in (14). For this aim, we recall that the well-known beta function can be expressed as

$$B(z,w) = \int_0^{+\infty} \frac{t^{z-1}}{(t+1)^{z+w}} \mathrm{d}t, \tag{17}$$

which converges if $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(w) > 0$ (see, for instance, the equation 6.2.1 in [1]).

Proposition 3. *Under the above assumptions, for $i = 1, 2$ it follows that*

$$\mathbb{P}(\delta = i) = \sum_{k=1}^{+\infty} p_M(k) \sum_{(x_1,x_2)\in R_k^{(i)}} \binom{x_1+x_2}{x_1} 2^{-x_1-x_2-1}. \tag{18}$$

Proof. Without loss of generality we prove the result for $i = 1$. From (4) and (15) we have

$$\mathbb{P}(\delta = 1) = \sum_{k=1}^{+\infty} p_M(k) \sum_{(x_1,x_2)\in R_k^{(1)}} (x_1+x_2+1) \binom{x_1+x_2}{x_1} \varphi(x_1+x_2),$$

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where

$$\varphi(x) := \int_0^{+\infty} \frac{\lambda t^x}{(2t+\lambda)^{x+2}} \mathrm{d}t = 2^{-x-1} B(x+1,\, 1),$$

having set $y = 2t/\lambda$ in the integral and by recalling (17). Then, since $B(x,\, 1) = 1/x$, (18) follows by straightforward calculations. □

Note that, from (18), the probability of the cause of failure does not depend on the respective intensities λ_1 and λ_2 of the geometric counting processes.

Hereafter we prove that, in the proposed competing risks framework, the conditioned moments of the failure time T given in (5) diverge for all $s \geq 1$.

Proposition 4. *Under the above assumptions, for $i = 1,\, 2$ it follows that*

$$\mathbb{E}(T^s | \delta = i) = +\infty, \qquad s \geq 1.$$

Proof. For $i = 1$, from (5) and (15) we have

$$\begin{aligned}
&\mathbb{P}(\delta = 1)\mathbb{E}(T^s | \delta = 1) \\
&= \sum_{k=1}^{+\infty} p_M(k) \sum_{(x_1,x_2)\in R_k^{(1)}} (x_1 + x_2 + 1) \binom{x_1+x_2}{x_1} \\
&\quad \times \left[g(x_1 + x_2; s; \lambda_1) + g(x_1 + x_2; s; \lambda_2) - g(x_1 + x_2; s; \lambda_1 + \lambda_2) \right] \\
&= \sum_{k=1}^{+\infty} p_M(k) \sum_{(x_1,x_2)\in R_k^{(1)}} (x_1 + x_2 + 1) \binom{x_1+x_2}{x_1} \\
&\quad \times \frac{[\lambda_1^s + \lambda_2^s - (\lambda_1 + \lambda_2)^s] B(x_1 + x_2 + s + 1,\, 1 - s)}{2^{x_1+x_2+s+1}},
\end{aligned}$$

where

$$g(x; s; \lambda) := \int_0^{+\infty} \frac{\lambda t^{x+s}}{(2t+\lambda)^{x+2}} \mathrm{d}t = \frac{\lambda^s B(x + s + 1,\, 1 - s)}{2^{x+s+1}},$$

having set $y = 2t/\lambda$ in the integral and by recalling (17). We can do the same similarly for $i = 2$. Hence, from (5), the thesis follows since $B(x_1 + x_2 + s + 1,\, 1 - s) = +\infty$ for all $s \geq 1$. □

Clearly, from the above results, the nature of the shock model is completely specified when the structures of the failure sets and of the risky sets are fixed. Hence, in the rest of this article, we consider a special case for the failure scheme. Specifically, by recalling (1), we assume that $\Psi(x_1, x_2) = x_1 + x_2$.

4. Failures due to the Sum of Shocks

In this section we assume that the system fails when the sum of shocks of type 1 and of type 2 reaches a random threshold that takes values in $\mathbb{N}$, keeping in mind that shocks are driven by the bivariate geometric counting process having probability law given in (14). Hence, from (1) we have

$$S_k = \{(x_1, x_2) \in \mathbb{N}_0^2 : x_1 + x_2 = k\}, \qquad k \in \mathbb{N}_0, \tag{19}$$

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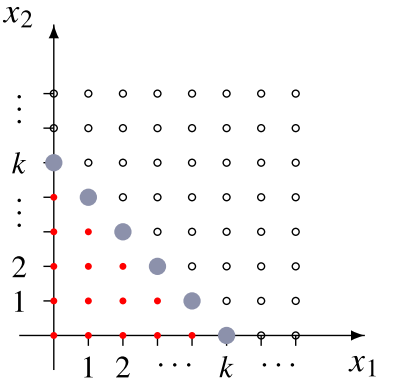


FIGURE 1. Failure scheme due to the sum of shocks.

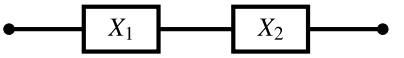


FIGURE 2. Example of a series system made of two units.

and

$$R_k^{(1)} = R_k^{(2)} = S_{k-1}, \qquad k \in \mathbb{N}; \tag{20}$$

see Figure 1. Note that $S_0 = \{(0, 0)\}$, which corresponds to no shocks of any kind having occurred during the system's working period.

Remark 3. From (14) and (19), the assumptions of Theorem 1 are satisfied, and thus T and δ are independent. It also holds that $f_1 = f_2$ and $\mathbb{P}(\delta = 1) = \mathbb{P}(\delta = 2) = 1/2$.

In reliability theory, such a model may be appropriate to describe, for instance, the failure of systems made of units connected in series, as described below.

Remark 4. Let X_1 and X_2 describe the respective random failure time of two components in a series system having random lifetime denoted by $T = \min\{X_1, X_2\}$. We assume that the component with lifetime X_i is subject to a shock of type i for $i = 1, 2$. If δ denotes the related cause of failure, then

$$\mathbb{P}(\delta = i) = \mathbb{P}(X_i = T), \qquad i = 1, 2.$$

An example of this kind of system is given in Figure 2. For more details about system reliability theory, we refer the reader to [30].

Hereafter, from (15), we specify the expression for the PDF of T according to the failure scheme of the sum.

Proposition 5. *Under the assumptions leading to* (19) *and* (20), *it follows that*

$$f_T(t) = \sum_{k=1}^{+\infty} 2k(2t)^{k-1} p_M(k) \left[\frac{\lambda_1}{(2t+\lambda_1)^{k+1}} + \frac{\lambda_2}{(2t+\lambda_2)^{k+1}} - \frac{\lambda_1+\lambda_2}{(2t+\lambda_1+\lambda_2)^{k+1}} \right] \tag{21}$$

for $t \geq 0$.

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Proof. By using (20) in (15), it holds that

$$f_1(t) = \sum_{k=1}^{+\infty} k p_M(k) t^{k-1} \left[\frac{\lambda_1}{(2t+\lambda_1)^{k+1}} + \frac{\lambda_2}{(2t+\lambda_2)^{k+1}} - \frac{\lambda_1+\lambda_2}{(2t+\lambda_1+\lambda_2)^{k+1}} \right] \left[\sum_{x_1=0}^{k-1} \binom{k-1}{x_1} \right].$$

Then the thesis follows from

$$\sum_{x_1=0}^{k-1} \binom{k-1}{x_1} = 2^{k-1}$$

and by taking into account that, from Remark 3, it holds that $f_T = 2f_1$. □

A similar result can be obtained for the SF of T given in (16), the proof of which is omitted for brevity.

Proposition 6. *Under the assumptions leading to* (19)*, it holds that*

$$\overline{F}_T(t) = \sum_{k=0}^{+\infty} \overline{F}_M(k) (2t)^k \left[\frac{\lambda_1}{(2t+\lambda_1)^{k+1}} + \frac{\lambda_2}{(2t+\lambda_2)^{k+1}} - \frac{\lambda_1+\lambda_2}{(2t+\lambda_1+\lambda_2)^{k+1}} \right] \tag{22}$$

for $t \geq 0$.

In the following we provide some examples of applications of Propositions 5 and 6 according to different choices for the random threshold M. We also consider the hazard (or failure) rate function of a random lifetime T, which is defined by

$$h_T(t) := \lim_{\varepsilon \to 0^+} \frac{\mathbb{P}(t < T < t+\varepsilon \mid T > t)}{\varepsilon} = \frac{f_T(t)}{\overline{F}_T(t)} \tag{23}$$

for all t such that $\overline{F}_T(t) > 0$. Such an ageing function has an interpretation different from that of the hazard rates introduced in (6), since it represents the average probability of failure in the interval $[t, t+\varepsilon]$ when $\varepsilon \to 0^+$ for an item that is working at time t (see Section 2.2 in [30]).

Example 1. Let M be a geometric distributed random variable with parameter $p \in (0, 1)$, i.e. $p_M(k) = p(1-p)^{k-1}$ for $k \in \mathbb{N}$. From (21), by applying the arithmetic–geometric series

$$\sum_{k=0}^{+\infty} (a+kr) y^k = \frac{a}{1-y} + \frac{ry}{(1-y)^2}$$

for $|y| < 1$, with a few calculations it follows that

$$f_T(t) = 2p \left[\frac{\lambda_1}{(\lambda_1+2pt)^2} + \frac{\lambda_2}{(\lambda_2+2pt)^2} - \frac{\lambda_1+\lambda_2}{(\lambda_1+\lambda_2+2pt)^2} \right], \qquad t \geq 0. \tag{24}$$

Similarly, from (22), we have

$$\overline{F}_T(t) = \frac{\lambda_1}{\lambda_1+2pt} + \frac{\lambda_2}{\lambda_2+2pt} - \frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+2pt}, \qquad t \geq 0, \tag{25}$$

which is a generalized mixture between three Pareto type-II (Lomax) distributions, having the same unitary shape parameter and scale parameters $2p/\lambda_1$, $2p/\lambda_2$, and $2p/(\lambda_1+\lambda_2)$, respectively. We recall that the generalized mixtures are defined as the classic ones but they might contain negative coefficients, as in (25). Moreover, from (23), it holds that

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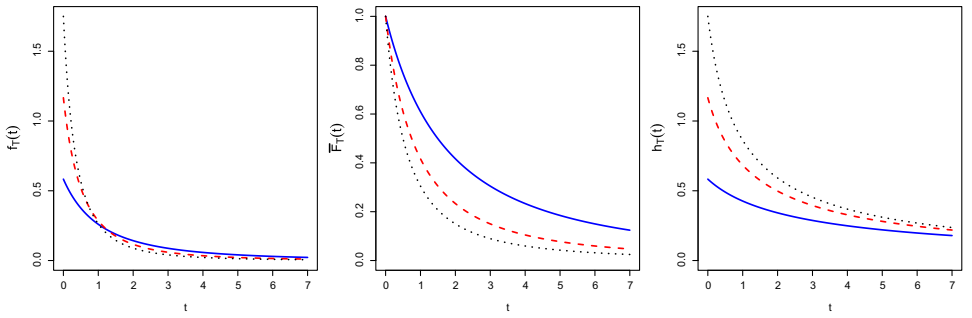


FIGURE 3. Plots of f_T (left) given in (24), $\overline{F}_T$ (middle) given in (25), and $h_T(t)$ (right) given in (26) when $\lambda_1 = 1$ and $\lambda_2 = 2$ for $p = 0.25$, $p = 0.5$, and $p = 0.75$ (full, dashed, and dotted lines, respectively).

$$h_T(t) = \frac{2p}{2pt+\lambda_1} + \frac{2p}{2pt+\lambda_2} + \frac{2p}{2pt+\lambda_1+\lambda_2} - \frac{4p}{4pt+\lambda_1+\lambda_2}, \qquad t \geq 0. \tag{26}$$

In Figure 3 we plot the PDF f_T given in (24) on the left-hand side, the SF $\overline{F}_T$ given in (25) in the middle, and the hazard rate function $h_T(t)$ given in (26) on the right-hand side for suitable choices of the parameters $\lambda_1, \lambda_2 > 0$ and $p \in (0, 1)$.

Note that, from Example 1, the random thresholds considered in Figure 3 are ordered in the usual stochastic sense. Then, according to Remark 1, the corresponding random lifetimes are ordered in the usual stochastic sense too. Moreover, the hazard rate functions depicted on the right-hand side in Figure 3 are decreasing in $t \geq 0$. Then the corresponding random lifetimes possess the property of decreasing failure rate (DFR); see, for instance, Section 4 in [30]. In addition, we recall that the reversed hazard (or failure) rate function of a random lifetime T is defined by

$$\tilde{h}_T(t) := \lim_{\varepsilon \to 0^+} \frac{\mathbb{P}(t-\varepsilon < T \leq t \mid T \leq t)}{\varepsilon} = \frac{f_T(t)}{1-\overline{F}_T(t)} \tag{27}$$

for all t such that $\overline{F}_T(t) < 1$, representing the instantaneous probability of failure at t for an item that has failed in the interval $[0, t]$ (see Section 2.2 in [30]). Hence, since the PDFs depicted on the left-hand side in Figure 3 are decreasing in $t \geq 0$, the corresponding random lifetimes have the property of decreasing reversed failure rate (DRFR); see, for instance, Section 4 in [30].

Note that, from (23) and (27), if T is DFR, then it is also DRFR. In general, the reverse implication does not hold. Indeed, in the next example, we consider a suitable choice for the random threshold M such that the corresponding random lifetime T is DRFR but not DFR. In this case the hazard rate function turns out to be increasing up to an instant close to 0, representing a high probability of an item's failure when it starts working.

Example 2. Let M be uniformly distributed with $p_M(k) = 1/3$ for $k = 1, 2, 3$. Then, from (21), it follows that

$$\begin{aligned} f_T(t) = {} & 8t^2 \left(\frac{\lambda_1}{(2t+\lambda_1)^4} + \frac{\lambda_2}{(2t+\lambda_2)^4} - \frac{\lambda_1+\lambda_2}{(2t+\lambda_1+\lambda_2)^4} \right) \\ & + \frac{8}{3} t \left(\frac{\lambda_1}{(2t+\lambda_1)^3} + \frac{\lambda_2}{(2t+\lambda_2)^3} - \frac{\lambda_1+\lambda_2}{(2t+\lambda_1+\lambda_2)^3} \right) \\ & + \frac{2}{3} \left(\frac{\lambda_1}{(2t+\lambda_1)^2} + \frac{\lambda_2}{(2t+\lambda_2)^2} - \frac{\lambda_1+\lambda_2}{(2t+\lambda_1+\lambda_2)^2} \right) \end{aligned} \tag{28}$$

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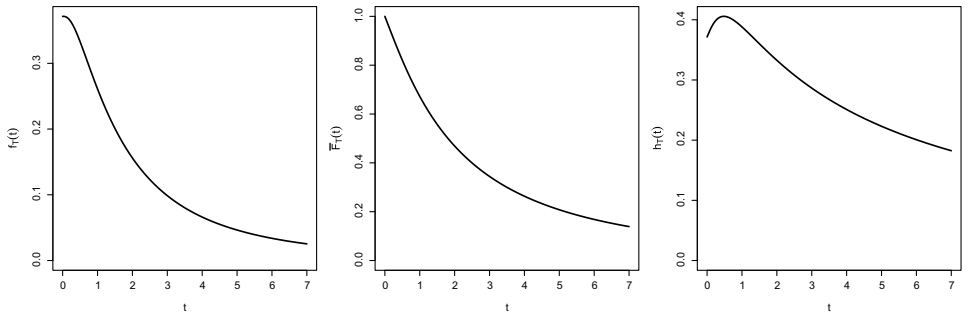


FIGURE 4. Plots of f_T (left) given in (28), $\overline{F}_T$ (middle) given in (29), and $h_T(t)$ (right) given in (30) when $\lambda_1 = 5$ and $\lambda_2 = 2$.

for $t \geq 0$, and, from (22), we get

$$\begin{aligned}\overline{F}_T(t) = {} & \frac{\lambda_1}{2t+\lambda_1} + \frac{\lambda_2}{2t+\lambda_2} - \frac{\lambda_1+\lambda_2}{2t+\lambda_1+\lambda_2} \\ & + \frac{4}{3}t^2\left(\frac{\lambda_1}{(2t+\lambda_1)^3} + \frac{\lambda_2}{(2t+\lambda_2)^3} - \frac{\lambda_1+\lambda_2}{(2t+\lambda_1+\lambda_2)^3}\right) \\ & + \frac{4}{3}t\left(\frac{\lambda_1}{(2t+\lambda_1)^2} + \frac{\lambda_2}{(2t+\lambda_2)^2} - \frac{\lambda_1+\lambda_2}{(2t+\lambda_1+\lambda_2)^2}\right)\end{aligned} \tag{29}$$

for $t \geq 0$. In Figure 4, for $\lambda_1 = 5$ and $\lambda_2 = 2$, we plot the PDF f_T given in (28) on the left-hand side, the SF $\overline{F}_T$ given in (29) in the middle, and the corresponding hazard rate function

$$\begin{aligned}h_T(t) = {} & \frac{3}{1+t} + \frac{6}{5+2t} + \frac{6}{7+2t} \\ & - \frac{4\left(30\,135 + 116\,361t + 176\,859t^2 + 136\,948t^3 + 57\,660t^4 + 12\,600t^5 + 1120t^6\right)}{25\,725 + 120\,540t + 232\,722t^2 + 235\,812t^3 + 136\,948t^4 + 46\,128t^5 + 8400t^6 + 640t^7}\end{aligned} \tag{30}$$

on the right-hand side, having used (23).

We conclude this section by providing an example in which M is a degenerate random variable, describing the case in which the sum of shocks that determines the item's failure is fixed.

Example 3. Let M be a degenerate random variable in $k = 2$, i.e. $p_M(2) = 1$ and $p_M(k) = 0$ for $k \in \mathbb{N} \setminus \{2\}$. Then, from (21), we have

$$f_T(t) = \frac{8\lambda_1 t}{(2t+\lambda_1)^3} + \frac{8\lambda_2 t}{(2t+\lambda_2)^3} - \frac{8(\lambda_1+\lambda_2)t}{(2t+\lambda_1+\lambda_2)^3}, \qquad t \geq 0, \tag{31}$$

and from (22) it holds that

$$\overline{F}_T(t) = \frac{\lambda_1(4t+\lambda_1)}{(2t+\lambda_1)^2} - \frac{4t^2}{(2t+\lambda_2)^2} + \frac{4t^2}{(2t+\lambda_1+\lambda_2)^2}, \qquad t \geq 0. \tag{32}$$

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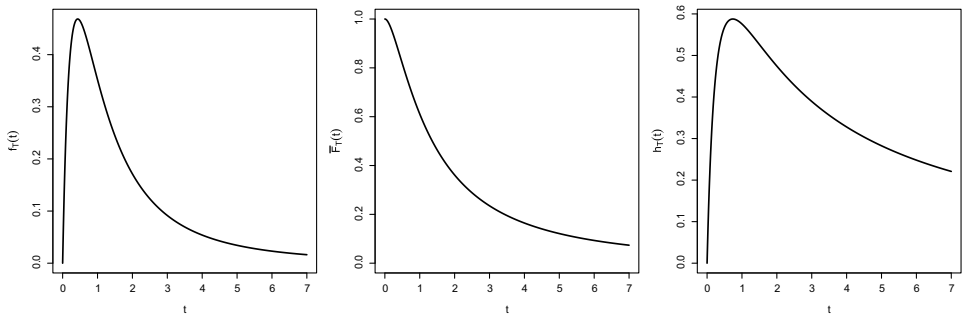


FIGURE 5. Plots of f_T (left) given in (31), $\overline{F}_T$ (middle) given in (32), and $h_T(t)$ (right) given in (33) when $\lambda_1 = \lambda_2 = 2$.

In Figure 5, for $\lambda_1 = \lambda_2 = 2$, we plot the (unimodal) PDF f_T given in (31) on the left-hand side, the SF $\overline{F}_T$ given in (32) in the middle, and the corresponding hazard rate function

$$h_T(t) = \frac{2t\,(7 + 3t(3 + t))}{(1 + t)(2 + t)(2 + 3t(2 + t))}, \qquad t \geq 0, \tag{33}$$

on the right-hand side, having used (23).

5. Concluding Remarks and Future Developments

Shock models describe how systems or living organisms fail due to random stresses over time. An item starts working and experiences random shocks, with failure that occurs after a random number of these shocks. The geometric counting process represents a relevant option in the context of counting processes in order to handle random occurrences between events that are not independent. In this article shock models driven by the bivariate geometric counting process have been studied within a competing risks framework. We assume that the mixing distribution is given by the maximum of two independent exponential distributions. In addition, the dependence between the marginal geometric counting processes comes from the hypothesis that intensities of baseline Poisson processes are shared. We have assumed that the failures are due to one of two mutually exclusive causes. In particular, failures occur at the first instant in which a random threshold M is reached by a suitable function Ψ of the two type of shock, given in (1). Under these assumptions we have obtained FDs, SFs, the probability of the cause of failure, and the moments of the failure time conditioned on a specific cause. Moreover, these functions have been laid down by considering $\Psi(x_1, x_2) = x_1 + x_2$. Particular cases have been analyzed for fixed distributions of M.

Other functions Ψ can be considered as long as they create certain partitions of the state space analogous to (1), such as (i) $\Psi(x_1, x_2) = \min\{x_1, x_2\}$ and (ii) $\Psi(x_1, x_2) = \max\{x_1, x_2\}$. In addition, aiming to deal with more general settings, we can make alternative choices of counting processes that govern the shocks as well (see [9, 28]). Note that each shock can be governed by a different type of counting process (for instance, one by the Poisson counting process and the other by the geometric counting process).

Further dependency structures related to the counting processes involved can be considered and described by copulas (see, for instance, [18]).

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Another option for future research would be to develop a comprehensive inferential framework for the proposed shock models within competing risks, including methods for parameter estimation, model validation, and statistical testing.

Finally, the proposed competing risks model can be extended to the case of an arbitrary number r of types of shock. Let $(N_1(t), \ldots, N_r(t))$ be a multidimensional counting process, where $N_i(t)$ describes the number of shocks of the ith type that occurred in $[0, t]$ for $i = 1, 2, \ldots, r$. Then the SF of T can be expressed as

$$\overline{F}_T(t) = \sum_{k=0}^{+\infty} p_M(k)\, \mathbb{P}\{\overline{\Psi}(N_1(t), \ldots, N_r(t)) < k\}, \qquad t \geq 0,$$

where $\overline{\Psi}$ extends the function Ψ defined in (1) to the case of r components.

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Competing interests

There are no competing interests to declare that arose during the preparation of or the publication process for this article.

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